Chapter 4

Solving Linear Systems

4.1 Solving Linear Systems by Graphing

LEARNING OBJECTIVES

1. Check solutions to systems of linear equations.
2. Solve linear systems using the graphing method.
3. Identify dependent and inconsistent systems.

Definition of a Linear System

Real-world applications are often modeled using more than one variable and more than one equation. A **system of equations**[1] consists of a set of two or more equations with the same variables. In this section, we will study **linear systems**[2] consisting of two linear equations each with two variables. For example,

$$\begin{cases} 2x - 3y = 0 \\ -4x + 2y = -8 \end{cases}$$

A **solution to a linear system**[3], or **simultaneous solution**[4], to a linear system is an ordered pair (x, y) that solves both of the equations. In this case, $(3, 2)$ is the only solution. To check that an ordered pair is a solution, substitute the corresponding x- and y-values into each equation and then simplify to see if you obtain a true statement for both equations.

Check: $(3, 2)$

Equation 1: $2x - 3y = 0$	*Equation 2*: $-4x + 2y = -8$
$2(3) - 3(2) = 0$	$-4(3) + 2(2) = -8$
$6 - 6 = 0$	$-12 + 4 = -8$
$0 = 0$ ✓	$-8 = -8$ ✓

1. A set of two or more equations with the same variables.
2. In this section, we restrict our study to systems of two linear equations with two variables.
3. An ordered pair that satisfies both equations and corresponds to a point of intersection.
4. Used when referring to a solution of a system of equations.

Example 1: Determine whether (1, 0) is a solution to the system $\begin{cases} x - y = 1 \\ -2x + 3y = 5 \end{cases}$.

Solution: Substitute the appropriate values into both equations.

Check: (1, 0)

Equation 1: $x - y = 1$	*Equation 2*: $-2x + 3y = 5$
$(1) - (0) = 1$	$-2(1) + 3(0) = 5$
$1 - 0 = 1$	$-2 + 0 = 5$
$1 = 1$ ✓	$-2 = 5$ ✗

Answer: Since (1, 0) does not satisfy *both* equations, it is not a solution.

Try this! Is (−2, 4) a solution to the system $\begin{cases} x - y = -6 \\ -2x + 3y = 16 \end{cases}$?

Answer: Yes

Video Solution

(click to see video)

Solve by Graphing

Geometrically, a linear system consists of two lines, where a solution is a point of intersection. To illustrate this, we will graph the following linear system with a solution of (3, 2):

$$\begin{cases} 2x - 3y = 0 \\ -4x + 2y = -8 \end{cases}$$

First, rewrite the equations in slope-intercept form so that we may easily graph them.

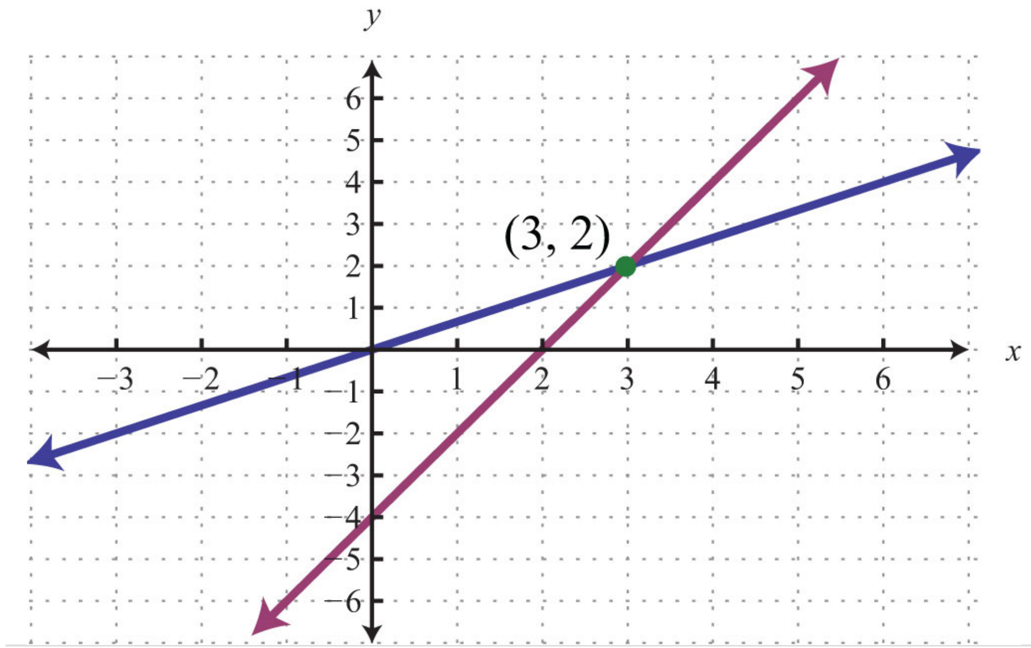
$2x-3y=0$	$-4x+2y=-8$
$2x-3y-2x=0-2x$	$-4x+2y+4x=-8+4x$
$-3y=-2x$	$2y=4x-8$
$\frac{-3y}{-3}=\frac{-2x}{-3}$	$\frac{2y}{2}=\frac{4x-8}{2}$
$y=\frac{2}{3}x$	$y=2x-4$

Next, replace these forms of the original equations in the system to obtain what is called an **equivalent system**[5]. Equivalent systems share the same solution set.

$$\begin{array}{ccc} \textit{Original system} & & \textit{Equivalent system} \\ \begin{cases} 2x-3y=0 \\ -4x+2y=-8 \end{cases} & \Rightarrow & \begin{cases} y=\frac{2}{3}x \\ y=2x-4 \end{cases} \end{array}$$

If we graph both of the lines on the same set of axes, then we can see that the point of intersection is indeed (3, 2), the solution to the system.

5. A system consisting of equivalent equations that share the same solution set.



To summarize, linear systems described in this section consist of two linear equations each with two variables. A solution is an ordered pair that corresponds to a point where the two lines in the rectangular coordinate plane intersect. Therefore, we can solve linear systems by graphing both lines on the same set of axes and determining the point where they cross. When graphing the lines, take care to choose a good scale and use a straightedge to draw the line through the points; accuracy is very important here. The steps for solving linear systems using the **graphing method**[6] are outlined in the following example.

Example 2: Solve by graphing: $\left\{\begin{array}{l} x - y = -4 \\ 2x + y = 1 \end{array}\right.$.

Solution:

Step 1: Rewrite the linear equations in slope-intercept form.

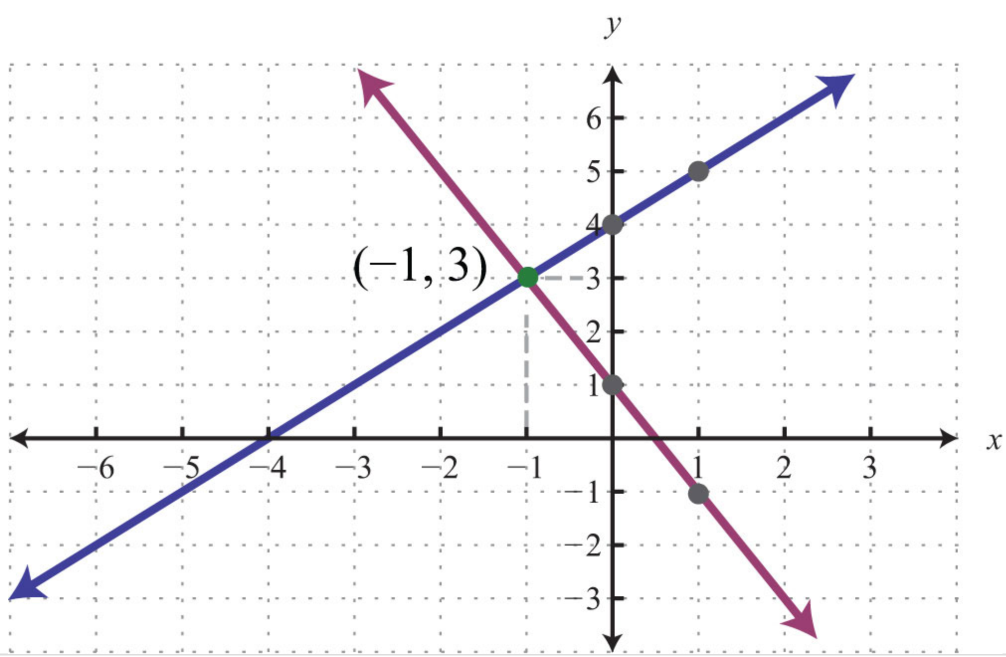
6. A means of solving a system by graphing the equations on the same set of axes and determining where they intersect.

$x-y=-4$	$2x+y=1$
$x-y-x=-4-x$	$2x+y-2x=1-2x$
$-y=-x-4$	$y=-2x+1$
$\frac{-y}{-1}=\frac{-x-4}{-1}$	
$y=x+4$	

Step 2: Write the equivalent system and graph the lines on the same set of axes.

$$\begin{cases} x-y=-4 \\ 2x+y=1 \end{cases} \Rightarrow \begin{cases} y=x+4 \\ y=-2x+1 \end{cases}$$

Line 1: $y=x+4$	*Line 2:* $y=-2x+1$
y-intercept: $(0,4)$	*y-intercept:* $(0,1)$
slope: $m=1=\frac{1}{1}=\frac{rise}{run}$	*slope:* $m=-2=\frac{-2}{1}=\frac{rise}{run}$



Step 3: Use the graph to estimate the point where the lines intersect and check to see if it solves the original system. In the above graph, the point of intersection appears to be (−1, 3).

Check: (−1, 3)

Line 1: $x - y = -4$	*Line 2*: $2x + y = 1$
$(-1) - (3) = -4$	$2(-1) + (3) = 1$
$-1 - 3 = -4$	$-2 + 3 = 1$
$-4 = -4$ ✓	$1 = 1$ ✓

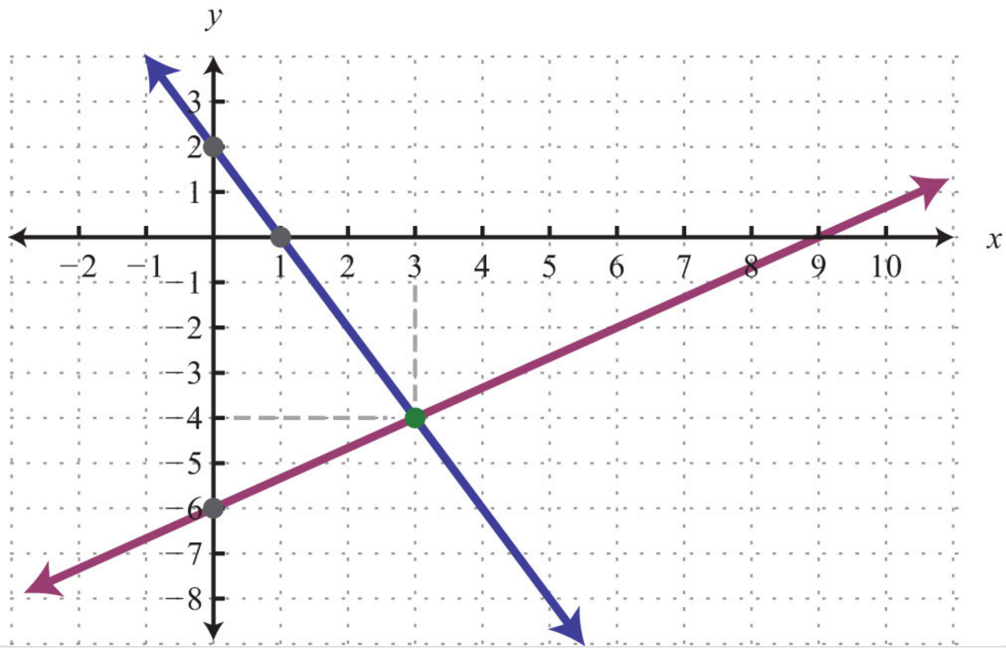
Answer: (−1, 3)

Example 3: Solve by graphing: $\begin{cases} 2x + y = 2 \\ -2x + 3y = -18 \end{cases}$.

Solution: We first solve each equation for *y* to obtain an equivalent system where the lines are in slope-intercept form.

$$\begin{cases} 2x + y = 2 \\ -2x + 3y = -18 \end{cases} \Rightarrow \begin{cases} y = -2x + 2 \\ y = \frac{2}{3}x - 6 \end{cases}$$

Graph the lines and determine the point of intersection.



Check: (3, –4)

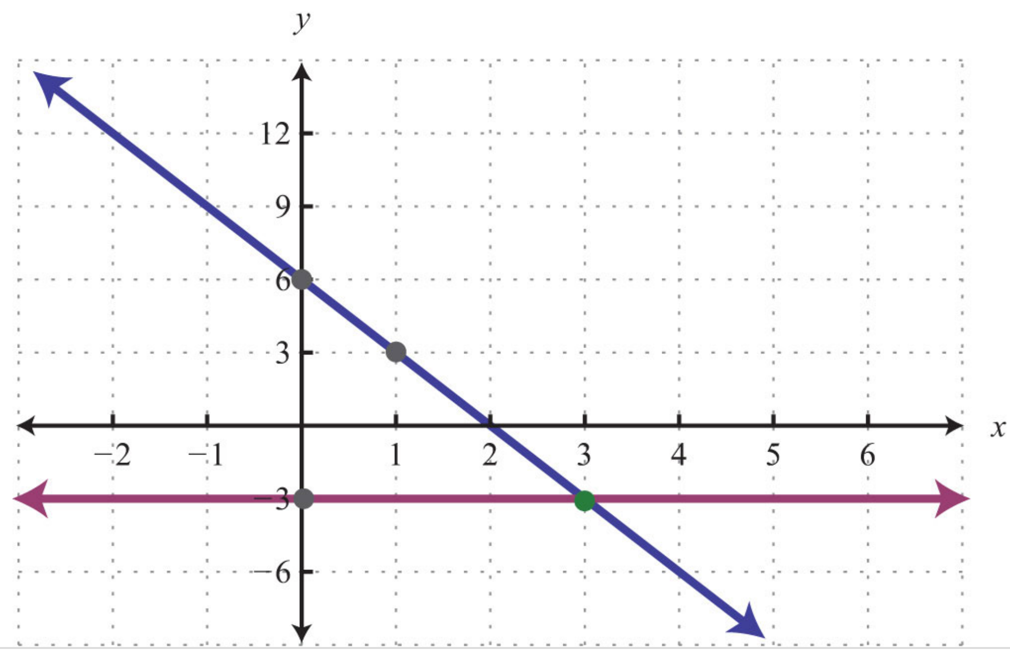
$2x + y = 2$	$-2x + 3y = -18$
$2(3) + (-4) = 2$	$-2(3) + 3(-4) = -18$
$6 - 4 = 2$	$-6 - 12 = -18$
$2 = 2$ ✓	$-18 = -18$ ✓

Answer: (3, -4)

Example 4: Solve by graphing: $\begin{cases} 3x + y = 6 \\ y = -3 \end{cases}$.

Solution:

$$\begin{cases} 3x + y = 6 \\ y = -3 \end{cases} \Rightarrow \begin{cases} y = -3x + 6 \\ y = -3 \end{cases}$$



Check: (3, –3)

$3x + y = 6$	$y = -3$
$3(3) + (-3) = 6$	$(-3) = -3$
$9 - 3 = 6$	$-3 = -3$ ✓
$6 = 6$ ✓	

Answer: (3, -3)

The graphing method for solving linear systems is not ideal when the solution consists of coordinates that are not integers. There will be more accurate algebraic methods in sections to come, but for now, the goal is to understand the geometry involved when solving systems. It is important to remember that the solutions to a system correspond to the point, or points, where the graphs of the equations intersect.

Try this! Solve by graphing: $\begin{cases} -x + y = 6 \\ 5x + 2y = -2 \end{cases}$.

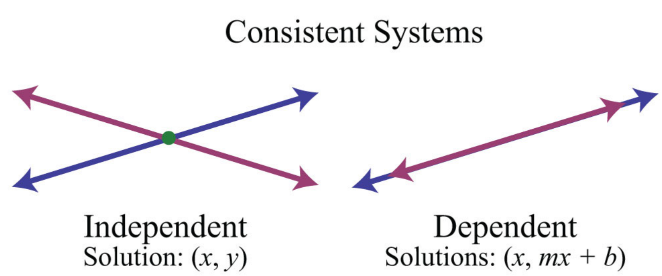
Answer: (−2, 4)

Video Solution

(click to see video)

Dependent and Inconsistent Systems

Systems with at least one solution are called **consistent systems**[7]. Up to this point, all of the examples have been of consistent systems with exactly one ordered pair solution. It turns out that this is not always the case. Sometimes systems consist of two linear equations that are equivalent. If this is the case, the two lines are the same and when graphed will coincide. Hence the solution set consists of all the points on the line. This is a **dependent system**[8]. Given a consistent linear system with two variables, there are two possible results:



The solutions to **independent systems**[9] are ordered pairs (x, y). We need some way to express the solution sets to dependent systems, since these systems have infinitely many solutions, or points of intersection. Recall that any line can be written in slope-intercept form, $y = mx + b$. Here, y depends on x. So we may express all the ordered pair solutions $(x, \ y)$ in the form $(x, \ mx + b)$, where x is any real number.

7. A system with at least one solution.

8. A system that consists of equivalent equations with infinitely many ordered pair solutions, denoted by $(x, mx + b)$.

9. A system of equations with one ordered pair solution (x, y).

Example 5: Solve by graphing: $\begin{cases} -2x + 3y = -9 \\ 4x - 6y = 18 \end{cases}$.

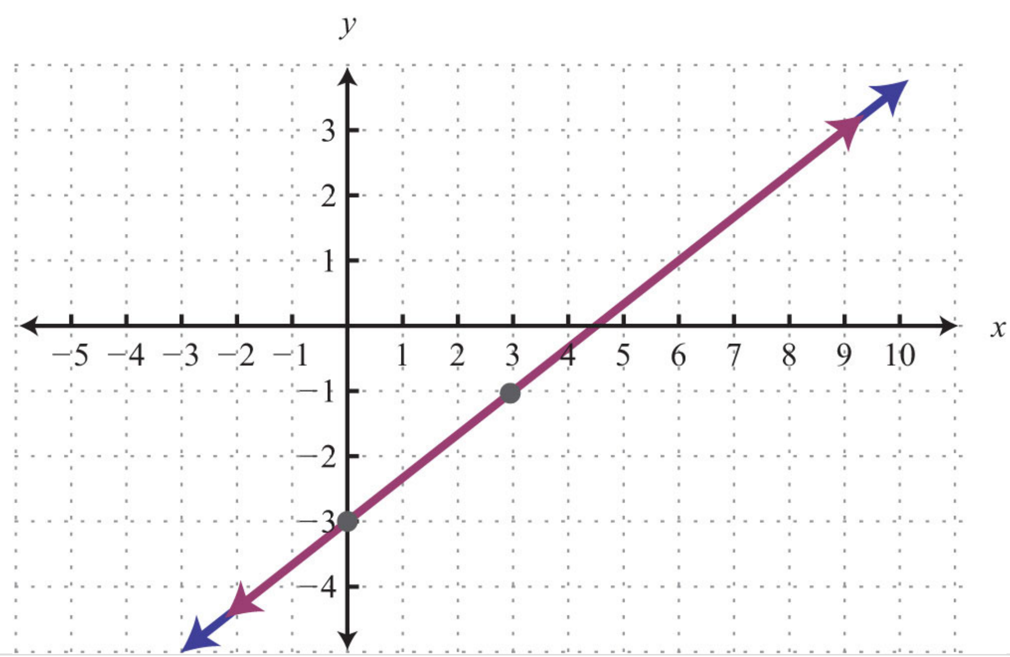
Solution: Determine slope-intercept form for each linear equation in the system.

$$\begin{aligned} -2x+3y&=-9 \\ -2x+3y+2x&=-9+2x \\ 3y&=2x-9 \\ \frac{3y}{3}&=\frac{2x-9}{3} \\ y&=\frac{2}{3}x-3 \end{aligned}$$

$$\begin{aligned} 4x-6y&=18 \\ 4x-6y-4x&=18-4x \\ -6y&=-4x+18 \\ \frac{-6y}{-6}&=\frac{-4x+18}{-6} \\ y&=\frac{-4}{-6}x+\frac{18}{-6} \\ y&=\frac{2}{3}x-3 \end{aligned}$$

$$\begin{cases} -2x+3y=-9 \\ 4x-6y=18 \end{cases} \Rightarrow \begin{cases} y=\frac{2}{3}x-3 \\ y=\frac{2}{3}x-3 \end{cases}$$

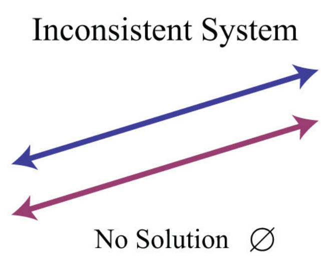
In slope-intercept form, we can easily see that the system consists of two lines with the same slope and same *y*-intercept. They are, in fact, the same line. And the system is dependent.



Answer: $\left(x, \ \frac{2}{3}x - 3\right)$

In this example, it is important to notice that the two lines have the same slope and same y-intercept. This tells us that the two equations are equivalent and that the simultaneous solutions are all the points on the line $y = \frac{2}{3}x - 3$. This is a dependent system, and the infinitely many solutions are expressed using the form $(x, \ mx + b)$. Other resources may express this set using set notation, $\{(x, y) \mid y = \frac{2}{3}x - 3\}$, which reads "the set of all ordered pairs (x, y) such that y equals two-thirds x minus 3."

Sometimes the lines do not cross and there is no point of intersection. Such systems have no solution, Ø, and are called **inconsistent systems**[10].



Example 6: Solve by graphing: $\begin{cases} -2x + 5y = -15 \\ -4x + 10y = 10 \end{cases}$.

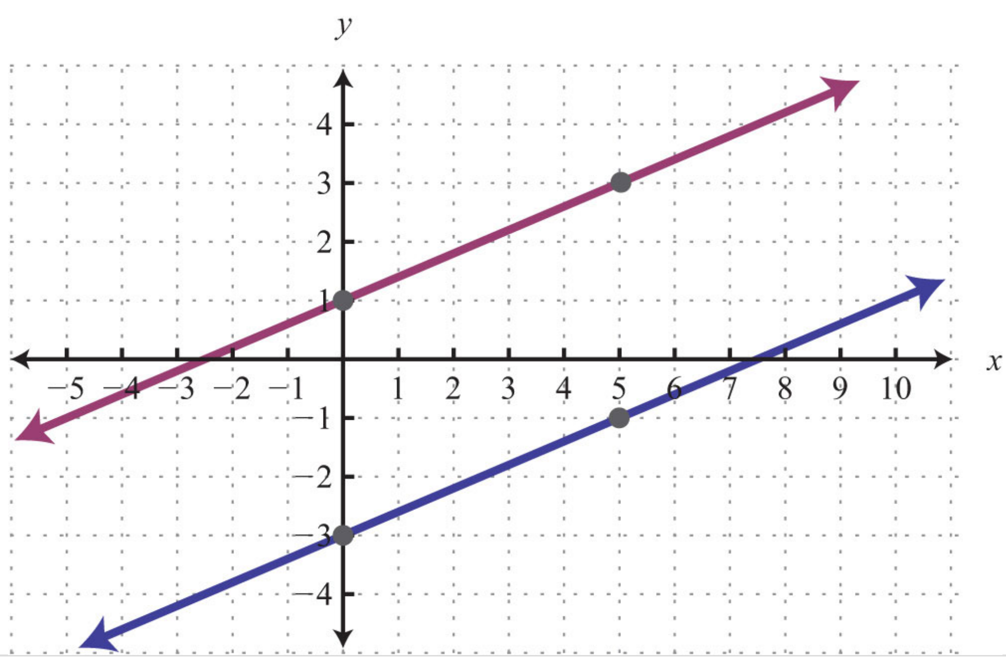
Solution: Determine slope-intercept form for each linear equation.

10. A system with no simultaneous solution.

$$\begin{aligned} -2x+5y &= -15 \\ -2x+5y+2x &= -15+2x \\ 5y &= 2x-15 \\ \frac{5y}{5} &= \frac{2x-15}{5} \\ y &= \frac{2}{5}x-3 \end{aligned} \qquad \begin{aligned} -4x+10y &= 10 \\ -4x+10y+4x &= 10+4x \\ 10y &= 4x+10 \\ \frac{10y}{10} &= \frac{4x+10}{10} \\ y &= \frac{2}{5}x+1 \end{aligned}$$

$$\begin{cases} -2x+5y=-15 \\ -4x+10y=10 \end{cases} \Rightarrow \begin{cases} y=\frac{2}{5}x-3 \\ y=\frac{2}{5}x+1 \end{cases}$$

In slope-intercept form, we can easily see that the system consists of two lines with the same slope and different *y*-intercepts. Therefore, they are parallel and will never intersect.



Answer: There is no simultaneous solution, Ø.

Try this! Solve by graphing: $\begin{cases} x + y = -1 \\ -2x - 2y = 2 \end{cases}$.

Answer: $(x, \ -x - 1)$

Video Solution

(click to see video)

KEY TAKEAWAYS

- In this section, we limit our study to systems of two linear equations with two variables. Solutions to such systems, if they exist, consist of ordered pairs that satisfy both equations. Geometrically, solutions are the points where the graphs intersect.
- The graphing method for solving linear systems requires us to graph both of the lines on the same set of axes as a means to determine where they intersect.
- The graphing method is not the most accurate method for determining solutions, particularly when the solutions have coordinates that are not integers. It is a good practice to always check your solutions.
- Some linear systems have no simultaneous solution. These systems consist of equations that represent parallel lines with different *y*-intercepts and do not intersect in the plane. They are called inconsistent systems and the solution set is the empty set, Ø.
- Some linear systems have infinitely many simultaneous solutions. These systems consist of equations that are equivalent and represent the same line. They are called dependent systems and their solutions are expressed using the notation $(x, \ mx + b)$, where x is any real number.

TOPIC EXERCISES

Part A: Solutions to Linear Systems

Determine whether the given ordered pair is a solution to the given system.

1. (3, −2); $\begin{cases} x + y = -1 \\ -2x - 2y = 2 \end{cases}$

2. (−5, 0); $\begin{cases} x + y = -1 \\ -2x - 2y = 2 \end{cases}$

3. (−2, −6); $\begin{cases} -x + y = -4 \\ 3x - y = -12 \end{cases}$

4. (2, −7); $\begin{cases} 3x + 2y = -8 \\ -5x - 3y = 11 \end{cases}$

5. (0, −3); $\begin{cases} 5x - 5y = 15 \\ -13x + 2y = -6 \end{cases}$

6. $\left(-\frac{1}{2}, \frac{1}{4}\right)$; $\begin{cases} x + y = -\frac{1}{4} \\ -2x - 4y = 0 \end{cases}$

7. $\left(\frac{3}{4}, \frac{1}{4}\right)$; $\begin{cases} -x - y = -1 \\ -4x - 8y = 5 \end{cases}$

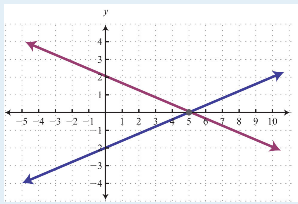
8. (−3, 4); $\begin{cases} \frac{1}{3}x + \frac{1}{2}y = 1 \\ \frac{2}{3}x - \frac{3}{2}y = -8 \end{cases}$

9. (−5, −3); $\begin{cases} y = -3 \\ 5x - 10y = 5 \end{cases}$

10. (4, 2); $\begin{cases} x = 4 \\ -7x + 4y = 8 \end{cases}$

Given the graph, determine the simultaneous solution.

11.



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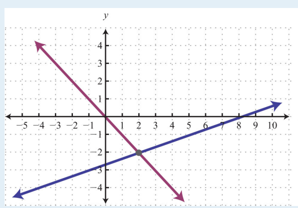

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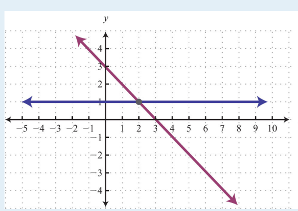

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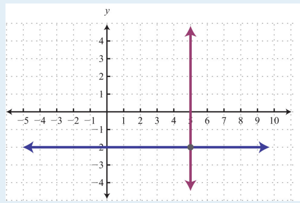

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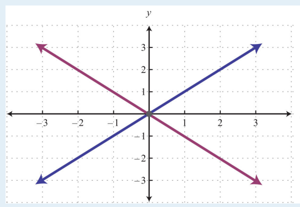

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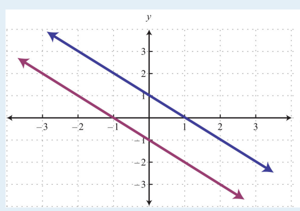

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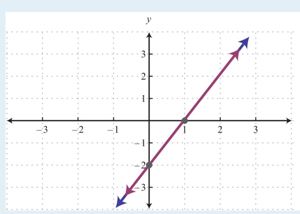

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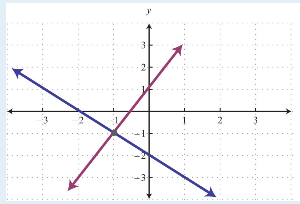

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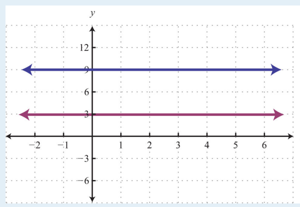

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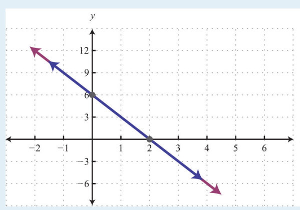


Part B: Solving Linear Systems

Solve by graphing.

21. $\begin{cases} y = \frac{3}{2}x + 6 \\ y = -x + 1 \end{cases}$

22. $\begin{cases} y = \frac{3}{4}x + 2 \\ y = -\frac{1}{4}x - 2 \end{cases}$

23. $\begin{cases} y = x - 4 \\ y = -x + 2 \end{cases}$

24. $\begin{cases} y = -5x + 4 \\ y = 4x - 5 \end{cases}$

25. $\begin{cases} y = \frac{2}{5}x + 1 \\ y = \frac{3}{5}x \end{cases}$

26. $\begin{cases} y = -\frac{2}{5}x + 6 \\ y = \frac{2}{5}x + 10 \end{cases}$

27. $\begin{cases} y = -2 \\ y = x + 1 \end{cases}$

28. $\begin{cases} y = 3 \\ x = -3 \end{cases}$

29. $\begin{cases} y = 0 \\ y = \frac{2}{5}x - 4 \end{cases}$

30. $\begin{cases} x = 2 \\ y = 3x \end{cases}$

31. $\begin{cases} y = \frac{3}{5}x - 6 \\ y = \frac{3}{5}x - 3 \end{cases}$

32. $\begin{cases} y = -\frac{1}{2}x + 1 \\ y = -\frac{1}{2}x + 1 \end{cases}$

33. $\begin{cases} 2x + 3y = 18 \\ -6x + 3y = -6 \end{cases}$

34. $\begin{cases} -3x + 4y = 20 \\ 2x + 8y = 8 \end{cases}$

35. $\begin{cases} -2x + y = 1 \\ 2x - 3y = 9 \end{cases}$

36. $\begin{cases} x + 2y = -8 \\ 5x + 4y = -4 \end{cases}$

37. $\begin{cases} 4x + 6y = 36 \\ 2x - 3y = 6 \end{cases}$

38. $\begin{cases} 2x - 3y = 18 \\ 6x - 3y = -6 \end{cases}$

39. $\begin{cases} 3x + 5y = 30 \\ -6x - 10y = -10 \end{cases}$

40. $\begin{cases} -x + 3y = 3 \\ 5x - 15y = -15 \end{cases}$

41. $\begin{cases} x - y = 0 \\ -x + y = 0 \end{cases}$

42. $\begin{cases} y = x \\ y - x = 1 \end{cases}$

43. $\begin{cases} 3x + 2y = 0 \\ x = 2 \end{cases}$

44. $\begin{cases} 2x + \frac{1}{3}y = \frac{2}{3} \\ -3x + \frac{1}{2}y = -2 \end{cases}$

45. $\begin{cases} \frac{1}{10}x + \frac{1}{5}y = 2 \\ -\frac{1}{5}x + \frac{1}{5}y = -1 \end{cases}$

46. $\begin{cases} \frac{1}{3}x - \frac{1}{2}y = 1 \\ \frac{1}{3}x + \frac{1}{5}y = 1 \end{cases}$

47. $\begin{cases} \frac{1}{9}x + \frac{1}{6}y = 0 \\ \frac{1}{9}x + \frac{1}{4}y = \frac{1}{2} \end{cases}$

48. $\begin{cases} \frac{5}{16}x - \frac{1}{2}y = 5 \\ -\frac{5}{16}x + \frac{1}{2}y = \frac{5}{2} \end{cases}$

49. $\begin{cases} \frac{1}{6}x - \frac{1}{2}y = \frac{9}{2} \\ -\frac{1}{18}x + \frac{1}{6}y = -\frac{3}{2} \end{cases}$

50. $\begin{cases} \frac{1}{2}x - \frac{1}{4}y = -\frac{1}{2} \\ \frac{1}{3}x - \frac{1}{2}y = 3 \end{cases}$

51. $\begin{cases} y = 4 \\ x = -5 \end{cases}$

52. $\begin{cases} y = -3 \\ x = 2 \end{cases}$

53. $\begin{cases} y = 0 \\ x = 0 \end{cases}$

54. $\begin{cases} y = -2 \\ y = 3 \end{cases}$

55. $\begin{cases} y = 5 \\ y = -5 \end{cases}$

56. $\begin{cases} y = 2 \\ y - 2 = 0 \end{cases}$

57. $\begin{cases} x = -5 \\ x = 1 \end{cases}$

58. $\begin{cases} y = x \\ x = 0 \end{cases}$

59. $\begin{cases} 4x + 6y = 3 \\ -x + y = -2 \end{cases}$

60. $\begin{cases} -2x + 20y = 20 \\ 3x + 10y = -10 \end{cases}$

Set up a linear system of two equations and two variables and solve it using the graphing method.

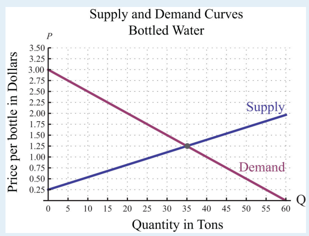
61. The sum of two numbers is 20. The larger number is 10 less than five times the smaller.

62. The difference between two numbers is 12 and their sum is 4.

63. Where on the graph of $3x - 2y = 6$ does the *x*-coordinate equal the *y*-coordinate?

64. Where on the graph of $-5x + 2y = 30$ does the *x*-coordinate equal the *y*-coordinate?

A regional bottled water company produces and sells bottled water. The following graph depicts the supply and demand curves of bottled water in the region. The horizontal axis represents the weekly tonnage of product produced, Q. The vertical axis represents the price per bottle in dollars, P.



Use the graph to answer the following questions.

65. Determine the price at which the quantity demanded is equal to the quantity supplied.

66. If production of bottled water slips to 20 tons, then what price does the demand curve predict for a bottle of water?

67. If production of bottled water increases to 40 tons, then what price does the demand curve predict for a bottle of water?

68. If the price of bottled water is set at $2.50 dollars per bottle, what quantity does the demand curve predict?

Part C: Discussion Board Topics

69. Discuss the weaknesses of the graphing method for solving systems.

70. Explain why the solution set to a dependent linear system is denoted by $(x, mx + b)$.

ANSWERS

1: No

3: No

5: Yes

7: No

9: Yes

11: (5, 0)

13: (2, 1)

15: (0, 0)

17: $(x,\ 2x-2)$

19: $\varnothing$

21: (−2, 3)

23: (3, −1)

25: (5, 3)

27: (−3, −2)

29: (10, 0)

31: $\varnothing$

33: (3, 4)

35: (−3, −5)

37: (6, 2)

39: $\varnothing$

41: $(x,\ x)$

43: (2, -3)

45: (10, 5)

47: (−9, 6)

49: $(x,\ \frac{1}{3}x - 9)$

51: (−5, 4)

53: (0, 0)

55: $\varnothing$

57: $\varnothing$

59: (3/2, -1/2)

61: The two numbers are 5 and 15.

63: (6, 6)

65: $1.25

67: $1.00

4.2 Solving Linear Systems by Substitution

LEARNING OBJECTIVE

1. Solve linear systems using the substitution method.

The Substitution Method

In this section, we will define a completely algebraic technique for solving systems. The idea is to solve one equation for one of the variables and substitute the result into the other equation. After performing this substitution step, we will be left with a single equation with one variable, which can be solved using algebra. This is called the **substitution method**[11], and the steps are outlined in the following example.

Example 1: Solve by substitution: $\begin{cases} 2x + y = 7 \\ 3x - 2y = -7 \end{cases}$.

Solution:

Step 1: Solve for either variable in either equation. If you choose the first equation, you can isolate y in one step.

$$\begin{aligned} 2x + y &= 7 \\ 2x + y - 2x &= 7 - 2x \\ y &= -2x + 7 \end{aligned}$$

Step 2: Substitute the expression $-2x + 7$ for the y variable in the *other* equation.

11. A means of solving a linear system by solving for one of the variables and substituting the result into the other equation.

$$\begin{cases} 2x + y = 7 \\ 3x - 2y = -7 \end{cases} \quad \Rightarrow \quad y = -2x + 7$$

$$3x - 2(-2x + 7) = -7$$

This leaves you with an equivalent equation with one variable, which can be solved using the techniques learned up to this point.

Step 3: Solve for the remaining variable. To solve for *x*, first distribute −2:

$$\begin{aligned} 3x - 2(-2x + 7) &= -7 \\ 3x + 4x - 14 &= -7 \\ 7x - 14 &= -7 \\ 7x - 14 + 14 &= -7 + 14 \\ 7x &= 7 \\ \frac{7x}{7} &= \frac{7}{7} \\ x &= 1 \end{aligned}$$

Step 4: Back substitute[12] to find the value of the other coordinate. Substitute $x = 1$ into either of the original equations or their equivalents. Typically, we use the equivalent equation that we found when isolating a variable in step 1.

$$\begin{aligned} y &= -2x + 7 \\ &= -2(1) + 7 \\ &= -2 + 7 \\ &= 5 \end{aligned}$$

The solution to the system is (1, 5). Be sure to present the solution as an ordered pair.

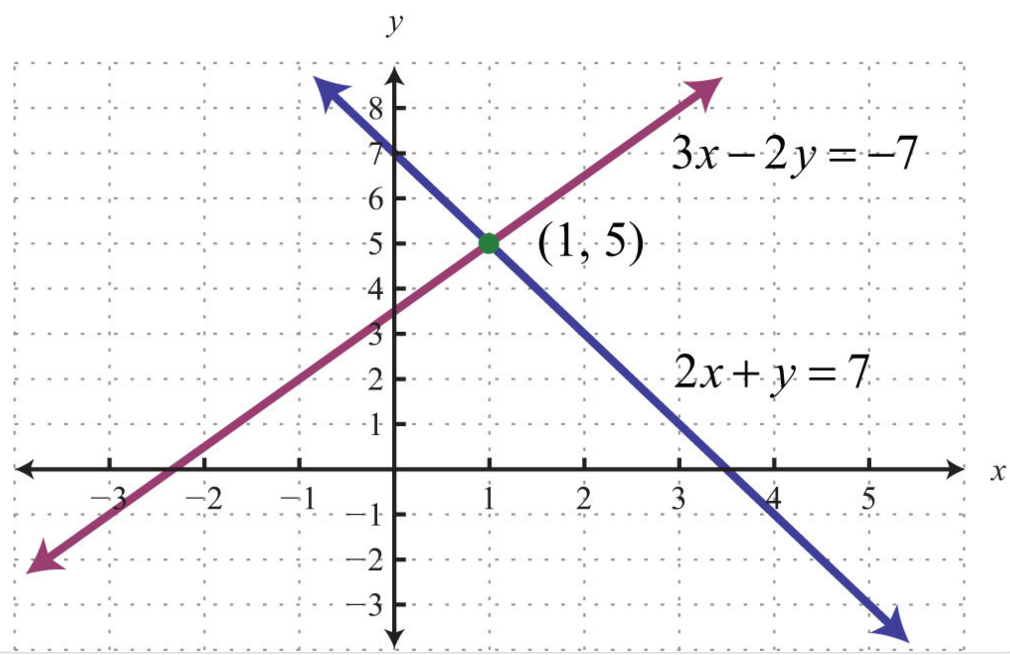
12. Once a value is found for a variable, substitute it back into one of the original equations, or their equivalent equations, to determine the corresponding value of the other variable.

Step 5: Check. Verify that these coordinates solve both equations of the original system:

Check: (1, 5)

Equation 1:	*Equation 2*:
$2x + y = 7$	$3x - 2y = -7$
$2(1) + (5) = 7$	$3(1) - 2(5) = -7$
$2 + 5 = 7$	$3 - 10 = -7$
$7 = 7$ ✓	$-7 = -7$ ✓

The graph of this linear system follows:



The substitution method for solving systems is a completely algebraic method. Thus graphing the lines is not required.

Answer: (1, 5)

Example 2: Solve by substitution: $\begin{cases} 2x - y = 12 \\ x - y = 3 \end{cases}$.

Solution: In this example, we can see that x has a coefficient of 1 in the second equation. This indicates that it can be isolated in one step as follows:

$$\begin{aligned} x - y &= 3 \\ x - y + y &= 3 + y \\ x &= 3 + y \end{aligned}$$

$$\begin{cases} 2x - y = 12 \\ x - y = 3 \end{cases} \quad \Rightarrow \quad x = 3 + y$$

Substitute $3 + y$ for x in the first equation. Use parentheses and take care to distribute.

$$\begin{aligned} 2x - y &= 12 \\ 2(3 + y) - y &= 12 \\ 6 + 2y - y &= 12 \\ 6 + y &= 12 \\ 6 + y - 6 &= 12 - 6 \\ y &= 6 \end{aligned}$$

Use $x = 3 + y$ to find x.

$$\begin{aligned} x &= 3 + y \\ &= 3 + 6 \\ &= 9 \end{aligned}$$

Answer: $(9, 6)$. The check is left to the reader.

Example 3: Solve by substitution: $\begin{cases} 3x - 5y = 17 \\ x = -1 \end{cases}$.

Solution: In this example, the variable *x* is already isolated. Hence we can substitute $x = -1$ into the first equation.

$$\begin{aligned} 3x - 5y &= 17 \\ 3(-1) - 5y &= 17 \\ -3 - 5y + 3 &= 17 + 3 \\ -5y &= 20 \\ \frac{-5y}{-5} &= \frac{20}{-5} \\ y &= -4 \end{aligned}$$

Answer: (−1, −4). It is a good exercise to graph this particular system to compare the substitution method to the graphing method for solving systems.

Try this! Solve by substitution: $\begin{cases} 3x + y = 4 \\ 8x + 2y = 10 \end{cases}$.

Answer: (1, 1)

Video Solution

(click to see video)

Solving systems algebraically frequently requires work with fractions.

Example 4: Solve by substitution: $\begin{cases} 2x + 8y = 5 \\ 24x - 4y = -15 \end{cases}$.

Solution: Begin by solving for x in the first equation.

$$\begin{aligned} 2x+8y&=5 \\ 2x+8y-8y&=5-8y \\ \frac{2x}{2}&=\frac{-8y+5}{2} \\ x&=\frac{-8y}{2}+\frac{5}{2} \\ x&=-4y+\frac{5}{2} \end{aligned}$$

$$\begin{cases} 2x+8y=5 \quad \Rightarrow \quad x=-4y+\dfrac{5}{2} \\ 24x-4y=-15 \end{cases}$$

Next, substitute into the second equation and solve for y.

$$\begin{aligned} 24x-4y&=-15 \\ 24\left(-4y+\frac{5}{2}\right)-4y&=-15 \\ -96y+60-4y&=-15 \\ -100y+60-60&=-15-60 \\ \frac{-100y}{-100}&=\frac{-75}{-100} \\ y&=\frac{3}{4} \end{aligned}$$

Back substitute into the equation used in the substitution step:

$$\begin{aligned} x &= -4y + \frac{5}{2} \\ &= -4\left(\frac{3}{4}\right) + \frac{5}{2} \\ &= -3 + \frac{5}{2} \\ &= -\frac{6}{2} + \frac{5}{2} \\ &= -\frac{1}{2} \end{aligned}$$

Answer: (−1/2, 3/4)

As we know, not all linear systems have only one ordered pair solution. Recall that some systems have infinitely many ordered pair solutions and some do not have any solutions. Next, we explore what happens when using the substitution method to solve a dependent system.

Example 5: Solve by substitution: $\begin{cases} -5x + y = -1 \\ 10x - 2y = 2 \end{cases}$.

Solution: Since the first equation has a term with coefficient 1, we choose to solve for that first.

$$\begin{aligned} -5x + y &= -1 \\ -5x + y + 5x &= -1 + 5x \\ y &= 5x - 1 \end{aligned}$$

$$\begin{cases} -5x + y = -1 & \Rightarrow \quad y = 5x - 1 \\ 10x - 2y = 2 \end{cases}$$

Next, substitute this expression in for y in the second equation.

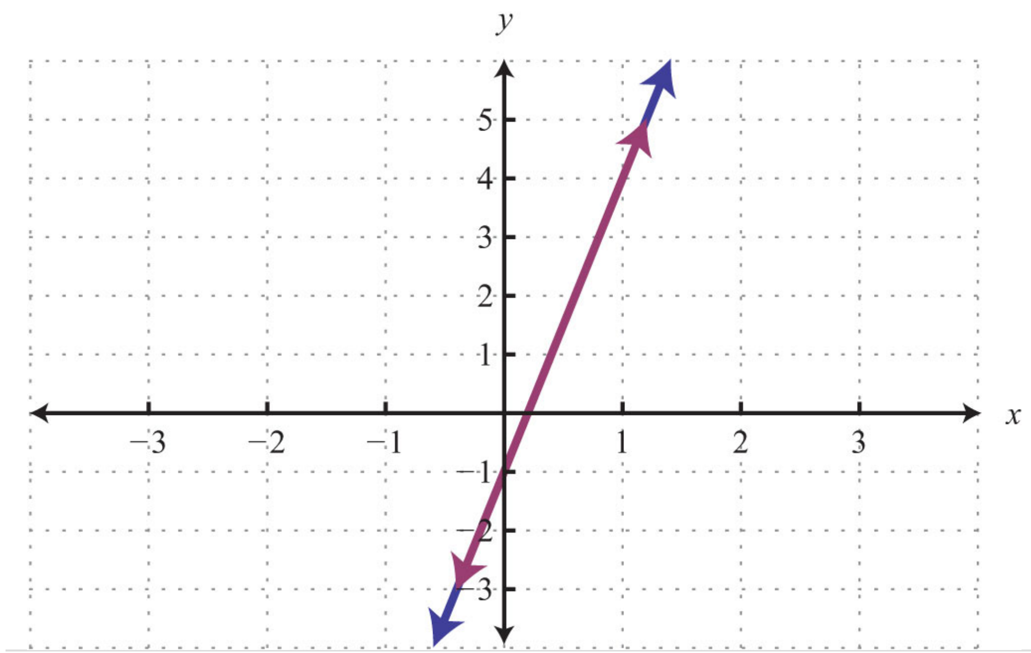
$$\begin{aligned} 10x - 2y &= 2 \\ 10x - 2\,(5x - 1) &= 2 \\ 10x - 10x + 2 &= 2 \\ 2 &= 2 \quad \textit{True} \end{aligned}$$

This process led to a true statement; hence the equation is an identity and any real number is a solution. This indicates that the system is dependent. The simultaneous solutions take the form (x, $mx + b$), or in this case, (x, $5x - 1$), where x is any real number.

Answer: $(x,\ 5x - 1)$

To have a better understanding of the previous example, rewrite both equations in slope-intercept form and graph them on the same set of axes.

$$\begin{cases} -5x + y = -1 \\ 10x - 2y = 2 \end{cases} \Rightarrow \begin{cases} y = 5x - 1 \\ y = 5x - 1 \end{cases}$$



We can see that both equations represent the same line, and thus the system is dependent. Now explore what happens when solving an inconsistent system using the substitution method.

Example 6: Solve by substitution: $\begin{cases} -7x + 3y = 3 \\ 14x - 6y = -16 \end{cases}$.

Solution: Solve for y in the first equation.

$$
\begin{aligned}
-7x+3y&=3\\
-7x+3y+7x&=3+7x\\
3y&=7x+3\\
\frac{3y}{3}&=\frac{7x+3}{3}\\
y&=\frac{7}{3}x+1
\end{aligned}
$$

$$
\begin{cases}-7x+3y=3 & \Rightarrow \quad y=\frac{7}{3}x+1\\ 14x-6y=-16\end{cases}
$$

Substitute into the second equation and solve.

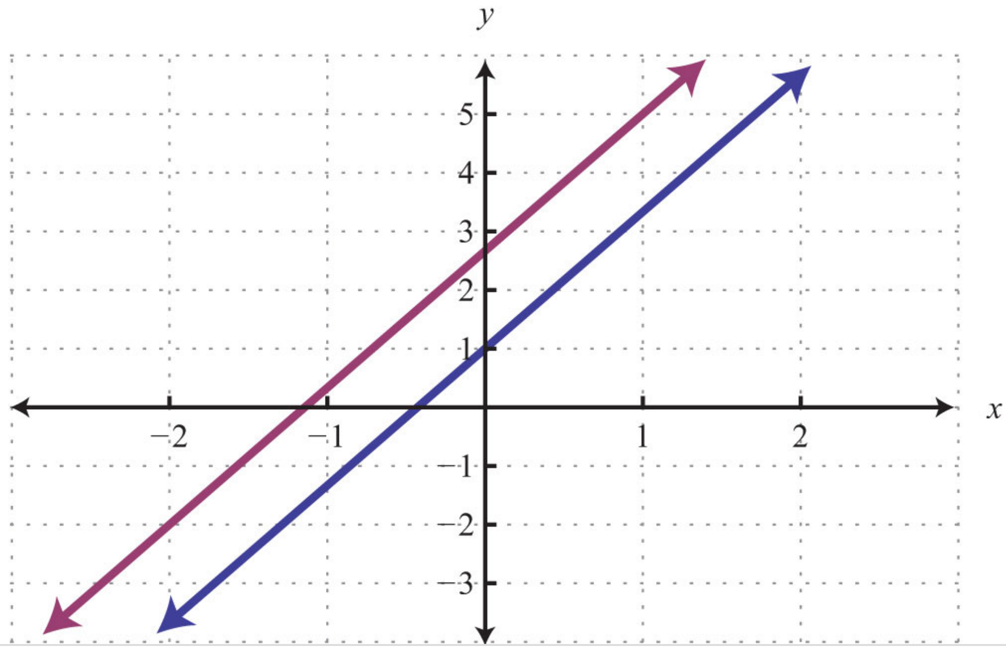
$$
\begin{aligned}
14x-6y&=-16\\
14x-6\left(\frac{7}{3}x+1\right)&=-16\\
14x-\overset{2}{\cancel{6}}\cdot\frac{7}{\underset{1}{\cancel{3}}}x-6&=-16\\
14x-14x-6&=-16\\
-6&=-16 \quad \textit{False}
\end{aligned}
$$

Solving leads to a false statement. This indicates that the equation is a contradiction. There is no solution for *x* and hence no solution to the system.

Answer: No solution, Ø

A false statement indicates that the system is inconsistent, or in geometric terms, that the lines are parallel and do not intersect. To illustrate this, determine the slope-intercept form of each line and graph them on the same set of axes.

$$\begin{cases} -7x+3y=3 \\ 14x-6y=-16 \end{cases} \Rightarrow \begin{cases} y=\frac{7}{3}x+1 \\ y=\frac{7}{3}x+\frac{8}{3} \end{cases}$$



In slope-intercept form, it is easy to see that the two lines have the same slope but different y-intercepts.

Try this! Solve by substitution: $\begin{cases} 2x-5y=3 \\ 4x-10y=6 \end{cases}$.

Answer: $\left(x,\ \frac{2}{5}x-\frac{3}{5}\right)$

Video Solution

(click to see video)

KEY TAKEAWAYS

- The substitution method is a completely algebraic method for solving a system of equations.
- The substitution method requires that we solve for one of the variables and then substitute the result into the other equation. After performing the substitution step, the resulting equation has one variable and can be solved using the techniques learned up to this point.
- When the value of one of the variables is determined, go back and substitute it into one of the original equations, or their equivalent equations, to determine the corresponding value of the other variable.
- Solutions to systems of two linear equations with two variables, if they exist, are ordered pairs (x, y).
- If the process of solving a system of equations leads to a false statement, then the system is inconsistent and there is no solution, Ø.
- If the process of solving a system of equations leads to a true statement, then the system is dependent and there are infinitely many solutions that can be expressed using the form $(x, mx + b)$.

TOPIC EXERCISES

Part A: Substitution Method

Solve by substitution.

1. $\begin{cases} y = 4x - 1 \\ -3x + y = 1 \end{cases}$

2. $\begin{cases} y = 3x - 8 \\ 4x - y = 2 \end{cases}$

3. $\begin{cases} x = 2y - 3 \\ x + 3y = -8 \end{cases}$

4. $\begin{cases} x = -4y + 1 \\ 2x + 3y = 12 \end{cases}$

5. $\begin{cases} y = 3x \\ -5x + 2y = 2 \end{cases}$

6. $\begin{cases} y = x \\ 2x + 3y = 10 \end{cases}$

7. $\begin{cases} y = 4x + 1 \\ -4x + y = 2 \end{cases}$

8. $\begin{cases} y = -3x + 5 \\ 3x + y = 5 \end{cases}$

9. $\begin{cases} y = 2x + 3 \\ 2x - y = -3 \end{cases}$

10. $\begin{cases} y = 5x - 1 \\ x - 2y = 5 \end{cases}$

11. $\begin{cases} y = -7x + 1 \\ 3x - y = 4 \end{cases}$

12. $\begin{cases} x = 6y + 2 \\ 5x - 2y = 0 \end{cases}$

13. $\begin{cases} y = -2 \\ -2x - y = -6 \end{cases}$

14. $\begin{cases} x = -3 \\ x - 4y = -3 \end{cases}$

15. $\begin{cases} y = -\frac{1}{5}x + 3 \\ 7x - 5y = 9 \end{cases}$

16. $\begin{cases} y = \frac{2}{3}x - 1 \\ 6x - 9y = 0 \end{cases}$

17. $\begin{cases} y = \frac{1}{2}x + \frac{1}{3} \\ x - 6y = 4 \end{cases}$

18. $\begin{cases} y = -\frac{3}{8}x + \frac{1}{2} \\ 2x + 4y = 1 \end{cases}$

19. $\begin{cases} x + y = 6 \\ 2x + 3y = 16 \end{cases}$

20. $\begin{cases} x - y = 3 \\ -2x + 3y = -2 \end{cases}$

21. $\begin{cases} 2x + y = 2 \\ 3x - 2y = 17 \end{cases}$

22. $\begin{cases} x - 3y = -11 \\ 3x + 5y = -5 \end{cases}$

23. $\begin{cases} x + 2y = -3 \\ 3x - 4y = -2 \end{cases}$

24. $\begin{cases} 5x - y = 12 \\ 9x - y = 10 \end{cases}$

25. $\begin{cases} x + 2y = -6 \\ -4x - 8y = 24 \end{cases}$

26. $\begin{cases} x + 3y = -6 \\ -2x - 6y = -12 \end{cases}$

27. $\begin{cases} -3x + y = -4 \\ 6x - 2y = -2 \end{cases}$

28. $\begin{cases} x - 5y = -10 \\ 2x - 10y = -20 \end{cases}$

29. $\begin{cases} 3x - y = 9 \\ 4x + 3y = -1 \end{cases}$

30. $\begin{cases} 2x - y = 5 \\ 4x + 2y = -2 \end{cases}$

31. $\begin{cases} -x + 4y = 0 \\ 2x - 5y = -6 \end{cases}$

32. $\begin{cases} 3y - x = 5 \\ 5x + 2y = -8 \end{cases}$

33. $\begin{cases} 2x - 5y = 1 \\ 4x + 10y = 2 \end{cases}$

34. $\begin{cases} 3x - 7y = -3 \\ 6x + 14y = 0 \end{cases}$

35. $\begin{cases} 10x - y = 3 \\ -5x + \frac{1}{2}y = 1 \end{cases}$

36. $\begin{cases} -\frac{1}{3}x + \frac{1}{6}y = \frac{2}{3} \\ \frac{1}{2}x - \frac{1}{3}y = -\frac{3}{2} \end{cases}$

37. $\begin{cases} \frac{1}{3}x + \frac{2}{3}y = 1 \\ \frac{1}{4}x - \frac{1}{3}y = -\frac{1}{12} \end{cases}$

38. $\begin{cases} \frac{1}{7}x - y = \frac{1}{2} \\ \frac{1}{4}x + \frac{1}{2}y = 2 \end{cases}$

39. $\begin{cases} -\frac{3}{5}x + \frac{2}{5}y = \frac{1}{2} \\ \frac{1}{3}x - \frac{1}{12}y = -\frac{1}{3} \end{cases}$

40. $\begin{cases} \frac{1}{2}x = \frac{2}{3}y \\ x - \frac{2}{3}y = 2 \end{cases}$

41. $\begin{cases} -\frac{1}{2}x + \frac{1}{2}y = \frac{5}{8} \\ \frac{1}{4}x + \frac{1}{2}y = \frac{1}{4} \end{cases}$

42. $\begin{cases} x - y = 0 \\ -x + 2y = 3 \end{cases}$

43. $\begin{cases} y = 3x \\ 2x - 3y = 0 \end{cases}$

44. $\begin{cases} 2x + 3y = 18 \\ -6x + 3y = -6 \end{cases}$

45. $\begin{cases} -3x + 4y = 20 \\ 2x + 8y = 8 \end{cases}$

46. $\begin{cases} 5x - 3y = -1 \\ 3x + 2y = 7 \end{cases}$

47. $\begin{cases} -3x + 7y = 2 \\ 2x + 7y = 1 \end{cases}$

48. $\begin{cases} y = 3 \\ y = -3 \end{cases}$

49. $\begin{cases} x = 5 \\ x = -2 \end{cases}$

50. $\begin{cases} y = 4 \\ y = 4 \end{cases}$

Set up a linear system and solve it using the substitution method.

51. The sum of two numbers is 19. The larger number is 1 less than three times the smaller.

52. The sum of two numbers is 15. The larger is 3 more than twice the smaller.

53. The difference of two numbers is 7 and their sum is 1.

54. The difference of two numbers is 3 and their sum is -7.

55. Where on the graph of $-5x + 3y = 30$ does the *x*-coordinate equal the *y*-coordinate?

56. Where on the graph of $\frac{1}{2}x - \frac{1}{3}y = 1$ does the *x*-coordinate equal the *y*-coordinate?

Part B: Discussion Board Topics

57. Describe what drives the choice of variable to solve for when beginning the process of solving by substitution.

58. Discuss the merits and drawbacks of the substitution method.

ANSWERS

1: (2, 7)

3: (−5, −1)

5: (2, 6)

7: $\varnothing$

9: $(x,\ 2x + 3)$

11: (1/2, −5/2)

13: (4, −2)

15: (3, 12/5)

17: (−3, −7/6)

19: (2, 4)

21: (3, −4)

23: (−8/5, −7/10)

25: $(x,\ -\frac{1}{2}x - 3)$

27: $\varnothing$

29: (2, −3)

31: (−8, −2)

33: (1/2, 0)

35: $\varnothing$

37: (1, 1)

39: (−11/10, −2/5)

41: (−1/2, 3/4)

43: (0, 0)

45: (−4, 2)

47: (−1/5, 1/5)

49: ∅

51: The two numbers are 5 and 14.

53: The two numbers are 4 and −3.

55: (−15, −15)

4.3 Solving Linear Systems by Elimination

LEARNING OBJECTIVES

1. Solve linear systems using the elimination method.
2. Solve linear systems with fractions and decimals.
3. Identify the weaknesses and strengths of each method for solving linear systems.

The Elimination Method

In this section, the goal is to develop another completely algebraic method for solving a system of linear equations. We begin by defining what it means to add equations together. In the following example, notice that if we add the expressions on both sides of the equal sign, we obtain another true statement.

$$\begin{array}{rr} & 2+3=5 \\ + & 1+7=8 \\ \hline & 3+10=13 \\ & 13=13 \quad \checkmark \end{array}$$

This is true in general: if A, B, C, and D are algebraic expressions, then we have the following **addition property of equations**[13]:

$$\text{If } \ A=B \text{ and } C=D, \text{ then } A+C=B+D.$$

For the system

$$\begin{cases} x+y=5 \\ x-y=1 \end{cases}$$

13. If A, B, C, and D are algebraic expressions, where $A = B$ and $C = D$, then $A + C = B + D$.

we add the two equations together:

$$\begin{array}{rr} & x+y=5 \\ + & x-y=1 \\ \hline & 2x \quad =6 \end{array}$$

The sum of y and $-y$ is zero and that term is eliminated. This leaves us with a linear equation with one variable that can be easily solved:

$$\begin{aligned} \frac{2x}{2} &= \frac{6}{2} \\ x &= 3 \end{aligned}$$

At this point, we have the x coordinate of the simultaneous solution, so all that is left to do is back substitute to find the corresponding y-value.

$$\begin{aligned} x+y&=5 \\ 3+y&=5 \\ y&=2 \end{aligned}$$

Hence the solution to the system is (3, 2). This process describes the **elimination (or addition) method**[14] for solving linear systems. Of course, the variable is not always so easily eliminated. Typically, we have to find an equivalent system by applying the multiplication property of equality to one or both of the equations as a means to line up one of the variables to eliminate. The goal is to arrange that either the x terms or the y terms are opposites, so that when the equations are added, the terms eliminate. The steps for the elimination method are outlined in the following example.

14. A means of solving a system by adding equivalent equations in such a way as to eliminate a variable.

Example 1: Solve by elimination: $\begin{cases} 2x+y=7 \\ 3x-2y=-7 \end{cases}.$

Solution:

Step 1: Multiply one, or both, of the equations to set up the elimination of one of the variables. In this example, we will eliminate the variable y by multiplying both sides of the first equation by 2. Take care to distribute.

$$2(2x+y)=2(7)$$
$$4x+2y=14$$

This leaves us with an equivalent system where the variable y is lined up to eliminate.

$$\begin{cases} 2x+y=7 \\ 3x-2y=-7 \end{cases} \stackrel{\times 2}{\Rightarrow} \begin{cases} 4x+2y=14 \\ 3x-2y=-7 \end{cases}$$

Step 2: Add the equations together to eliminate one of the variables.

$$\begin{array}{rr} & 4x+2y=14 \\ + & 3x-2y=-7 \\ \hline & 7x \quad\;\; =7 \end{array}$$

Step 3: Solve for the remaining variable.

$$\frac{7x}{7}=\frac{7}{7}$$
$$x=1$$

Step 3: Back substitute into either equation or its equivalent equation.

$$\begin{aligned} 2x + y &= 7 \\ 2(1) + y &= 7 \\ 2 + y - 2 &= 7 - 2 \\ y &= 5 \end{aligned}$$

Step 4: Check. Remember that the solution must solve both of the original equations.

Check: (1, 5)

Equation 1:	*Equation 2*:
$2x + y = 7$	$4x + 2y = 14$
$2(1) + (5) = 7$	$4(1) + 2(5) = 14$
$2 + 5 = 7$	$4 + 10 = 14$
$7 = 7$ ✓	$14 = 14$ ✓

Answer: (1, 5)

Occasionally, we will have to multiply both equations to line up one of the variables to eliminate. We want the resulting equivalent equations to have terms with opposite coefficients.

Example 2: Solve by elimination: $\begin{cases} 5x - 3y = -1 \\ 3x + 2y = 7 \end{cases}$.

Solution: We choose to eliminate the terms with variable y because the coefficients have different signs. To do this, we first determine the least common multiple of the coefficients; in this case, the LCM(3, 2) is 6. Therefore, multiply both sides of both equations by the appropriate values to obtain coefficients of −6 and 6.

$$2(5x-3y)=2(-1)$$
$$10x-6y=-2$$
Multiply both sides of the first equation by 2.

$$3(3x+2y)=3(7)$$
$$9x+6y=21$$
Multiply both sides of the second equation by 3.

This results in the following equivalent system:

$$\begin{cases}5x-3y=-1 \\ 3x+2y=7\end{cases} \begin{matrix}\xRightarrow{\times 2} \\ \xRightarrow[\times 3]{}\end{matrix} \begin{cases}10x-6y=-2 \\ 9x+6y=21\end{cases}$$

The y terms are now lined up to eliminate.

$$\begin{array}{rl} & 10x-6y=-2 \\ + & 9x+6y=21 \\ \hline & 19x \quad =19 \\ & \frac{19x}{19}=\frac{19}{19} \\ & x=1 \end{array}$$

Back substitute.

$$\begin{aligned} 3x+2y&=7 \\ 3(1)+2y&=7 \\ 3+2y-3&=7-3 \\ 2y&=4 \\ y&=2 \end{aligned}$$

Answer: (1, 2)

Sometimes linear systems are not given in standard form. When this is the case, it is best to first rearrange the equations before beginning the steps to solve by elimination.

Example 3: Solve by elimination: $\begin{cases} 5x + 12y = 11 \\ 3y = 4x + 1 \end{cases}$.

Solution: First, rewrite the second equation in standard form.

$$\begin{aligned} 3y &= 4x+1 \\ 3y-4x &= 4x+1-4x \\ -4x+3y &= 1 \end{aligned}$$

This results in the following equivalent system where like terms are aligned in columns:

$$\begin{cases} 5x+12y=11 \\ 3y=4x+1 \end{cases} \Rightarrow \begin{cases} 5x+12y=11 \\ -4x+3y=1 \end{cases}$$

We can eliminate the term with variable y if we multiply the second equation by −4.

$$\begin{cases} 5x+12y=11 \\ -4x+3y=1 \end{cases} \underset{\times(-4)}{\Rightarrow} \begin{cases} 5x+12y=11 \\ 16x-12y=-4 \end{cases}$$

Next, we add the equations together,

$$\begin{array}{rl} 5x+12y&=11 \\ +\quad 16x-12y&=-4 \\ \hline 21x\quad\quad &=7 \end{array}$$

$$\frac{21x}{21}=\frac{7}{21}$$

$$x=\frac{1}{3}$$

Back substitute.

$$\begin{aligned} 3y&=4x+1 \\ 3y&=4\left(\frac{1}{3}\right)+1 \\ 3y&=\frac{4}{3}+\frac{3}{3} \\ 3y&=\frac{7}{3} \\ \frac{3y}{3}&=\frac{\frac{7}{3}}{3} \\ y&=\frac{7}{3}\cdot\frac{1}{3} \\ y&=\frac{7}{9} \end{aligned}$$

Answer: (1/3, 7/9)

Try this! Solve by elimination: $\begin{cases} 2x+y=-3 \\ -3x-2y=4 \end{cases}$.

Answer: (−2, 1)

Video Solution

(click to see video)

At this point, we explore what happens when solving dependent and inconsistent systems using the elimination method.

Example 4: Solve by elimination: $\begin{cases} 3x - y = 7 \\ 6x - 2y = 14 \end{cases}$.

Solution: To eliminate the variable *x*, we could multiply the first equation by −2.

$$\begin{cases} 3x - y = 7 \\ 6x - 2y = 14 \end{cases} \overset{\times(-2)}{\Rightarrow} \begin{cases} -6x + 2y = -14 \\ 6x - 2y = 14 \end{cases}$$

Now adding the equations we have

$$\begin{array}{rr} & -6x + 2y = -14 \\ + & 6x - 2y = 14 \\ \hline & 0 = 0 \end{array} \quad \textit{True}$$

A true statement indicates that this is a dependent system. The lines coincide, and we need *y* in terms of *x* to present the solution set in the form $(x,\ mx + b)$. Choose one of the original equations and solve for *y*. Since the equations are equivalent, it does not matter which one we choose.

$$\begin{aligned} 3x - y &= 7 \\ 3x - y - 3x &= 7 - 3x \\ -y &= -3x + 7 \\ -1(-y) &= -1(-3x + 7) \\ y &= 3x - 7 \end{aligned}$$

Answer: $(x,\ 3x-7)$

Example 5: Solve by elimination: $\begin{cases} -x+3y=9 \\ 2x-6y=12 \end{cases}$.

Solution: We can eliminate x by multiplying the first equation by 2.

$$\begin{cases} -x+3y=9 \\ 2x-6y=12 \end{cases} \quad \stackrel{\times 2}{\Rightarrow} \quad \begin{cases} -2x+6y=18 \\ \ \ 2x-6y=12 \end{cases}$$

Now adding the equations we have

$$\begin{array}{rr} & -2x+6y=18 \\ + & 2x-6y=12 \\ \hline & 0=30 \end{array} \qquad \textit{False}$$

A false statement indicates that the system is inconsistent. The lines are parallel and do not intersect.

Answer: No solution, ∅

Try this! Solve by elimination: $\begin{cases} 3x+15y=-15 \\ 2x+10y=30 \end{cases}$.

Answer: No solution, ∅

Video Solution

(click to see video)

Clearing Fractions and Decimals

Given a linear system where the equations have fractional coefficients, it is usually best to clear the fractions before beginning the elimination method.

Example 6: Solve: $\begin{cases} -\frac{1}{10}x + \frac{1}{2}y = \frac{4}{5} \\ \frac{1}{7}x + \frac{1}{3}y = -\frac{2}{21} \end{cases}$.

Solution: Recall that we can clear fractions by multiplying both sides of an equation by the least common denominator (LCD). Take care to distribute and then simplify.

Equation 1:	*Equation 2*:
$10\left(-\frac{1}{10}x+\frac{1}{2}y\right)=10\left(\frac{4}{5}\right)$	$21\left(\frac{1}{7}x+\frac{1}{3}y\right)=21\left(-\frac{2}{21}\right)$
$10\cdot\left(-\frac{1}{10}x\right)+10\cdot\frac{1}{2}y=10\cdot\frac{4}{5}$	$21\cdot\frac{1}{7}x+21\cdot\frac{1}{3}y=21\left(-\frac{2}{21}\right)$
$-x+5y=8$	$3x+7y=-2$

This results in an equivalent system where the equations have integer coefficients,

$$\begin{cases} -\frac{1}{10}x+\frac{1}{2}y=\frac{4}{5} \\ \frac{1}{7}x+\frac{1}{3}y=-\frac{2}{21} \end{cases} \begin{matrix} \overset{\times 10}{\Rightarrow} \\ \overset{\times 21}{\Rightarrow} \end{matrix} \begin{cases} -x+5y=8 \\ 3x+7y=-2 \end{cases}$$

Solve using the elimination method.

$$\begin{cases} -x+5y=8 \\ 3x+7y=-2 \end{cases} \overset{\times 3}{\Rightarrow} \begin{cases} -3x+15y=24 \\ 3x+7y=-2 \end{cases}$$

$$\begin{aligned} -3x+15y&=24 \\ +\quad 3x+7y&=-2 \\ \hline 22y&=22 \\ \frac{22y}{22}&=\frac{22}{22} \\ y&=1 \end{aligned}$$

Back substitute.

$$\begin{aligned} 3x+7y&=-2 \\ 3x+7(1)&=-2 \\ 3x+7-7&=-2-7 \\ 3x&=-9 \\ \frac{3x}{3}&=\frac{-9}{3} \\ x&=-3 \end{aligned}$$

Answer: (−3, 1)

We can use a similar technique to clear decimals before solving.

Example 7: Solve: $\begin{cases} 3x-0.6y=-0.9 \\ -0.5x+0.12y=0.16 \end{cases}$.

Solution: Multiply each equation by the lowest power of 10 necessary to result in integer coefficients. In this case, multiply the first equation by 10 and the second equation by 100.

Equation 1:	*Equation 2*:
$10(3x-0.6y)=10(-0.9)$	$100(-0.5x+0.12y)=100(0.16)$
$30x-6y=-9$	$-50x+12y=16$

This results in an equivalent system where the equations have integer coefficients:

$$\begin{cases} 3x-0.6y=-0.9 \\ -0.5x+0.12y=0.16 \end{cases} \begin{array}{c} \overset{\times 10}{\Rightarrow} \\ \overset{\times 100}{\Rightarrow} \end{array} \begin{cases} 30x-6y=-9 \\ -50x+12y=16 \end{cases}$$

Solve using the elimination method.

$$\begin{cases} 30x-6y=-9 \\ -50x+12y=16 \end{cases} \overset{\times 2}{\Rightarrow} \begin{cases} 60x-12y=-18 \\ -50x+12y=16 \end{cases}$$

$$\begin{array}{rrl} & 60x-12y & =-18 \\ + & -50x+12y & =16 \\ \hline & 10x & =-2 \end{array}$$

$$\frac{10x}{10}=-\frac{2}{10}$$

$$x=-\frac{1}{5}=-0.2$$

Back substitute.

$$\begin{aligned} 3x - 0.6y &= -0.9 \\ 3(-0.2) - 0.6y &= -0.9 \\ -0.6 - 0.6y + 0.6 &= -0.9 + 0.6 \\ \frac{-0.6y}{-0.6} &= \frac{-0.3}{-0.6} \\ y &= 0.5 \end{aligned}$$

Answer: (−0.2, 0.5)

Try this! Solve using elimination: $\begin{cases} \frac{1}{3}x - \frac{2}{3}y = 3 \\ \frac{1}{3}x - \frac{1}{2}y = \frac{8}{3} \end{cases}$

Answer: (5, −2)

Video Solution

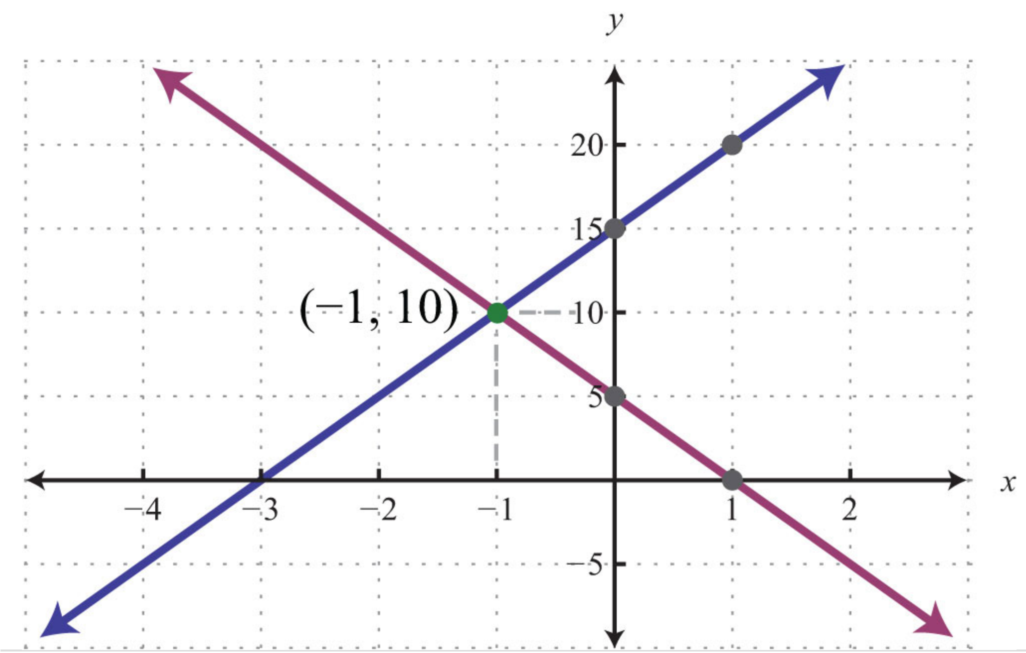
(click to see video)

Summary of the Methods for Solving Linear Systems

We have developed three methods for solving linear systems of two equations with two variables. In this section, we summarize the strengths and weaknesses of each method.

The graphing method is useful for understanding what a system of equations is and what the solutions must look like. When the equations of a system are graphed on the same set of axes, we can see that the solution is the point where the graphs intersect. The graphing is made easy when the equations are in slope-intercept form. For example,

$$\begin{cases} y = 5x + 15 \\ y = -5x + 5 \end{cases}$$



The simultaneous solution (−1, 10) corresponds to the point of intersection. One drawback of this method is that it is very inaccurate. When the coordinates of the solution are not integers, the method is practically unusable. If we have a choice, we typically avoid this method in favor of the more accurate algebraic techniques.

The substitution method, on the other hand, is a completely algebraic method. It requires you to solve for one of the variables and substitute the result into the other equation. The resulting equation has one variable for which you can solve. This method is particularly useful when there is a variable within the system with coefficient of 1. For example,

$$\begin{cases} 10x + y = 20 \\ 7x + 5y = 14 \end{cases} \qquad \textit{Choose the substitution method.}$$

In this case, it is easy to solve for *y* in the first equation and then substitute the result into the other equation. One drawback of this method is that it often leads to equivalent equations with fractional coefficients, which are tedious to work with. If there is not a coefficient of 1, then it usually is best to choose the elimination method.

The elimination method is a completely algebraic method that makes use of the addition property of equations. We multiply one or both of the equations to obtain equivalent equations where one of the variables is eliminated if we add them together. For example,

$$\begin{cases} 2x - 3y = 9 \\ 5x - 8y = -16 \end{cases}$$ *Choose the elimination method.*

Here we multiply both sides of the first equation by 5 and both sides of the second equation by −2. This results in an equivalent system where the variable x is eliminated when we add the equations together. Of course, there are other combinations of numbers that achieve the same result. We could even choose to eliminate the variable y. No matter which variable is eliminated first, the solution will be the same. Note that the substitution method, in this case, would require tedious calculations with fractional coefficients. One weakness of the elimination method, as we will see later in our study of algebra, is that it does not always work for nonlinear systems.

KEY TAKEAWAYS

- The elimination method is a completely algebraic method for solving a system of equations.
- Multiply one or both of the equations in a system by certain numbers to obtain an equivalent system consisting of like terms with opposite coefficients. Adding these equivalent equations together eliminates a variable, and the resulting equation has one variable for which you can solve.
- It is a good practice to first rewrite the equations in standard form before beginning the elimination method.
- When the value of one of the variables is determined, back substitute into one of the original equations, or their equivalent equations, and determine the corresponding value of the other variable.

TOPIC EXERCISES

Part A: Elimination Method

Solve by elimination.

1. $\begin{cases} x + y = 3 \\ 2x - y = 9 \end{cases}$

2. $\begin{cases} x - y = -6 \\ 5x + y = -18 \end{cases}$

3. $\begin{cases} x + 3y = 5 \\ -x - 2y = 0 \end{cases}$

4. $\begin{cases} -x + 4y = 4 \\ x - y = -7 \end{cases}$

5. $\begin{cases} -x + y = 2 \\ x - y = -3 \end{cases}$

6. $\begin{cases} 3x - y = -2 \\ 6x + 4y = 2 \end{cases}$

7. $\begin{cases} 5x + 2y = -3 \\ 10x - y = 4 \end{cases}$

8. $\begin{cases} -2x + 14y = 28 \\ x - 7y = 21 \end{cases}$

9. $\begin{cases} -2x + y = 4 \\ 12x - 6y = -24 \end{cases}$

10. $\begin{cases} x + 8y = 3 \\ 3x + 12y = 6 \end{cases}$

11. $\begin{cases} 2x - 3y = 15 \\ 4x + 10y = 14 \end{cases}$

12. $\begin{cases} 4x + 3y = -10 \\ 3x - 9y = 15 \end{cases}$

13. $\begin{cases} -4x - 5y = -3 \\ 8x + 3y = -15 \end{cases}$

14. $\begin{cases} -2x + 7y = 56 \\ 4x - 2y = -112 \end{cases}$

15. $\begin{cases} -9x - 15y = -15 \\ 3x + 5y = -10 \end{cases}$

16. $\begin{cases} 6x - 7y = 4 \\ 2x + 6y = -7 \end{cases}$

17. $\begin{cases} 4x + 2y = 4 \\ -5x - 3y = -7 \end{cases}$

18. $\begin{cases} 5x - 3y = -1 \\ 3x + 2y = 7 \end{cases}$

19. $\begin{cases} 7x + 3y = 9 \\ 2x + 5y = -14 \end{cases}$

20. $\begin{cases} 9x - 3y = 3 \\ 7x + 2y = -15 \end{cases}$

21. $\begin{cases} 5x - 3y = -7 \\ -7x + 6y = 11 \end{cases}$

22. $\begin{cases} 2x + 9y = 8 \\ 3x + 7y = -1 \end{cases}$

23. $\begin{cases} 2x + 2y = 5 \\ 3x + 3y = -5 \end{cases}$

24. $\begin{cases} -3x + 6y = -12 \\ 2x - 4y = 8 \end{cases}$

25. $\begin{cases} 25x + 15y = -1 \\ 15x + 10y = -1 \end{cases}$

26. $\begin{cases} 2x - 3y = 2 \\ 18x - 12y = 5 \end{cases}$

27. $\begin{cases} y = -2x - 3 \\ -3x - 2y = 4 \end{cases}$

28. $\begin{cases} 28x + 6y = 9 \\ 6y = 4x - 15 \end{cases}$

29. $\begin{cases} y = 5x + 15 \\ y = -5x + 5 \end{cases}$

30. $\begin{cases} 2x - 3y = 9 \\ 5x - 8y = -16 \end{cases}$

31. $\begin{cases} \frac{1}{2}x - \frac{1}{3}y = \frac{1}{6} \\ \frac{5}{2}x + y = \frac{7}{2} \end{cases}$

32. $\begin{cases} \frac{1}{4}x - \frac{1}{9}y = 1 \\ x + y = \frac{3}{4} \end{cases}$

33. $\begin{cases} \frac{1}{2}x - \frac{1}{4}y = \frac{1}{3} \\ \frac{1}{4}x + \frac{1}{2}y = -\frac{19}{6} \end{cases}$

34. $\begin{cases} -\frac{14}{3}x + 2y = 4 \\ -\frac{1}{3}x + \frac{1}{7}y = \frac{4}{21} \end{cases}$

35. $\begin{cases} 0.025x + 0.1y = 0.5 \\ 0.11x + 0.04y = -0.2 \end{cases}$

36. $\begin{cases} 1.3x + 0.1y = 0.35 \\ 0.5x + y = -2.75 \end{cases}$

37. $\begin{cases} x + y = 5 \\ 0.02x + 0.03y = 0.125 \end{cases}$

38. $\begin{cases} x + y = 30 \\ 0.05x + 0.1y = 2.4 \end{cases}$

Set up a linear system and solve it using the elimination method.

39. The sum of two numbers is 14. The larger number is 1 less than two times the smaller.

40. The sum of two numbers is 30. The larger is 2 more than three times the smaller.

41. The difference of two numbers is 13 and their sum is 11.

42. The difference of two numbers is 2 and their sum is −12.

Part B: Mixed Exercises

Solve using any method.

43. $\begin{cases} y = 2x - 3 \\ 3x + y = 12 \end{cases}$

44. $\begin{cases} x + 3y = -5 \\ y = \frac{1}{3}x + 5 \end{cases}$

45. $\begin{cases} x = -1 \\ y = 3 \end{cases}$

46. $\begin{cases} y = \frac{1}{2} \\ x + 9 = 0 \end{cases}$

47. $\begin{cases} y = x \\ -x + y = 1 \end{cases}$

48. $\begin{cases} y = 5x \\ y = -10 \end{cases}$

49. $\begin{cases} 3y = 2x - 24 \\ 3x + 4y = 2 \end{cases}$

50. $\begin{cases} y = -\frac{3}{2}x + 1 \\ -2y + 2 = 3x \end{cases}$

51. $\begin{cases} 7y = -2x - 1 \\ 7x = 2y + 23 \end{cases}$

52. $\begin{cases} 5x + 9y - 14 = 0 \\ 3x + 2y - 5 = 0 \end{cases}$

53. $\begin{cases} y = -\frac{5}{16}x + 10 \\ y = \frac{5}{16}x - 10 \end{cases}$

54. $\begin{cases} y = -\frac{6}{5}x + 12 \\ x = 6 \end{cases}$

55. $\begin{cases} 2(x - 3) + y = 0 \\ 3(2x + y - 1) = 15 \end{cases}$

56. $\begin{cases} 3 - 2(x - y) = -3 \\ 4x - 3(y + 1) = 8 \end{cases}$

57. $\begin{cases} 2(x + 1) = 3(2y - 1) - 21 \\ 3(x + 2) = 1 - (3y - 2) \end{cases}$

58. $\begin{cases} \frac{x}{2} - \frac{y}{3} = -7 \\ \frac{x}{3} - \frac{y}{2} = -8 \end{cases}$

59. $\begin{cases} \frac{x}{4} - \frac{y}{2} = \frac{3}{4} \\ \frac{x}{3} + \frac{y}{6} = \frac{1}{6} \end{cases}$

60. $\begin{cases} \frac{1}{3}x - \frac{2}{3}y = 3 \\ \frac{1}{3}x - \frac{1}{2}y = \frac{8}{3} \end{cases}$

61. $\begin{cases} -\frac{1}{10}x + \frac{1}{2}y = \frac{4}{5} \\ \frac{1}{7}x + \frac{1}{3}y = -\frac{2}{21} \end{cases}$

62. $\begin{cases} y = -\frac{5}{3}x + \frac{1}{2} \\ \frac{1}{3}x + \frac{1}{5}y = \frac{1}{10} \end{cases}$

63. $\begin{cases} -\frac{1}{7}x + y = -\frac{2}{3} \\ -\frac{1}{14}x + \frac{1}{2}y = \frac{1}{3} \end{cases}$

64. $\begin{cases} \frac{1}{15}x - \frac{1}{12}y = \frac{1}{3} \\ -\frac{3}{10}x + \frac{3}{8}y = -\frac{3}{2} \end{cases}$

65. $\begin{cases} x + y = 4{,}200 \\ 0.03x + 0.0525y = 193.5 \end{cases}$

66. $\begin{cases} x + y = 350 \\ 0.2x + 0.1y = 52.5 \end{cases}$

67. $\begin{cases} 0.2x - 0.05y = 0.43 \\ 0.3x + 0.1y = -0.3 \end{cases}$

68. $\begin{cases} 0.1x + 0.3y = 0.3 \\ 0.05x - 0.5y = -0.63 \end{cases}$

69. $$\begin{cases} 0.15x - 0.25y = -0.3 \\ -0.75x + 1.25y = -4 \end{cases}$$

70. $$\begin{cases} -0.15x + 1.25y = 0.4 \\ -0.03x + 0.25y = 0.08 \end{cases}$$

Part C: Discussion Board Topics

71. How do we choose the best method for solving a linear system?

72. What does it mean for a system to be dependent? How can we tell if a given system is dependent?

ANSWERS

1: (4, –1)

3: (–10, 5)

5: ∅

7: (1/5, –2)

9: $(x,\ 2x+4)$

11: (6, –1)

13: (–3, 3)

15: ∅

17: (–1, 4)

19: (3, –4)

21: (–1, 2/3)

23: ∅

25: (1/5, –2/5)

27: (–2, 1)

29: (–1, 10)

31: (1, 1)

33: (–2, –16/3)

35: (–4, 6)

37: (2.5, 2.5)

39: The two numbers are 5 and 9.

41: The two numbers are 12 and −1.

43: (3, 3)

45: (−1, 3)

47: Ø

49: (6, −4)

51: (3, −1)

53: (32, 0)

55: $(x, \ -2x+6)$

57: (−4, 3)

59: (1, −1)

61: (−3, 1)

63: $\varnothing$

65: (1,200, 3,000)

67: (0.8, −5.4)

69: Ø

4.4 Applications of Linear Systems

LEARNING OBJECTIVES

1. Set up and solve applications involving relationships between numbers.
2. Set up and solve applications involving interest and money.
3. Set up and solve mixture problems.
4. Set up and solve uniform motion problems (distance problems).

Problems Involving Relationships between Real Numbers

We now have the techniques needed to solve linear systems. For this reason, we are no longer limited to using one variable when setting up equations that model applications. If we translate an application to a mathematical setup using two variables, then we need to form a linear system with two equations.

Example 1: The sum of two numbers is 40 and their difference is 8. Find the numbers.

Solution:

Identify variables.

Let x represent one of the unknown numbers.
Let y represent the other unknown number.

Set up equations: When using two variables, we need to set up two equations. The first key phrase, "the *sum* of the two numbers is 40," translates as follows:

$$x + y = 40$$

And the second key phrase, "the *difference* is 8," leads us to the second equation:

$$x - y = 8$$

Therefore, our algebraic setup consists of the following system:

$$\begin{cases} x + y = 40 \\ x - y = 8 \end{cases}$$

Solve: We can solve the resulting system using any method of our choosing. Here we choose to solve by elimination. Adding the equations together eliminates the variable y.

$$\begin{array}{rl} x + y &= 40 \\ + \quad x - y &= 8 \\ \hline 2x \quad &= 48 \\ x &= 24 \end{array}$$

Once we have x, back substitute to find y.

$$\begin{aligned} x + y &= 40 \\ 24 + y &= 40 \\ 24 + y - 24 &= 40 - 24 \\ y &= 16 \end{aligned}$$

Check: The sum of the two numbers should be 42 and their difference 8.

$$\begin{aligned} 24 + 16 &= 40 \\ 24 - 16 &= 8 \end{aligned}$$

Answer: The two numbers are 24 and 16.

Example 2: The sum of 9 times a larger number and twice a smaller is 6. The difference of 3 times the larger and the smaller is 7. Find the numbers.

Solution: Begin by assigning variables to the larger and smaller number.

Let x represent the larger number.
Let y represent the smaller number.

The first sentence describes a sum and the second sentence describes a difference.

$$\overbrace{9x}^{\text{9 times a larger}} + \overbrace{2y}^{\text{twice a smaller}} = 6$$

$$\overbrace{3x}^{\text{3 times the larger}} - \overbrace{y}^{\text{the smaller}} = 7$$

This leads to the following system:

$$\begin{cases} 9x + 2y = 6 \\ 3x - y = 7 \end{cases}$$

Solve using the elimination method. Multiply the second equation by 2 and add.

$$\begin{cases} 9x+2y=6 \\ 3x-y=7 \end{cases} \underset{\times 2}{\Rightarrow} \begin{cases} 9x+2y=6 \\ 6x-2y=14 \end{cases}$$

$$\begin{array}{rl} & 9x+2y=6 \\ + & 6x-2y=14 \\ \hline & 15x \quad\; =20 \\ & x=\frac{20}{15} \\ & x=\frac{4}{3} \end{array}$$

Back substitute to find *y*.

$$\begin{aligned} 3x-y&=7 \\ 3\left(\frac{4}{3}\right)-y&=7 \\ 4-y&=7 \\ -y&=3 \\ y&=-3 \end{aligned}$$

Answer: The larger number is 4/3 and the smaller number is −3.

Try this! The sum of two numbers is 3. When twice the smaller number is subtracted from 6 times the larger the result is 22. Find the numbers.

Answer: The two numbers are −1/2 and 7/2.

Video Solution

(click to see video)

Interest and Money Problems

In this section, the **interest and money problems**[15] should seem familiar. The difference is that we will be making use of two variables when setting up the algebraic equations.

Example 3: A roll of 32 bills contains only \$5 bills and \$10 bills. If the value of the roll is \$220, then how many of each bill are in the roll?

Solution: Begin by identifying the variables.

Let x represent the number of \$5 bills.
Let y represent the number of \$10 bills.

When using two variables, we need to set up two equations. The first equation is created from the fact that there are 32 bills.

$$x + y = 32$$

The second equation sums the value of each bill: the total value is \$220.

$$\$5 \cdot x + \$10 \cdot y = \$220$$

Present both equations as a system; this is our algebraic setup.

$$\begin{cases} x + y = 32 \\ 5x + 10y = 220 \end{cases}$$

15. Applications involving simple interest and money.

Here we choose to solve by elimination, although substitution would work just as well. Eliminate *x* by multiplying the first equation by −5.

$$\begin{cases} x + y = 32 \\ 5x + 10y = 220 \end{cases} \overset{\times(-5)}{\Rightarrow} \begin{cases} -5x - 5y = -160 \\ 5x + 10y = 220 \end{cases}$$

Now add the equations together:

$$\begin{array}{rr} & -5x - 5y = -160 \\ + & 5x + 10y = 220 \\ \hline & 5y = 60 \\ & \frac{5y}{5} = \frac{60}{5} \\ & y = 12 \end{array}$$

Once we have *y*, the number of $10 bills, back substitute to find *x*.

$$\begin{aligned} x + y &= 32 \\ x + 12 &= 32 \\ x + 12 - 12 &= 32 - 12 \\ x &= 20 \end{aligned}$$

Answer: There are twenty $5 bills and twelve $10 bills. The check is left to the reader.

Example 4: A total of $6,300 was invested in two accounts. Part was invested in a CD at a $4\frac{1}{2}$ %annual interest rate and part was invested in a money market fund at a $3\frac{3}{4}$ %annual interest rate. If the total simple interest for one year was $267.75, then how much was invested in each account?

Solution:

Let x represent the amount invested at $4\frac{1}{2}\% = 4.5\% = 0.045$

Let y represent the amount invested at $3\frac{3}{4}\% = 3.75\% = 0.0375$

The total amount in both accounts can be expressed as

$$x + y = 6,300$$

To set up a second equation, use the fact that the total interest was $267.75. Recall that the interest for one year is the interest rate times the principal $(I = prt = pr \cdot 1 = pr)$. Use this to add the interest in both accounts. Be sure to use the decimal equivalents for the interest rates given as percentages.

$$\begin{array}{ccccc} \textit{interest from the CD} & + & \textit{interest from the fund} & = & \textit{total interest} \\ 0.045x & + & 0.0375y & = & 267.75 \end{array}$$

These two equations together form the following linear system:

$$\begin{cases} x + y = 6,300 \\ 0.045x + 0.0375y = 267.75 \end{cases}$$

Eliminate y by multiplying the first equation by -0.0375.

$$\begin{cases} x + y = 6,300 \\ 0.045x + 0.0375y = 267.75 \end{cases} \overset{\times(-0.0375)}{\Rightarrow} \begin{cases} -0.0375x - 0.0375y = -236.25 \\ 0.045x + 0.0375y = 267.75 \end{cases}$$

Next, add the equations together to eliminate the variable y.

$$\begin{array}{rrcl} & -0.0375x - 0.0375y & = & -236.25 \\ + & 0.045x + 0.0375y & = & 267.75 \\ \hline & 0.0075x & = & 31.5 \end{array}$$

$$\frac{0.0075x}{0.0075} = \frac{31.5}{0.0075}$$

$$x = 4,200$$

Back substitute.

$$\begin{aligned} x + y &= 6,300 \\ 4,200 + y &= 6,300 \\ 4,200 + y - 4,200 &= 6,300 - 4,200 \\ y &= 2,100 \end{aligned}$$

Answer: \$4,200 was invested at $4\frac{1}{2}\%$and \$2,100 was invested at $3\frac{3}{4}\%$

At this point, we should be able to solve these types of problems in two ways: with one variable and now with two variables. Setting up word problems with two variables often simplifies the entire process, particularly when the relationships between the variables are not so clear.

Try this! On the first day of a two-day meeting, 10 coffees and 10 doughnuts were purchased for a total of \$20.00. Since nobody drank the coffee and all the doughnuts were eaten, the next day only 2 coffees and 14 doughnuts were purchased for a total of \$13.00. How much did each coffee and each doughnut cost?

Answer: Coffee: \$1.25; doughnut: \$0.75

Video Solution

(click to see video)

Mixture Problems

Mixture problems[16] often include a percentage and some total amount. It is important to make a distinction between these two types of quantities. For example, if a problem states that a 20-ounce container is filled with a 2% saline (salt) solution, then this means that the container is filled with a mixture of salt and water as follows:

	Percentage	Amount
Salt	2% = 0.02	0.02(20 ounces) = 0.4 ounces
Water	98% = 0.98	0.98(20 ounces) = 19.6 ounces

In other words, we multiply the percentage times the total to get the amount of each part of the mixture.

Example 5: A 2% saline solution is to be combined and mixed with a 5% saline solution to produce 72 ounces of a 2.5% saline solution. How much of each is needed?

Solution:

Let x represent the amount of 2% saline solution needed.
Let y represent the amount of 5% saline solution needed.

The total amount of saline solution needed is 72 ounces. This leads to one equation,

$$x + y = 72$$

The second equation adds up the amount of salt in the correct percentages. The amount of salt is obtained by multiplying the percentage times the amount, where the variables x and y represent the amounts of the solutions.

16. Applications involving a mixture of amounts usually given as a percentage of some total.

$$\begin{array}{ccccc} \textit{salt in } 2\% \textit{ solution} & + & \textit{salt in } 5\% \textit{ solution} & = & \textit{salt in the end solution} \\ 0.02x & + & 0.05y & = & 0.025(72) \end{array}$$

The algebraic setup consists of both equations presented as a system:

$$\begin{cases} x + y = 72 \\ 0.02x + 0.05y = 0.025(72) \end{cases}$$

Solve.

$$\begin{cases} x + y = 72 \\ 0.02x + 0.05y = 0.025(72) \end{cases} \overset{\times(-0.02)}{\Rightarrow} \begin{cases} -0.02x - 0.02y = -1.44 \\ 0.02x + 0.05y = 1.8 \end{cases}$$

$$\begin{aligned} -0.02x - 0.02y &= -1.44 \\ +\quad 0.02x + 0.05y &= 1.8 \\ \hline 0.03y &= 0.36 \\ \frac{0.03y}{0.03} &= \frac{0.36}{0.03} \\ y &= 12 \end{aligned}$$

Back substitute.

$$\begin{aligned} x + y &= 72 \\ x + 12 &= 72 \\ x + 12 - 12 &= 72 - 12 \\ x &= 60 \end{aligned}$$

Answer: We need 60 ounces of the 2% saline solution and 12 ounces of the 5% saline solution.

Example 6: A 50% alcohol solution is to be mixed with a 10% alcohol solution to create an 8-ounce mixture of a 32% alcohol solution. How much of each is needed?

Solution:

Let x represent the amount of 50% alcohol solution needed.
Let y represent the amount of 10% alcohol solution needed.

The total amount of the mixture must be 8 ounces.

$$x + y = 8$$

The second equation adds up the amount of alcohol from each solution in the correct percentages. The amount of alcohol in the end result is 32% of 8 ounces, or 0.032(8).

$$\begin{array}{ccccc} \textit{alcohol in } 50\% \textit{ solution} & + & \textit{alcohol in } 10\% \textit{ solution} & = & \textit{alcohol in the end solution} \\ 0.50x & + & 0.10y & = & 0.32(8) \end{array}$$

Now we can form a system of two linear equations and two variables as follows:

$$\begin{cases} x + y = 8 \\ 0.50x + 0.10y = 0.32(8) \end{cases}$$

In this example, multiply the second equation by 100 to eliminate the decimals. In addition, multiply the first equation by −10 to line up the variable y to eliminate.

Equation 1:	*Equation 2*:
$-10(x+y)=-10(8)$	$100\ \ 0.50x+0.10y\ \ =100(0.32)(8)$
$-10x-10y=-80$	$50x+10y=256$

We obtain the following equivalent system:

$$\begin{cases} x+y=8 \\ 0.50x+0.10y=0.32(8) \end{cases} \begin{array}{c} \xRightarrow{\times(-10)} \\ \xRightarrow[\times 100]{} \end{array} \begin{cases} -10x-10y=-80 \\ 50x+10y=256 \end{cases}$$

Add the equations and then solve for *x*:

$$\begin{aligned} -10x-10y&=-80 \\ +\quad 50x+10y&=256 \\ \hline 40x\qquad\ &=176 \\ \frac{40x}{40}&=\frac{176}{40} \\ x&=4.4 \end{aligned}$$

Back substitute.

$$\begin{aligned} x+y&=8 \\ 4.4+y&=8 \\ 4.4+y-4.4&=8-4.4 \\ x&=3.6 \end{aligned}$$

Answer: To obtain 8 ounces of a 32% alcohol mixture we need to mix 4.4 ounces of the 50% alcohol solution and 3.6 ounces of the 10% solution.

Try this! A 70% antifreeze concentrate is to be mixed with water to produce a 5-gallon mixture containing 28% antifreeze. How much water and antifreeze concentrate is needed?

Answer: We need to mix 3 gallons of water with 2 gallons of antifreeze concentrate.

Video Solution

(click to see video)

Uniform Motion Problems (Distance Problems)

Recall that the distance traveled is equal to the average rate times the time traveled at that rate, $D = r \cdot t$. These **uniform motion problems**[17] usually have a lot of data, so it helps to first organize that data in a chart and then set up a linear system. In this section, you are encouraged to use two variables.

Example 7: An executive traveled a total of 8 hours and 1,930 miles by car and by plane. Driving to the airport by car, she averaged 60 miles per hour. In the air, the plane averaged 350 miles per hour. How long did it take her to drive to the airport?

Solution: We are asked to find the time it takes her to drive to the airport; this indicates that time is the unknown quantity.

Let x represent the time it took to drive to the airport.
Let y represent the time spent in the air.

	Distance =	*Rate* ×	*Time*
Travel by car		60 mph	x
Travel by air		350 mph	y
Total	1,930 mi		8 hours

Use the formula $D = r \cdot t$ to fill in the unknown distances.

17. Applications relating distance, average rate, and time.

$$\textit{Distance traveled in the car :} \quad D = r \cdot t = 60 \cdot x$$
$$\textit{Distance traveled in the air :} \quad D = r \cdot t = 350 \cdot y$$

	Distance =	***Rate*** ×	***Time***
Travel by car	$60x$	60 mph	x
Travel by air	$350y$	350 mph	y
Total	1,930 mi		8 hours

The distance column and the time column of the chart help us to set up the following linear system.

	Distance =	***Rate*** ×	***Time***
Travel by car	$60x$	60 mph	x
Travel by air	$350y$	350 mph	y
Total	1,930 mi		8 hours

$$60x + 350y = 1{,}930 \qquad x + y = 8$$

$$\begin{cases} x + y = 8 & \leftarrow \textit{total time traveled} \\ 60x + 350y = 1{,}930 & \leftarrow \textit{total distance traveled} \end{cases}$$

Solve.

$$\begin{cases} x+y=8 \\ 60x+350y=1,930 \end{cases} \xRightarrow{\times(-60)} \begin{cases} -60x-60y=-480 \\ 60x+350y=1,930 \end{cases}$$

$$\begin{array}{rr} & -60x-60y=-480 \\ + & 60x+350y=1,930 \\ \hline & 290y=1450 \\ & \dfrac{290y}{290}=\dfrac{1450}{290} \\ & y=5 \end{array}$$

Now back substitute to find the time it took to drive to the airport *x*:

$$\begin{aligned} x+y&=8 \\ x+5&=8 \\ x&=3 \end{aligned}$$

Answer: It took her 3 hours to drive to the airport.

It is not always the case that time is the unknown quantity. Read the problem carefully and identify what you are asked to find; this defines your variables.

Example 8: Flying with the wind, an airplane traveled 1,365 miles in 3 hours. The plane then turned against the wind and traveled another 870 miles in 2 hours. Find the speed of the airplane and the speed of the wind.

Solution: There is no obvious relationship between the speed of the plane and the speed of the wind. For this reason, use two variables as follows:

Let x represent the speed of the airplane.
Let w represent the speed of the wind.

Use the following chart to organize the data:

	Distance =	Rate ×	Time
Flight with wind	1,365 mi		3 hrs
Flight against wind	870 mi		2 hrs
Total			

With the wind, the airplane's total speed is $x + w$. Flying against the wind, the total speed is $x - w$.

	Distance =	Rate ×	Time
Flight with wind	1,365 mi	$x + w$	3 hrs
Flight against wind	870 mi	$x - w$	2 hrs
Total	2,235 mi		5 hrs

Use the rows of the chart along with the formula $D = r \cdot t$ to construct a linear system that models this problem. Take care to group the quantities that represent the rate in parentheses.

	Distance =	Rate ×	Time	
Flight with wind	1,365 mi	$x + w$	3 hrs	$1{,}365 = (x + w) \cdot 3$
Flight against wind	870 mi	$x - w$	2 hrs	$870 = (x - w) \cdot 2$
Total	2,235 mi		5 hrs	

$$\begin{cases} 1{,}365 = (x + w) \cdot 3 & \leftarrow \textit{distance traveled with the wind} \\ 870 = (x - w) \cdot 2 & \leftarrow \textit{distance traveled against the wind} \end{cases}$$

If we divide both sides of the first equation by 3 and both sides of the second equation by 2, then we obtain the following equivalent system:

$$\begin{cases} 1{,}365 = (x+w)\cdot 3 \\ 870 = (x-w)\cdot 2 \end{cases} \begin{matrix} \overset{\div 3}{\Rightarrow} \\ \underset{\div 2}{\Rightarrow} \end{matrix} \begin{cases} 455 = x+w \\ 435 = x-w \end{cases}$$

$$\begin{aligned} x + w &= 455 \\ + \quad x - w &= 435 \\ \hline 2x \quad &= 890 \\ \frac{2x}{2} &= \frac{890}{2} \\ x &= 445 \end{aligned}$$

Back substitute.

$$\begin{aligned} x + w &= 455 \\ 445 + w &= 455 \\ w &= 10 \end{aligned}$$

Answer: The speed of the airplane is 445 miles per hour and the speed of the wind is 10 miles per hour.

Try this! A boat traveled 24 miles downstream in 2 hours. The return trip, which was against the current, took twice as long. What are the speeds of the boat and of the current?

Answer: The speed of the boat is 9 miles per hour and the speed of the current is 3 miles per hour.

Video Solution

(click to see video)

KEY TAKEAWAYS

- Use two variables as a means to simplify the algebraic setup of applications where the relationship between unknowns is unclear.
- Carefully read the problem several times. If two variables are used, then remember that you need to set up two linear equations in order to solve the problem.
- Be sure to answer the question in sentence form and include the correct units for the answer.

TOPIC EXERCISES

Part A: Applications Involving Numbers

Set up a linear system and solve.

1. The sum of two integers is 54 and their difference is 10. Find the integers.

2. The sum of two integers is 50 and their difference is 24. Find the integers.

3. The sum of two positive integers is 32. When the smaller integer is subtracted from twice the larger, the result is 40. Find the two integers.

4. The sum of two positive integers is 48. When twice the smaller integer is subtracted from the larger, the result is 12. Find the two integers.

5. The sum of two integers is 74. The larger is 26 more than twice the smaller. Find the two integers.

6. The sum of two integers is 45. The larger is 3 less than three times the smaller. Find the two integers.

7. The sum of two numbers is zero. When 4 times the smaller number is added to 8 times the larger, the result is 1. Find the two numbers.

8. The sum of a larger number and 4 times a smaller number is 5. When 8 times the smaller is subtracted from twice the larger, the result is −2. Find the numbers.

9. The sum of 12 times the larger number and 11 times the smaller is −36. The difference of 12 times the larger and 7 times the smaller is 36. Find the numbers.

10. The sum of 4 times the larger number and 3 times the smaller is 7. The difference of 8 times the larger and 6 times the smaller is 10. Find the numbers.

Part B: Interest and Money Problems

Set up a linear system and solve.

11. A $7,000 principal is invested in two accounts, one earning 3% interest and another earning 7% interest. If the total interest for the year is $262, then how much is invested in each account?

12. Mary has her total savings of $12,500 in two different CD accounts. One CD earns 4.4% interest and another earns 3.2% interest. If her total interest for the year is $463, then how much does she have in each CD account?

13. Sally's $1,800 savings is in two accounts. One account earns 6% annual interest and the other earns 3%. Her total interest for the year is $93. How much does she have in each account?

14. Joe has two savings accounts totaling $4,500. One account earns $3\frac{3}{4}\%$ annual interest and the other earns $2\frac{5}{8}\%$. If his total interest for the year is $141.75, then how much is in each account?

15. Millicent has $10,000 invested in two accounts. For the year, she earns $535 more in interest from her 7% mutual fund account than she does from her 4% CD. How much does she have in each account?

16. A small business has $85,000 invested in two accounts. If the account earning 3% annual interest earns $825 more in interest than the account earning 4.5% annual interest, then how much is invested in each account?

17. Jerry earned a total of $284 in simple interest from two separate accounts. In an account earning 6% interest, Jerry invested $1,000 more than twice the amount he invested in an account earning 4%. How much did he invest in each account?

18. James earned a total of $68.25 in simple interest from two separate accounts. In an account earning 2.6% interest, James invested one-half as much as he did in the other account that earned 5.2%. How much did he invest in each account?

19. A cash register contains $10 bills and $20 bills with a total value of $340. If there are 23 bills total, then how many of each does the register contain?

20. John was able to purchase a pizza for $10.80 with quarters and dimes. If he uses 60 coins to buy the pizza, then how many of each did he have?

21. Dennis mowed his neighbor's lawn for a jar of dimes and nickels. Upon completing the job, he counted the coins and found that there were 4 less than twice as many dimes as there were nickels. The total value of all the coins is $6.60. How many of each coin did he have?

22. Two families bought tickets for the big football game. One family ordered 2 adult tickets and 3 children's tickets for a total of $26.00. Another family ordered 3 adult tickets and 4 children's tickets for a total of $37.00. How much did each adult ticket cost?

23. Two friends found shirts and shorts on sale at a flea market. One bought 5 shirts and 3 shorts for a total of $51.00. The other bought 3 shirts and 7 shorts for a total of $80.00. How much was each shirt and each pair of shorts?

24. On Monday Joe bought 10 cups of coffee and 5 doughnuts for his office at a cost of $16.50. It turns out that the doughnuts were more popular than the coffee. Therefore, on Tuesday he bought 5 cups of coffee and 10 doughnuts for a total of $14.25. How much was each cup of coffee?

Part C: Mixture Problems

Set up a linear system and solve.

25. A 15% acid solution is to be mixed with a 25% acid solution to produce 12 gallons of a 20% acid solution. How much of each is needed?

26. One alcohol solution contains 12% alcohol and another contains 26% alcohol. How much of each should be mixed together to obtain 5 gallons of a 14.8% alcohol solution?

27. A nurse wishes to obtain 40 ounces of a 1.2% saline solution. How much of a 1% saline solution must she mix with a 2% saline solution to achieve the desired result?

28. A customer ordered 20 pounds of fertilizer that contains 15% nitrogen. To fill the customer's order, how much of the stock 30% nitrogen fertilizer must be mixed with the 10% nitrogen fertilizer?

29. A customer ordered 2 pounds of a mixed peanut product containing 15% cashews. The inventory consists of only two mixes containing 10% and 30% cashews. How much of each type must be mixed to fill the order?

30. How many pounds of pure peanuts must be combined with a 20% peanut mix to produce 10 pounds of a 32% peanut mix?

31. How much cleaning fluid with 20% alcohol content, must be mixed with water to obtain a 24-ounce mixture with 10% alcohol content?

32. A chemist wishes to create a 32-ounce solution with 12% acid content. He uses two types of stock solutions, one with 30% acid content and another with 10% acid content. How much of each does he need?

33. A concentrated cleaning solution that contains 50% ammonia is mixed with another solution containing 10% ammonia. How much of each is mixed to obtain 8 ounces of a 32% ammonia cleaning formula?

34. A 50% fruit juice concentrate can be purchased wholesale. Best taste is achieved when water is mixed with the concentrate in such a way as to obtain a 12% fruit juice mixture. How much water and concentrate is needed to make a 50-ounce fruit juice drink?

35. A 75% antifreeze concentrate is to be mixed with water to obtain 6 gallons of a 25% antifreeze solution. How much water is needed?

36. Pure sugar is to be mixed with a fruit salad containing 10% sugar to produce 48 ounces of a salad containing 16% sugar. How much pure sugar is required?

Part D: Uniform Motion Problems

Set up a linear system and solve.

37. An airplane averaged 460 miles per hour on a trip with the wind behind it and 345 miles per hour on the return trip against the wind. If the total round trip took 7 hours, then how long did the airplane spend on each leg of the trip?

38. The two legs of a 330-mile trip took 5 hours. The average speed for the first leg of the trip was 70 miles per hour and the average speed for the second leg of the trip was 60 miles per hour. How long did each leg of the trip take?

39. An executive traveled 1,200 miles, part by helicopter and part by private jet. The jet averaged 320 miles per hour while the helicopter averaged 80 miles per hour. If the total trip took 4½ hours, then how long did she spend in the private jet?

40. Joe took two buses on the 463-mile trip from San Jose to San Diego. The first bus averaged 50 miles per hour and the second bus was able to average 64 miles per hour. If the total trip took 8 hours, then how long was spent in each bus?

41. Billy canoed downstream to the general store at an average rate of 9 miles per hour. His average rate canoeing back upstream was 4 miles per hour. If the total trip took 6½ hours, then how long did it take Billy to get back on the return trip?

42. Two brothers drove the 2,793 miles from Los Angeles to New York. One of the brothers, driving in the day, was able to average 70 miles per hour, and the other, driving at night, was able to average 53 miles per hour. If the total trip took 45 hours, then how many hours did each brother drive?

43. A boat traveled 24 miles downstream in 2 hours. The return trip took twice as long. What was the speed of the boat and the current?

44. A helicopter flying with the wind can travel 525 miles in 5 hours. On the return trip, against the wind, it will take 7 hours. What are the speeds of the helicopter and of the wind?

45. A boat can travel 42 miles with the current downstream in 3 hours. Returning upstream against the current, the boat can only travel 33 miles in 3 hours. Find the speed of the current.

46. A light aircraft flying with the wind can travel 180 miles in 1½ hours. The aircraft can fly the same distance against the wind in 2 hours. Find the speed of the wind.

Part E: Discussion Board

47. Compose a number or money problem that can be solved with a system of equations of your own and share it on the discussion board.

48. Compose a mixture problem that can be solved with a system of equations of your own and share it on the discussion board.

49. Compose a uniform motion problem that can be solved with a system of equations of your own and share it on the discussion board.

ANSWERS

1: The integers are 22 and 32.

3: The integers are 8 and 24.

5: The integers are 16 and 58.

7: The two numbers are −1/4 and 1/4.

9: The smaller number is −4 and the larger is 2/3.

11: \$5,700 at 3% and \$1,300 at 7%

13: \$1,300 at 6% and \$500 at 3%

15: \$8,500 at 7% and \$1,500 at 4%

17: \$1,400 at 4% and \$3,800 at 6%

19: 12 tens and 11 twenties

21: 52 dimes and 28 nickels

23: Shirts: \$4.50; shorts: \$9.50

25: 6 gallons of each

27: 32 ounces of the 1% saline solution and 8 ounces of the 2% saline solution

29: 1.5 pounds of the 10% cashew mix and 0.5 pounds of the 30% cashew mix

31: 12 ounces of cleaning fluid

33: 4.4 ounces of the 50% ammonia solution and 3.6 ounces of the 10% ammonia solution

35: 4 gallons

37: The airplane flew 3 hours with the wind and 4 hours against the wind.

39: 3.5 hours

41: 4.5 hours

43: Boat: 9 miles per hour; current: 3 miles per hour

45: 1.5 miles per hour

4.5 Solving Systems of Linear Inequalities (Two Variables)

LEARNING OBJECTIVES

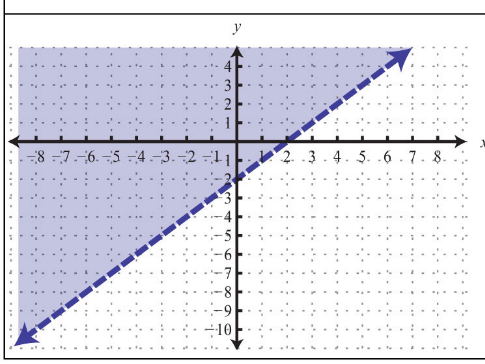
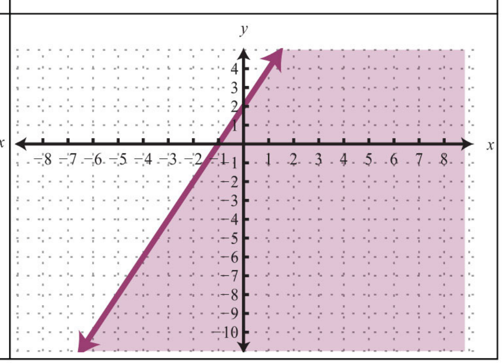
1. Check solutions to systems of linear inequalities with two variables.
2. Solve systems of linear inequalities.

Solutions to Systems of Linear Inequalities

A **system of linear inequalities**[18] consists of a set of two or more linear inequalities with the same variables. The inequalities define the conditions that are to be considered simultaneously. For example,

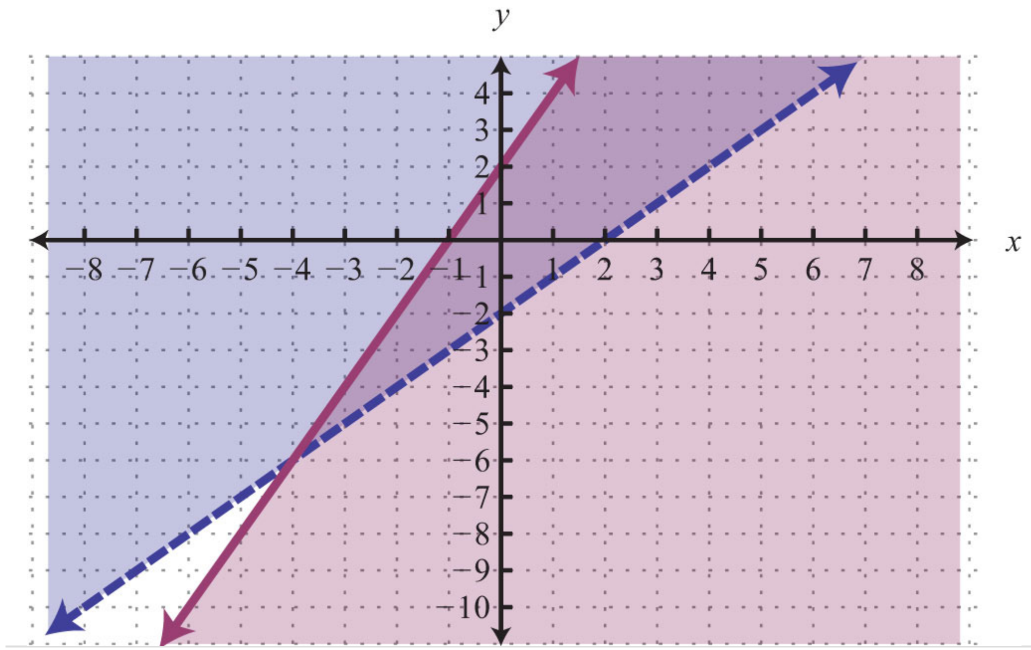
$$\begin{cases} y > x - 2 \\ y \leq 2x + 2 \end{cases}$$

We know that each inequality in the set contains infinitely many ordered pair solutions defined by a region in a rectangular coordinate plane. When considering two of these inequalities together, the intersection of these sets defines the set of simultaneous ordered pair solutions. When we graph each of the above inequalities separately, we have

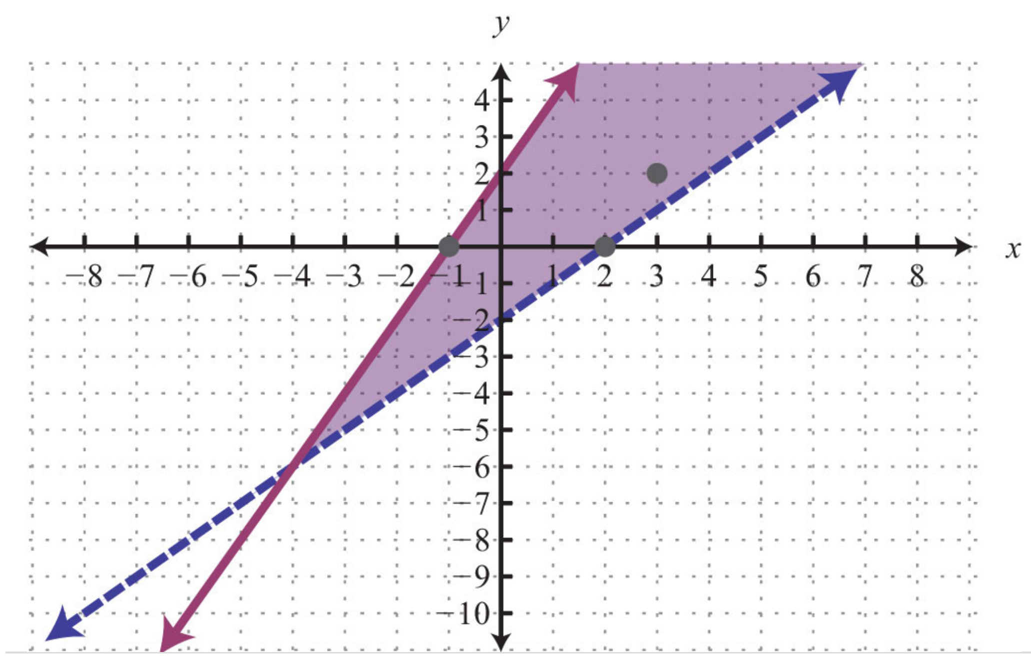
$y > x - 2$	$y \leq 2x + 2$
	

When graphed on the same set of axes, the intersection can be determined.

18. A set of two or more linear inequalities that define the conditions to be considered simultaneously.



The intersection is shaded darker and the final graph of the solution set is presented as follows:



The graph suggests that (3, 2) is a solution because it is in the intersection. To verify this, show that it solves both of the original inequalities:

Check: (3, 2)

Inequality 1: $y > x - 2$	*Inequality 2:* $y \leq 2x + 2$
$2 > 3 - 2$	$2 \leq 2(3) + 2$
$2 > 1$ ✓	$2 \leq 8$ ✓

Points on the solid boundary are included in the set of simultaneous solutions and points on the dashed boundary are not. Consider the point (−1, 0) on the solid boundary defined by $y = 2x + 2$ and verify that it solves the original system:

Check: (−1, 0)

Inequality 1: $y > x - 2$	*Inequality 2:* $y \leq 2x + 2$
$0 > -1 - 2$	$0 \leq 2(-1) + 2$
$0 > -3$ ✓	$0 \leq 0$ ✓

Notice that this point satisfies both inequalities and thus is included in the solution set. Now consider the point (2, 0) on the dashed boundary defined by $y = x - 2$ and verify that it does not solve the original system:

Check: (2, 0)

Inequality 1: $y > x - 2$	*Inequality 2:* $y \leq 2x + 2$
$0 > 2 - 2$	$0 \leq 2(2) + 2$
$0 > 0$ ✗	$0 \leq 6$ ✓

This point does not satisfy both inequalities and thus is not included in the solution set.

Solving Systems of Linear Inequalities

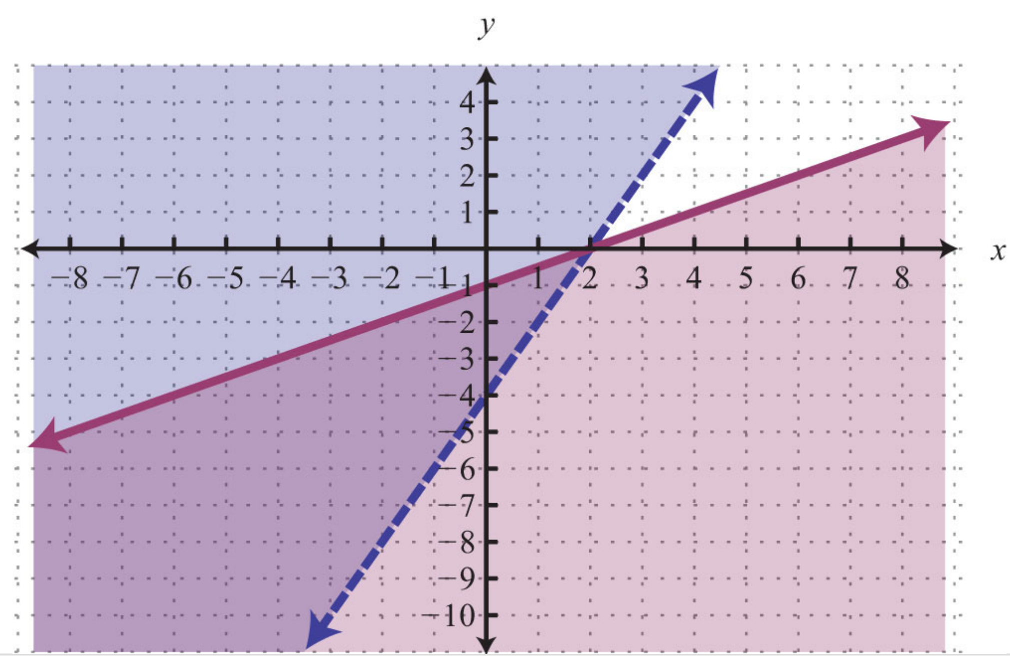
Solutions to a system of linear inequalities are the ordered pairs that solve all the inequalities in the system. Therefore, to solve these systems, graph the solution sets of the inequalities on the same set of axes and determine where they intersect. This intersection, or overlap, defines the region of common ordered pair solutions.

Example 1: Graph the solution set: $\begin{cases} -2x + y > -4 \\ 3x - 6y \geq 6 \end{cases}$.

Solution: To facilitate the graphing process, we first solve for *y*.

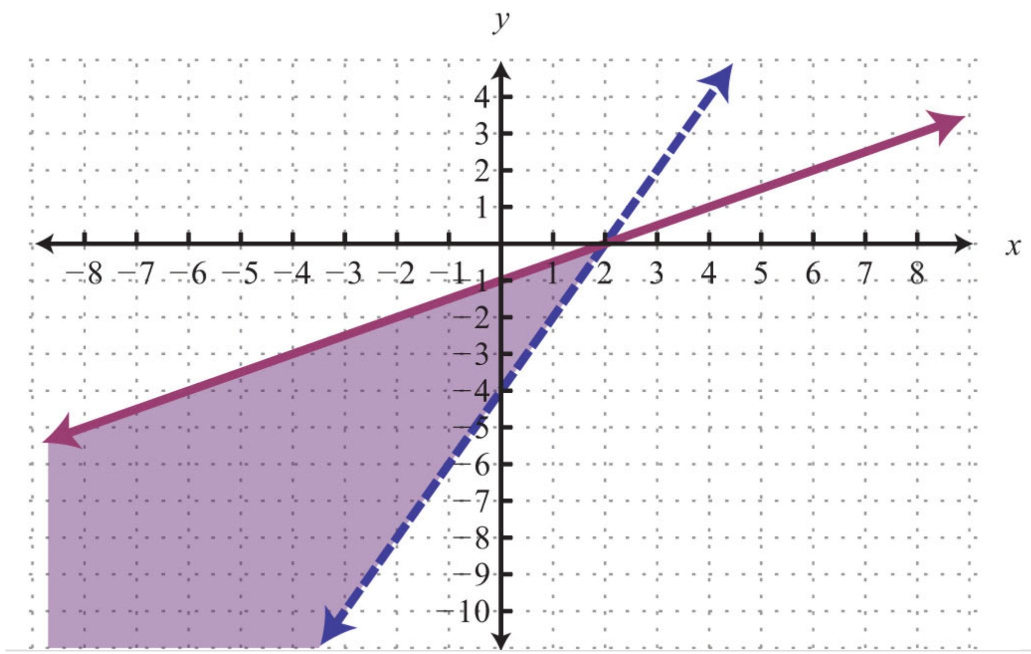
$$\begin{cases} -2x + y > -4 \\ 3x - 6y \geq 6 \end{cases} \Rightarrow \begin{cases} y > 2x - 4 \\ y \leq \dfrac{1}{2}x - 1 \end{cases}$$

For the first inequality, we use a dashed boundary defined by $y = 2x - 4$ and shade all points above the line. For the second inequality, we use a solid boundary defined by $y = \frac{1}{2}x - 1$ and shade all points below. The intersection is darkened.



Now we present the solution with only the intersection shaded.

Answer:

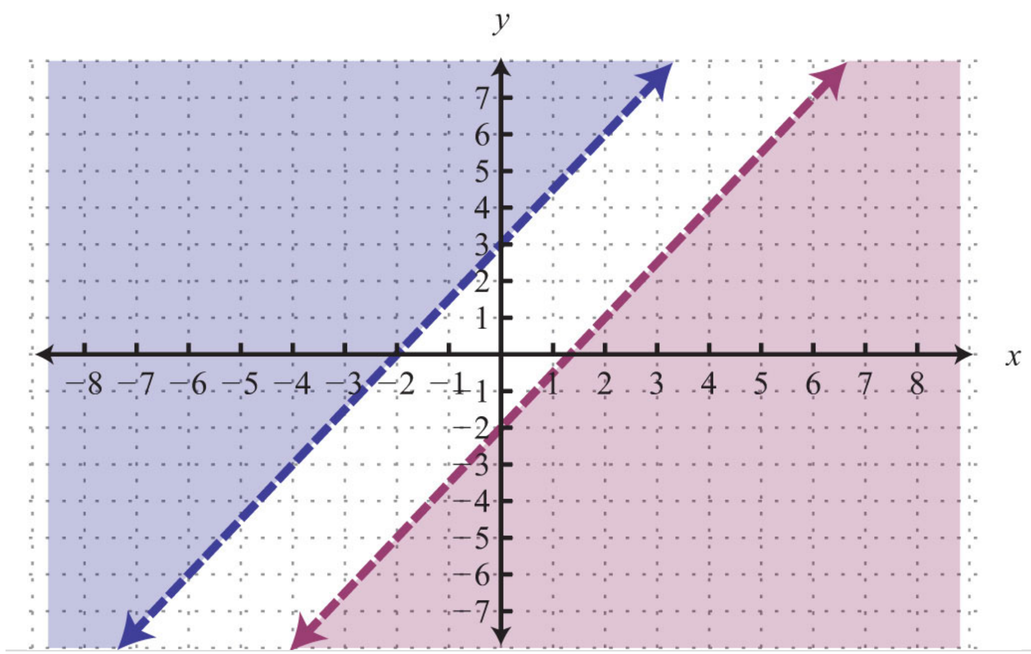


Example 2: Graph the solution set: $\begin{cases} -2x + 3y > 6 \\ 4x - 6y > 12 \end{cases}$.

Solution: Begin by solving both inequalities for *y*.

$$\begin{cases} -3x + 2y > 6 \\ 6x - 4y > 8 \end{cases} \Rightarrow \begin{cases} y > \dfrac{3}{2}x + 3 \\ y < \dfrac{3}{2}x - 2 \end{cases}$$

Use a dashed line for each boundary. For the first inequality, shade all points above the boundary. For the second inequality, shade all points below the boundary.

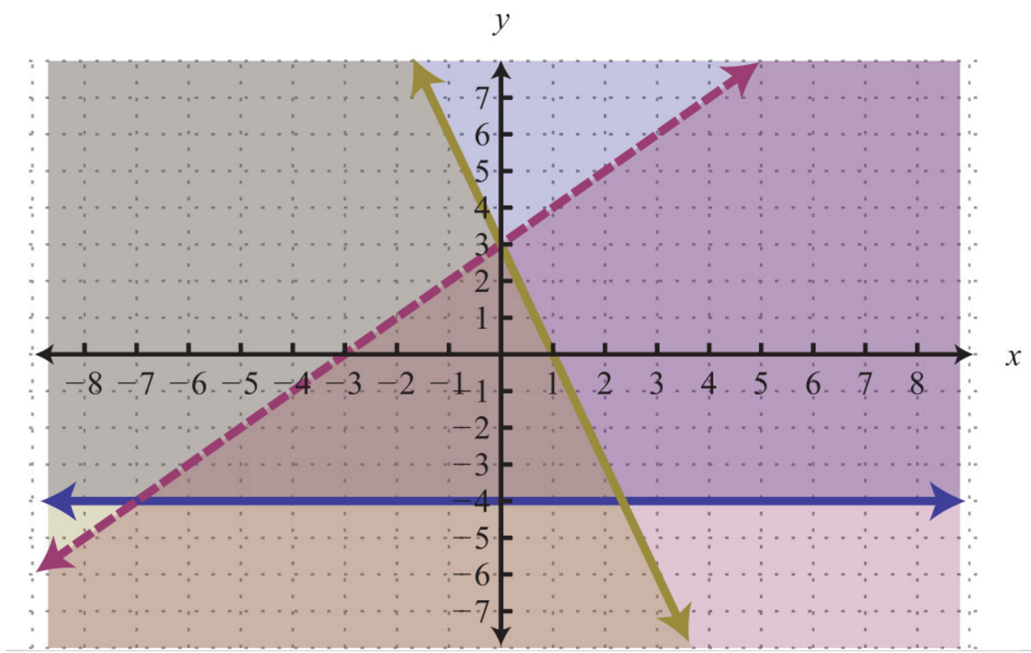


As you can see, there is no intersection of these two shaded regions. Therefore, there are no simultaneous solutions.

Answer: No solution, $\varnothing$

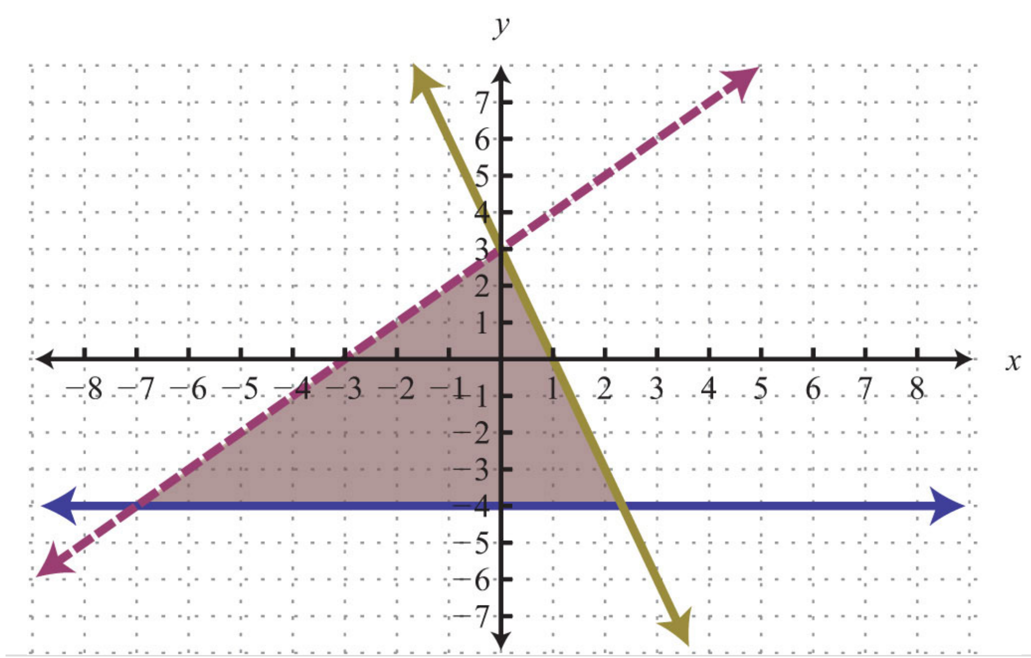
Example 3: Graph the solution set: $\begin{cases} y \geq -4 \\ y < x + 3 \\ y \leq -3x + 3 \end{cases}$.

Solution: The intersection of all the shaded regions forms the triangular region as pictured darkened below:



After graphing all three inequalities on the same set of axes, we determine that the intersection lies in the triangular region pictured.

Answer:



The graphic suggests that (−1, 1) is a common point. As a check, substitute that point into the inequalities and verify that it solves all three conditions.

Check: $(-1, 1)$

Inequality 1:	*Inequality 2:*	*Inequality 3:*
$y \geq -4$	$y < x + 3$	$y \leq -3x + 3$
$1 \geq -4$ ✓	$1 < -1 + 3$	$1 \leq -3(-1) + 3$
	$1 < 2$ ✓	$1 \leq 3 + 3$
		$1 \leq 6$ ✓

KEY TAKEAWAY

- To solve systems of linear inequalities, graph the solution sets of each inequality on the same set of axes and determine where they intersect.

TOPIC EXERCISES

Part A: Solving Systems of Linear Inequalities

Determine whether the given point is a solution to the given system of linear equations.

1. (3, 2); $\begin{cases} y \leq x + 3 \\ y \geq -x + 3 \end{cases}$

2. (−3, −2); $\begin{cases} y < -3x + 4 \\ y \geq 2x - 1 \end{cases}$

3. (5, 0); $\begin{cases} y > -x + 5 \\ y \leq \frac{3}{4}x - 2 \end{cases}$

4. (0, 1); $\begin{cases} y < \frac{2}{3}x + 1 \\ y \geq \frac{5}{2}x - 2 \end{cases}$

5. $\left(-1, \frac{8}{3}\right)$; $\begin{cases} -4x + 3y \geq -12 \\ 2x + 3y < 6 \end{cases}$

6. (−1, −2); $\begin{cases} -x + y < 0 \\ x + y < 0 \\ x + y < -2 \end{cases}$

Part B: Solving Systems of Linear Inequalities

Graph the solution set.

7. $\begin{cases} y \leq x + 3 \\ y \geq -x + 3 \end{cases}$

8. $\begin{cases} y < -3x + 4 \\ y \geq 2x - 1 \end{cases}$

9. $\begin{cases} y > x \\ y < -1 \end{cases}$

10. $\begin{cases} y < \frac{2}{3}x + 1 \\ y \geq \frac{5}{2}x - 2 \end{cases}$

11. $\begin{cases} y > -x + 5 \\ y \leq \frac{3}{4}x - 2 \end{cases}$

12. $\begin{cases} y > \frac{3}{5}x + 3 \\ y < \frac{3}{5}x - 3 \end{cases}$

13. $\begin{cases} x + 4y < 12 \\ -3x + 12y \geq -12 \end{cases}$

14. $\begin{cases} -x + y \leq 6 \\ 2x + y \geq 1 \end{cases}$

15. $\begin{cases} -2x + 3y > 3 \\ 4x - 3y < 15 \end{cases}$

16. $\begin{cases} -4x + 3y \geq -12 \\ 2x + 3y < 6 \end{cases}$

17. $\begin{cases} 5x + y \leq 4 \\ -4x + 3y < -6 \end{cases}$

18. $\begin{cases} 3x + 5y < 15 \\ -x + 2y \leq 0 \end{cases}$

19. $\begin{cases} x \geq 0 \\ 5x + y > 5 \end{cases}$

20. $\begin{cases} x \geq -2 \\ y \geq 1 \end{cases}$

21. $\begin{cases} x - 3 < 0 \\ y + 2 \geq 0 \end{cases}$

22. $\begin{cases} 5y \geq 2x + 5 \\ -2x < -5y - 5 \end{cases}$

23. $\begin{cases} x - y \geq 0 \\ -x + y < 1 \end{cases}$

24. $\begin{cases} -x + y \geq 0 \\ y - x < 1 \end{cases}$

25. $\begin{cases} x > -2 \\ x \leq 2 \end{cases}$

26. $\begin{cases} y > -1 \\ y < 2 \end{cases}$

27. $\begin{cases} -x + 2y > 8 \\ 3x - 6y \geq 18 \end{cases}$

28. $\begin{cases} -3x + 4y \leq 4 \\ 6x - 8y > -8 \end{cases}$

29. $\begin{cases} 2x + y < 3 \\ -x \leq \frac{1}{2}y \end{cases}$

30. $\begin{cases} 2x + 6y \leq 6 \\ -\frac{1}{3}x - y \leq 3 \end{cases}$

31. $\begin{cases} y < 3 \\ y > x \\ x > -4 \end{cases}$

32. $\begin{cases} y < 1 \\ y \geq x - 1 \\ y < -3x + 3 \end{cases}$

33. $\begin{cases} -4x + 3y > -12 \\ y \geq 2 \\ 2x + 3y > 6 \end{cases}$

34. $\begin{cases} -x + y < 0 \\ x + y \leq 0 \\ x + y > -2 \end{cases}$

35. $\begin{cases} x + y < 2 \\ x < 3 \\ -x + y \leq 2 \end{cases}$

36. $\begin{cases} y + 4 \geq 0 \\ \frac{1}{2}x + \frac{1}{3}y \leq 1 \\ -\frac{1}{2}x + \frac{1}{3}y \leq 1 \end{cases}$

37. Construct a system of linear inequalities that describes all points in the first quadrant.

38. Construct a system of linear inequalities that describes all points in the second quadrant.

39. Construct a system of linear inequalities that describes all points in the third quadrant.

40. Construct a system of linear inequalities that describes all points in the fourth quadrant.

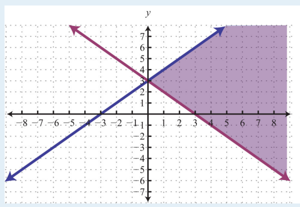
ANSWERS

1: Yes

3: No

5: No

7:



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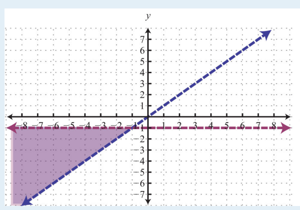

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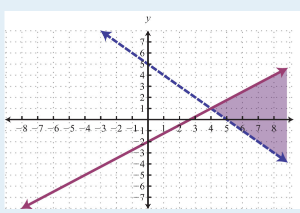

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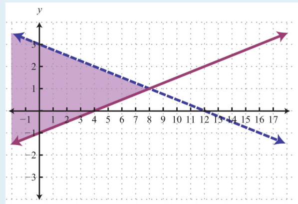

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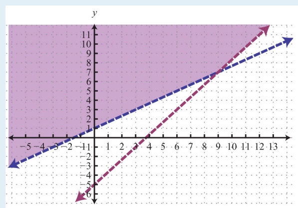

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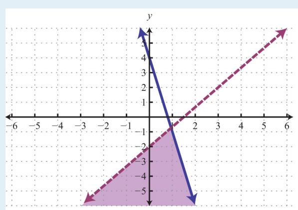

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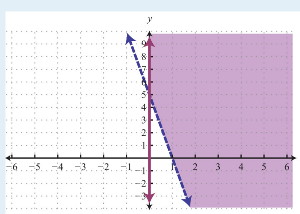

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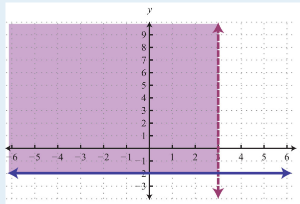

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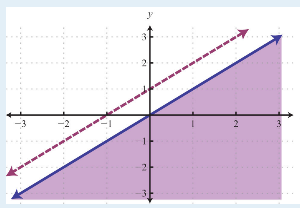

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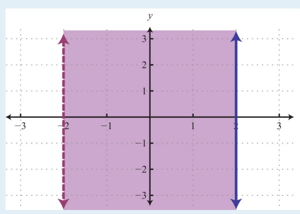
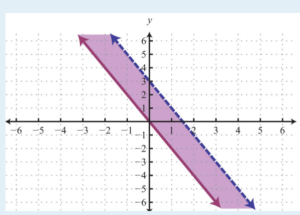


27: No solution, ∅

29:



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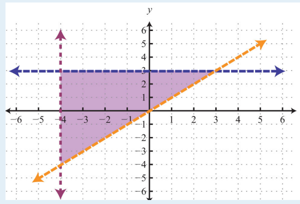

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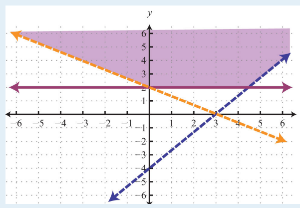

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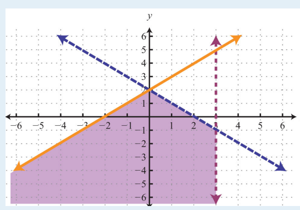


37: $\begin{cases} x > 0 \\ y > 0 \end{cases}$

39: $\begin{cases} x < 0 \\ y < 0 \end{cases}$

4.6 Review Exercises and Sample Exam

REVIEW EXERCISES

Solving Linear Systems by Graphing

Determine whether the given ordered pair is a solution to the given system.

1. (1, -3); $\begin{cases} 5x - y = 8 \\ -3x + y = -6 \end{cases}$

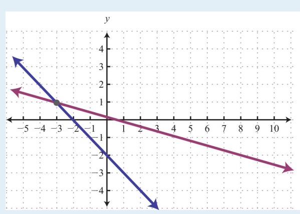
2. (-3, -4); $\begin{cases} 4x - \frac{1}{2}y = -10 \\ 6x - 5y = -2 \end{cases}$

3. (-1, 1/5); $\begin{cases} \frac{3}{5}x - \frac{1}{3}y = -\frac{2}{3} \\ -\frac{1}{5}x - \frac{1}{2}y = \frac{1}{10} \end{cases}$

4. (1/2, -1); $\begin{cases} x + \frac{3}{4}y = -\frac{1}{4} \\ \frac{2}{3}x - y = \frac{4}{3} \end{cases}$

Given the graph, determine the simultaneous solution.

5.



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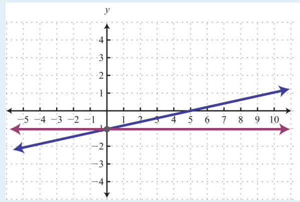

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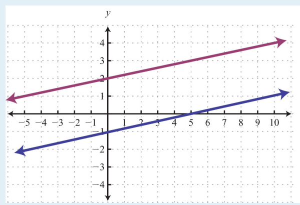

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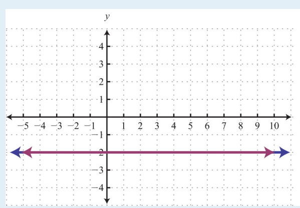


Solve by graphing.

9. $\begin{cases} y = \frac{1}{2}x - 3 \\ y = -\frac{3}{4}x + 2 \end{cases}$

10. $\begin{cases} y = 5 \\ y = -\frac{4}{5}x + 1 \end{cases}$

11. $\begin{cases} x - 2y = 0 \\ 2x - 3y = 3 \end{cases}$

12. $\begin{cases} 5x - y = -11 \\ -4x + 2y = 16 \end{cases}$

13. $\begin{cases} \frac{5}{2}x + 2y = 6 \\ 5x + 4y = 12 \end{cases}$

14. $\begin{cases} 6x - 10y = -2 \\ 3x - 5y = 5 \end{cases}$

Solving Linear Systems by Substitution

Solve by substitution.

15. $\begin{cases} y = 7x - 2 \\ x + y = 6 \end{cases}$

16. $\begin{cases} 2x - 4y = 10 \\ x = -2y - 1 \end{cases}$

17. $\begin{cases} x - y = 0 \\ 5x - 7y = -8 \end{cases}$

18. $\begin{cases} 9x + 2y = -41 \\ -x + y = 7 \end{cases}$

19. $\begin{cases} 6x - 3y = 4 \\ 2x - 9y = 4 \end{cases}$

20. $\begin{cases} 8x - y = 7 \\ 12x + 3y = 6 \end{cases}$

21. $\begin{cases} 20x - 4y = -3 \\ -5x + y = -\frac{1}{2} \end{cases}$

22. $\begin{cases} 3x - y = 6 \\ x - \frac{1}{3}y = 2 \end{cases}$

23. $\begin{cases} x = -1 \\ 8x - 4y = -10 \end{cases}$

24. $\begin{cases} y = -7 \\ 14x - 4y = 0 \end{cases}$

Solving Linear Systems by Elimination

Solve by elimination.

25. $\begin{cases} x - y = 5 \\ 3x - 8y = 5 \end{cases}$

26. $\begin{cases} 7x + 2y = -10 \\ 9x + 4y = -30 \end{cases}$

27. $\begin{cases} 9x - 6y = -6 \\ 2x - 5y = 17 \end{cases}$

28. $\begin{cases} 4x - 2y = 30 \\ 3x + 7y = 14 \end{cases}$

29. $\begin{cases} \frac{5}{2}x - 2y = -\frac{11}{4} \\ \frac{1}{6}x - \frac{1}{3}y = -\frac{1}{3} \end{cases}$

30. $\begin{cases} 2x - \frac{3}{2}y = \frac{20}{3} \\ \frac{3}{2}x - \frac{1}{3}y = \frac{11}{6} \end{cases}$

31. $\begin{cases} 0.1x - 0.3y = 0.17 \\ 0.6x + 0.5y = -0.13 \end{cases}$

32. $\begin{cases} -1.25x - 0.45y = -12.23 \\ 0.5x - 1.5y = 5.9 \end{cases}$

33. $\begin{cases} 6x - \frac{5}{2}y = -5 \\ -12x + 5y = 10 \end{cases}$

34. $\begin{cases} 27x + 12y = -2 \\ 9x + 4y = 3 \end{cases}$

35. $\begin{cases} 6x - 5y = 0 \\ 4x - 3y = 2 \end{cases}$

36. $\begin{cases} 5x = 1 \\ 10x + 3y = 6 \end{cases}$

37. $\begin{cases} 8y = -2x + 6 \\ 3x = 6y - 18 \end{cases}$

38. $\begin{cases} 6y = 3x + 1 \\ 9x - 27y - 3 = 0 \end{cases}$

Applications of Linear Systems

Set up a linear system and solve.

39. The sum of two numbers is 74 and their difference is 38. Find the numbers.

40. The sum of two numbers is 34. When the larger is subtracted from twice the smaller, the result is 8. Find the numbers.

41. A jar full of 40 coins consisting of dimes and nickels has a total value of $2.90. How many of each coin are in the jar?

42. A total of $9,600 was invested in two separate accounts earning 5.5% and 3.75% annual interest. If the total simple interest earned for the year was $491.25, then how much was invested in each account?

43. A 1% saline solution is to be mixed with a 3% saline solution to produce 6 ounces of a 1.8% saline solution. How much of each is needed?

44. An 80% fruit juice concentrate is to be mixed with water to produce 10 gallons of a 20% fruit juice mixture. How much of each is needed?

45. An executive traveled a total of 4½ hours and 435 miles to a conference by car and by light aircraft. Driving to the airport by car, he averaged 50 miles per hour. In the air, the light aircraft averaged 120 miles per hour. How long did it take him to drive to the airport?

46. Flying with the wind, an airplane traveled 1,065 miles in 3 hours. On the return trip, against the wind, the airplane traveled 915 miles in 3 hours. What is the speed of the wind?

Systems of Linear Inequalities (Two Variables)

Determine whether the given point is a solution to the system of linear inequalities.

47. (5, −2); $\left\{\begin{array}{r} 5x - y > 8 \\ -3x + y \leq -6 \end{array}\right.$

48. (2, 3); $\left\{\begin{array}{r} 2x - 3y > -10 \\ -5x + y > 1 \end{array}\right.$

49. (2, −10); $\begin{cases} y < -10 \\ x - y \geq 0 \end{cases}$

50. (0, −2); $\begin{cases} y > \frac{1}{2}x - 4 \\ y < -\frac{3}{4}x + 2 \end{cases}$

Graph the solution set.

51. $\begin{cases} 8x + 3y \leq 24 \\ 2x + 3y < 12 \end{cases}$

52. $\begin{cases} x + y \geq 7 \\ 4x - y \geq 0 \end{cases}$

53. $\begin{cases} x - 3y > -12 \\ -2x + 6y > -6 \end{cases}$

54. $\begin{cases} y \leq 7 \\ x - y > 0 \end{cases}$

55. $\begin{cases} y < 4 \\ y \geq \frac{4}{3}x + 1 \\ y > -x - 1 \end{cases}$

56. $\begin{cases} x - y \geq -3 \\ x - y \leq 3 \\ x + y < 1 \end{cases}$

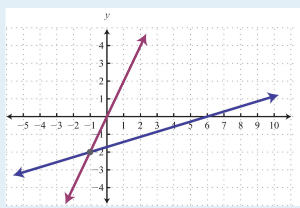
SAMPLE EXAM

1. Is (−3, 2) a solution to the system $\begin{cases} 2x - 3y = -12 \\ -4x + y = 14 \end{cases}$?

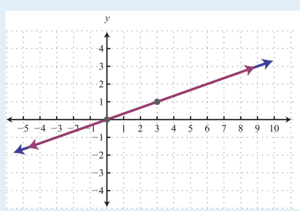
2. Is (−2, 9) a solution to the system $\begin{cases} x + y \geq 7 \\ 4x - y < 0 \end{cases}$?

Given the graph, determine the simultaneous solution.

3.



4.



Solve using the graphing method.

5. $\begin{cases} y = x - 3 \\ y = -\frac{1}{2}x + 3 \end{cases}$

6. $\begin{cases} 2x + 3y = 6 \\ -x + 6y = -18 \end{cases}$

7. $\begin{cases} y = 2 \\ x + y = 3 \end{cases}$

8. $\begin{cases} y = x \\ x = -5 \end{cases}$

Solve using the substitution method.

9. $\begin{cases} 5x + y = -14 \\ 2x - 3y = -9 \end{cases}$

10. $\begin{cases} 4x - 3y = 1 \\ x - 2y = 2 \end{cases}$

11. $\begin{cases} 5x + y = 1 \\ 10x + 2y = 4 \end{cases}$

12. $\begin{cases} x - 2y = 4 \\ 3x - 6y = 12 \end{cases}$

Solve using the elimination method.

13. $\begin{cases} 4x - y = 13 \\ -5x + 2y = -17 \end{cases}$

14. $\begin{cases} 7x - 3y = -23 \\ 4x + 5y = 7 \end{cases}$

15. $\begin{cases} -3x + 18y = 18 \\ x - 6y = 6 \end{cases}$

16. $\begin{cases} -4x + 3y = -3 \\ 8x - 6y = 6 \end{cases}$

17. $\begin{cases} \frac{1}{2}x + \frac{3}{4}y = \frac{7}{4} \\ 4x - \frac{1}{3}y = \frac{4}{3} \end{cases}$

18. $\begin{cases} 0.2x - 0.1y = -0.24 \\ -0.3x + 0.5y = 0.08 \end{cases}$

Graph the solution set.

19. $\begin{cases} 3x + 4y < 24 \\ x - 4y < 8 \end{cases}$

20. $\begin{cases} x \leq 8 \\ 3x - 8y \leq 0 \end{cases}$

Set up a linear system of two equations and two variables and solve it using any method.

21. The sum of two integers is 23. If the larger integer is one less than twice the smaller, then find the two integers.

22. James has $2,400 saved in two separate accounts. One account earns 3% annual interest and the other earns 4%. If his interest for the year totals $88, then how much is in each account?

23. Mary drives 110 miles to her grandmother's house in a total of 2 hours. On the freeway, she averages 62 miles per hour. In the city she averages 34 miles per hour. How long does she spend on the freeway?

24. A 15% acid solution is to be mixed with a 35% acid solution to produce 12 ounces of a 22% acid solution. How much of each is needed?

25. Joey has bag full of 52 dimes and quarters with a total value of $8.35. How many of each coin does Joey have?

REVIEW EXERCISES ANSWERS

1: Yes

3: Yes

5: (−3, 1)

7: Ø

9: (4, −1)

11: (6, 3)

13: $\left(x, -\frac{5}{4}x + 3\right)$

15: (1, 5)

17: (4, 4)

19: (1/2, −1/3)

21: Ø

23: (−1, 1/2)

25: (7, 2)

27: (−4, −5)

29: (−1/2, 3/4)

31: (0.2, −0.5)

33: $\left(x, \frac{12}{5}x + 2\right)$

35: (5, 6)

37: (−3, 3/2)

39: 18 and 56

41: 18 dimes and 22 nickels

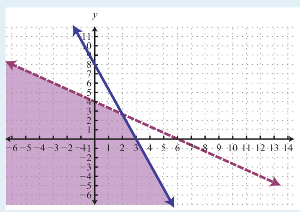
43: 3.6 ounces of the 1% saline solution and 2.4 ounces of the 3% saline solution

45: It took him 1½ hours to drive to the airport.

47: Yes

49: No

51:



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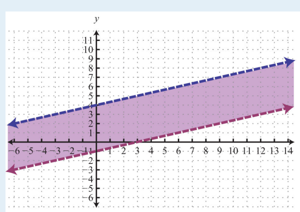

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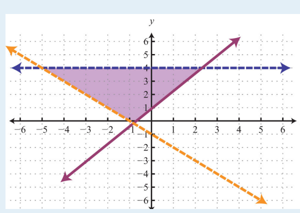


SAMPLE EXAM ANSWERS

1: Yes

3: (−1, −2)

5: (4, 1)

7: (1, 2)

9: (−3, 1)

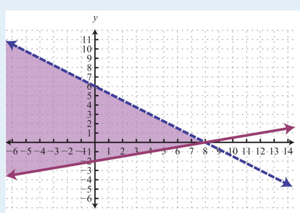
11: Ø

13: (3, −1)

15: Ø

17: (1/2, 2)

19:



21: 8 and 15

23: She drives 1½ hours on the freeway.

25: 21 quarters and 31 dimes