Advanced Algebra

v. 1.0

This is the book Advanced Algebra (v. 1.0).

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About the Author

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John Redden earned his degrees at California State University–Northridge and Glendale Community College. He is now a professor of mathematics at the College of the Sequoias, located in Visalia, California. With over a decade of experience working with students to develop their algebra skills, he knows just where they struggle and how to present complex techniques in more understandable ways. His student-friendly and commonsense approach carries over to his writing of *Intermediate Algebra* and various other open-source learning resources.

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Preface

Intermediate Algebra is the second part of a two-part course in Algebra. Written in a clear and concise manner, it carefully builds on the basics learned in Elementary Algebra and introduces the more advanced topics required for further study of applications found in most disciplines. Used as a standalone textbook, it offers plenty of review as well as something new to engage the student in each chapter. Written as a blend of the traditional and graphical approaches to the subject, this textbook introduces functions early and stresses the geometry behind the algebra. While CAS independent, a standard scientific calculator will be required and further research using technology is encouraged.

Intermediate Algebra clearly lays out the steps required to build the skills needed to solve a variety of equations and interpret the results. With robust and diverse exercise sets, students have the opportunity to solve plenty of practice problems. In addition to embedded video examples and other online learning resources, the importance of practice with pencil and paper is stressed. This text respects the traditional approaches to algebra pedagogy while enhancing it with the technology available today. In addition, Intermediate Algebra was written from the ground up in an open and modular format, allowing the instructor to modify it and leverage their individual expertise as a means to maximize the student experience and success.

The importance of Algebra cannot be overstated; it is the basis for all mathematical modeling used in all disciplines. After completing a course sequence based on Elementary and Intermediate Algebra, students will be on firm footing for success in higher-level studies at the college level.

Chapter 1

Algebra Fundamentals

1.1 Review of Real Numbers and Absolute Value

LEARNING OBJECTIVES

1. Review the set of real numbers.
2. Review the real number line and notation.
3. Define the geometric and algebraic definition of absolute value.

Real Numbers

Algebra is often described as the generalization of arithmetic. The systematic use of **variables**[1], letters used to represent numbers, allows us to communicate and solve a wide variety of real-world problems. For this reason, we begin by reviewing real numbers and their operations.

A **set**[2] is a collection of objects, typically grouped within braces { }, where each object is called an **element**[3]. When studying mathematics, we focus on special sets of numbers.

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\} \qquad \textit{Natural Numbers}$$
$$W = \{0, 1, 2, 3, 4, 5, \ldots\} \qquad \textit{Whole Numbers}$$
$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \qquad \textit{Integers}$$

The three periods (...) are called an ellipsis and indicate that the numbers continue without bound. A **subset**[4], denoted ⊆, is a set consisting of elements that belong to a given set. Notice that the sets of **natural**[5] and **whole numbers**[6] are both subsets of the set of integers and we can write:

$$\mathbb{N} \subseteq \mathbb{Z} \text{ and } W \subseteq \mathbb{Z}$$

A set with no elements is called the **empty set**[7] and has its own special notation:

1. Letters used to represent numbers.
2. Any collection of objects.
3. An object within a set.
4. A set consisting of elements that belong to a given set.
5. The set of counting numbers: {1, 2, 3, 4, 5, ...}.
6. The set of natural numbers combined with zero: {0, 1, 2, 3, 4, 5, ...}.
7. A subset with no elements, denoted Ø or { }.

$$\{\ \ \} = \emptyset \quad \textit{Empty Set}$$

Rational numbers[8], denoted $\mathbb{Q}$, are defined as any number of the form $\frac{a}{b}$ where a and b are integers and b is nonzero. We can describe this set using **set notation**[9]:

$$\mathbb{Q} = \left\{ \frac{a}{b} \middle| a, b \in \mathbb{Z},\ b \neq 0 \right\} \quad \textit{Rational Numbers}$$

The vertical line | inside the braces reads, "*such that*" and the symbol ∈ indicates set membership and reads, "*is an element of*." The notation above in its entirety reads, "*the set of all numbers* $\frac{a}{b}$ *such that a and b are elements of the set of integers and b is not equal to zero.*" Decimals that terminate or repeat are rational. For example,

$$0.05 = \frac{5}{100} \text{ and } 0.\overline{6} = 0.6666\ldots = \frac{2}{3}$$

The set of integers is a subset of the set of rational numbers, $\mathbb{Z} \subseteq \mathbb{Q}$, because every integer can be expressed as a ratio of the integer and 1. In other words, any integer can be written over 1 and can be considered a rational number. For example,

$$7 = \frac{7}{1}$$

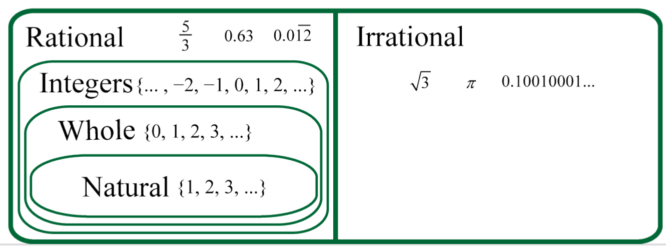
Irrational numbers[10] are defined as any numbers that cannot be written as a ratio of two integers. Nonterminating decimals that do not repeat are irrational. For example,

8. Numbers of the form $\frac{a}{b}$, where a and b are integers and b is nonzero.
9. Notation used to describe a set using mathematical symbols.
10. Numbers that cannot be written as a ratio of two integers.

$$\pi = 3.14159\ldots \text{ and } \sqrt{2} = 1.41421\ldots$$

Finally, the set of **real numbers**[11], denoted $\mathbb{R}$, is defined as the set of all rational numbers combined with the set of all irrational numbers. Therefore, all the numbers defined so far are subsets of the set of real numbers. In summary,

Real Numbers



The set of **even integers**[12] is the set of all integers that are evenly divisible by 2. We can obtain the set of even integers by multiplying each integer by 2.

$$\{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\} \quad \textit{Even Integers}$$

The set of **odd integers**[13] is the set of all nonzero integers that are not evenly divisible by 2.

$$\{\ldots, -5, -3, -1, 1, 3, 5, \ldots\} \textit{Odd Integers}$$

A **prime number**[14] is an integer greater than 1 that is divisible only by 1 and itself. The smallest prime number is 2 and the rest are necessarily odd.

$$\{2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots\} \textit{Prime Numbers}$$

11. The set of all rational and irrational numbers.

12. Integers that are divisible by 2.

13. Nonzero integers that are not divisible by 2.

14. Integer greater than 1 that is divisible only by 1 and itself.

Any integer greater than 1 that is not prime is called a **composite number**[15] and can be uniquely written as a product of primes. When a composite number, such as 42, is written as a product, $42 = 2 \cdot 21$, we say that $2 \cdot 21$ is a **factorization**[16] of 42 and that 2 and 21 are **factors**[17]. Note that factors divide the number evenly. We can continue to write composite factors as products until only a product of primes remains.

$$\begin{aligned} &42 \\ &= 2 \cdot 21 \\ &= 2 \cdot 3 \cdot 7 \end{aligned}$$

Therefore, the **prime factorization**[18] of 42 is $2 \cdot 3 \cdot 7$.

Example 1

Determine the prime factorization of 210.

Solution:

Begin by writing 210 as a product with 10 as a factor. Then continue factoring until only a product of primes remains.

$$\begin{aligned} 210 &= 10 \cdot 21 \\ &= 2 \cdot 5 \cdot 3 \cdot 7 \\ &= 2 \cdot 3 \cdot 5 \cdot 7 \end{aligned}$$

Since the prime factorization is unique, it does not matter how we choose to initially factor the number; the end result will be the same.

Answer: $2 \cdot 3 \cdot 5 \cdot 7$

15. Integers greater than 1 that are not prime.
16. Any combination of factors, multiplied together, resulting in the product.
17. Any of the numbers that form a product.
18. The unique factorization of a natural number written as a product of primes.

A **fraction**[19] is a rational number written as a quotient, or ratio, of two integers a and b where $b \neq 0$.

$$\begin{array}{r} \text{Numerator} \rightarrow \\ \text{Denominator} \rightarrow \end{array} \frac{a}{b}$$

The integer above the fraction bar is called the **numerator**[20] and the integer below is called the **denominator**[21]. Two equal ratios expressed using different numerators and denominators are called **equivalent fractions**[22]. For example,

$$\frac{50}{100} = \frac{1}{2}$$

Consider the following factorizations of 50 and 100:

$$\begin{aligned} 50 &= 2 \cdot 25 \\ 100 &= 4 \cdot 25 \end{aligned}$$

The numbers 50 and 100 share the factor 25. A shared factor is called a **common factor**[23]. Making use of the fact that $\frac{25}{25} = 1$, we have

$$\frac{50}{100} = \frac{2 \cdot \cancel{25}}{4 \cdot \cancel{25}} = \frac{2}{4} \cdot 1 = \frac{2}{4}$$

Dividing $\frac{25}{25}$ and replacing this factor with a 1 is called **cancelling**[24]. Together, these basic steps for finding equivalent fractions define the process of **reducing**[25]. Since factors divide their product evenly, we achieve the same result by dividing both the numerator and denominator by 25 as follows:

19. A rational number written as a quotient of two integers: $\frac{a}{b}$, where $b \neq 0$.

20. The number above the fraction bar.

21. The number below the fraction bar.

22. Two equal fractions expressed using different numerators and denominators.

23. A factor that is shared by more than one real number.

24. The process of dividing out common factors in the numerator and the denominator.

25. The process of finding equivalent fractions by dividing the numerator and the denominator by common factors.

$$\frac{50 \div 25}{100 \div 25} = \frac{2}{4}$$

Finding equivalent fractions where the numerator and denominator are **relatively prime**[26], or have no common factor other than 1, is called **reducing to lowest terms**[27]. This can be done by dividing the numerator and denominator by the **greatest common factor (GCF).**[28] The GCF is the largest number that divides a set of numbers evenly. One way to find the GCF of 50 and 100 is to list all the factors of each and identify the largest number that appears in both lists. Remember, each number is also a factor of itself.

$$\begin{aligned} &\{\mathbf{1}, \mathbf{2}, \mathbf{5}, \mathbf{10}, \mathbf{25}, \mathbf{50}\} && \textit{Factors of } 50 \\ &\{\mathbf{1}, \mathbf{2}, 4, \mathbf{5}, \mathbf{10}, 20, \mathbf{25}, \mathbf{50}, 100\} && \textit{Factors of } 100 \end{aligned}$$

Common factors are listed in bold, and we see that the greatest common factor is 50. We use the following notation to indicate the GCF of two numbers: GCF(50, 100) = 50. After determining the GCF, reduce by dividing both the numerator and the denominator as follows:

$$\frac{50 \div 50}{100 \div 50} = \frac{1}{2}$$

26. Numbers that have no common factor other than 1.

27. Finding equivalent fractions where the numerator and the denominator share no common integer factor other than 1.

28. The largest shared factor of any number of integers.

Example 2

Reduce to lowest terms: $\frac{108}{72}$.

Solution:

A quick way to find the GCF of the numerator and denominator requires us to first write each as a product of primes. The GCF will be the product of all the common prime factors.

$$\left.\begin{aligned} 108 &= 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \\ 72 &= 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \end{aligned}\right\} \text{GCF}(108, 72) = 2 \cdot 2 \cdot 3 \cdot 3 = 36$$

In this case, the product of the common prime factors is 36.

$$\frac{108}{72} = \frac{108 \div 36}{72 \div 36} = \frac{3}{2}$$

We can convert the improper fraction $\frac{3}{2}$ to a mixed number $1\ \frac{1}{2}$; however, it is important to note that converting to a mixed number is not part of the reducing process. We consider improper fractions, such as $\frac{3}{2}$, to be reduced to lowest terms. In algebra it is often preferable to work with improper fractions, although in some applications, mixed numbers are more appropriate.

Answer: $\frac{3}{2}$

Recall the relationship between multiplication and division:

$$\begin{matrix} \textit{dividend} \rightarrow \\ \textit{divisor} \rightarrow \end{matrix} \frac{12}{6} = 2 \quad \leftarrow \quad \textit{quotient} \text{ because } 6 \cdot 2 = 12$$

In this case, the **dividend**[29] 12 is evenly divided by the **divisor**[30] 6 to obtain the **quotient**[31] 2. It is true in general that if we multiply the divisor by the quotient we obtain the dividend. Now consider the case where the dividend is zero and the divisor is nonzero:

$$\frac{0}{6} = 0 \text{ since } 6 \cdot 0 = 0$$

This demonstrates that zero divided by any nonzero real number must be zero. Now consider a nonzero number divided by zero:

$$\frac{12}{0} = \ ? \text{ or } 0 \cdot \ ? = 12$$

Zero times anything is zero and we conclude that there is no real number such that $0 \cdot \ ? = 12$. Thus, the quotient $12 \div 0$ is **undefined**[32]. Try it on a calculator, what does it say? For our purposes, we will simply write "undefined." To summarize, given any real number $a \neq 0$, then

$$0 \div a = \frac{0}{a} = 0 \quad \textit{zero} \text{ and } a \div 0 = \frac{a}{0} \quad \textit{undefined}$$

We are left to consider the case where the dividend and divisor are both zero.

29. A number to be divided by another number.

30. The number that is divided into the dividend.

31. The result of division.

32. A quotient such as $\frac{5}{0}$ is left without meaning and is not assigned an interpretation.

$$\frac{0}{0} = ? \text{ or } 0 \cdot ? = 0$$

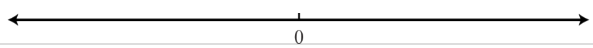
Here, any real number seems to work. For example, $0 \cdot 5 = 0$ and also, $0 \cdot 3 = 0$. Therefore, the quotient is uncertain or **indeterminate**[33].

$$0 \div 0 = \frac{0}{0} \quad \textit{indeterminate}$$

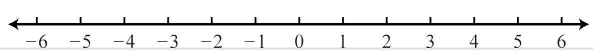
In this course, we state that $0 \div 0$ is undefined.

The Number Line and Notation

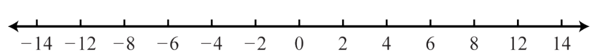
A **real number line**[34], or simply **number line**, allows us to visually display real numbers by associating them with unique points on a line. The real number associated with a point is called a **coordinate**[35]. A point on the real number line that is associated with a coordinate is called its **graph**[36]. To construct a number line, draw a horizontal line with arrows on both ends to indicate that it continues without bound. Next, choose any point to represent the number zero; this point is called the **origin**[37].

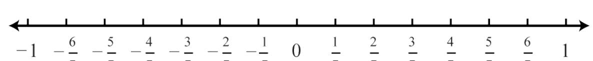


Positive real numbers lie to the right of the origin and negative real numbers lie to the left. The number zero (0) is neither positive nor negative. Typically, each tick represents one unit.



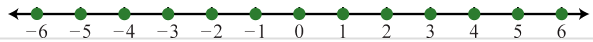
As illustrated below, the scale need not always be one unit. In the first number line, each tick mark represents two units. In the second, each tick mark represents $\frac{1}{7}$:





33. A quotient such as $\frac{0}{0}$ is a quantity that is uncertain or ambiguous.
34. A line that allows us to visually represent real numbers by associating them with points on the line.
35. The real number associated with a point on a number line.
36. A point on the number line associated with a coordinate.
37. The point on the number line that represents zero.

The graph of each real number is shown as a dot at the appropriate point on the number line. A partial graph of the set of integers $\mathbb{Z}$, follows:



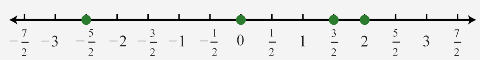
Example 3

Graph the following set of real numbers: $\left\{-\frac{5}{2},\ 0,\ \frac{3}{2},\ 2\right\}$.

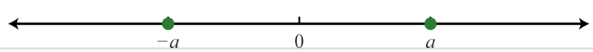
Solution:

Graph the numbers on a number line with a scale where each tick mark represents $\frac{1}{2}$ unit.

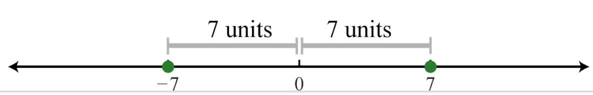
Answer:



The **opposite**[38] of any real number *a* is −*a*. Opposite real numbers are the same distance from the origin on a number line, but their graphs lie on opposite sides of the origin and the numbers have opposite signs.



Given the integer −7, the integer the same distance from the origin and with the opposite sign is +7, or just 7.



Therefore, we say that the opposite of −7 is −(−7) = 7. This idea leads to what is often referred to as the **double-negative property**[39]. For any real number *a*,

38. Real numbers whose graphs are on opposite sides of the origin with the same distance to the origin.

39. The opposite of a negative number is positive: $-(-a) = a$.

$$-(-a) = a$$

Example 4

Calculate: $-\left(-\left(-\frac{3}{8}\right)\right)$.

Solution:

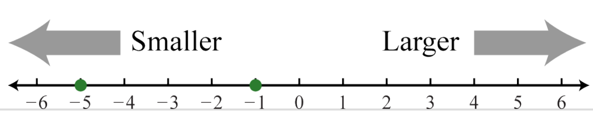
Here we apply the double-negative within the innermost parentheses first.

$$\begin{aligned} -\left(-\left(-\frac{3}{8}\right)\right) &= -\left(\frac{3}{8}\right) \\ &= -\frac{3}{8} \end{aligned}$$

Answer: $-\frac{3}{8}$

In general, an odd number of sequential negative signs results in a negative value and an even number of sequential negative signs results in a positive value.

When comparing real numbers on a number line, the larger number will always lie to the right of the smaller one. It is clear that 15 is greater than 5, but it may not be so clear to see that −1 is greater than −5 until we graph each number on a number line.



We use symbols to help us efficiently communicate relationships between numbers on the number line.

Equality Relationships		*Order Relationships*	
$=$	*"is equal to"*	$<$	*"is less than"*
$\neq$	*"is not equal to"*	$>$	*"is greater than"*
$\approx$	*"is approximately equal to"*	$\leq$	*"is less than or equal to"*
		$\geq$	*"is greater than or equal to"*

The relationship between the **integers**[40] in the previous illustration can be expressed two ways as follows:

$$-5 < -1 \quad \text{"Negative five is less than negative one."}$$

or

$$-1 > -5 \quad \text{"Negative one is greater than negative five."}$$

The symbols < and > are used to denote **strict inequalities**[41], and the symbols $\leq$ and $\geq$ are used to denote **inclusive inequalities**[42]. In some situations, more than one symbol can be correctly applied. For example, the following two statements are both true:

$$-10 < 0 \text{ and } -10 \leq 0$$

In addition, the "*or equal to*" component of an inclusive inequality allows us to correctly write the following:

$$-10 \leq -10$$

40. The set of positive and negative whole numbers combined with zero: {..., −3, −2, −1, 0, 1, 2, 3, ...}.

41. Express ordering relationships using the symbol < for "less than" and > for "greater than."

42. Use the symbol $\leq$ to express quantities that are "less than or equal to" and $\geq$ for quantities that are "greater than or equal to" each other.

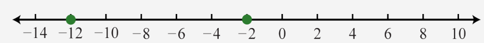
The logical use of the word "*or*" requires that only one of the conditions need be true: the "*less than*" or the "*equal to*."

Example 5

Fill in the blank with <, =, or >: $-2\_\_\_\_ - 12$.

Solution:

Use > because the graph of −2 is to the right of the graph of −12 on a number line. Therefore, −2 > −12, which reads, "*negative two is greater than negative twelve.*"

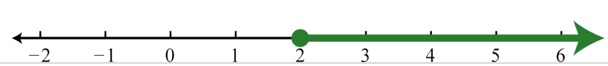


Answer: −2 > −12

An **algebraic inequality**[43], such as $x \geq 2$, is read, "*x is greater than or equal to 2*." Here the letter *x* is a variable, which can represent any real number. However, the statement $x \geq 2$ imposes a condition on the variable. **Solutions**[44] are the values for *x* that satisfy the condition. This inequality has infinitely many solutions for *x*, some of which are 2, 3, 4.1, 5, 20, and 20.001. Since it is impossible to list all of the solutions, a system is needed that allows a clear communication of this infinite set. Common ways of expressing solutions to an inequality are by graphing them on a number line, using interval notation, or using set notation.

To express the solution graphically, draw a number line and shade in all the values that are solutions to the inequality. This is called the **graph of the solution set**[45]. Interval and set notation follow:

"*x is greater than or equal to* 2"$x \geq 2$



43. Algebraic expressions related with the symbols $\leq$, <, $\geq$, and >.

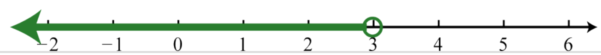
44. Values that can be used in place of the variable to satisfy the given condition.

45. Solutions to an algebraic expression expressed on a number line.

$$\textit{Interval notation}: \quad [2, \infty)$$
$$\textit{Set notation}: \quad \{x \in \mathbb{R} | x \geq 2\}$$

In this example, there is an inclusive inequality, which means that the lower-bound 2 is included in the solution set. Denote this with a closed dot on the number line and a square bracket in interval notation. The symbol ∞ is read as "**infinity**[46]" and indicates that the set is unbounded to the right on a number line. If using a standard keyboard, use (inf) as a shortened form to denote infinity. Now compare the notation in the previous example to that of the strict, or noninclusive, inequality that follows:

$$\text{" } x \textit{ is less than } 3 \text{ "} \quad x < 3$$



$$\textit{Interval notation}: \quad (-\infty, 3)$$
$$\textit{Set notation}: \quad \{x \in \mathbb{R} | x < 3\}$$

Strict inequalities imply that solutions may get very close to the boundary point, in this case 3, but not actually include it. Denote this idea with an open dot on the number line and a round parenthesis in interval notation. The symbol −∞ is read as "**negative infinity**[47]" and indicates that the set is unbounded to the left on a number line. Infinity is a bound to the real numbers, but is not itself a real number: it cannot be included in the solution set and thus is always enclosed with a parenthesis.

Interval notation is textual and is determined after graphing the solution set on a number line. The numbers in interval notation should be written in the same order as they appear on the number line, with smaller numbers in the set appearing first. Set notation, sometimes called set-builder notation, allows us to describe the set using familiar mathematical notation. For example,

46. The symbol ∞ indicates the interval is unbounded to the right.

47. The symbol −∞ indicates the interval is unbounded to the left.

$$\{x \in \mathbb{R} | x \geq 2\}$$

Here, $x \in \mathbb{R}$ describes the type of number. This implies that the variable x represents a real number. The statement $x \geq 2$ is the condition that describes the set using mathematical notation. At this point in our study of algebra, it is assumed that all variables represent real numbers. For this reason, you can omit the "$\in \mathbb{R}$", and write

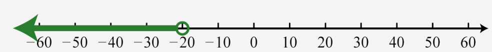
$$\{x | x \geq 2\}$$

Example 6

Graph the solution set and give the interval and set notation equivalents: $x < -20$.

Solution:

Use an open dot at −20, because of the strict inequality <, and shade all real numbers to the left.



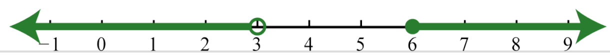
Answer: Interval notation: $(-\infty, -20)$; set notation: $\{x | x < -20\}$

A **compound inequality**[48] is actually two or more inequalities in one statement joined by the word "and" or by the word "or". Compound inequalities with the logical "or" require that either condition must be satisfied. Therefore, the solution set of this type of compound inequality consists of all the elements of the solution sets of each inequality. When we join these individual solution sets it is called the **union**[49], denoted $\cup$. For example,

48. Two or more inequalities in one statement joined by the word "and" or by the word "or."

49. The set formed by joining the individual solution sets indicated by the logical use of the word "or" and denoted with the symbol ∪.

$$x < 3 \text{ or } x \geq 6$$



$$\textit{Interval notation}: \quad (-\infty, 3) \cup [6, \infty)$$
$$\textit{Set notation}: \quad \{x \mid x < 3 \text{ or } x \geq 6\}$$

An inequality such as,

$$-1 \leq x < 3$$

reads, "*negative one is less than or equal to x and x is less than three*." This is actually a compound inequality because it can be decomposed as follows:

$$-1 \leq x \text{ and } x < 3$$

The logical "and" requires that both conditions must be true. Both inequalities will be satisfied by all the elements in the **intersection**[50], denoted $\cap$, of the solution sets of each.

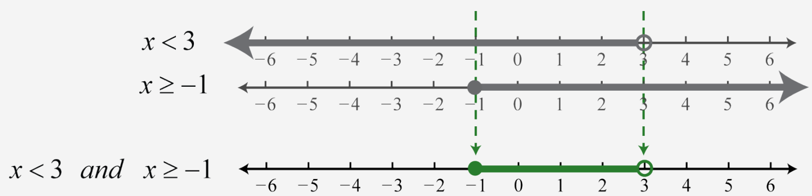
50. The set formed by the shared values of the individual solution sets that is indicated by the logical use of the word "and," denoted with the symbol $\cap$.

Example 7

Graph and give the interval notation equivalent: $-1 \leq x < 3$.

Solution:

Determine the intersection, or overlap, of the two solution sets to $x < 3$ and $x \geq -1$. The solutions to each inequality are sketched above the number line as a means to determine the intersection, which is graphed on the number line below.



Here, 3 is not a solution because it solves only one of the inequalities. Alternatively, we may interpret $-1 \leq x < 3$ as all possible values for x between, or bounded by, -1 and 3 where -1 is included in the solution set.

Answer: Interval notation: $[-1, \ 3)$; set notation: $\{x| -1 \leq x < 3\}$

In this text, we will often point out the equivalent notation used to express mathematical quantities electronically using the standard symbols available on a keyboard.

$$\begin{array}{llll} \times & "*" & \geq & ">=" \\ \div & "/" & \leq & "<=" \\ & & \neq & "!=" \end{array}$$

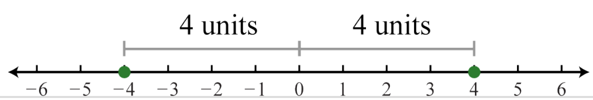
Many calculators, computer algebra systems, and programming languages use the notation presented above, in quotes.

Absolute Value

The **absolute value**[51] of a real number *a*, denoted $|a|$, is defined as the distance between zero (the origin) and the graph of that real number on the number line. Since it is a distance, it is always positive. For example,

$$|-4| = 4 \text{ and } |4| = 4$$

Both 4 and −4 are four units from the origin, as illustrated below:



Also, it is worth noting that,

$$|0| = 0$$

The algebraic definition of the absolute value of a real number *a* follows:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

This is called a **piecewise definition**[52]. The result depends on the quantity *a*. If *a* is nonnegative, as indicated by the inequality $a \geq 0$, then the absolute value will be that number *a*. If *a* is negative, as indicated by the inequality $a < 0$, then the absolute value will be the opposite of that number, −*a*. The results will be the same as the geometric definition. For example, to determine $|-4|$ we make note that the value is negative and use the second part of the definition. The absolute value will be the opposite of −4.

51. The absolute value of a number represents the distance from the graph of the number to zero on a number line.

52. A definition that changes depending on the value of the variable.

$$\begin{aligned} |-4| &= -(-4) \\ &= 4 \end{aligned}$$

At this point, we can determine what real numbers have certain absolute values.

Example 8

Determine the values represented by *x*: $|x| = 6$.

Solution:

Think of a real number whose distance to the origin is 6 units. There are two solutions: the distance to the right of the origin and the distance to the left of the origin, namely $\{\pm 6\}$. The symbol ± is read "*plus or minus*" and indicates that there are two answers, one positive and one negative.

$$|-6| = 6 \text{ and } |6| = 6$$

Answer: $x = \pm 6$

Example 9

Determine the values represented by x: $|x| = -6$.

Solution:

Here we wish to find a value where the distance to the origin is negative. Since negative distance is not defined, this equation has no solution. Use the empty set Ø to denote this.

Answer: Ø

The absolute value can be expressed textually using the notation abs(a). We often encounter negative absolute values, such as $-|3|$ or -abs(3). Notice that the negative sign is in front of the absolute value symbol. In this case, work the absolute value first and then find the opposite of the result.

$$\begin{array}{ccc} -|3| & & -|-3| \\ \downarrow & \text{and} & \downarrow \\ =-3 & & =-3 \end{array}$$

Try not to confuse this with the double negative property, which states that $-(-3) = +3$.

Example 10

Simplify: $-\left(-\left|-50\right|\right)$.

Solution:

First, find the absolute value of -50 and then apply the double-negative property.

$$\begin{aligned} -\left(-\left|-50\right|\right) &= -\left(-50\right) \\ &= 50 \end{aligned}$$

Answer: 50

KEY TAKEAWAYS

- Algebra is often described as the generalization of arithmetic. The systematic use of variables, used to represent real numbers, allows us to communicate and solve a wide variety of real-world problems. Therefore, it is important to review the subsets of real numbers and their properties.
- The number line allows us to visually display real numbers by associating them with unique points on a line.
- Special notation is used to communicate equality and order relationships between numbers on a number line.
- The absolute value of a real number is defined geometrically as the distance between zero and the graph of that number on a number line. Alternatively, the absolute value of a real number is defined algebraically in a piecewise manner. If a real number a is nonnegative, then the absolute value will be that number a. If a is negative, then the absolute value will be the opposite of that number, $-a$.

TOPIC EXERCISES

PART A: REAL NUMBERS

Use set notation to list the described elements.

1. Every other positive odd number up to 21.
2. Every other positive even number up to 22.
3. The even prime numbers.
4. Rational numbers that are also irrational.
5. The set of negative integers.
6. The set of negative even integers.
7. Three consecutive odd integers starting with 13.
8. Three consecutive even integers starting with 22.

Determine the prime factorization of the given composite number.

9. 195
10. 78
11. 330
12. 273
13. 180
14. 350

Reduce to lowest terms.

15. $\frac{42}{30}$
16. $\frac{105}{70}$
17. $\frac{84}{120}$
18. $\frac{315}{420}$

19. $\frac{60}{45}$

20. $\frac{144}{120}$

21. $\frac{64}{128}$

22. $\frac{72}{216}$

23. $\frac{0}{25}$

24. $\frac{33}{0}$

PART B: NUMBER LINE AND NOTATION

Graph the following sets of numbers.

25. {−5, 5, 10, 15}

26. {−4, −2, 0, 2, 4}

27. $\left\{-\frac{3}{2}, -\frac{1}{2}, 0, 1, 2\right\}$

28. $\left\{-\frac{3}{4}, -\frac{1}{4}, 0, \frac{1}{2}, \frac{3}{4}\right\}$

29. {−5,−4,−3,−1, 1}

30. {−40, −30, −20, 10, 30}

Simplify.

31. -(−10)

32. $-\left(-\frac{3}{5}\right)$

33. -(-(−12))

34. $-\left(-\left(-\frac{5}{3}\right)\right)$

35. $-\left(-\left(-\left(-\frac{1}{2}\right)\right)\right)$

36. $-\left(-\left(-\left(-\left(-\frac{3}{4}\right)\right)\right)\right)$

Fill in the blank with <, =, or >.

37. −10 _____ −15

38. −101 _____ −100

39. −33 _____ 0

40. 0 _____ −50

41. −(−(−2)) _____ −(−3)

42. $-\left(-\left(-\frac{1}{2}\right)\right)$ _____ $-\frac{1}{4}$

43. $-\left(-\left(-\frac{2}{3}\right)\right)$ _____ $-\left(-\frac{1}{2}\right)$

44. $-\left(-\frac{2}{3}\right)$ _____ $-\left(-\left(-\left(-\frac{2}{3}\right)\right)\right)$

True or False.

45. $0 = 0$

46. $5 \leq 5$

47. $1.0\overline{3}2$ is irrational.

48. 0 is a nonnegative number.

49. Any integer is a rational number.

50. The constant π is rational.

Graph the solution set and give the interval notation equivalent.

51. $x < -1$

52. $x > -3$

53. $x \geq -8$

54. $x \leq 6$

55. $-10 \leq x < 4$

56. $3 < x \leq 7$

57. $-40 < x < 0$

58. $-12 \leq x \leq -4$

59. $x < 5$ and $x \geq 0$

60. $x \leq -10$ and $x \geq -40$
61. $x \leq 7$ and $x < 10$
62. $x < 1$ and $x > 3$
63. $x < -2$ or $x \geq 5$
64. $x \leq 0$ or $x \geq 4$
65. $x < 6$ or $x > 2$
66. $x < 0$ or $x \leq 5$

Write an equivalent inequality.

67. All real numbers less than –15.
68. All real numbers greater than or equal to –7.
69. All real numbers less than 6 and greater than zero.
70. All real numbers less than zero and greater than –5.
71. All real numbers less than or equal to 5 or greater than 10.
72. All real numbers between –2 and 2.

Determine the inequality given the answers expressed in interval notation.

73. $(-\infty, 12)$
74. $[-8, \infty)$
75. $(-\infty, 0]$
76. $(0, \infty)$
77. $(-6, 14)$
78. $(0, 12]$
79. $[5, 25)$
80. $[-30, -10]$

81. $(-\infty, 2) \cup [3, \infty)$

82. $(-\infty, -19] \cup [-12, \infty)$

83. $(-\infty, -2) \cup (0, \infty)$

84. $(-\infty, -15] \cup (-5, \infty)$

PART C: ABSOLUTE VALUE

Simplify.

85. $|-9|$

86. $|14|$

87. $-|-4|$

88. $-|8|$

89. $-\left|-\frac{5}{8}\right|$

90. $-\left(-\left|\frac{7}{2}\right|\right)$

91. $-|-(-7)|$

92. $-|-(-10)|$

93. $-(-|-2|)$

94. $-(-|-10|)$

95. $-\left(-\left|-(-5)\right|\right)$

96. $-(-(-|-20|))$

Determine the values represented by *a*.

97. $|a| = 10$

98. $|a| = 7$

99. $|a| = \frac{1}{2}$

100. $|a| = \frac{9}{4}$

101. $|a| = 0$

102. $|a| = -1$

PART D: DISCUSSION BOARD

103. Research and discuss the origins and evolution of algebra.

104. Research and discuss reasons why algebra is a required subject today.

105. Solution sets to inequalities can be expressed using a graph, interval notation, or set notation. Discuss the merits and drawbacks of each method. Which do you prefer?

106. Research and discuss the Fundamental Theorem of Algebra. Illustrate its idea with an example and share your results.

ANSWERS

1. $\{1, 5, 9, 13, 17, 21\}$

3. $\{2\}$

5. $\{\ldots, -3, -2, -1\}$

7. $\{13, 15, 17\}$

9. $3 \cdot 5 \cdot 13$

11. $2 \cdot 3 \cdot 5 \cdot 11$

13. $2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$

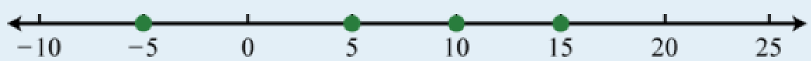
15. $\frac{7}{5}$

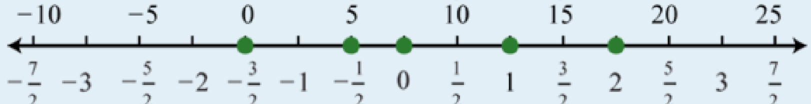
17. $\frac{7}{10}$

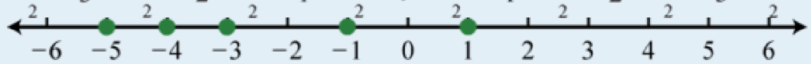
19. $\frac{4}{3}$

21. $\frac{1}{2}$

23. 0

25. 

27. 

29. 

31. 10

33. −12

35. $\frac{1}{2}$

37. $>$

39. $<$

41. $<$

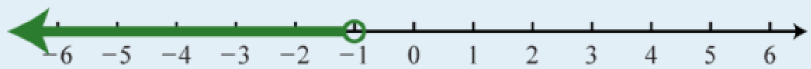
43. $<$

45. True

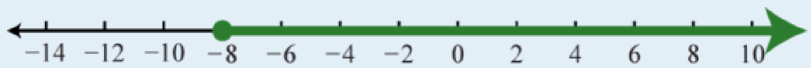
47. False

49. True

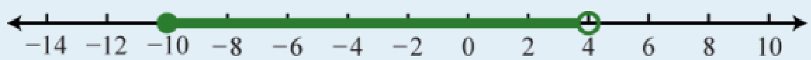
51. (-∞, -1);



53. $[8, \infty)$;



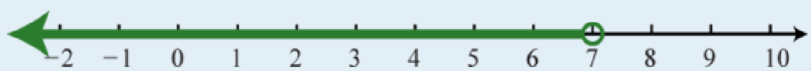
55. $[-10, 4)$;



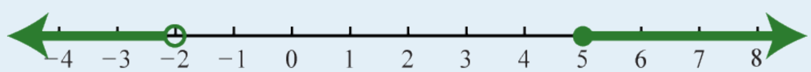
57. $(-40, 0)$;

59. $[0, 5)$;

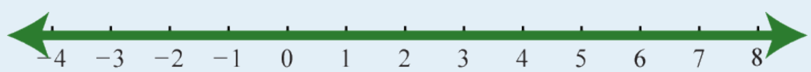
61. $(-\infty, 7)$;



63. $(-\infty, -2) \cup [5, \infty)$;



65. $(-\infty, \infty) = \mathbb{R}$;



67. $x < -15$

69. $0 < x < 6$

71. $x \leq 5$ or $x > 10$

73. $x < 12$

75. $x \leq 0$

77. $-6 < x < 14$

79. $5 \leq x < 25$

81. $x < 2$ or $x \geq 3$

83. $x < -2$ or $x > 0$

85. 9

87. −4

89. $-\frac{5}{8}$

91. −7

93. 2

95. 5

97. $a = \pm 10$

99. $a = \pm\frac{1}{2}$

101. $a = 0$

103. Answer may vary

105. Answer may vary

1.2 Operations with Real Numbers

LEARNING OBJECTIVES

1. Review the properties of real numbers.
2. Simplify expressions involving grouping symbols and exponents.
3. Simplify using the correct order of operations.

Working with Real Numbers

In this section, we continue to review the properties of real numbers and their operations. The result of adding real numbers is called the **sum**[53] and the result of subtracting is called the **difference**[54]. Given any real numbers *a*, *b*, and *c*, we have the following properties of addition:

Additive Identity Property:	$a + 0 = 0 + a = a$
Additive Inverse Property:	$a + (-a) = (-a) + a = 0$
Associative Property:	$(a + b) + c = a + (b + c)$
Commutative Property:	$a + b = b + a$

53. The result of adding.

54. The result of subtracting.

55. Given any real number *a*, $a + 0 = 0 + a = a$.

56. Given any real number *a*, $a + (-a) = (-a) + a = 0$.

57. Given real numbers *a*, *b* and *c*, $(a + b) + c = a + (b + c)$.

[58]

It is important to note that addition is commutative and subtraction is not. In other words, the order in which we add does not matter and will yield the same result. However, this is not true of subtraction.

$$\begin{aligned} 5+10 &= 10+5 \\ 15 &= 15 \end{aligned} \qquad\qquad \begin{aligned} 5-10 &\neq 10-5 \\ -5 &\neq 5 \end{aligned}$$

We use these properties, along with the double-negative property for real numbers, to perform more involved sequential operations. To simplify things, make it a general rule to first replace all sequential operations with either addition or subtraction and then perform each operation in order from left to right.

Example 1

Simplify: $-10-\ (-10)+\ (-5)$.

Solution:

Replace the sequential operations and then perform them from left to right.

$$\begin{aligned} -10-\ (-10)+\ (-5) &= -10+10-5 && \textit{Replace } -(-) \textit{ with addition } (+). \\ & && \textit{Replace } +(-) \textit{ with subtraction } (-). \\ &= 0-5 \\ &= -5 \end{aligned}$$

Answer: -5

58. Given real numbers *a* and *b*, $a+b=b+a$.

Adding or subtracting fractions requires a **common denominator**[59]. Assume the common denominator c is a nonzero integer and we have

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \text{ and } \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

59. A denominator that is shared by more than one fraction.

Example 2

Simplify: $\frac{2}{9} - \frac{1}{15} + \frac{8}{45}$.

Solution:

First determine the least common multiple (LCM) of 9, 15, and 45. The least common multiple of all the denominators is called the **least common denominator**[60] (LCD). We begin by listing the multiples of each given denominator:

$$\begin{array}{ll} \{9, 18, 27, 36, \mathbf{45}, 54, 63, 72, 81, \mathbf{90}, \ldots\} & \textit{Multiples of } 9 \\ \{15, 30, \mathbf{45}, 60, 75, \mathbf{90}, \ldots\} & \textit{Multiples of } 15 \\ \{\mathbf{45}, \mathbf{90}, \mathbf{135} \ldots\} & \textit{Multiples of } 45 \end{array}$$

Here we see that the LCM(9, 15, 45) = 45. Multiply the numerator and the denominator of each fraction by values that result in equivalent fractions with the determined common denominator.

$$\begin{aligned} \frac{2}{9} - \frac{1}{15} + \frac{8}{45} &= \frac{2}{9} \cdot \frac{5}{5} - \frac{1}{15} \cdot \frac{3}{3} + \frac{8}{45} \\ &= \frac{10}{45} - \frac{3}{45} + \frac{8}{45} \end{aligned}$$

Once we have equivalent fractions, with a common denominator, we can perform the operations on the numerators and write the result over the common denominator.

60. The least common multiple of a set of denominators.

$$= \frac{10 - 3 + 8}{45}$$
$$= \frac{15}{45}$$

And then reduce if necessary,

$$= \frac{15 \div 15}{45 \div 15}$$
$$= \frac{1}{3}$$

Answer: $\frac{1}{3}$

Finding the LCM using lists of multiples, as described in the previous example, is often very cumbersome. For example, try making a list of multiples for 12 and 81. We can streamline the process of finding the LCM by using prime factors.

$$12 = 2^2 \cdot 3$$
$$81 = 3^4$$

The least common multiple is the product of each prime factor raised to the highest power. In this case,

$$\text{LCM}(12, 81) = 2^2 \cdot 3^4 = 324$$

Often we will find the need to translate English sentences involving addition and subtraction to mathematical statements. Below are some common translations.

$$n+2 \text{The sum of a number and 2.}$$
$$2-n \text{The difference of 2 and a number.}$$
$$n-2 \text{Here 2 is subtracted from a number.}$$

Example 3

What is 8 subtracted from the sum of 3 and $\frac{1}{2}$?

Solution:

We know that subtraction is not commutative; therefore, we must take care to subtract in the correct order. First, add 3 and $\frac{1}{2}$ and then subtract 8 as follows:

$$\overbrace{\left(3 + \frac{1}{2}\right)}^{\text{"the sum of 3 and } \frac{1}{2}\text{"}} \overbrace{- \quad 8}^{\text{"subtract 8 from the sum"}}$$

Perform the indicated operations.

$$\begin{aligned}
\left(3+\frac{1}{2}\right)-8 &= \left(\frac{3}{1}\cdot\frac{2}{2}+\frac{1}{2}\right)-8 \\
&= \left(\frac{6+1}{2}\right)-8 \\
&= \frac{7}{2}-\frac{8}{1}\cdot\frac{2}{2} \\
&= \frac{7-16}{2} \\
&= -\frac{9}{2}
\end{aligned}$$

Answer: $-\frac{9}{2}$

The result of multiplying real numbers is called the **product**[61] and the result of dividing is called the **quotient**[62]. Given any real numbers a, b, and c, we have the following properties of multiplication:

61. The result of multiplying.

62. The result of dividing.

Zero Factor Property:	$a \cdot 0 = 0 \cdot a = 0$
Multiplicative Identity Property:	$a \cdot 1 = 1 \cdot a = a$
Associative Property:	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
Commutative Property:	$a \cdot b = b \cdot a$

63

64

65

66

It is important to note that multiplication is commutative and division is not. In other words, the order in which we multiply does not matter and will yield the same result. However, this is not true of division.

$$5 \cdot 10 = 10 \cdot 5 \qquad 5 \div 10 \neq 10 \div 5$$
$$50 = 50 \qquad 0.5 \neq 2$$

We will use these properties to perform sequential operations involving multiplication and division. Recall that the product of a positive number and a negative number is negative. Also, the product of two negative numbers is positive.

63. Given any real number *a*, $a \cdot 0 = 0 \cdot a = 0$.

64. Given any real number *a*, $a \cdot 1 = 1 \cdot a = a$.

65. Given any real numbers *a*, *b* and *c*, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

66. Given any real numbers *a* and *b*, $a \cdot b = b \cdot a$.

Example 4

Multiply: $5(-3)(-2)(-4)$.

Solution:

Multiply two numbers at a time as follows:

$$
\begin{aligned}
5(-3)(-2)(-4) &= \underbrace{5(-3)}(-2)(-4) \\
&= \underbrace{-15(-2)}(-4) \\
&= \underbrace{30(-4)} \\
&= -120
\end{aligned}
$$

Answer: −120

Because multiplication is commutative, the order in which we multiply does not affect the final answer. However, when sequential operations involve multiplication and division, order does matter; hence we must work the operations from *left to right* to obtain a correct result.

Example 5

Simplify: $10 \div (-2)(-5)$.

Solution:

Perform the division first; otherwise the result will be incorrect.

Correct!

$$\begin{aligned} 10 \div (-2)(-5) &= \underbrace{10 \div (-2)}_{\text{division first}}(-5) \\ &= -5(-5) \\ &= 25 \checkmark \end{aligned}$$

Incorrect!

$$\begin{aligned} 10 \div (-2)(-5) &= 10 \div \underbrace{(-2)(-5)}_{\text{multiplication first}} \\ &= 10 \div 10 \\ &= 1 \times \end{aligned}$$

Notice that the order in which we multiply and divide does affect the result. Therefore, it is important to perform the operations of multiplication and division as they appear from left to right.

Answer: 25

The product of two fractions is the fraction formed by the product of the numerators and the product of the denominators. In other words, to multiply fractions, multiply the numerators and multiply the denominators:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Example 6

Multiply: $-\frac{4}{5} \cdot \frac{25}{12}$.

Solution:

Multiply the numerators and multiply the denominators. Reduce by dividing out any common factors.

$$
\begin{aligned}
-\frac{4}{5} \cdot \frac{25}{12} &= -\frac{4 \cdot 25}{5 \cdot 12} \\
&= -\frac{\overset{1}{\cancel{4}} \cdot \overset{5}{\cancel{25}}}{\underset{1}{\cancel{5}} \cdot \underset{3}{\cancel{12}}} \\
&= -\frac{5}{3}
\end{aligned}
$$

Answer: $-\frac{5}{3}$

Two real numbers whose product is 1 are called **reciprocals**[67]. Therefore, $\frac{a}{b}$ and $\frac{b}{a}$ are reciprocals because $\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ab} = 1$For example,

$$\frac{2}{3} \cdot \frac{3}{2} = \frac{6}{6} = 1$$

Because their product is 1, $\frac{2}{3}$ and $\frac{3}{2}$ are reciprocals. Some other reciprocals are listed below:

67. Two real numbers whose product is 1.

$$\frac{5}{8} \text{ and } \frac{8}{5} \qquad 7 \text{ and } \frac{1}{7} \qquad -\frac{4}{5} \text{ and } -\frac{5}{4}$$

This definition is important because dividing fractions requires that you multiply the dividend by the reciprocal of the divisor.

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} \cdot \frac{\frac{d}{c}}{\frac{d}{c}} = \frac{\frac{a}{b} \cdot \frac{d}{c}}{1} = \frac{a}{b} \cdot \frac{d}{c}$$

In general,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

Example 7

Simplify: $\frac{5}{4} \div \frac{3}{5} \cdot \frac{1}{2}$.

Solution:

Perform the multiplication and division from left to right.

$$\begin{aligned} \frac{5}{4} \div \frac{3}{5} \cdot \frac{1}{2} &= \frac{5}{4} \cdot \frac{5}{3} \cdot \frac{1}{2} \\ &= \frac{5 \cdot 5 \cdot 1}{4 \cdot 3 \cdot 2} \\ &= \frac{25}{24} \end{aligned}$$

In algebra, it is often preferable to work with improper fractions. In this case, we leave the answer expressed as an improper fraction.

Answer: $\frac{25}{24}$

Try this! Simplify: $\frac{1}{2} \cdot \frac{3}{4} \div \frac{1}{8}$.

Answer: 3

(click to see video)

Grouping Symbols and Exponents

In a computation where more than one operation is involved, grouping symbols help tell us which operations to perform first. The **grouping symbols**[68] commonly used in algebra are:

$(\ )$	*Parentheses*
$[\ ]$	*Brackets*
$\{\ \}$	*Braces*
$—$	*Fraction bar*

All of the above grouping symbols, as well as absolute value, have the same order of precedence. Perform operations inside the innermost grouping symbol or absolute value first.

68. Parentheses, brackets, braces, and the fraction bar are the common symbols used to group expressions and mathematical operations within a computation.

Example 8

Simplify: $2 - \left(\frac{4}{5} - \frac{2}{15}\right)$.

Solution:

Perform the operations within the parentheses first.

$$\begin{aligned} 2 - \left(\frac{4}{5} - \frac{2}{15}\right) &= 2 - \left(\frac{4}{5} \cdot \frac{3}{3} - \frac{2}{15}\right) \\ &= 2 - \left(\frac{12}{15} - \frac{2}{15}\right) \\ &= 2 - \left(\frac{10}{15}\right) \\ &= \frac{2}{1} \cdot \frac{3}{3} - \frac{2}{3} \\ &= \frac{6-2}{3} \\ &= \frac{4}{3} \end{aligned}$$

Answer: $\frac{4}{3}$

Example 9

Simplify: $\frac{5-|4-(-3)|}{|-3|-(5-7)}$.

Solution:

The fraction bar groups the numerator and denominator. Hence, they should be simplified separately.

$$\begin{aligned}\frac{5-\ |4-\ (-3)|}{|-3|-\ \left(5-7\right)} &= \frac{5-\ |4+3|}{|-3|-\ (-2)} \\ &= \frac{5-\ |7|}{|-3|+2} \\ &= \frac{5-7}{3+2} \\ &= \frac{-2}{5} \\ &= -\frac{2}{5}\end{aligned}$$

Answer: $-\frac{2}{5}$

If a number is repeated as a factor numerous times, then we can write the product in a more compact form using **exponential notation**[69]. For example,

$$5 \cdot 5 \cdot 5 \cdot 5 = 5^4$$

The **base**[70] is the factor and the positive integer **exponent**[71] indicates the number of times the base is repeated as a factor. In the above example, the base is 5 and the

69. The compact notation a^n used when a factor a is repeated n times.

70. The factor a in the exponential notation a^n .

71. The positive integer n in the exponential notation a^n that indicates the number of times the base is used as a factor.

exponent is 4. Exponents are sometimes indicated with the caret (^) symbol found on the keyboard, 5^4 = 5*5*5*5. In general, if *a* is the base that is repeated as a factor *n* times, then

$$a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ factors of } a}$$

When the exponent is 2 we call the result a **square**[72], and when the exponent is 3 we call the result a **cube**[73]. For example,

$$\begin{aligned} 5^2 &= 5 \cdot 5 = 25 && \text{“5 } \textit{squared}\text{”} \\ 5^3 &= 5 \cdot 5 \cdot 5 = 125 && \text{“5 } \textit{cubed}\text{”} \end{aligned}$$

If the exponent is greater than 3, then the notation a^n is read, "*a raised to the nth power*." The base can be any real number,

$$(2.5)^2 = (2.5)\ (2.5) = 6.25$$
$$\left(-\frac{2}{3}\right)^3 = \left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right) = -\frac{8}{27}$$
$$(-2)^4 = (-2)\ (-2)\ (-2)\ (-2) = 16$$
$$-2^4 = -1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = -16$$

Notice that the result of a negative base with an even exponent is positive. The result of a negative base with an odd exponent is negative. These facts are often confused when negative numbers are involved. Study the following four examples carefully:

72. The result when the exponent of any real number is 2.

73. The result when the exponent of any real number is 3.

The base is (−3).	*The base is 3.*
$(-3)^4 = (-3)\ (-3)\ (-3)\ (-3) = +81$ $(-3)^3 = (-3)\ (-3)\ (-3) = -27$	$-3^4 = -1 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = -81$ $-3^3 = -1 \cdot 3 \cdot 3 \cdot 3 = -27$

The parentheses indicate that the negative number is to be used as the base.

Example 10

Calculate:

a. $\left(-\frac{1}{3}\right)^3$
b. $\left(-\frac{1}{3}\right)^4$

Solution:

Here $-\frac{1}{3}$ is the base for both problems.

a. Use the base as a factor three times.

$$\begin{aligned}\left(-\frac{1}{3}\right)^3 &= \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\\ &= -\frac{1}{27}\end{aligned}$$

b. Use the base as a factor four times.

$$\begin{aligned}\left(-\frac{1}{3}\right)^4 &= \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\\ &= +\frac{1}{81}\end{aligned}$$

Answers:

a. $-\frac{1}{27}$
b. $\frac{1}{81}$

Try this! Simplify:

a. -2^4
b. $(-2)^4$

Answers:

a. −16
b. 16

(click to see video)

Order of Operations

When several operations are to be applied within a calculation, we must follow a specific order to ensure a single correct result.

1. Perform all calculations within the innermost **parentheses** or grouping symbol first.
2. Evaluate all **exponents**.
3. Apply **multiplication and division** from left to right.
4. Perform all remaining **addition and subtraction** operations last from left to right.

Note that multiplication and division *should* be worked from *left to right*. Because of this, it is often reasonable to perform division before multiplication.

Example 11

Simplify: $5^3 - 24 \div 6 \cdot \frac{1}{2} + 2$.

Solution:

First, evaluate 5^3 and then perform multiplication and division as they appear from left to right.

$$\begin{aligned}
5^3 - 24 \div 6 \cdot \frac{1}{2} + 2 &= 5^3 - 24 \div 6 \cdot \frac{1}{2} + 2 \\
&= 125 - 24 \div 6 \cdot \frac{1}{2} + 2 \\
&= 125 - 4 \cdot \frac{1}{2} + 2 \\
&= 125 - 2 + 2 \\
&= 123 + 2 \\
&= 125
\end{aligned}$$

Multiplying first would have led to an incorrect result.

$$\begin{aligned}
5^3 - 24 \div 6 \cdot \frac{1}{2} + 2 &= 125 - 24 \div \underbrace{6 \cdot \frac{1}{2}}_{\textit{Incorrect}} + 2 \\
&= 125 - \underbrace{24 \div 3} + 2 \\
&= \underbrace{125 - 8} + 2 \\
&= \underbrace{117 + 2} \\
&= 119 \quad \times
\end{aligned}$$

Answer: 125

Example 12

Simplify: $-10-5^2+(-3)^4$.

Solution:

Take care to correctly identify the base when squaring.

$$\begin{aligned} -10-5^2+(-3)^4 &= -10-25+81 \\ &= -35+81 \\ &= 46 \end{aligned}$$

Answer: 46

We are less likely to make a mistake if we work one operation at a time. Some problems may involve an absolute value, in which case we assign it the same order of precedence as parentheses.

Example 13

Simplify: $7 - 5\left|-2^2 + (-3)^2\right|$.

Solution:

Begin by performing the operations within the absolute value first.

$$\begin{aligned}7 - 5\left|-2^2 + (-3)^2\right| &= 7 - 5\,|-4 + 9| \\ &= 7 - 5\,|5| \\ &= 7 - 5 \cdot 5 \\ &= 7 - 25 \\ &= -18\end{aligned}$$

Subtracting $7 - 5$ first will lead to incorrect results.

$$\begin{aligned}7-5\left|-2^2+(-3)^2\right| &= \underbrace{7\ -\ 5}_{\textit{Incorrect}}\left|-2^2+(-3)^2\right| \\ &= 2\left|-4+9\right| \\ &= 2\left|5\right| \\ &= 10 \quad \times\end{aligned}$$

Answer: -18

Try this! Simplify: $-6^2 - \left[-15 - (-2)^3\right] - (-2)^4$.

Answer: -45

(click to see video)

KEY TAKEAWAYS

- Addition is commutative and subtraction is not. Furthermore, multiplication is commutative and division is not.
- Adding or subtracting fractions requires a common denominator; multiplying or dividing fractions does not.
- Grouping symbols indicate which operations to perform first. We usually group mathematical operations with parentheses, brackets, braces, and the fraction bar. We also group operations within absolute values. All groupings have the same order of precedence: the operations within the innermost grouping are performed first.
- When using exponential notation a^n, the base *a* is used as a factor *n* times. Parentheses indicate that a negative number is to be used as the base. For example, $(-5)^2$ is positive and -5^2 is negative.
- To ensure a single correct result when applying operations within a calculation, follow the order of operations. First, perform operations in the innermost parentheses or groupings. Next, simplify all exponents. Perform multiplication and division operations from left to right. Finally, perform addition and subtraction operations from left to right.

TOPIC EXERCISES

PART A: WORKING WITH REAL NUMBERS

Perform the operations. Reduce all fractions to lowest terms.

1. $33 - \left(-15\right) + (-8)$
2. $-10 - 9 + \left(-6\right)$
3. $-23 + (-7) - (-10)$
4. $-1 - (-1) - 1$
5. $\frac{1}{2} + \frac{1}{3} - \frac{1}{6}$
6. $-\frac{1}{5} + \frac{1}{2} - \frac{1}{10}$
7. $\frac{2}{3} - \left(-\frac{1}{4}\right) - \frac{1}{6}$
8. $-\frac{3}{2} - \left(-\frac{2}{9}\right) - \frac{5}{6}$
9. $\frac{3}{4} - \left(-\frac{1}{2}\right) - \frac{5}{8}$
10. $-\frac{1}{5} - \frac{3}{2} - \left(-\frac{7}{10}\right)$
11. Subtract 3 from 10.
12. Subtract −2 from 16.
13. Subtract $-\frac{5}{6}$ from 4.
14. Subtract $-\frac{1}{2}$ from $\frac{3}{2}$.
15. Calculate the sum of −10 and 25.
16. Calculate the sum of −30 and −20.
17. Find the difference of 10 and 5.
18. Find the difference of −17 and −3.

The formula $d = |b - a|$ gives the distance between any two points on a number line. Determine the distance between the given numbers on a number line.

19. 10 and 15
20. 6 and 22
21. 0 and 12
22. −8 and 0
23. −5 and −25
24. −12 and −3

Determine the reciprocal of the following.

25. $\frac{1}{3}$
26. $\frac{2}{5}$
27. $-\frac{3}{4}$
28. −12
29. *a* where $a \neq 0$
30. $\frac{1}{a}$
31. $\frac{a}{b}$ where $a \neq 0$
32. $\frac{1}{ab}$

Perform the operations.

33. $-4\left(-5\right) \div 2$
34. $\left(-15\right)\ (-3) \div\ (-9)$
35. $-22 \div\ (-11)\ (-2)$
36. $50 \div\ \left(-25\right)\ (-4)$
37. $\frac{2}{3}\ \left(-\frac{9}{10}\right)$

38. $-\frac{5}{8}\left(-\frac{16}{25}\right)$

39. $\frac{7}{6}\left(-\frac{6}{7}\right)$

40. $-\frac{15}{9}\left(\frac{9}{5}\right)$

41. $\frac{4}{5}\left(-\frac{2}{5}\right)\div\frac{16}{25}$

42. $\left(-\frac{9}{2}\right)\left(-\frac{3}{2}\right)\div\frac{27}{16}$

43. $\frac{8}{5}\div\frac{5}{2}\cdot\frac{15}{40}$

44. $\frac{3}{16}\div\frac{5}{8}\cdot\frac{1}{2}$

45. Find the product of 12 and 7.

46. Find the product of $-\frac{2}{3}$ and 12.

47. Find the quotient of −36 and 12.

48. Find the quotient of $-\frac{3}{4}$ and 9.

49. Subtract 10 from the sum of 8 and −5.

50. Subtract −2 from the sum of −5 and −3.

51. Joe earns $18.00 per hour and "time and a half" for every hour he works over 40 hours. What is his pay for 45 hours of work this week?

52. Billy purchased 12 bottles of water at $0.75 per bottle, 5 pounds of assorted candy at $4.50 per pound, and 15 packages of microwave popcorn costing $0.50 each for his party. What was his total bill?

53. James and Mary carpooled home from college for the Thanksgiving holiday. They shared the driving, but Mary drove twice as far as James. If Mary drove for 210 miles, then how many miles was the entire trip?

54. A $6\frac{3}{4}$ foot plank is to be cut into 3 pieces of equal length. What will be the length of each piece?

55. A student earned 72, 78, 84, and 90 points on her first four algebra exams. What was her average test score? (Recall that the average is calculated by adding all the values in a set and dividing that result by the number of elements in the set.)

56. The coldest temperature on Earth, −129° F, was recorded in 1983 at Vostok Station, Antarctica. The hottest temperature on Earth, 136° F, was recorded in 1922 at Al' Aziziyah, Libya. Calculate the temperature range on Earth.

PART B: GROUPING SYMBOLS AND EXPONENTS

Perform the operations.

57. $7-\{3-[-6-(10)]\}$

58. $-(9-12)-[6-(-8-3)]$

59. $\frac{1}{2}\{5-(10-3)\}$

60. $\frac{2}{3}\{-6+(6-9)\}$

61. $5\left\{2\left[3\left(4-\frac{3}{2}\right)\right]\right\}$

62. $\frac{1}{2}\left\{-6\left[-\left(\frac{1}{2}-\frac{5}{3}\right)\right]\right\}$

63. $\frac{5-|5-(-6)|}{|-5|-|-3|}$

64. $\frac{|9-12|-(-3)}{|-16|-3(4)}$

65. $\frac{-|-5-(-7)|-(-2)}{|-2|+|-3|}$

66. $\frac{1-|9-(3-4)|}{-|-2|+(-8-(-10))}$

Perform the operations.

67. 12^2

68. $(-12)^2$

69. -12^2

70. $-(-12)^2$

71. -5^4

72. $(-5)^4$

73. $\left(-\frac{1}{2}\right)^3$

74. $-\left(-\frac{1}{2}\right)^3$

75. $-\left(-\frac{3}{4}\right)^2$

76. $-\left(-\frac{5}{2}\right)^3$

77. $(-1)^{22}$

78. $(-1)^{13}$

79. $-(-1)^{12}$

80. $-(-1)^5$

81. -10^2

82. -10^4

PART C: ORDER OF OPERATIONS

Simplify.

83. $5 - 3\left(4 - 3^2\right)$

84. $8 - 5\left(3 - 3^2\right)$

85. $(-5)^2 + 3\left(2 - 4^2\right)$

86. $6 - 2\left(-5^2 + 4 \cdot 7\right)$

87. $5 - 3\left[3\left(2 - 3^2\right) + (-3)^2\right]$

88. $10 - 5\left[(2 - 5)^2 - 3\right]$

89. $\left[5^2 - 3^2\right] - \left[2 - \left(5 + (-4)^2\right)\right]$

90. $-7^2 - \left[(2 - 7)^2 - (-8)^2\right]$

91. $\frac{3}{16} \div \left(\frac{5}{12} - \frac{1}{2} + \frac{2}{3}\right) \cdot 4$

92. $6 \cdot \left[\left(\frac{2}{3}\right)^2 - \left(\frac{1}{2}\right)^2\right] \div (-2)^2$

93. $\frac{3-2\cdot5+4}{2^2-3^2}$

94. $\frac{\left(3+(-2)^2\right)\cdot4-3}{-4^2+1}$

95. $\frac{-5^2+(-3)^2\cdot 2-3}{8^2+6(-10)}$

96. $\frac{(-4)^2+(-3)^3}{-9^2-\left(-12+2^2\right)*10}$

97. $-5^2-2\,|-5|$

98. $-2^4+6\left|2^4-5^2\right|$

99. $-\left(4-\left|7^2-8^2\right|\right)$

100. $-3\left(5-2\,|-6|\right)$

101. $(-3)^2-\left|-2+(-3)^3\right|-4^2$

102. $-5^2-2\left|3^3-2^4\right|-(-2)^5$

103. $5\cdot\ |-5|-(2-\ |-7|)^3$

104. $10^2+2\left(|-5|^3-6^3\right)$

105. $\frac{2}{3}-\left|\frac{1}{2}-\left(-\frac{4}{3}\right)^2\right|$

106. $-24\left|\frac{10}{3}-\frac{1}{2}\div\frac{1}{5}\right|$

107. Calculate the sum of the squares of the first three consecutive positive odd integers.

108. Calculate the sum of the squares of the first three consecutive positive even integers.

109. What is 6 subtracted from the sum of the squares of 5 and 8?

110. What is 5 subtracted from the sum of the cubes of 2 and 3?

PART D: DISCUSSION BOARD

111. What is PEMDAS and what is it missing?

112. Does 0 have a reciprocal? Explain.

113. Explain why we need a common denominator in order to add or subtract fractions.

114. Explain why $(-10)^4$ is positive and -10^4 is negative.

ANSWERS

1. 40

3. −20

5. $\frac{2}{3}$

7. $\frac{3}{4}$

9. $\frac{5}{8}$

11. 7

13. $\frac{29}{6}$

15. 15

17. 5

19. 5 units

21. 12 units

23. 20 units

25. 3

27. $-\frac{4}{3}$

29. $\frac{1}{a}$

31. $\frac{b}{a}$

33. 10

35. −4

37. $-\frac{3}{5}$

39. −1

41. $-\frac{1}{2}$

43. $\frac{6}{25}$

45. 84

47. −3

49. −7

51. $855

53. 315 miles

55. 81 points

57. −12

59. −1

61. 75

63. −3

65. 0

67. 144

69. −144

71. −625

73. $-\frac{1}{8}$

75. $-\frac{9}{16}$

77. 1

79. −1

81. −100

83. 20

85. −17

87. 41

89. 35

91. $\frac{9}{7}$

93. $\frac{3}{5}$

95. $-\frac{5}{2}$

97. -35

99. 11

101. -36

103. 150

105. $-\frac{11}{18}$

107. 35

109. 83

111. Answer may vary

113. Answer may vary

1.3 Square and Cube Roots of Real Numbers

LEARNING OBJECTIVES

1. Calculate the exact and approximate value of the square root of a real number.
2. Calculate the exact and approximate value of the cube root of a real number.
3. Simplify the square and cube root of a real number.
4. Apply the Pythagorean theorem.

The Definition of Square and Cube Roots

A **square root**[74] of a number is a number that when multiplied by itself yields the original number. For example, 4 is a square root of 16, because $4^2 = 16$. Since $(-4)^2 = 16$, we can say that −4 is a square root of 16 as well. Every positive real number has two square roots, one positive and one negative. For this reason, we use the **radical sign**[75] $\sqrt{\ }$ to denote the **principal (nonnegative) square root**[76] and a negative sign in front of the radical $-\sqrt{\ }$ to denote the negative square root.

$$\sqrt{16}=4 \quad \textit{Positive square root of } 16$$
$$-\sqrt{16}=-4 \quad \textit{Negative square root of } 16$$

Zero is the only real number with exactly one square root.

$$\sqrt{0} = 0$$

If the **radicand**[77], the number inside the radical sign, is nonzero and can be factored as the square of another nonzero number, then the square root of the number is apparent. In this case, we have the following property:

74. That number that when multiplied by itself yields the original number.

75. The symbol $\sqrt{\ }$ used to denote a square root.

76. The non-negative square root.

77. The number within a radical.

$$\sqrt{a^2} = a, \text{ if } a \geq 0$$

It is important to point out that *a* is required to be nonnegative. Note that $\sqrt{(-3)^2} \neq -3$ because the radical denotes the principal square root. Instead,

$$\sqrt{(-3)^2} = \sqrt{9} = 3$$

This distinction will be carefully considered later in the course.

Example 1

Find the square root:

a. $\sqrt{121}$
b. $\sqrt{0.25}$
c. $\sqrt{\frac{4}{9}}$

Solution:

a. $\sqrt{121} = \sqrt{11^2} = 11$
b. $\sqrt{0.25} = \sqrt{0.5^2} = 0.5$
c. $\sqrt{\frac{4}{9}} = \sqrt{\left(\frac{2}{3}\right)^2} = \frac{2}{3}$

Example 2

Find the negative square root:

a. $-\sqrt{64}$
b. $-\sqrt{1}$

Solution:

a. $-\sqrt{64} = -\sqrt{8^2} = -8$
b. $-\sqrt{1} = -\sqrt{1^2} = -1$

The radicand may not always be a perfect square. If a positive integer is not a perfect square, then its square root will be irrational. Consider $\sqrt{5}$, we can obtain an approximation by bounding it using the perfect squares 4 and 9 as follows:

$$\sqrt{4} < \sqrt{5} < \sqrt{9}$$
$$2 < \sqrt{5} < 3$$

With this we conclude that $\sqrt{5}$ is somewhere between 2 and 3. This number is better approximated on most calculators using the square root button, $\boxed{\sqrt{\ }}$.

$$\sqrt{5} \approx 2.236 \text{because} 2.236 \wedge 2 \approx 5$$

Next, consider the square root of a negative number. To determine the square root of −9, you must find a number that when squared results in −9,

$$\sqrt{-9} = ? \text{ or } (\ ?\ )^2 = -9$$

However, any real number squared always results in a positive number,

$$(3)^2 = 9 \text{ and } (-3)^2 = 9$$

The square root of a negative number is currently left undefined. Try calculating $\sqrt{-9}$ on your calculator; what does it say? For now, we will state that $\sqrt{-9}$ is not a real number. The square root of a negative number is defined later in the course.

A **cube root**[78] of a number is a number that when multiplied by itself three times yields the original number. Furthermore, we denote a cube root using the symbol $\sqrt[3]{}$, where 3 is called the **index**[79]. For example,

$$\sqrt[3]{8} = 2, \text{ because } 2^3 = 8$$

The product of three equal factors will be positive if the factor is positive, and negative if the factor is negative. For this reason, any real number will have only one real cube root. Hence the technicalities associated with the principal root do not apply. For example,

$$\sqrt[3]{-8} = -2, \text{ because } (-2)^3 = -8$$

In general, given any real number a, we have the following property:

78. The number that when multiplied by itself three times yields the original number, denoted by $\sqrt[3]{}$.

79. The positive integer n in the notation $\sqrt[n]{}$ that is used to indicate an nth root.

$$\sqrt[3]{a^3} = a$$

When simplifying cube roots, look for factors that are perfect cubes.

Example 3

Find the cube root:

a. $\sqrt[3]{125}$
b. $\sqrt[3]{0}$
c. $\sqrt[3]{\frac{8}{27}}$

Solution:

a. $\sqrt[3]{125} = \sqrt[3]{5^3} = 5$
b. $\sqrt[3]{0} = \sqrt[3]{0^3} = 0$
c. $\sqrt[3]{\frac{8}{27}} = \sqrt[3]{\left(\frac{2}{3}\right)^3} = \frac{2}{3}$

Example 4

Find the cube root:

a. $\sqrt[3]{-27}$
b. $\sqrt[3]{-1}$

Solution:

a. $\sqrt[3]{-27} = \sqrt[3]{(-3)^3} = -3$
b. $\sqrt[3]{-1} = \sqrt[3]{(-1)^3} = -1$

It may be the case that the radicand is not a perfect cube. If this is the case, then its cube root will be irrational. For example, $\sqrt[3]{2}$ is an irrational number, which can be approximated on most calculators using the root button $\boxed{\sqrt[x]{\ }}$. Depending on the calculator, we typically type in the index prior to pushing the button and then the radicand as follows:

$$3 \ \boxed{\sqrt[x]{y}} \ 2 \ \boxed{=}$$

Therefore, we have

$$\sqrt[3]{2} \approx 1.260, \text{ because } 1.260 \wedge 3 \approx 2$$

We will extend these ideas using any integer as an index later in this course. It is important to point out that a square root has index 2; therefore, the following are equivalent:

$$\sqrt[2]{a} = \sqrt{a}$$

In other words, if no index is given, it is assumed to be the square root.

Simplifying Square and Cube Roots

It will not always be the case that the radicand is a perfect square. If not, we use the following two properties to simplify the expression. Given real numbers $\sqrt[n]{A}$ and $\sqrt[n]{B}$ where $B \neq 0$,

Product Rule for Radicals:	$\sqrt[n]{A \cdot B} = \sqrt[n]{A} \cdot \sqrt[n]{B}$
Quotient Rule for Radicals:	$\sqrt[n]{\frac{A}{B}} = \frac{\sqrt[n]{A}}{\sqrt[n]{B}}$

80

81

A **simplified radical**[82] is one where the radicand does not consist of any factors that can be written as perfect powers of the index. Given a square root, the idea is to identify the largest square factor of the radicand and then apply the property shown above. As an example, to simplify $\sqrt{12}$, notice that 12 is not a perfect square. However, 12 does have a perfect square factor, $12 = 4 \cdot 3$. Apply the property as follows:

$$\begin{aligned} \sqrt{12} &= \sqrt{4 \cdot 3} && \textit{Apply the product rule for radicals.} \\ &= \sqrt{4} \cdot \sqrt{3} && \textit{Simplify.} \\ &= 2 \cdot \sqrt{3} \end{aligned}$$

80. Given real numbers $\sqrt[n]{A}$ and $\sqrt[n]{B}$, $\sqrt[n]{A \cdot B} = \sqrt[n]{A} \cdot \sqrt[n]{B}$.

81. Given real numbers $\sqrt[n]{A}$ and $\sqrt[n]{B}$, $\sqrt[n]{\frac{A}{B}} = \frac{\sqrt[n]{A}}{\sqrt[n]{B}}$.

82. A radical where the radicand does not consist of any factors that can be written as perfect powers of the index.

The number $2\sqrt{3}$ is a simplified irrational number. You are often asked to find an approximate answer rounded off to a certain decimal place. In that case, use a calculator to find the decimal approximation using either the original problem or the simplified equivalent.

$$\sqrt{12} = 2\sqrt{3} \approx 3.46$$

As a check, calculate $\sqrt{12}$ and $2\sqrt{3}$ on a calculator and verify that the results are both approximately 3.46.

Example 5

Simplify: $\sqrt{135}$.

Solution:

Begin by finding the largest perfect square factor of 135.

$$\begin{aligned}135 &= 3^3 \cdot 5 \\ &= 3^2 \cdot 3 \cdot 5 \\ &= 9 \cdot 15\end{aligned}$$

Therefore,

$$\begin{aligned}\sqrt{135} &= \sqrt{9 \cdot 15} && \textit{Apply the product rule for radicals.} \\ &= \sqrt{9} \cdot \sqrt{15} && \textit{Simplify.} \\ &= 3 \cdot \sqrt{15}\end{aligned}$$

Answer: $3\sqrt{15}$

Example 6

Simplify: $\sqrt{\frac{108}{169}}$.

Solution:

We begin by finding the prime factorizations of both 108 and 169. This will enable us to easily determine the largest perfect square factors.

$$\begin{aligned} 108 &= 2^2 \cdot 3^3 = 2^2 \cdot 3^2 \cdot 3 \\ 169 &= 13^2 \end{aligned}$$

Therefore,

$$\begin{aligned} \sqrt{\frac{108}{169}} &= \sqrt{\frac{2^2 \cdot 3^2 \cdot 3}{13^2}} && \textit{Apply the product and quotient rule for radicals.} \\ &= \frac{\sqrt{2^2} \cdot \sqrt{3^2} \cdot \sqrt{3}}{\sqrt{13^2}} && \textit{Simplify.} \\ &= \frac{2 \cdot 3 \cdot \sqrt{3}}{13} \\ &= \frac{6\sqrt{3}}{13} \end{aligned}$$

Answer: $\frac{6\sqrt{3}}{13}$

Example 7

Simplify: $-5\sqrt{162}$.

Solution:

$$\begin{aligned}-5\sqrt{162} &= -5 \cdot \sqrt{81 \cdot 2}\\ &= -5 \cdot \sqrt{81} \cdot \sqrt{2}\\ &= -5 \cdot 9 \cdot \sqrt{2}\\ &= -45 \cdot \sqrt{2}\\ &= -45\sqrt{2}\end{aligned}$$

Answer: $-45\sqrt{2}$

Try this! Simplify: $4\sqrt{150}$.

Answer: $20\sqrt{6}$

(click to see video)

A cube root is simplified if it does not contain any factors that can be written as perfect cubes. The idea is to identify the largest cube factor of the radicand and then apply the product or quotient rule for radicals. As an example, to simplify $\sqrt[3]{80}$, notice that 80 is not a perfect cube. However, $80 = 8 \cdot 10$ and we can write,

$$\begin{aligned} \sqrt[3]{80} &= \sqrt[3]{8 \cdot 10} && \textit{Apply the product rule for radicals.} \\ &= \sqrt[3]{8} \cdot \sqrt[3]{10} && \textit{Simplify.} \\ &= 2 \cdot \sqrt[3]{10} \end{aligned}$$

Example 8

Simplify: $\sqrt[3]{162}$.

Solution:

Begin by finding the largest perfect cube factor of 162.

$$\begin{aligned} 162 &= 3^4 \cdot 2 \\ &= 3^3 \cdot 3 \cdot 2 \\ &= 27 \cdot 6 \end{aligned}$$

Therefore,

$$\begin{aligned} \sqrt[3]{162} &= \sqrt[3]{27 \cdot 6} && \textit{Apply the product rule for radicals.} \\ &= \sqrt[3]{27} \cdot \sqrt[3]{6} && \textit{Simplify.} \\ &= 3 \cdot \sqrt[3]{6} \end{aligned}$$

Answer: $3\sqrt[3]{6}$

Example 9

Simplify: $\sqrt[3]{-\frac{16}{343}}$.

Solution:

$$\begin{aligned}\sqrt[3]{-\frac{16}{343}} &= \frac{\sqrt[3]{-1\cdot 8\cdot 2}}{\sqrt[3]{7^3}}\\ &= \frac{\sqrt[3]{-1}\cdot\sqrt[3]{8}\cdot\sqrt[3]{2}}{\sqrt[3]{7^3}}\\ &= \frac{-1\cdot 2\cdot\sqrt[3]{2}}{7}\\ &= \frac{-2\sqrt[3]{2}}{7}\end{aligned}$$

Answer: $\frac{-2\sqrt[3]{2}}{7}$

Try this! Simplify: $-2\sqrt[3]{-256}$.

Answer: $8\sqrt[3]{4}$

(click to see video)

Consider the following two calculations,

$$\sqrt{81} = \sqrt{9^2} = 9$$
$$\sqrt{81} = \sqrt{9^2} = \left(\sqrt{9}\right)^2 = (3)^2 = 9$$

Notice that it does not matter if we apply the exponent first or the square root first. This is true for any positive real number. We have the following,

$$\sqrt{a^2} = \left(\sqrt{a}\right)^2 = a, \text{ if } a \geq 0$$

Example 10

Simplify: $\left(\sqrt{10}\right)^2$.

Solution:

Apply the fact that $\left(\sqrt{a}\right)^2 = a$ if a is nonnegative.

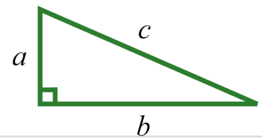
$$\left(\sqrt{10}\right)^2 = 10$$

Pythagorean Theorem

A **right triangle**[83] is a triangle where one of the angles measures 90°. The side opposite the right angle is the longest side, called the **hypotenuse**[84], and the other two sides are called **legs**[85]. Numerous real-world applications involve this geometric figure. The **Pythagorean theorem**[86] states that given any right triangle with legs measuring a and b units, the square of the measure of the hypotenuse c is equal to the sum of the squares of the measures of the legs, $a^2 + b^2 = c^2$. In other words,

83. A triangle with an angle that measures 90°.
84. The longest side of a right triangle; it will always be the side opposite the right angle.
85. The sides of a right triangle that are not the hypotenuse.
86. The hypotenuse of any right triangle is equal to the square root of the sum of the squares of the lengths of the triangle's legs.

the hypotenuse of any right triangle is equal to the square root of the sum of the squares of its legs.



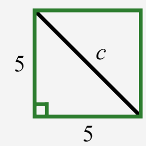
$$a^2 + b^2 = c^2$$
or
$$c = \sqrt{a^2 + b^2}$$

Example 11

Calculate the diagonal of a square with sides measuring 5 units.

Solution:

The diagonal of a square will form an isosceles right triangle where the two equal legs measure 5 units each.



We can use the Pythagorean theorem to determine the length of the hypotenuse.

$$\begin{aligned} c &= \sqrt{a^2 + b^2} \\ &= \sqrt{5^2 + 5^2} \\ &= \sqrt{25 + 25} \\ &= \sqrt{50} \\ &= \sqrt{25 \cdot 2} \\ &= \sqrt{25} \cdot \sqrt{2} \\ &= 5 \cdot \sqrt{2} \end{aligned}$$

Answer: $5\sqrt{2}$ units

The Pythagorean theorem actually states that having side lengths satisfying the property $a^2 + b^2 = c^2$ is a necessary and sufficient condition of right triangles. In other words, if we can show that the sum of the squares of the lengths of the legs of the triangle is equal to the square of the hypotenuse, then it must be a right triangle.

Example 12

Determine whether or not a triangle with legs $a = 1$ cm and $b = 2$ cm and hypotenuse $b = \sqrt{5}$ cm is a right triangle.

Solution:

If the legs satisfy the condition $a^2 + b^2 = c^2$ then the Pythagorean theorem guarantees that the triangle is a right triangle.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ (1)^2 + (2)^2 &\stackrel{?}{=} \left(\sqrt{5}\right)^2 \\ 1 + 4 &= 5 \\ 5 &= 5 \quad ✓ \end{aligned}$$

Answer: Yes, the described triangle is a right triangle.

KEY TAKEAWAYS

- The square root of a number is a number that when squared results in the original number. The principal square root of a positive real number is the positive square root. The square root of a negative number is currently left undefined.
- When simplifying the square root of a number, look for perfect square factors of the radicand. Apply the product or quotient rule for radicals and then simplify.
- The cube root of a number is a number that when cubed results in the original number. Every real number has only one real cube root.
- When simplifying cube roots, look for perfect cube factors of the radicand. Apply the product or quotient rule for radicals and then simplify.
- The Pythagorean theorem gives us a necessary and sufficient condition of right triangles: $a^2 + b^2 = c^2$ if and only if a, b and c represent the lengths of the sides of a right triangle.

TOPIC EXERCISES

PART A: SQUARE AND CUBE ROOTS

Simplify.

1. $\sqrt{81}$
2. $\sqrt{49}$
3. $-\sqrt{16}$
4. $-\sqrt{100}$
5. $\sqrt{\frac{25}{16}}$
6. $\sqrt{\frac{9}{64}}$
7. $\sqrt{\frac{1}{4}}$
8. $\sqrt{\frac{1}{100}}$
9. $\sqrt{-1}$
10. $\sqrt{-25}$
11. $\sqrt{0.36}$
12. $\sqrt{1.21}$
13. $\sqrt{(-5)^2}$
14. $\sqrt{(-6)^2}$
15. $2\sqrt{64}$
16. $3\sqrt{36}$

17. $-10\sqrt{4}$

18. $-8\sqrt{25}$

19. $\sqrt[3]{64}$

20. $\sqrt[3]{125}$

21. $\sqrt[3]{-27}$

22. $\sqrt[3]{-1}$

23. $\sqrt[3]{0}$

24. $\sqrt[3]{0.008}$

25. $\sqrt[3]{0.064}$

26. $-\sqrt[3]{-8}$

27. $-\sqrt[3]{1000}$

28. $\sqrt[3]{(-8)^3}$

29. $\sqrt[3]{(-15)^3}$

30. $\sqrt[3]{\frac{1}{216}}$

31. $\sqrt[3]{\frac{27}{64}}$

32. $\sqrt[3]{-\frac{1}{8}}$

33. $\sqrt[3]{-\frac{1}{27}}$

34. $5\sqrt[3]{343}$

35. $4\sqrt[3]{512}$

36. $-10\sqrt[3]{8}$

37. $-6\sqrt[3]{-64}$

38. $8\sqrt[3]{-8}$

Use a calculator to approximate to the nearest hundredth.

39. $\sqrt{3}$
40. $\sqrt{10}$
41. $\sqrt{19}$
42. $\sqrt{7}$
43. $3\sqrt{5}$
44. $-2\sqrt{3}$
45. $\sqrt[3]{3}$
46. $\sqrt[3]{6}$
47. $\sqrt[3]{28}$
48. $\sqrt[3]{9}$
49. $4\sqrt[3]{10}$
50. $-3\sqrt[3]{12}$
51. Determine the set consisting of the squares of the first twelve positive integers.
52. Determine the set consisting of the cubes of the first twelve positive integers.

PART B: SIMPLIFYING SQUARE ROOTS AND CUBE ROOTS

Simplify.

53. $\sqrt{18}$
54. $\sqrt{50}$
55. $\sqrt{24}$
56. $\sqrt{40}$

57. $\sqrt{\frac{50}{81}}$

58. $\sqrt{\frac{54}{25}}$

59. $4\sqrt{72}$

60. $3\sqrt{27}$

61. $-5\sqrt{80}$

62. $-6\sqrt{128}$

63. $3\sqrt{-40}$

64. $5\sqrt{-160}$

65. $\sqrt[3]{16}$

66. $\sqrt[3]{54}$

67. $\sqrt[3]{81}$

68. $\sqrt[3]{24}$

69. $\sqrt[3]{\frac{48}{125}}$

70. $\sqrt[3]{\frac{135}{64}}$

71. $7\sqrt[3]{500}$

72. $25\sqrt[3]{686}$

73. $-2\sqrt[3]{-162}$

74. $5\sqrt[3]{-96}$

75. $\left(\sqrt{64}\right)^2$

76. $\left(\sqrt{25}\right)^2$

77. $\left(\sqrt{2}\right)^2$

78. $\left(\sqrt{6}\right)^2$

PART C: PYTHAGOREAN THEOREM

79. If the two legs of a right triangle measure 3 units and 4 units, then find the length of the hypotenuse.

80. If the two legs of a right triangle measure 6 units and 8 units, then find the length of the hypotenuse.

81. If the two equal legs of an isosceles right triangle measure 7 units, then find the length of the hypotenuse.

82. If the two equal legs of an isosceles right triangle measure 10 units, then find the length of the hypotenuse.

83. Calculate the diagonal of a square with sides measuring 3 centimeters.

84. Calculate the diagonal of a square with sides measuring 10 centimeters.

85. Calculate the diagonal of a square with sides measuring $\sqrt{6}$ centimeters.

86. Calculate the diagonal of a square with sides measuring $\sqrt{10}$ centimeters.

87. Calculate the length of the diagonal of a rectangle with dimensions 4 centimeters by 8 centimeters.

88. Calculate the length of the diagonal of a rectangle with dimensions 8 meters by 10 meters.

89. Calculate the length of the diagonal of a rectangle with dimensions $\sqrt{3}$ meters by 2 meters.

90. Calculate the length of the diagonal of a rectangle with dimensions $\sqrt{6}$ meters by $\sqrt{10}$ meters.

91. To ensure that a newly built gate is square, the measured diagonal must match the distance calculated using the Pythagorean theorem. If the gate measures 4 feet by 4 feet, what must the diagonal measure in inches? (Round off to the nearest tenth of an inch.)

92. If a doorframe measures 3.5 feet by 6.6 feet, what must the diagonal measure to ensure that the frame is a perfect rectangle?

Determine whether or not the given triangle with legs *a* and *b* and hypotenuse *c* is a right triangle or not.

93. $a = 3, b = 7$, and $c = 10$
94. $a = 5, b = 12$, and $c = 13$
95. $a = 8, b = 15$, and $c = 17$
96. $a = 7, b = 24$, and $c = 30$
97. $a = 3, b = 2$, and $c = \sqrt{13}$
98. $a = \sqrt{7}, b = 4$, and $c = \sqrt{11}$
99. $a = 4, b = \sqrt{3}$, and $c = \sqrt{19}$
100. $a = \sqrt{6}, b = \sqrt{15}$, and $c = \sqrt{21}$

PART D: DISCUSSION BOARD

101. What does your calculator say after taking the square root of a negative number? Share your results on the discussion board and explain why it says that.
102. Research and discuss the history of the Pythagorean theorem.
103. Research and discuss the history of the square root.
104. Discuss the importance of the principal square root. Why is it that the same issue does not come up with cube roots? Provide some examples with your explanation.

ANSWERS

1. 9
3. −4
5. $\frac{5}{4}$
7. $\frac{1}{2}$
9. Not a real number.
11. 0.6
13. 5
15. 16
17. −20
19. 4
21. −3
23. 0
25. 0.4
27. −10
29. −15
31. $\frac{3}{4}$
33. $-\frac{1}{3}$
35. 32
37. 24
39. 1.73
41. 4.36
43. 6.71
45. 1.44
47. 3.04

49. 8.62

51. {1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144}

53. $3\sqrt{2}$

55. $2\sqrt{6}$

57. $\frac{5\sqrt{2}}{9}$

59. $24\sqrt{2}$

61. $-20\sqrt{5}$

63. Not a real number.

65. $2\sqrt[3]{2}$

67. $3\sqrt[3]{3}$

69. $\frac{2\sqrt[3]{6}}{5}$

71. $35\sqrt[3]{4}$

73. $6\sqrt[3]{6}$

75. 64

77. 2

79. 5 units

81. $7\sqrt{2}$ units

83. $3\sqrt{2}$ centimeters

85. $2\sqrt{3}$ centimeters

87. $4\sqrt{5}$ centimeters

89. $\sqrt{7}$ meters

91. The diagonal must measure approximately 67.9 inches.

93. Not a right triangle.

95. Right triangle.
97. Right triangle.
99. Right triangle.
101. Answer may vary
103. Answer may vary

1.4 Algebraic Expressions and Formulas

LEARNING OBJECTIVES

1. Identify the parts of an algebraic expression.
2. Apply the distributive property.
3. Evaluate algebraic expressions.
4. Use formulas that model common applications.

Algebraic Expressions and the Distributive Property

In algebra, letters called variables are used to represent numbers. Combinations of variables and numbers along with mathematical operations form **algebraic expressions**[87], or just **expressions**. The following are some examples of expressions with one variable, x:

$2x + 3$	$x^2 - 9$	$\frac{1}{x} + \frac{x}{x+2}$	$3\sqrt{x} + x$

Terms[88] in an algebraic expression are separated by addition operators and **factors**[89] are separated by multiplication operators. The numerical factor of a term is called the **coefficient**[90]. For example, the algebraic expression $x^2y^2 + 6xy - 3$ can be thought of as $x^2y^2 + 6xy + (-3)$ and has three terms. The first term, x^2y^2, represents the quantity $1x^2y^2 = 1 \cdot x \cdot x \cdot y \cdot y$ where 1 is the coefficient and *x* and *y* are the variables. All of the variable factors with their exponents form the **variable part of a term**[91]. If a term is written without a variable factor, then it is called a **constant term**[92]. Consider the components of $x^2y^2 + 6xy - 3$,

87. Combinations of variables and numbers along with mathematical operations used to generalize specific arithmetic operations.

88. Components of an algebraic expression separated by addition operators.

89. Components of a term separated by multiplication operators.

90. The numerical factor of a term.

91. All the variable factors with their exponents.

92. A term written without a variable factor.

Terms	*Coefficient*	*Variable Part*
x^2y^2	1	x^2y^2
$6xy$	6	xy
−3	−3	

The third term in this expression, −3, is called a constant term because it is written without a variable factor. While a variable represents an unknown quantity and may change, the constant term does not change.

Example 1

List all coefficients and variable parts of each term: $10a^2 - 5ab - b^2$.

Solution:

We want to think of the third term in this example $-b^2$ as $-1b^2$.

Terms	*Coefficient*	*Variable Part*
$10a^2$	10	a^2
$-5ab$	−5	ab
$-b^2$	−1	b^2

Answer: Coefficients: $\{-5, -1, 10\}$; Variable parts: $\{a^2, ab, b^2\}$

In our study of algebra, we will encounter a wide variety of algebraic expressions. Typically, expressions use the two most common variables, x and y. However, expressions may use any letter (or symbol) for a variable, even Greek letters, such as alpha (α) and beta (β). Some letters and symbols are reserved for constants, such as $\pi \approx 3.14159$ and $e \approx 2.71828$. Since there is only a limited number of letters, you will also use subscripts, $x_1, x_2, x_3, x_4, \ldots$, to indicate different variables.

The properties of real numbers are important in our study of algebra because a variable is simply a letter that represents a real number. In particular, the **distributive property**[93] states that if given any real numbers *a*, *b* and *c*, then,

$$a\left(b + c\right) = ab + ac$$

This property is one that we apply often when simplifying algebraic expressions. To demonstrate how it will be used, we simplify $2(5 - 3)$ in two ways, and observe the same correct result.

Working parenthesis first.	*Using the distributive property.*
$\begin{aligned} 2\left(5-3\right) &= 2\left(2\right) \\ &= 4 \end{aligned}$	$\begin{aligned} 2\left(5-3\right) &= 2\cdot5 - 2\cdot3 \\ &= 10-6 \\ &= 4 \end{aligned}$

Certainly, if the contents of the parentheses can be simplified we should do that first. On the other hand, when the contents of parentheses cannot be simplified any further, we multiply every term within it by the factor outside of it using the distributive property. Applying the distributive property allows us to multiply and remove the parentheses.

93. Given any real numbers *a*, *b*, and *c*, $a\left(b + c\right) = ab + ac$ or $\left(b + c\right)a = ba + ca$.

Example 2

Simplify: $5\left(-2a+5b\right)-2c$.

Solution:

Multiply only the terms grouped within the parentheses for which we are applying the distributive property.

$$5(-2a+5b)-2c$$

$$\begin{aligned} &= 5 \cdot (-2a) + 5 \cdot 5b - 2c \\ &= -10a + 25b - 2c \end{aligned}$$

Answer: $-10a+25b-2c$

Recall that multiplication is commutative and therefore we can write the distributive property in the following manner, $\left(b+c\right)a = ba + ca$.

Example 3

Simplify: $\left(3x - 4y + 1\right) \cdot 3.$

Solution:

Multiply all terms within the parenthesis by 3.

$$\begin{aligned}\left(3x - 4y + 1\right) \cdot 3 &= 3x \cdot 3 - 4y \cdot 3 + 1 \cdot 3 \\ &= 9x - 12y + 3\end{aligned}$$

Answer: $9x - 12y + 3$

Terms whose variable parts have the same variables with the same exponents are called **like terms**[94], or **similar terms**[95]. Furthermore, constant terms are considered to be like terms. If an algebraic expression contains like terms, apply the distributive property as follows:

$$\begin{aligned}5x + 7x &= (5 + 7)x = 12x \\ 4x^2 + 5x^2 - 7x^2 &= (4 + 5 - 7)x^2 = 2x^2\end{aligned}$$

In other words, if the variable parts of terms are *exactly the same*, then we can add or subtract the coefficients to obtain the coefficient of a single term with the same variable part. This process is called **combining like terms**[96]. For example,

$$12x^2y^3 + 3x^2y^3 = 15x^2y^3$$

94. Constant terms or terms whose variable parts have the same variables with the same exponents.

95. Used when referring to like terms.

96. Adding or subtracting like terms within an algebraic expression to obtain a single term with the same variable part.

Notice that the variable factors and their exponents do not change. Combining like terms in this manner, so that the expression contains no other similar terms, is called **simplifying the expression**[97]. Use this idea to simplify algebraic expressions with multiple like terms.

Example 4

Simplify: $x^2 - 10x + 8 + 5x^2 - 6x - 1$.

Solution:

Identify the like terms and add the corresponding coefficients.

$$\begin{aligned} &\underline{1x^2} - \underline{\underline{10x}} + \underline{\underline{\underline{8}}} + \underline{5x^2} - \underline{\underline{6x}} - \underline{\underline{\underline{1}}} \quad \textit{Combine like terms.} \\ &= 6x^2 - 16x + 7 \end{aligned}$$

Answer: $6x^2 - 16x + 7$

97. The process of combining like terms until the expression contains no more similar terms.

Example 5

Simplify: $a^2b^2 - ab - 2\left(2a^2b^2 - 5ab + 1\right)$.

Solution:

Distribute −2 and then combine like terms.

$$\begin{aligned} a^2b^2 - ab - 2\left(2a^2b^2 - 5ab + 1\right) &= a^2b^2 - ab - 4a^2b^2 + 10ab - 2 \\ &= -3a^2b^2 + 9ab - 2 \end{aligned}$$

Answer: $-3a^2b^2 + 9ab - 2$

Evaluating Algebraic Expressions

An algebraic expression can be thought of as a generalization of particular arithmetic operations. Performing these operations after substituting given values for variables is called **evaluating**[98]. In algebra, a variable represents an unknown value. However, if the problem specifically assigns a value to a variable, then you can replace that letter with the given number and evaluate using the order of operations.

98. The process of performing the operations of an algebraic expression for given values of the variables.

Example 6

Evaluate:

a. $5x - 2$ where $x = \frac{2}{3}$
b. $y^2 - y - 6$ where $y = -4$

Solution:

To avoid common errors, it is a best practice to first replace all variables with parentheses, and then replace, or **substitute**[99], the appropriate given value.

a.

$$\begin{aligned} 5x - 2 &= 5\left(\quad\right) - 2 \\ &= 5\left(\frac{2}{3}\right) - 2 \\ &= \frac{10}{3} - \frac{2}{1} \cdot \frac{3}{3} \\ &= \frac{10 - 6}{3} \\ &= \frac{4}{3} \end{aligned}$$

b.

99. The act of replacing a variable with an equivalent quantity.

$$\begin{aligned} y^2 - y - 6 &= (\quad)^2 - (\quad) - 6 \\ &= (-4)^2 - (-4) - 6 \\ &= 16 + 4 - 6 \\ &= 14 \end{aligned}$$

Answer:

a. $\frac{4}{3}$
b. 14

Often algebraic expressions will involve more than one variable.

Example 7

Evaluate $a^3 - 8b^3$ where $a = -1$ and $b = \frac{1}{2}$.

Solution:

After substituting in the appropriate values, we must take care to simplify using the correct order of operations.

$$\begin{aligned} a^3 - 8b^3 &= (\quad)^3 - 8(\quad)^3 && \textit{Replace variables with parentheses.} \\ &= (-1)^3 - 8\left(\frac{1}{2}\right)^3 && \textit{Substitute in the appropriate values.} \\ &= -1 - 8\left(\frac{1}{8}\right) && \textit{Simplify.} \\ &= -1 - 1 \\ &= -2 \end{aligned}$$

Answer: -2

Example 8

Evaluate $\frac{x^2-y^2}{2x-1}$ where $x=-\frac{3}{2}$ and $y=-3$.

Solution:

$$\begin{aligned}\frac{x^2-y^2}{2x-1} &= \frac{(\quad)^2-(\quad)^2}{2(\quad)-1}\\ &= \frac{\left(-\frac{3}{2}\right)^2-(-3)^2}{2\left(-\frac{3}{2}\right)-1}\\ &= \frac{\frac{9}{4}-9}{-3-1}\end{aligned}$$

At this point we have a complex fraction. Simplify the numerator and then multiply by the reciprocal of the denominator.

$$\begin{aligned} &= \frac{\frac{9}{4}-\frac{9}{1}\cdot\frac{4}{4}}{-4}\\ &= \frac{\frac{-27}{4}}{\frac{-4}{1}}\\ &= \frac{-27}{4}\left(-\frac{1}{4}\right)\\ &= \frac{27}{16}\end{aligned}$$

Answer: $\frac{27}{16}$

The answer to the previous example can be written as a mixed number, $\frac{27}{16} = 1\,\frac{11}{16}$. Unless the original problem has mixed numbers in it, or it is an answer to a real-world application, solutions will be expressed as reduced improper fractions.

Example 9

Evaluate $\sqrt{b^2 - 4ac}$ where $a = -1, b = -7$, and $c = \frac{1}{4}$.

Solution:

Substitute in the appropriate values and then simplify.

$$\begin{aligned}\sqrt{b^2 - 4ac} &= \sqrt{(\quad)^2 - 4(\quad)(\quad)}\\ &= \sqrt{(-7)^2 - 4(-1)\left(\frac{1}{4}\right)}\\ &= \sqrt{49 + 4\left(\frac{1}{4}\right)}\\ &= \sqrt{49 + 1}\\ &= \sqrt{50}\\ &= \sqrt{25 \cdot 2}\\ &= 5\sqrt{2}\end{aligned}$$

Answer: $5\sqrt{2}$

Try this! Evaluate $\frac{\sqrt{3\pi V h}}{\pi h}$ where $V = 25\pi$ and $h = 3$.

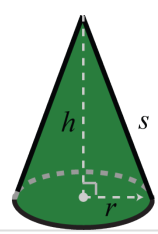
Answer: 5

(click to see video)

Using Formulas

The main difference between algebra and arithmetic is the organized use of variables. This idea leads to reusable **formulas**[100], which are mathematical models using algebraic expressions to describe common applications. For example, the volume of a right circular cone depends on its radius *r* and height *h* and is modeled by the formula:

$$V = \frac{1}{3}\pi r^2 h$$



In this equation, variables and constants are used to describe the relationship between volume and the length of the base and height. If the radius of the base measures 3 meters and the height measures 5 meters, then the volume can be calculated using the formula as follows:

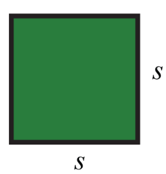
100. A reusable mathematical model using algebraic expressions to describe a common application.

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi(3\,m)^2\,(5\,m) \\ &= \frac{1}{\cancel{3}}\pi\cdot\overset{3}{\cancel{9}}\cdot 5\ \ m^3 \\ &= 15\pi\ \ m^3 \end{aligned}$$

Using $\pi \approx 3.14$, we can approximate the volume: $V \approx 15\,(3.14) = 47.1$ cubic meters.

A list of formulas that describe the area and perimeter of common plane figures follows. The letter *P* represents perimeter and is measured in linear units. The letter *A* represents area and is measured in square units.

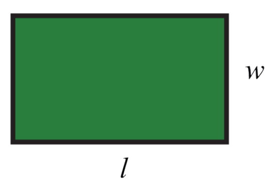
Square



$P = 4s$

$A = s^2$

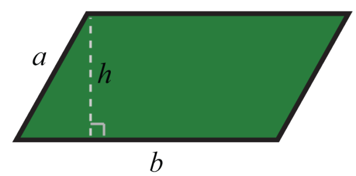
Rectangle



$P = 2l + 2w$

$A = lw$

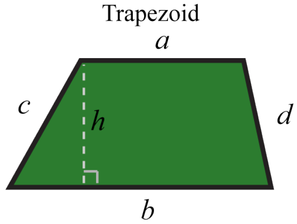
Parallelogram



$P = 2a + 2b$

$A = bh$

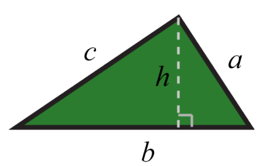
Trapezoid



$P = a + b + c + d$

$A = \frac{1}{2}h(a + b)$

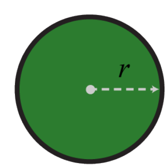
Triangle



$P = a + b + c$

$A = \frac{1}{2}bh$

Circle

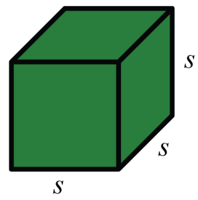


$C = 2\pi r$

$A = \pi r^2$

A list of formulas that describe the surface area and volume of common figures follows. Here *SA* represents surface area and is measured in square units. The letter *V* represents volume and is measured in cubic units.

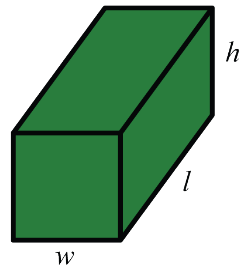
Cube



$SA = 6s^2$

$V = s^3$

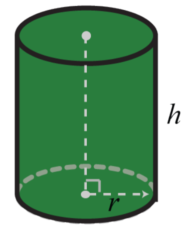
Rectangular Solid



$SA = 2lw + 2lh + 2wh$

$V = lwh$

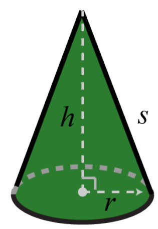
Right Circular Cylinder



$SA = 2\pi r^2 + 2\pi rh$

$V = \pi r^2 h$

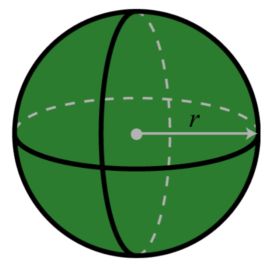
Right Circular Cone



$SA = \pi r^2 + \pi rs$

$V = \frac{1}{3}\pi r^2 h$

Sphere



$SA = 4\pi r^2$

$V = \frac{4}{3}\pi r^3$

Example 10

The diameter of a spherical balloon is 10 inches. Determine the volume rounded off to the nearest hundredth.

Solution:

The formula for the volume of a sphere is

$$V = \frac{4}{3}\pi r^3$$

This formula gives the volume in terms of the radius, r. Therefore, divide the diameter by 2 and then substitute into the formula. Here, $r = \frac{10}{2} = 5$inches and we have

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi(5\text{ in})^3 \\ &= \frac{4}{3}\pi \cdot 125\text{ in}^3 \\ &= \frac{500\pi}{3}\text{ in}^3 \approx 523.60\text{ in}^3 \end{aligned}$$

Answer: The volume of the balloon is approximately 523.60 cubic inches.

Formulas can be found in a multitude of subjects. For example, **uniform motion**[101] is modeled by the formula $D = rt$, which expresses distance D, in terms of the

101. The distance D after traveling at an average rate r for some time t can be calculated using the formula $D = rt$.

average rate, or speed, r and the time traveled at that rate, t. This formula, $D = rt$, is used often and is read, "*distance equals rate times time.*"

Example 11

Jim's road trip took $2\,\frac{1}{2}$ hours at an average speed of 66 miles per hour. How far did he travel?

Solution:

Substitute the appropriate values into the formula and then simplify.

$$\begin{aligned} D &= r \cdot t \\ &= \left(66\,\frac{\text{mi}}{\text{hr}}\right) \cdot \left(2\,\frac{1}{2}\ \text{hr}\right) \\ &= \frac{66}{1} \cdot \frac{5}{2}\ \text{mi} \\ &= 33 \cdot 5\ \text{mi} \\ &= 165\ \text{mi} \end{aligned}$$

Answer: Jim traveled 165 miles.

Simple interest[102] I is given by the formula $I = prt$, where p represents the principal amount invested at an annual interest rate r for t years.

102. Modeled by the formula $I = prt$, where p represents the principal amount invested at an annual interest rate r for t years.

Example 12

Calculate the simple interest earned on a 2-year investment of $1,250 at an annual interest rate of $3\frac{3}{4}\%$.

Solution:

Convert $3\frac{3}{4}\%$to a decimal number before using it in the formula.

$$r = 3\frac{3}{4}\% = 3.75\% = 0.0375$$

Use this and the fact that p = $1,250 and t = 2 years to calculate the simple interest.

$$\begin{aligned} I &= prt \\ &= (1,250)\,(0.0375)\,(2) \\ &= 93.75 \end{aligned}$$

Answer: The simple interest earned is $93.75.

KEY TAKEAWAYS

- Think of algebraic expressions as generalizations of common arithmetic operations that are formed by combining numbers, variables, and mathematical operations.
- The distributive property $a\left(b + c\right) = ab + ac$ is used when multiplying grouped algebraic expressions. Applying the distributive property allows us to remove parentheses.
- Combine like terms, or terms whose variable parts have the same variables with the same exponents, by adding or subtracting the coefficients to obtain the coefficient of a single term with the same variable part. Remember that the variable factors and their exponents do not change.
- To avoid common errors when evaluating, it is a best practice to replace all variables with parentheses and then substitute the appropriate values.
- The use of algebraic expressions allows us to create useful and reusable formulas that model common applications.

TOPIC EXERCISES

PART A: ALGEBRAIC EXPRESSIONS AND THE DISTRIBUTIVE PROPERTY

List all of the coefficients and variable parts of each term.

1. $-5x^2 + x - 1$
2. $y^2 - 9y + 3$
3. $5x^2 - 3xy + y^2$
4. $a^2b^2 + 2ab - 4$
5. $x^2y + xy^2 - 3xy + 9$
6. $x^4 - x^3 + x^2 - x + 2$

Multiply.

7. $5\left(3x - 5\right)$
8. $3\left(4x - 1\right)$
9. $-2\left(2x^2 - 5x + 1\right)$
10. $-5\left(6x^2 - 3x - 1\right)$
11. $\frac{2}{3}\left(9y^2 + 12y - 3\right)$
12. $-\frac{3}{4}\left(8y^2 + 20y + 4\right)$
13. $12\left(\frac{1}{3}a^2 - \frac{5}{6}a + \frac{7}{12}\right)$
14. $-9\left(\frac{1}{9}a^2 - \frac{5}{3}a + 1\right)$
15. $9\left(a^2 - 2b^2\right)$
16. $-5\left(3x^2 - y^2\right)$

17. $\left(5a^2 - 3ab + b^2\right) \cdot 6$
18. $\left(a^2 b^2 - 9ab - 3\right) \cdot 7$
19. $-\left(5x^2 - xy + y^2\right)$
20. $-\left(x^2 y^2 - 6xy - 1\right)$

Combine like terms.

21. $18x - 5x + 3x$
22. $30x - 50x + 10x$
23. $3y - 4 + 2y - 12$
24. $12y + 7 - 15y - 6$
25. $2x^2 - 3x + 2 + 5x^2 - 6x + 1$
26. $9x^2 + 7x - 5 - 10x^2 - 8x + 6$
27. $\frac{3}{5}a^2 - \frac{1}{2} + \frac{1}{3}a^2 + \frac{4}{5}$
28. $\frac{1}{6}a^2 + \frac{2}{3} - \frac{4}{3}a^2 - \frac{1}{9}$
29. $\frac{1}{2}y^2 + \frac{2}{3}y - 3 + \frac{3}{5}y^2 + \frac{1}{3}y - \frac{7}{3}$
30. $\frac{5}{6}x^2 + \frac{1}{8}x - 1 - \frac{1}{2}x^2 + \frac{3}{4}x - \frac{4}{5}$
31. $a^2 b^2 + 5ab - 2 + 7a^2 b^2 - 6ab + 12$
32. $a^2 - 12ab + 4b^2 - 6a^2 + 10ab - 5b^2$
33. $3x^2 y + 12xy - 5xy^2 + 5xy - 8x^2 y + 2xy^2$
34. $10x^2 y + 2xy - 4xy^2 + 2x^2 y - 8xy + 5xy^2$
35. $7m^2 n - 9mn + mn^2 - 6m^2 n + mn - 2mn^2$
36. $m^2 n - 5mn + 5mn^2 - 3m^2 n + 5mn + 2mn^2$
37. $x^{2n} - 3x^n + 5 + 2x^{2n} - 4x^n - 3$
38. $5y^{2n} - 3y^n + 1 - 3y^{2n} - 2y^n - 1$

Simplify.

39. $5 - 2(4x + 8)$
40. $8 - 6(2x - 1)$
41. $2\left(x^2 - 7x + 1\right) + 3x - 7$
42. $-5\left(x^2 + 4x - 1\right) + 8x^2 - 5$
43. $5ab - 4\left(ab + 5\right)$
44. $5\left(7 - ab\right) + 2ab$
45. $2 - a^2 + 3\left(a^2 + 4\right)$
46. $7 - 3y + 2\left(y^2 - 3y - 2\right)$
47. $8x^2 - 3x - 5\left(x^2 + 4x - 1\right)$
48. $2 - 5y - 6\left(y^2 - y + 2\right)$
49. $a^2b^2 - 5 + 3\left(a^2b^2 - 3ab + 2\right)$
50. $a^2 - 3ab - 2\left(a^2 - ab + 1\right)$
51. $10y^2 + 6 - \left(3y^2 + 2y + 4\right)$
52. $4m^2 - 3mn - \left(m^2 - 3mn + n^2\right)$
53. $x^{2n} - 3x^n + 5\left(x^{2n} - x^n + 1\right)$
54. $-3\left(y^{2n} - 2y^n + 1\right) + 4y^{2n} - 5$

PART B: EVALUATING ALGEBRAIC EXPRESSIONS

Evaluate.

55. $-2x + 3$ where $x = -2$
56. $8x - 5$ where $x = -1$
57. $x^2 - x + 5$ where $x = -5$

58. $2x^2 - 8x + 1$ where $x = 3$

59. $\frac{x^2-x+2}{2x-1}$ where $x = -\frac{1}{2}$

60. $\frac{9x^2+x-2}{3x-4}$ where $x = -\frac{2}{3}$

61. $(3y-2)(y+5)$ where $y = \frac{2}{3}$

62. $(3x+2)(5x+1)$ where $x = -\frac{1}{5}$

63. $(3x-1)(x-8)$ where $x = -1$

64. $(7y+5)(y+1)$ where $y = -2$

65. $y^6 - y^3 + 2$ where $y = -1$

66. $y^5 + y^3 - 3$ where $y = -2$

67. $a^2 - 5b^2$ where $a = -2$ and $b = -1$

68. $a^3 - 2b^3$ where $a = -3$ and $b = 2$

69. $(x-2y)(x+2y)$ where $x = 2$ and $y = -5$

70. $(4x-3y)(x-y)$ where $x = -4$ and $y = -3$

71. $a^2 - ab + b^2$ where $a = -1$ and $b = -2$

72. $x^2y^2 - xy + 2$ where $x = -3$ and $y = -2$

73. $a^4 - b^4$ where $a = -2$ and $b = -3$

74. $a^6 - 2a^3b^3 - b^6$ where $a = 2$ and $b = -1$

Evaluate $\sqrt{b^2 - 4ac}$ given the following values.

75. $a = 6, b = 1$ and $c = -1$

76. $a = 15, b = 4$ and $c = -4$

77. $a = \frac{3}{4}, b = -2$ and $c = -4$

78. $a = \frac{1}{2}, b = -2$ and $c = -30$

79. $a = 1, b = 2$ and $c = -1$

80. $a = 1, b = -4$ and $c = -50$

81. $a = 1, b = -1$ and $c = -\frac{1}{16}$

82. $a = -2, b = -\frac{1}{3}$ and $c = 1$

PART C: USING FORMULAS

Convert the following temperatures to degrees Celsius given $C = \frac{5}{9}(F - 32)$, where *F* represents degrees Fahrenheit.

83. 95°F

84. 86°F

85. 32°F

86. −40°F

87. Calculate the perimeter and area of a rectangle with dimensions 12 feet by 5 feet.

88. Calculate the perimeter and area of a rectangle with dimensions 5 meters by 1 meter.

89. Calculate the surface area and volume of a sphere with radius 6 centimeters.

90. The radius of the base of a right circular cylinder measures 4 inches and the height measures 10 inches. Calculate the surface area and volume.

91. Calculate the volume of a sphere with a diameter of 18 centimeters.

92. The diameter of the base of a right circular cone measures 6 inches. If the height is $1\frac{1}{2}$ feet, then calculate its volume.

93. Given that the height of a right circular cylinder is equal to the radius of the base, derive a formula for the surface area in terms of the radius of the base.

94. Given that the area of the base of a right circular cylinder is 25π square inches, find the volume if the height is 1 foot.

95. Jose was able to drive from Tucson to Phoenix in 2 hours at an average speed of 58 mph. How far is Phoenix from Tucson?

96. If a bullet train can average 152 mph, then how far can it travel in $\frac{3}{4}$ of an hour?

97. Margaret traveled for $1\frac{3}{4}$ hour at an average speed of 68 miles per hour. How far did she travel?

98. The trip from Flagstaff, AZ to the Grand Canyon national park took $1\frac{1}{2}$ hours at an average speed of 54 mph. How far is the Grand Canyon national park from Flagstaff?

99. Calculate the simple interest earned on a 3-year investment of $2,500 at an annual interest rate of $5\frac{1}{4}\%$.

100. Calculate the simple interest earned on a 1-year investment of $5,750 at an annual interest rate of $2\frac{5}{8}\%$.

101. What is the simple interest earned on a 5-year investment of $20,000 at an annual interest rate of 6%?

102. What is the simple interest earned on a 1-year investment of $50,000 at an annual interest rate of 4.5%?

103. The time *t* in seconds an object is in free fall is given by the formula $t = \frac{\sqrt{s}}{4}$, where *s* represents the distance in feet the object has fallen. How long does it take an object to fall 32 feet? (Give the exact answer and the approximate answer to the nearest hundredth.)

104. The current *I* measured in amperes, is given by the formula $I = \sqrt{\frac{P}{R}}$, where *P* is the power usage measured in watts, and *R* is the resistance measured in ohms. If a light bulb uses 60 watts of power and has 240 ohms of resistance, then how many amperes of current are required?

PART D: DISCUSSION BOARD

105. Find and post a useful mathematical model. Demonstrate its use with some values.

106. Research and discuss the history of the variable. What can we use if we run out of letters?

107. Find and post a link to a useful resource describing the Greek alphabet.

108. Given the algebraic expression $5 - 3\,(9x - 1)$, explain why we do not subtract 5 and 3 first.

109. Do we need a separate distributive property for more than two terms? For example, $a\left(b + c + d\right) = ab + ac + ad$Explain.

110. How can we check to see if we have simplified an expression correctly?

ANSWERS

1. Coefficients: $\{-5, 1, -1\}$; variable parts: $\{x^2, x\}$
3. Coefficients: $\{5, -3, 1\}$; variable parts: $\{x^2, xy, y^2\}$
5. Coefficients: $\{1, -3, 9\}$; variable parts: $\{x^2y, xy^2, xy\}$
7. $15x - 25$
9. $-4x^2 + 10x - 2$
11. $6y^2 + 8y - 2$
13. $4a^2 - 10a + 7$
15. $9a^2 - 18b^2$
17. $30a^2 - 18ab + 6b^2$
19. $-5x^2 + xy - y^2$
21. $16x$
23. $5y - 16$
25. $7x^2 - 9x + 3$
27. $\frac{14}{15}a^2 + \frac{3}{10}$
29. $\frac{11}{10}y^2 + y - \frac{16}{3}$
31. $8a^2b^2 - ab + 10$
33. $-5x^2y + 17xy - 3xy^2$
35. $m^2n - 8mn - mn^2$
37. $3x^{2n} - 7x^n + 2$
39. $-8x - 11$
41. $2x^2 - 11x - 5$
43. $ab - 20$

45. $2a^2 + 14$

47. $3x^2 - 23x + 5$

49. $4a^2b^2 - 9ab + 1$

51. $7y^2 - 2y + 2$

53. $6x^{2n} - 8x^n + 5$

55. 7

57. 35

59. $-\frac{11}{8}$

61. 0

63. 36

65. 4

67. −1

69. −96

71. 3

73. −65

75. 5

77. 4

79. $2\sqrt{2}$

81. $\frac{\sqrt{5}}{2}$

83. 35°C

85. 0°C

87. P = 34 feet; A = 60 square feet

89. SA = 144π square centimeters; V = 288π cubic centimeters

91. 972π cubic centimeters

93. $SA = 4\pi r^2$

95. 116 miles
97. 119 miles
99. $393.75
101. $6,000
103. $\sqrt{2} \approx 1.41$ seconds
105. Answer may vary
107. Answer may vary
109. Answer may vary

1.5 Rules of Exponents and Scientific Notation

LEARNING OBJECTIVES

1. Review the rules of exponents.
2. Review the definition of negative exponents and zero as an exponent.
3. Work with numbers using scientific notation.

Review of the Rules of Exponents

In this section, we review the rules of exponents. Recall that if a factor is repeated multiple times, then the product can be written in exponential form x^n. The positive integer exponent *n* indicates the number of times the base *x* is repeated as a factor.

$$x^n = \underbrace{x \cdot x \cdot \ldots \cdot x}_{n\,times}$$

Consider the product of x^4 and x^6,

$$x^4 \cdot x^6 = \underbrace{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}_{10\ factors\ of\ x} = x^{10}$$

Expanding the expression using the definition produces multiple factors of the base which is quite cumbersome, particularly when *n* is large. For this reason, we have useful rules to help us simplify expressions with exponents. In this example, notice that we could obtain the same result by adding the exponents.

$$x^4 \cdot x^6 = x^{4+6} = x^{10} \quad \textit{Product rule for exponents}$$

In general, this describes the **product rule for exponents**[103]. In other words, when multiplying two expressions with the same base we add the exponents. Compare this to raising a factor involving an exponent to a power, such as $\left(x^6\right)^4$.

103. $x^m \cdot x^n = x^{m+n}$; the product of two expressions with the same base can be simplified by adding the exponents.

$$\begin{aligned}\left(x^6\right)^4 &= \underbrace{x^6 \cdot x^6 \cdot x^6 \cdot x^6}_{4\ factors\ of\ x^6} \\ &= x^{6+6+6+6} \\ &= x^{24}\end{aligned}$$

Here we have 4 factors of x^6, which is equivalent to multiplying the exponents.

$$\left(x^6\right)^4 = x^{6 \cdot 4} = x^{24} \quad \textit{Power rule for exponents}$$

This describes the **power rule for exponents**[104]. Now we consider raising grouped products to a power. For example,

$$\begin{aligned}\left(x^2y^3\right)^4 &= x^2y^3 \cdot x^2y^3 \cdot x^2y^3 \cdot x^2y^3 \\ &= x^2 \cdot x^2 \cdot x^2 \cdot x^2 \cdot y^3 \cdot y^3 \cdot y^3 \cdot y^3 \quad \textit{Commutative property} \\ &= x^{2+2+2+2} \cdot y^{3+3+3+3} \\ &= x^8y^{12}\end{aligned}$$

After expanding, we are left with four factors of the product x^2y^3. This is equivalent to raising each of the original grouped factors to the fourth power and applying the power rule.

$$\left(x^2y^3\right)^4 = \left(x^2\right)^4\left(y^3\right)^4 = x^8y^{12}$$

In general, this describes the use of the power rule for a product as well as the power rule for exponents. In summary, the rules of exponents streamline the process of working with algebraic expressions and will be used extensively as we move through our study of algebra. Given any positive integers m and n where $x, y \neq 0$ we have

104. $(x^m)^n = x^{mn}$; a power raised to a power can be simplified by multiplying the exponents.

Product rule for exponents:	$x^m \cdot x^n = x^{m+n}$
Quotient rule for exponents:	$\frac{x^m}{x^n} = x^{m-n}$
Power rule for exponents:	$(x^m)^n = x^{m \cdot n}$
Power rule for a product:	$(xy)^n = x^n y^n$
Power rule for a quotient:	$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$

105

106

These rules allow us to efficiently perform operations with exponents.

105. $(xy)^n = x^n y^n$; if a product is raised to a power, then apply that power to each factor in the product.

106. $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$; if a quotient is raised to a power, then apply that power to the numerator and the denominator.

Example 1

Simplify: $\frac{10^4 \cdot 10^{12}}{10^3}$.

Solution:

$$\begin{aligned} \frac{10^4 \cdot 10^{12}}{10^3} &= \frac{10^{16}}{10^3} && \textit{Product rule} \\ &= 10^{16-3} && \textit{Quotient rule} \\ &= 10^{13} \end{aligned}$$

Answer: 10^{13}

In the previous example, notice that we did not multiply the base 10 times itself. When applying the product rule, add the exponents and leave the base unchanged.

Example 2

Simplify: $\left(x^5 \cdot x^4 \cdot x\right)^2$.

Solution:

Recall that the variable *x* is assumed to have an exponent of one, $x = x^1$.

$$\begin{aligned}\left(x^5 \cdot x^4 \cdot x\right)^2 &= \left(x^{5+4+1}\right)^2 \\ &= \left(x^{10}\right)^2 \\ &= x^{10\cdot 2} \\ &= x^{20}\end{aligned}$$

Answer: x^{20}

The base could in fact be any algebraic expression.

Example 3

Simplify: $(x+y)^9\,(x+y)^{13}$.

Solution:

Treat the expression $(x+y)$ as the base.

$$\begin{aligned}(x+y)^9\,(x+y)^{13} &= (x+y)^{9+13}\\ &= (x+y)^{22}\end{aligned}$$

Answer: $(x+y)^{22}$

The commutative property of multiplication allows us to use the product rule for exponents to simplify factors of an algebraic expression.

Example 4

Simplify: $-8x^5y \cdot 3x^7y^3$.

Solution:

Multiply the coefficients and add the exponents of variable factors with the same base.

$$\begin{aligned} -8x^5y \cdot 3x^7y^3 &= -8 \cdot 3 \cdot x^5 \cdot x^7 \cdot y^1 \cdot y^3 && \color{blue}{\textit{Commutative property}} \\ &= -24 \cdot x^{5+7} \cdot y^{1+3} && \color{blue}{\textit{Power rule for exponents}} \\ &= -24x^{12}y^4 \end{aligned}$$

Answer: $-24x^{12}y^4$

Division involves the quotient rule for exponents.

Example 5

Simplify: $\frac{33x^7y^5(x-y)^{10}}{11x^6y(x-y)^3}$.

Solution:

$$\begin{aligned}\frac{33x^7y^5(x-y)^{10}}{11x^6y(x-y)^3} &= \frac{33}{11} \cdot x^{7-6} \cdot y^{5-1} \cdot (x-y)^{10-3} \\ &= 3x^1y^4(x-y)^7\end{aligned}$$

Answer: $3xy^4(x-y)^7$

The power rule for a quotient allows us to apply that exponent to the numerator and denominator. This rule requires that the denominator is nonzero and so we will make this assumption for the remainder of the section.

Example 6

Simplify: $\left(\frac{-4a^2b}{c^4}\right)^3$.

Solution:

First apply the power rule for a quotient and then the power rule for a product.

$$\begin{aligned}\left(\frac{-4a^2b}{c^4}\right)^3 &= \frac{\left(-4a^2b\right)^3}{\left(c^4\right)^3} && \textit{Power rule for a quotient}\\ &= \frac{(-4)^3\left(a^2\right)^3(b)^3}{\left(c^4\right)^3} && \textit{Power rule for a product}\\ &= \frac{-64a^6b^3}{c^{12}}\end{aligned}$$

Answer: $-\frac{64a^6b^3}{c^{12}}$

Using the quotient rule for exponents, we can define what it means to have zero as an exponent. Consider the following calculation:

$$1 = \frac{25}{25} = \frac{5^2}{5^2} = 5^{2-2} = 5^0$$

Twenty-five divided by twenty-five is clearly equal to one, and when the quotient rule for exponents is applied, we see that a zero exponent results. In general, given any nonzero real number x and integer n,

$$1 = \frac{x^n}{x^n} = x^{n-n} = x^0$$

This leads us to the definition of **zero as an exponent**[107],

$$x^0 = 1 \quad x \neq 0$$

It is important to note that 0^0 is indeterminate. If the base is negative, then the result is still positive one. In other words, any nonzero base raised to the zero power is defined to be equal to one. In the following examples assume all variables are nonzero.

107. $x^0 = 1$; any nonzero base raised to the 0 power is defined to be 1.

Example 7

Simplify:

a. $(-2x)^0$
b. $-2x^0$

Solution:

a. Any nonzero quantity raised to the zero power is equal to 1.

$$(-2x)^0 = 1$$

b. In the example, $-2x^0$, the base is x, not $-2x$.

$$\begin{aligned} -2x^0 &= -2 \cdot x^0 \\ &= -2 \cdot 1 \\ &= -2 \end{aligned}$$

Noting that $2^0 = 1$ we can write,

$$\frac{1}{2^3} = \frac{2^0}{2^3} = 2^{0-3} = 2^{-3}$$

In general, given any nonzero real number x and integer n,

$$\frac{1}{x^n} = \frac{x^0}{x^n} = x^{0-n} = x^{-n} \; x \neq 0$$

This leads us to the definition of **negative exponents**[108]:

$$x^{-n} = \frac{1}{x^n} \quad x \neq 0$$

An expression is completely simplified if it does not contain any negative exponents.

108. $x^{-n} = \frac{1}{x^n}$, given any integer n, where x is nonzero.

Example 8

Simplify: $\left(-4x^2y\right)^{-2}$.

Solution:

Rewrite the entire quantity in the denominator with an exponent of 2 and then simplify further.

$$\begin{aligned}\left(-4x^2y\right)^{-2} &= \frac{1}{\left(-4x^2y\right)^2}\\ &= \frac{1}{(-4)^2\left(x^2\right)^2(y)^2}\\ &= \frac{1}{16x^4y^2}\end{aligned}$$

Answer: $\frac{1}{16x^4y^2}$

Sometimes negative exponents appear in the denominator.

Example 9

Simplify: $\frac{x^{-3}}{y^{-4}}$.

Solution:

$$\frac{x^{-3}}{y^{-4}} = \frac{\frac{1}{x^3}}{\frac{1}{y^4}} = \frac{1}{x^3} \cdot \frac{y^4}{1} = \frac{y^4}{x^3}$$

Answer: $\frac{y^4}{x^3}$

The previous example suggests a property of **quotients with negative exponents**[109]. Given any integers *m* and *n* where $x \neq 0$ and $y \neq 0$, then

$$\frac{x^{-n}}{y^{-m}} = \frac{\frac{1}{x^n}}{\frac{1}{y^m}} = \frac{1}{x^n} \cdot \frac{y^m}{1} = \frac{y^m}{x^n}$$

This leads us to the property

$$\frac{x^{-n}}{y^{-m}} = \frac{y^m}{x^n}$$

109. $\frac{x^{-n}}{y^{-m}} = \frac{y^m}{x^n}$, given any integers *m* and *n*, where $x \neq 0$ and $y \neq 0$.

In other words, negative exponents in the numerator can be written as positive exponents in the denominator and negative exponents in the denominator can be written as positive exponents in the numerator.

Example 10

Simplify: $\frac{-5x^{-3}y^3}{z^{-4}}$.

Solution:

Take care with the coefficient -5, recognize that this is the base and that the exponent is actually positive one: $-5 = (-5)^1$. Hence, the rules of negative exponents do not apply to this coefficient; leave it in the numerator.

$$\begin{aligned}\frac{-5x^{-3}y^3}{z^{-4}} &= \frac{-5\ x^{-3}\ y^3}{z^{-4}} \\ &= \frac{-5\,y^3\,z^4}{x^3}\end{aligned}$$

Answer: $\frac{-5y^3z^4}{x^3}$

In summary, given integers m and n where $x, y \neq 0$ we have

Zero exponent:	$x^0 = 1$
Negative exponent:	$x^{-n} = \frac{1}{x^n}$

Quotients with negative exponents:	$\frac{x^{-n}}{y^{-m}} = \frac{y^m}{x^n}$

Furthermore, all of the rules of exponents defined so far extend to any integer exponents. We will expand the scope of these properties to include any real number exponents later in the course.

Try this! Simplify: $\left(\frac{2x^{-2}y^3}{z}\right)^{-4}$.

Answer: $\frac{x^8z^4}{16y^{12}}$

(click to see video)

Scientific Notation

Real numbers expressed using **scientific notation**[110] have the form,

$$a \times 10^n$$

where *n* is an integer and $1 \leq a < 10$. This form is particularly useful when the numbers are very large or very small. For example,

$$\begin{aligned} 9{,}460{,}000{,}000{,}000{,}000 \text{ m} &= 9.46 \times 10^{15} \text{ m} && \textit{One light year} \\ 0.000000000025 \text{ m} &= 2.5 \times 10^{-11} \text{ m} && \textit{Radius of a hydrogen atom} \end{aligned}$$

It is cumbersome to write all the zeros in both of these cases. Scientific notation is an alternative, compact representation of these numbers. The factor 10^n indicates the power of ten to multiply the coefficient by to convert back to decimal form:

$$9.46 \times 10^{15} = 9.46 \times 1,\overbrace{000,000,000,000,000}^{15 \textit{ zeros}} = 9,460,000,000,000,000$$

110. Real numbers expressed the form $a \times 10^n$, where *n* is an integer and $1 \leq a < 10$.

This is equivalent to moving the decimal in the coefficient fifteen places to the right.

A negative exponent indicates that the number is very small:

$$2.5\times10^{-11} = 2.5\times\frac{1}{10^{11}} = \frac{2.5}{\underbrace{100,000,000,000}_{11\ zeros}} = 0.000000000025$$

This is equivalent to moving the decimal in the coefficient eleven places to the left.

Converting a decimal number to scientific notation involves moving the decimal as well. Consider all of the equivalent forms of 0.00563 with factors of 10 that follow:

$$\begin{aligned} 0.00563 &= 0.0563 \times 10^{-1} \\ &= 0.563 \times 10^{-2} \\ &= 5.63 \times 10^{-3} \\ &= 56.3 \times 10^{-4} \\ &= 563 \times 10^{-5} \end{aligned}$$

While all of these are equal, 5.63×10^{-3} is the only form expressed in correct scientific notation. This is because the coefficient 5.63 is between 1 and 10 as required by the definition. Notice that we can convert 5.63×10^{-3} back to decimal form, as a check, by moving the decimal three places to the left.

111

111. $\frac{x^m}{x^n} = x^{m-n}$; the quotient of two expressions with the same base can be simplified by subtracting the exponents.

Example 11

Write $1{,}075{,}000{,}000{,}000$ using scientific notation.

Solution:

Here we count twelve decimal places to the left of the decimal point to obtain the number 1.075.

$$1{,}075{,}000{,}000{,}000 = 1.075 \times 10^{12}$$

Answer: 1.075×10^{12}

Example 12

Write 0.000003045 using scientific notation.

Solution:

Here we count six decimal places to the right to obtain 3.045.

$$0.000003045 = 3.045 \times 10^{-6}$$

Answer: 3.045×10^{-6}

Often we will need to perform operations when using numbers in scientific notation. All the rules of exponents developed so far also apply to numbers in scientific notation.

Example 13

Multiply: $\left(4.36 \times 10^{-5}\right)\left(5.3 \times 10^{12}\right)$.

Solution:

Use the fact that multiplication is commutative, and apply the product rule for exponents.

$$\begin{aligned}\left(4.36 \times 10^{-5}\right)\left(5.30 \times 10^{12}\right) &= (4.36 \cdot 5.30) \times \left(10^{-5} \cdot 10^{12}\right)\\ &= 23.108 \times 10^{-5+12}\\ &= 2.3108 \times 10^{1} \times 10^{7}\\ &= 2.3108 \times 10^{1+7}\\ &= 2.3108 \times 10^{8}\end{aligned}$$

Answer: 2.3108×10^{8}

Example 14

Divide: $\left(3.24 \times 10^{8}\right) \div \left(9.0 \times 10^{-3}\right)$.

Solution:

$$\begin{aligned}\frac{\left(3.24 \times 10^{8}\right)}{\left(9.0 \times 10^{-3}\right)} &= \left(\frac{3.24}{9.0}\right) \times \left(\frac{10^{8}}{10^{-3}}\right) \\ &= 0.36 \times 10^{8-(-3)} \\ &= 0.36 \times 10^{8+3} \\ &= 3.6 \times 10^{-1} \times 10^{11} \\ &= 3.6 \times 10^{-1+11} \\ &= 3.6 \times 10^{10}\end{aligned}$$

Answer: 3.6×10^{10}

Example 15

The speed of light is approximately 6.7×10^8 miles per hour. Express this speed in miles per second.

Solution:

A unit analysis indicates that we must divide the number by 3,600.

$$\begin{aligned}
6.7 \times 10^8 \text{ miles per hour} &= \frac{6.7 \times 10^8 \text{ miles}}{1 \ \cancel{\text{hour}}} \cdot \left(\frac{1 \ \cancel{\text{hour}}}{60 \ \cancel{\text{minutes}}}\right) \cdot \left(\frac{1 \ \cancel{\text{minutes}}}{60 \text{ seconds}}\right) \\
&= \frac{6.7 \times 10^8 \text{ miles}}{3600 \text{ seconds}} \\
&= \left(\frac{6.7}{3600}\right) \times 10^8 \\
&\approx 0.0019 \times 10^8 \qquad \textit{rounded to two significant digits} \\
&= 1.9 \times 10^{-3} \times 10^8 \\
&= 1.9 \times 10^{-3+8} \\
&= 1.9 \times 10^5
\end{aligned}$$

Answer: The speed of light is approximately 1.9×10^5 miles per second.

Example 16

The Sun moves around the center of the galaxy in a nearly circular orbit. The distance from the center of our galaxy to the Sun is approximately 26,000 light-years. What is the circumference of the orbit of the Sun around the galaxy in meters?

Solution:

One light-year measures 9.46×10^{15} meters. Therefore, multiply this by 26,000 or 2.60×10^{4} to find the length of 26,000 light years in meters.

$$\begin{aligned}\left(9.46 \times 10^{15}\right)\left(2.60 \times 10^{4}\right) &= 9.46 \cdot 2.60 \times 10^{15} \cdot 10^{4} \\ &\approx 24.6 \times 10^{19} \\ &= 2.46 \times 10^{1} \cdot 10^{19} \\ &= 2.46 \times 10^{20}\end{aligned}$$

The radius *r* of this very large circle is approximately 2.46×10^{20} meters. Use the formula $C = 2\pi r$to calculate the circumference of the orbit.

$$\begin{aligned}C &= 2\pi r \\ &\approx 2\,(3.14)\left(2.46 \times 10^{20}\right) \\ &= 15.4 \times 10^{20} \\ &= 1.54 \times 10^{1} \cdot 10^{20} \\ &= 1.54 \times 10^{21}\end{aligned}$$

Answer: The circumference of the Sun’s orbit is approximately 1.54×10^{21} meters.

Try this! Divide: $\left(3.15 \times 10^{-5}\right) \div \left(12 \times 10^{-13}\right)$.

Answer: 2.625×10^{7}

(click to see video)

KEY TAKEAWAYS

- When multiplying two quantities with the same base, add exponents: $x^m \cdot x^n = x^{m+n}$.
- When dividing two quantities with the same base, subtract exponents: $\frac{x^m}{x^n} = x^{m-n}$.
- When raising powers to powers, multiply exponents: $(x^m)^n = x^{m \cdot n}$.
- When a grouped quantity involving multiplication and division is raised to a power, apply that power to all of the factors in the numerator and the denominator: $(xy)^n = x^n y^n$ and $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$.
- Any nonzero quantity raised to the 0 power is defined to be equal to 1: $x^0 = 1$.
- Expressions with negative exponents in the numerator can be rewritten as expressions with positive exponents in the denominator: $x^{-n} = \frac{1}{x^n}$.
- Expressions with negative exponents in the denominator can be rewritten as expressions with positive exponents in the numerator: $\frac{1}{x^{-m}} = x^m$.
- Take care to distinguish negative coefficients from negative exponents.
- Scientific notation is particularly useful when working with numbers that are very large or very small.

TOPIC EXERCISES

PART A: RULES OF EXPONENTS

Simplify. (Assume all variables represent nonzero numbers.)

1. $10^4 \cdot 10^7$
2. $7^3 \cdot 7^2$
3. $\dfrac{10^2 \cdot 10^4}{10^5}$
4. $\dfrac{7^5 \cdot 7^9}{7^2}$
5. $x^3 \cdot x^2$
6. $y^5 \cdot y^3$
7. $\dfrac{a^8 \cdot a^6}{a^5}$
8. $\dfrac{b^4 \cdot b^{10}}{b^8}$
9. $\dfrac{x^{2n} \cdot x^{3n}}{x^n}$
10. $\dfrac{x^n \cdot x^{8n}}{x^{3n}}$
11. $(x^5)^3$
12. $(y^4)^3$
13. $(x^4y^5)^3$
14. $(x^7y)^5$
15. $(x^2y^3z^4)^4$
16. $(xy^2z^3)^2$

17. $\left(-5x^2yz^3\right)^2$
18. $\left(-2xy^3z^4\right)^5$
19. $\left(x^2yz^5\right)^n$
20. $\left(xy^2z^3\right)^{2n}$
21. $\left(x \cdot x^3 \cdot x^2\right)^3$
22. $\left(y^2 \cdot y^5 \cdot y\right)^2$
23. $\dfrac{a^2 \cdot \left(a^4\right)^2}{a^3}$
24. $\dfrac{a \cdot a^3 \cdot a^2}{\left(a^2\right)^3}$
25. $(2x+3)^4(2x+3)^9$
26. $\left(3y-1\right)^7\left(3y-1\right)^2$
27. $\left(a+b\right)^3\left(a+b\right)^5$
28. $\left(x-2y\right)^7\left(x-2y\right)^3$
29. $5x^2y \cdot 3xy^2$
30. $-10x^3y^2 \cdot 2xy$
31. $-6x^2yz^3 \cdot 3xyz^4$
32. $2xyz^2\left(-4x^2y^2z\right)$
33. $3x^ny^{2n} \cdot 5x^2y$
34. $8x^{5n}y^n \cdot 2x^{2n}y$
35. $\dfrac{40x^5y^3z}{4x^2y^2z}$
36. $\dfrac{8x^2y^5z^3}{16x^2yz}$

37. $\dfrac{24a^8 b^3 \left(a-5b\right)^{10}}{8a^5 b^3 \left(a-5b\right)^2}$

38. $\dfrac{175m^9 n^5 (m+n)^7}{25m^8 n(m+n)^3}$

39. $\left(-2x^4 y^2 z\right)^6$

40. $\left(-3xy^4 z^7\right)^5$

41. $\left(\dfrac{-3ab^2}{2c^3}\right)^3$

42. $\left(\dfrac{-10a^3 b}{3c^2}\right)^2$

43. $\left(\dfrac{-2xy^4}{z^3}\right)^4$

44. $\left(\dfrac{-7x^9 y}{z^4}\right)^3$

45. $\left(\dfrac{xy^2}{z^3}\right)^n$

46. $\left(\dfrac{2x^2 y^3}{z}\right)^n$

47. $\left(-5x\right)^0$

48. $\left(3x^2 y\right)^0$

49. $-5x^0$

50. $3x^2 y^0$

51. $\left(-2a^2 b^0 c^3\right)^5$

52. $\left(-3a^4 b^2 c^0\right)^4$

53. $\dfrac{(9x^3y^2z^0)^2}{3xy^2}$

54. $\dfrac{(-5x^0y^5z)^3}{25y^2z^0}$

55. $-2x^{-3}$

56. $(-2x)^{-2}$

57. $a^4 \cdot a^{-5} \cdot a^2$

58. $b^{-8} \cdot b^3 \cdot b^4$

59. $\dfrac{a^8 \cdot a^{-3}}{a^{-6}}$

60. $\dfrac{b^{-10} \cdot b^4}{b^{-2}}$

61. $10x^{-3}y^2$

62. $-3x^{-5}y^{-2}$

63. $3x^{-2}y^2z^{-1}$

64. $-5x^{-4}y^{-2}z^2$

65. $\dfrac{25x^{-3}y^2}{5x^{-1}y^{-3}}$

66. $\dfrac{-9x^{-1}y^3z^{-5}}{3x^{-2}y^2z^{-1}}$

67. $(-5x^{-3}y^2z)^{-3}$

68. $(-7x^2y^{-5}z^{-2})^{-2}$

69. $\left(\dfrac{2x^{-3}z}{y^2}\right)^{-5}$

70. $\left(\dfrac{5x^5z^{-2}}{2y^{-3}}\right)^{-3}$

71. $\left(\dfrac{12x^3y^2z}{2x^7yz^8}\right)^3$

72. $$\left(\frac{150xy^8z^2}{90x^7y^2z}\right)^2$$

73. $$\left(\frac{-9a^{-3}b^4c^{-2}}{3a^3b^5c^{-7}}\right)^{-4}$$

74. $$\left(\frac{-15a^7b^5c^{-8}}{3a^{-6}b^2c^3}\right)^{-3}$$

The value in dollars of a new mobile phone can be estimated by using the formula $V = 210(2t + 1)^{-1}$, where *t* is the number of years after purchase.

75. How much was the phone worth new?
76. How much will the phone be worth in 1 year?
77. How much will the phone be worth in 3 years?
78. How much will the phone be worth in 10 years?
79. How much will the phone be worth in 100 years?
80. According to the formula, will the phone ever be worthless? Explain.
81. The height of a particular right circular cone is equal to the square of the radius of the base, $h = r^2$. Find a formula for the volume in terms of *r*.
82. A sphere has a radius $r = 3x^2$. Find the volume in terms of *x*.

PART B: SCIENTIFIC NOTATION

Convert to a decimal number.

83. 5.2×10^8
84. 6.02×10^9
85. 1.02×10^{-6}
86. 7.44×10^{-5}

Rewrite using scientific notation.

87. 7,050,000

88. 430,000,000,000

89. 0.00005001

90. 0.000000231

Perform the operations.

91. $\left(1.2 \times 10^{9}\right)\left(3 \times 10^{5}\right)$

92. $\left(4.8 \times 10^{-5}\right)\left(1.6 \times 10^{20}\right)$

93. $\left(9.1 \times 10^{23}\right)\left(3 \times 10^{10}\right)$

94. $\left(5.5 \times 10^{12}\right)\left(7 \times 10^{-25}\right)$

95. $\dfrac{9.6 \times 10^{16}}{1.2 \times 10^{-4}}$

96. $\dfrac{4.8 \times 10^{-14}}{2.4 \times 10^{-6}}$

97. $\dfrac{4 \times 10^{-8}}{8 \times 10^{10}}$

98. $\dfrac{2.3 \times 10^{23}}{9.2 \times 10^{-3}}$

99. $987{,}000{,}000{,}000{,}000 \times 23, 000, 000$

100. $0.00000000024 \times 0.00000004$

101. $0.000000000522 \div 0.0000009$

102. $81{,}000{,}000{,}000 \div 0.0000648$

103. The population density of Earth refers to the number of people per square mile of land area. If the total land area on Earth is 5.751×10^{7} square miles and the population in 2007 was estimated to be 6.67×10^{9} people, then calculate the population density of Earth at that time.

104. In 2008 the population of New York City was estimated to be 8.364 million people. The total land area is 305 square miles. Calculate the population density of New York City.

105. The mass of Earth is 5.97×10^{24} kilograms and the mass of the Moon is 7.35×10^{22} kilograms. By what factor is the mass of Earth greater than the mass of the Moon?

106. The mass of the Sun is 1.99×10^{30} kilograms and the mass of Earth is 5.97×10^{24} kilograms. By what factor is the mass of the Sun greater than the mass of Earth? Express your answer in scientific notation.

107. The radius of the Sun is 4.322×10^{5} miles and the average distance from Earth to the Moon is 2.392×10^{5} miles. By what factor is the radius of the Sun larger than the average distance from Earth to the Moon?

108. One light year, 9.461×10^{15} meters, is the distance that light travels in a vacuum in one year. If the distance from our Sun to the nearest star, Proxima Centauri, is estimated to be 3.991×10^{16} meters, then calculate the number of years it would take light to travel that distance.

109. It is estimated that there are about 1 million ants per person on the planet. If the world population was estimated to be 6.67 billion people in 2007, then estimate the world ant population at that time.

110. The radius of the earth is 6.3×10^{6} meters and the radius of the sun is 7.0×10^{8} meters. By what factor is the radius of the Sun larger than the radius of the Earth?

111. A gigabyte is 1×10^{9} bytes and a megabyte is 1×10^{6} bytes. If the average song in the MP3 format consumes about 4.5 megabytes of storage, then how many songs will fit on a 4-gigabyte memory card?

112. Water weighs approximately 18 grams per mole. If one mole is about 6×10^{23} molecules, then approximate the weight of each molecule of water.

PART C: DISCUSSION BOARD

113. Use numbers to show that $(x + y)^{n} \neq x^{n} + y^{n}$.

114. Why is 0^{0} indeterminate?

115. Explain to a beginning algebra student why $2^{2} \cdot 2^{3} \neq 4^{5}$.

116. René Descartes (1637) established the usage of exponential form: a^{2}, a^{3}, and so on. Before this, how were exponents denoted?

ANSWERS

1. 10^{11}
3. 10
5. x^5
7. a^9
9. x^{4n}
11. x^{15}
13. $x^{12}y^{15}$
15. $x^8y^{12}z^{16}$
17. $25x^4y^2z^6$
19. $x^{2n}y^nz^{5n}$
21. x^{18}
23. a^7
25. $(2x+3)^{13}$
27. $(a+b)^8$
29. $15x^3y^3$
31. $-18x^3y^2z^7$
33. $15x^{n+2}y^{2n+1}$
35. $10x^3y$
37. $3a^3(a-5b)^8$
39. $64x^{24}y^{12}z^6$
41. $-\dfrac{27a^3b^6}{8c^9}$
43. $\dfrac{16x^4y^{16}}{z^{12}}$

45. $\frac{x^{n}y^{2n}}{z^{3n}}$

47. 1

49. −5

51. $-32a^{10}c^{15}$

53. $27x^{5}y^{2}$

55. $-\frac{2}{x^{3}}$

57. a

59. a^{11}

61. $\frac{10y^{2}}{x^{3}}$

63. $\frac{3y^{2}}{x^{2}z}$

65. $\frac{5y^{5}}{x^{2}}$

67. $-\frac{x^{9}}{125y^{6}z^{3}}$

69. $\frac{x^{15}y^{10}}{32z^{5}}$

71. $\frac{216y^{3}}{x^{12}z^{21}}$

73. $\frac{a^{24}b^{4}}{81c^{20}}$

75. $210

77. $30

79. $1.04

81. $V = \frac{1}{3}\pi r^{4}$

83. 520,000,000

85. 0.00000102

87. 7.05×10^{6}

89. 5.001×10^{-5}

91. 3.6×10^{14}

93. 2.73×10^{34}

95. 8×10^{20}

97. 5×10^{-19}

99. 2.2701×10^{22}

101. 5.8×10^{-4}

103. About 116 people per square mile

105. 81.2

107. 1.807

109. 6.67×10^{15} ants

111. Approximately 889 songs

113. Answer may vary

115. Answer may vary

1.6 Polynomials and Their Operations

LEARNING OBJECTIVES

1. Identify a polynomial and determine its degree.
2. Add and subtract polynomials.
3. Multiply and divide polynomials.

Definitions

A **polynomial**[112] is a special algebraic expression with terms that consist of real number coefficients and variable factors with whole number exponents. Some examples of polynomials follow:

$3x^2$	$7xy + 5$	$\frac{3}{2}x^3 + 3x^2 - \frac{1}{2}x + 1$	$6x^2y - 4xy^3 + 7$

The **degree of a term**[113] in a polynomial is defined to be the exponent of the variable, or if there is more than one variable in the term, the degree is the sum of their exponents. Recall that $x^0 = 1$; any constant term can be written as a product of x^0 and itself. Hence the degree of a constant term is 0.

Term	*Degree*
$3x^2$	2
$6x^2y$	$2 + 1 = 3$

112. An algebraic expression consisting of terms with real number coefficients and variables with whole number exponents.

113. The exponent of the variable. If there is more than one variable in the term, the degree of the term is the sum their exponents.

Term	*Degree*
$7a^2b^3$	$2 + 3 = 5$
8	0, since $8 = 8x^0$
$2x$	1, since $2x = 2x^1$

The **degree of a polynomial**[114] is the largest degree of all of its terms.

Polynomial	*Degree*
$4x^5 - 3x^3 + 2x - 1$	5
$6x^2y - 5xy^3 + 7$	4, because $5xy^3$ has degree 4.
$\frac{1}{2}x + \frac{5}{4}$	1, because $\frac{1}{2}x = \frac{1}{2}x^1$

Of particular interest are **polynomials with one variable**[115], where each term is of the form $a_n x^n$. Here a_n is any real number and *n* is any whole number. Such polynomials have the standard form:

114. The largest degree of all of its terms.

115. A polynomial where each term has the form $a_n x^n$, where a_n is any real number and *n* is any whole number.

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Typically, we arrange terms of polynomials in descending order based on the degree of each term. The **leading coefficient**[116] is the coefficient of the variable with the highest power, in this case, a_n.

Example 1

Write in standard form: $3x - 4x^2 + 5x^3 + 7 - 2x^4$.

Solution:

Since terms are defined to be separated by addition, we write the following:

$$\begin{aligned} &3x - 4x^2 + 5x^3 + 7 - 2x^4 \\ &= 3x + (-4)\,x^2 + 5x^3 + 7 + (-2)\,x^4 \end{aligned}$$

In this form, we can see that the subtraction in the original corresponds to negative coefficients. Because addition is commutative, we can write the terms in descending order based on the degree as follows:

$$\begin{aligned} &= (-2)\,x^4 + 5x^3 + (-4)\,x^2 + 3x + 7 \\ &= -2x^4 + 5x^3 - 4x^2 + 3x + 7 \end{aligned}$$

Answer: $-2x^4 + 5x^3 - 4x^2 + 3x + 7$

We classify polynomials by the number of terms and the degree:

116. The coefficient of the term with the largest degree.

Expression	*Classification*	*Degree*
$5x^7$	**Monomial** (one term)	7
$8x^6 - 1$	**Binomial** (two terms)	6
$-3x^2 + x - 1$	**Trinomial** (three terms)	2
$5x^3 - 2x^2 + 3x - 6$	**Polynomial** (many terms)	3

117

118

119

We can further classify polynomials with one variable by their degree:

Polynomial	*Name*
5	**Constant** (degree 0)
$2x + 1$	**Linear** (degree 1)

117. Polynomial with one term.

118. Polynomial with two terms.

119. Polynomial with three terms.

Polynomial	*Name*
$3x^2 + 5x - 3$	**Quadratic** (degree 2)
$x^3 + x^2 + x + 1$	**Cubic** (degree 3)
$7x^4 + 3x^3 - 7x + 8$	Fourth-degree polynomial

120

121

122

123

In this text, we call any polynomial of degree $n \geq 4$ an *n*th-degree polynomial. In other words, if the degree is 4, we call the polynomial a fourth-degree polynomial. If the degree is 5, we call it a fifth-degree polynomial, and so on.

120. A polynomial with degree 0.

121. A polynomial with degree 1.

122. A polynomial with degree 2.

123. A polynomial with degree 3.

Example 2

State whether the following polynomial is linear or quadratic and give the leading coefficient: $25 + 4x - x^2$.

Solution:

The highest power is 2; therefore, it is a quadratic polynomial. Rewriting in standard form we have

$$-x^2 + 4x + 25$$

Here $-x^2 = -1x^2$ and thus the leading coefficient is −1.

Answer: Quadratic; leading coefficient: −1

Adding and Subtracting Polynomials

We begin by simplifying algebraic expressions that look like $+\left(a + b\right)$ or $-\left(a + b\right)$ .Here, the coefficients are actually implied to be +1 and −1 respectively and therefore the distributive property applies. Multiply each term within the parentheses by these factors as follows:

$$+\left(a + b\right) = +1\left(a + b\right) = (+1)\,a + (+1)\,b = a + b$$
$$-\left(a + b\right) = -1\left(a + b\right) = (-1)\,a + (-1)\,b = -a - b$$

Use this idea as a means to eliminate parentheses when adding and subtracting polynomials.

Example 3

Add: $9x^2 + (x^2 - 5)$.

Solution:

The property $+(a + b) = a + b$allows us to eliminate the parentheses, after which we can then combine like terms.

$$\begin{aligned} 9x^2 + (x^2 - 5) &= 9x^2 + x^2 - 5 \\ &= 10x^2 - 5 \end{aligned}$$

Answer: $10x^2 - 5$

Example 4

Add: $\left(3x^2y^2 - 4xy + 9\right) + \left(2x^2y^2 - 6xy - 7\right)$.

Solution:

Remember that the variable parts have to be exactly the same before we can add the coefficients.

$$\begin{aligned}
&\left(3x^2y^2 - 4xy + 9\right) + \left(2x^2y^2 - 6xy - 7\right) \\
&= 3x^2y^2 - 4xy + 9 + 2x^2y^2 - 6xy - 7 \\
&= 5x^2y^2 - 10xy + 2
\end{aligned}$$

Answer: $5x^2y^2 - 10xy + 2$

When subtracting polynomials, the parentheses become very important.

Example 5

Subtract: $4x^2 - \left(3x^2 + 5x\right)$.

Solution:

The property $-\left(a + b\right) = -a - b$allows us to remove the parentheses after subtracting each term.

$$\begin{aligned} 4x^2 - \left(3x^2 + 5x\right) &= 4x^2 - 3x^2 - 5x \\ &= x^2 - 5x \end{aligned}$$

Answer: $x^2 - 5x$

Subtracting a quantity is equivalent to multiplying it by −1.

Example 6

Subtract: $\left(3x^2 - 2xy + y^2\right) - \left(2x^2 - xy + 3y^2\right)$.

Solution:

Distribute the −1, remove the parentheses, and then combine like terms. Multiplying the terms of a polynomial by −1 changes all the signs.

$$\left(3x^2 - 2xy + y^2\right) - \left(2x^2 - xy + 3y^2\right)$$

$$\begin{aligned} &= 3x^2 - 2xy + y^2 - 2x^2 + xy - 3y^2 \\ &= x^2 - xy - 2y^2 \end{aligned}$$

Answer: $x^2 - xy - 2y^2$

Try this! Subtract: $\left(7a^2 - 2ab + b^2\right) - \left(a^2 - 2ab + 5b^2\right)$.

Answer: $6a^2 - 4b^2$

(click to see video)

Multiplying Polynomials

Use the product rule for exponents, $x^m \cdot x^n = x^{m+n}$, to multiply a monomial times a polynomial. In other words, when multiplying two expressions with the same base, add the exponents. To find the product of monomials, multiply the coefficients and add the exponents of variable factors with the same base. For example,

$$\begin{aligned} 7x^4 \cdot 8x^3 &= 7 \cdot 8 \cdot x^4 \cdot x^3 && \textit{Commutative property} \\ &= 56x^{4+3} && \textit{Product rule for exponents} \\ &= 56x^7 \end{aligned}$$

To multiply a polynomial by a monomial, apply the distributive property, and then simplify each term.

Example 7

Multiply: $5xy^2\left(2x^2y^2 - xy + 1\right)$.

Solution:

Apply the distributive property and then simplify.

$$5xy^2\left(2x^2y^2 - xy + 1\right)$$

$$\begin{aligned} &= 5xy^2 \cdot 2x^2y^2 - 5xy^2 \cdot xy + 5xy^2 \cdot 1 \\ &= 10x^3y^4 - 5x^2y^3 + 5xy^2 \end{aligned}$$

Answer: $10x^3y^4 - 5x^2y^3 + 5xy^2$

To summarize, multiplying a polynomial by a monomial involves the distributive property and the product rule for exponents. Multiply all of the terms of the polynomial by the monomial. For each term, multiply the coefficients and add exponents of variables where the bases are the same.

In the same manner that we used the distributive property to distribute a monomial, we use it to distribute a binomial.

$$\begin{aligned}(a+b)(c+d) &= (a+b)\cdot c + (a+b)\cdot d \\ &= ac + bc + ad + bd \\ &= ac + ad + bc + bd\end{aligned}$$

Here we apply the distributive property multiple times to produce the final result. This same result is obtained in one step if we apply the distributive property to *a* and *b* separately as follows:

$$(a+b)(c+d) = ac + ad + bc + bd$$

This is often called the FOIL method. Multiply the first, outer, inner, and then last terms.

Example 8

Multiply: $(6x - 1)(3x - 5)$.

Solution:

Distribute $6x$ and -1 and then combine like terms.

$$\begin{aligned}(6x-1)(3x-5) &= 6x \cdot 3x - 6x \cdot 5 + (-1) \cdot 3x - (-1) \cdot 5 \\ &= 18x^2 - 30x - 3x + 5 \\ &= 18x^2 - 33x + 5\end{aligned}$$

Answer: $18x^2 - 33x + 5$

Consider the following two calculations:

$\begin{aligned}(a+b)^2 &= (a+b)(a+b) \\ &= a^2 + ab + ba + b^2 \\ &= a^2 + ab + ab + b^2 \\ &= a^2 + 2ab + b^2\end{aligned}$	$\begin{aligned}(a-b)^2 &= (a-b)(a-b) \\ &= a^2 - ab - ba + b^2 \\ &= a^2 - ab - ab + b^2 \\ &= a^2 - 2ab + b^2\end{aligned}$

This leads us to two formulas that describe **perfect square trinomials**[124]:

124. The trinomials obtained by squaring the binomials $(a+b)^2 = a^2 + 2ab + b^2$ and $(a-b)^2 = a^2 - 2ab + b^2$.

$$\begin{aligned}(a+b)^2 &= a^2 + 2ab + b^2\\(a-b)^2 &= a^2 - 2ab + b^2\end{aligned}$$

We can use these formulas to quickly square a binomial.

Example 9

Multiply: $(3x+5)^2$.

Solution:

Here $a = 3x$ and $b = 5$. Apply the formula:

$$\begin{aligned}(a+b)^2 &= a^2 + 2\,a\cdot b + b^2\\(3x+5)^2 &= (3x)^2 + 2\cdot(3x)(5) + (5)^2\\&= 9x^2 + 30x + 25\end{aligned}$$

Answer: $9x^2 + 30x + 25$

This process should become routine enough to be performed mentally. Our third special product follows:

$$\begin{aligned}(a+b)\,(a-b) &= a^2 - ab + ba - b^2\\&= a^2 - ab + ab - b^2\\&= a^2 - b^2\end{aligned}$$

This product is called **difference of squares**[125]:

$$(a+b)\,(a-b) = a^2 - b^2$$

125. The special product obtained by multiplying conjugate binomials $(a+b)\,(a-b) = a^2 - b^2$.

The binomials $(a+b)$ and $(a-b)$ are called **conjugate binomials**[126]. When multiplying conjugate binomials the middle terms are opposites and their sum is zero; the product is itself a binomial.

Example 10

Multiply: $(3xy+1)\ (3xy-1)$.

Solution:

$$\begin{aligned}(3xy+1)\ (3xy-1) &= (3xy)^2 - 3xy + 3xy - 1^2 \\ &= 9x^2y^2 - 1\end{aligned}$$

Answer: $9x^2y^2 - 1$

Try this! Multiply: $(x^2+5y^2)\ (x^2-5y^2)$.

Answer: $(x^4 - 25y^4)$

(click to see video)

126. The binomials $(a+b)$ and $(a-b)$.

Example 11

Multiply: $(5x - 2)^3$.

Solution:

Here we perform one product at a time.

$$\begin{aligned}(5x-2)^3 &= (5x-2)\underbrace{(5x-2)(5x-2)}_{\textit{Multiply first.}}\\ &= (5x-2)(25x^2-10x-10x+4)\\ &= (5x-2)(25x^2-20x+4)\\ &= 125x^3-100x^2+20x-50x^2+40x-8\\ &= 125x^3-150x^2+60x-8\end{aligned}$$

Answer: $125x^2 - 150x^2 + 60x - 8$

Dividing Polynomials

Use the quotient rule for exponents, $\frac{x^m}{x^n} = x^{m-n}$, to divide a polynomial by a monomial. In other words, when dividing two expressions with the same base, subtract the exponents. In this section, we will assume that all variables in the denominator are nonzero.

Example 12

Divide: $\frac{24x^7y^5}{8x^3y^2}$.

Solution:

Divide the coefficients and apply the quotient rule by subtracting the exponents of the like bases.

$$\begin{aligned}\frac{24x^7y^5}{8x^3y^2} &= \frac{24}{8}x^{7-3}y^{5-2}\\ &= 3x^4y^3\end{aligned}$$

Answer: $3x^4y^3$

When dividing a polynomial by a monomial, we may treat the monomial as a common denominator and break up the fraction using the following property:

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

Applying this property will result in terms that can be treated as quotients of monomials.

Example 13

Divide: $\frac{-5x^4+25x^3-15x^2}{5x^2}$.

Solution:

Break up the fraction by dividing each term in the numerator by the monomial in the denominator, and then simplify each term.

$$\begin{aligned}\frac{-5x^4+25x^3-15x^2}{5x^2} &= -\frac{5x^4}{5x^2}+\frac{25x^3}{5x^2}-\frac{15x^2}{5x^2}\\ &= -\frac{5}{5}x^{4-2}+\frac{25}{5}x^{3-2}-\frac{15}{5}x^{2-2}\\ &= -1x^2+5x^1-3x^0\\ &= -x^2+5x-3\cdot 1\end{aligned}$$

Answer: $-x^2+5x-3$

We can check our division by multiplying our answer, the quotient, by the monomial in the denominator, the divisor, to see if we obtain the original numerator, the dividend.

$\frac{Dividend}{Divisor} = Quotient$	$\frac{-5x^4+25x^3-15x^2}{5x^2} = -x^2+5x-$
or	or

$Dividend = Divisor \cdot Quotient$	$-5x^4 + 25x^3 - 15x^2 = 5x^2 \left(-x^2 + 5\right.$

The same technique outlined for dividing by a monomial *does not* work for polynomials with two or more terms in the denominator. In this section, we will outline a process called **polynomial long division**[127], which is based on the division algorithm for real numbers. For the sake of clarity, we will assume that all expressions in the denominator are nonzero.

127. The process of dividing two polynomials using the division algorithm.

Example 14

Divide: $\frac{x^3+3x^2-8x-4}{x-2}$.

Solution:

Here $x-2$ is the divisor and x^3+3x^2-8x-4 is the dividend. To determine the first term of the quotient, divide the leading term of the dividend by the leading term of the divisor.

$$\begin{array}{r} ? \\ x-2\overline{)x^3+3x^2-8x-4} \end{array}$$

We begin by dividing the leading terms.

$$x^3 \div x = x^2$$

Multiply the first term of the quotient by the divisor, remembering to distribute, and line up like terms with the dividend.

$$\begin{array}{r} x^2 \qquad\qquad \\ x-2\overline{)x^3+3x^2-8x-4} \\ \underline{x^3-2x^2} \qquad\quad \end{array}$$

Distribute and line up like terms.

$$x^2\cdot(x-2)=x^3-2x^2$$

Subtract the resulting quantity from the dividend. Take care to subtract both terms.

$$\begin{array}{r} x^2 \qquad\qquad \\ x-2\overline{)x^3+3x^2-8x-4} \\ \underline{-(x^3-2x^2)} \qquad\quad \\ 5x^2 \qquad\quad \end{array}$$

Subtract.

$$-(x^3-2x^2)=-x^3+2x^2$$

Bring down the remaining terms and repeat the process.

$$\begin{array}{r} x^2 \quad ? \qquad \\ x-2\overline{)x^3+3x^2-8x-4} \\ \underline{-(x^3-2x^2)} \qquad\quad \\ 5x^2-8x-4 \end{array}$$

Bring down the remaining terms.

Notice that the leading term is eliminated and that the result has a degree that is one less. The complete process is illustrated below:

$$
\begin{array}{r}
x^2 + 5x + 2 \\
x-2\overline{\smash{)}\,x^3 + 3x^2 - 8x - 4} \\
\underline{-\left(x^3 - 2x^2\right)} \qquad\quad \\
5x^2 - 8x - 4 \\
\underline{-\left(5x^2 - 10x\right)} \quad \\
2x - 4 \\
\underline{-\left(2x - 4\right)} \\
0
\end{array}
$$

Polynomial long division ends when the degree of the remainder is less than the degree of the divisor. Here, the remainder is 0. Therefore, the binomial divides the polynomial evenly and the answer is the quotient shown above the division bar.

$$\frac{x^3 + 3x^2 - 8x - 4}{x - 2} = x^2 + 5x + 2$$

To check the answer, multiply the divisor by the quotient to see if you obtain the dividend as illustrated below:

$$x^3 + 3x^2 - 8x - 4 = (x - 2)\left(x^2 + 5x + 2\right)$$

This is left to the reader as an exercise.

Answer: $x^2 + 5x + 2$

Next, we demonstrate the case where there is a nonzero remainder.

$$
\begin{array}{r}
\textit{Quotient} \\
\textit{Divisor}\overline{\smash{)}\,\textit{Dividend}} \\
\vdots \\
\hline
\textit{Remainder}
\end{array}
$$

Just as with real numbers, the final answer adds to the quotient the fraction where the remainder is the numerator and the divisor is the denominator. In general, when dividing we have:

$$\frac{Dividend}{Divisor} = Quotient + \frac{Remainder}{Divisor}$$

If we multiply both sides by the divisor we obtain,

$$Dividend = Quotient \times Divisor + Remainder$$

Example 15

Divide: $\frac{6x^2-5x+3}{2x-1}$.

Solution:

Since the denominator is a binomial, begin by setting up polynomial long division.

$$\begin{array}{r} ? \\ 2x-1\,\overline{)\,6x^2-5x+3} \end{array}$$

To start, determine what monomial times $2x-1$ results in a leading term $6x^2$. This is the quotient of the given leading terms: $(6x^2) \div (2x) = 3x$. Multiply $3x$ times the divisor $2x-1$, and line up the result with like terms of the dividend.

$$\begin{array}{r} 3x \\ 2x-1\,\overline{)\,6x^2-5x+3} \\ 6x^2-3x \end{array}$$

Subtract the result from the dividend and bring down the constant term +3.

$$\begin{array}{r} 3x \quad ? \\ 2x-1\,\overline{)\,6x^2-5x+3} \\ \underline{-(6x^2-3x)} \\ -2x+3 \end{array}$$

Subtracting eliminates the leading term. Multiply $2x-1$ by -1 and line up the result.

$$\begin{array}{r} 3x-1 \\ 2x-1\,\overline{)\,6x^2-5x+3} \\ \underline{-(6x^2-3x)} \\ -2x+3 \\ -2x+1 \end{array}$$

Subtract again and notice that we are left with a remainder.

$$\begin{array}{r} 3x - 1 \\ 2x - 1 \overline{\smash{)}\,6x^2 - 5x + 3} \\ -\left(6x^2 - 3x\right) \\ \hline -2x + 3 \\ -(-2x + 1) \\ \hline 2 \end{array}$$

The constant term 2 has degree 0 and thus the division ends. Therefore,

$$\frac{6x^2 - 5x + 3}{2x - 1} = 3x - 1 + \frac{2}{2x - 1}$$

To check that this result is correct, we multiply as follows:

$$\begin{aligned} \textit{quotient} \times \textit{divisor} + \textit{remainder} &= (3x - 1)(2x - 1) \quad + \quad 2 \\ &= 6x^2 - 3x - 2x + 1 \quad + \quad 2 \\ &= 6x^2 - 5x + 2 \;=\; \textit{dividend} \quad \checkmark \end{aligned}$$

Answer: $3x - 1 + \frac{2}{2x-1}$

Occasionally, some of the powers of the variables appear to be missing within a polynomial. This can lead to errors when lining up like terms. Therefore, when first learning how to divide polynomials using long division, fill in the missing terms with zero coefficients, called **placeholders**[128].

128. Terms with zero coefficients used to fill in all missing exponents within a polynomial.

Example 16

Divide: $\frac{27x^3+64}{3x+4}$.

Solution:

Notice that the binomial in the numerator does not have terms with degree 2 or 1. The division is simplified if we rewrite the expression with placeholders:

$$27x^3+64=27x^3+0x^2+0x+64$$

Set up polynomial long division:

$$\begin{array}{r}
? \\
3x+4\overline{)27x^3+0x^2+0x+64}
\end{array}$$

We begin with $27x^3 \div 3x = 9x^2$ and work the rest of the division algorithm.

$$\begin{array}{r}
9x^2-12x+16 \\
3x+4\overline{)\;27x^3+0x^2+0x+64} \\
-\underline{(27x^3+36x^2)} \qquad\qquad \\
-36x^2+\;\;0x+64 \\
-\underline{(-36x^2-48x)} \qquad \\
48x+64 \\
-\underline{(48x+64)} \\
0
\end{array}$$

Answer: $9x^2-12x+16$

Example 17

Divide: $\frac{3x^4-2x^3+6x^2+23x-7}{x^2-2x+5}$.

Solution:

$$\begin{array}{r}
? \\
x^2-2x+5\overline{\smash{\big)}\,3x^4-2x^3+6x^2+23x-7}
\end{array}$$

Begin the process by dividing the leading terms to determine the leading term of the quotient $3x^4 \div x^2 = 3x^2$. Take care to distribute and line up the like terms. Continue the process until the remainder has a degree less than 2.

$$\begin{array}{r}
3x^2+4x-1 \\
x^2-2x+5\overline{\smash{\big)}\,3x^4-2x^3+6x^2+23x-7} \\
\underline{-\left(3x^4-6x^3+15x^2\right)} \\
4x^3-9x^2+23x-7 \\
\underline{-\left(4x^3-8x^2+20x\right)} \\
-x^2+3x-7 \\
\underline{-\left(-x^2+2x-5\right)} \\
x-2
\end{array}$$

The remainder is $x-2$. Write the answer with the remainder:

$$\frac{3x^4-2x^3+6x^2+23x-7}{x^2-2x+5} = 3x^2+4x-1+\frac{x-2}{x^2-2x+5}$$

Answer: $3x^2+4x-1+\frac{x-2}{x^2-2x+5}$

Polynomial long division takes time and practice to master. Work lots of problems and remember that you may check your answers by multiplying the quotient by the divisor (and adding the remainder if present) to obtain the dividend.

Try this! Divide: $\frac{6x^4 - 13x^3 + 9x^2 - 14x + 6}{3x - 2}$.

Answer: $2x^3 - 3x^2 + x - 4 - \frac{2}{3x - 2}$

(click to see video)

KEY TAKEAWAYS

- Polynomials are special algebraic expressions where the terms are the products of real numbers and variables with whole number exponents.
- The degree of a polynomial with one variable is the largest exponent of the variable found in any term. In addition, the terms of a polynomial are typically arranged in descending order based on the degree of each term.
- When adding polynomials, remove the associated parentheses and then combine like terms. When subtracting polynomials, distribute the −1, remove the parentheses, and then combine like terms.
- To multiply polynomials apply the distributive property; multiply each term in the first polynomial with each term in the second polynomial. Then combine like terms.
- When dividing by a monomial, divide all terms in the numerator by the monomial and then simplify each term.
- When dividing a polynomial by another polynomial, apply the division algorithm.

TOPIC EXERCISES

PART A: DEFINITIONS

Write the given polynomials in standard form.

1. $1 - x - x^2$
2. $y - 5 + y^2$
3. $y - 3y^2 + 5 - y^3$
4. $8 - 12a^2 + a^3 - a$
5. $2 - x^2 + 6x - 5x^3 + x^4$
6. $a^3 - 5 + a^2 + 2a^4 - a^5 + 6a$

Classify the given polynomial as a monomial, binomial, or trinomial and state the degree.

7. $x^2 - x + 2$
8. $5 - 10x^3$
9. $x^2y^2 + 5xy - 6$
10. $-2x^3y^2$
11. $x^4 - 1$
12. 5

State whether the polynomial is linear or quadratic and give the leading coefficient.

13. $1 - 9x^2$
14. $10x^2$
15. $2x - 3$
16. $100x$
17. $5x^2 + 3x - 1$

18. $x - 1$

19. $x - 6 - 2x^2$

20. $1 - 5x$

PART B: ADDING AND SUBTRACTING POLYNOMIALS

Simplify.

21. $(5x^2 - 3x - 2) + (2x^2 - 6x + 7)$

22. $(x^2 + 7x - 12) + (2x^2 - x + 3)$

23. $(x^2 + 5x + 10) + (x^2 - 10)$

24. $(x^2 - 1) + (4x + 2)$

25. $(10x^2 + 3x - 2) - (x^2 - 6x + 1)$

26. $(x^2 - 3x - 8) - (2x^2 - 3x - 8)$

27. $\left(\frac{2}{3}x^2 + \frac{3}{4}x - 1\right) - \left(\frac{1}{6}x^2 + \frac{5}{2}x - \frac{1}{2}\right)$

28. $\left(\frac{4}{5}x^2 - \frac{5}{8}x + \frac{10}{6}\right) - \left(\frac{3}{10}x^2 - \frac{2}{3}x + \frac{3}{5}\right)$

29. $(x^2y^2 + 7xy - 5) - (2x^2y^2 + 5xy - 4)$

30. $(x^2 - y^2) - (x^2 + 6xy + y^2)$

31. $(a^2b^2 + 5ab - 2) + (7ab - 2) - (4 - a^2b^2)$

32. $(a^2 + 9ab - 6b^2) - (a^2 - b^2) + 7ab$

33. $(10x^2y - 8xy + 5xy^2) - (x^2y - 4xy) + (xy^2 + 4xy)$

34. $(2m^2n - 6mn + 9mn^2) - (m^2n + 10mn) - m^2n$

35. $(8x^2y^2 - 5xy + 2) - (x^2y^2 + 5) + (2xy - 3)$

36. $(x^2 - y^2) - (5x^2 - 2xy - y^2) - (x^2 - 7xy)$

37. $\left(\frac{1}{6}a^2 - 2ab + \frac{3}{4}b^2\right) - \left(\frac{5}{3}a^2 + \frac{4}{5}b^2\right) + \frac{11}{8}ab$

38. $\left(\frac{5}{2}x^2 - 2y^2\right) - \left(\frac{7}{5}x^2 - \frac{1}{2}xy + \frac{7}{3}y^2\right) - \frac{1}{2}xy$

39. $\left(x^{2n} + 5x^n - 2\right) + \left(2x^{2n} - 3x^n - 1\right)$

40. $\left(7x^{2n} - x^n + 5\right) - \left(6x^{2n} - x^n - 8\right)$

41. Subtract $4y - 3$ from $y^2 + 7y - 10$.

42. Subtract $x^2 + 3x - 2$ from $2x^2 + 4x - 1$.

43. A right circular cylinder has a height that is equal to the radius of the base, $h = r$. Find a formula for the surface area in terms of h.

44. A rectangular solid has a width that is twice the height and a length that is 3 times that of the height. Find a formula for the surface area in terms of the height.

PART C: MULTIPLYING POLYNOMIALS

Multiply.

45. $-8x^2 \cdot 2x$

46. $-10x^2y \cdot 5x^3y^2$

47. $2x\left(5x - 1\right)$

48. $-4x\left(3x - 5\right)$

49. $7x^2\left(2x - 6\right)$

50. $-3x^2\left(x^2 - x + 3\right)$

51. $-5y^4\left(y^2 - 2y + 3\right)$

52. $\frac{5}{2}a^3\left(24a^2 - 6a + 4\right)$

53. $2xy\left(x^2 - 7xy + y^2\right)$

54. $-2a^2b\left(a^2 - 3ab + 5b^2\right)$

55. $x^n\left(x^2 + x + 1\right)$

56. $x^n\left(x^{2n} - x^n - 1\right)$

57. $(x+4)\left(x-5\right)$
58. $(x-7)\left(x-6\right)$
59. $(2x-3)\ (3x-1)$
60. $(9x+1)\ (3x+2)$
61. $\left(3x^2-y^2\right)\left(x^2-5y^2\right)$
62. $\left(5y^2-x^2\right)\left(2y^2-3x^2\right)$
63. $\left(3x+5\right)\left(3x-5\right)$
64. $\left(x+6\right)\left(x-6\right)$
65. $\left(a^2-b^2\right)\left(a^2+b^2\right)$
66. $\left(ab+7\right)\left(ab-7\right)$
67. $\left(4x-5y^2\right)\left(3x^2-y\right)$
68. $\left(xy+5\right)\left(x-y\right)$
69. $\left(x-5\right)\left(x^2-3x+8\right)$
70. $(2x-7)\left(3x^2-x+1\right)$
71. $\left(x^2+7x-1\right)\left(2x^2-3x-1\right)$
72. $\left(4x^2-x+6\right)\left(5x^2-4x-3\right)$
73. $(x+8)^2$
74. $(x-3)^2$
75. $\left(2x-5\right)^2$
76. $(3x+1)^2$
77. $\left(a-3b\right)^2$
78. $\left(7a-b\right)^2$

79. $\left(x^{2}+2y^{2}\right)^{2}$

80. $\left(x^{2}-6y\right)^{2}$

81. $\left(a^{2}-a+5\right)^{2}$

82. $\left(x^{2}-3x-1\right)^{2}$

83. $(x-3)^{3}$

84. $(x+2)^{3}$

85. $(3x+1)^{3}$

86. $(2x-3)^{3}$

87. $(x+2)^{4}$

88. $(x-3)^{4}$

89. $(2x-1)^{4}$

90. $(3x-1)^{4}$

91. $\left(x^{2n}+5\right)\left(x^{2n}-5\right)$

92. $\left(x^{n}-1\right)\left(x^{2n}+4x^{n}-3\right)$

93. $\left(x^{2n}-1\right)^{2}$

94. $\left(x^{3n}+1\right)^{2}$

95. Find the product of $3x-2$ and $x^{2}-5x-2$.

96. Find the product of $x^{2}+4$ and $x^{3}-1$.

97. Each side of a square measures $3x^{3}$ units. Determine the area in terms of *x*.

98. Each edge of a cube measures $2x^{2}$ units. Determine the volume in terms of *x*.

PART D: DIVIDING POLYNOMIALS

Divide.

99. $\dfrac{125x^{5}y^{2}}{25x^{4}y^{2}}$

100. $\dfrac{256x^2y^3z^5}{64x^2yz^2}$

101. $\dfrac{20x^3 - 12x^2 + 4x}{4x}$

102. $\dfrac{15x^4 - 75x^3 + 18x^2}{3x^2}$

103. $\dfrac{12a^2b + 28ab^2 - 4ab}{4ab}$

104. $\dfrac{-2a^4b^3 + 16a^2b^2 + 8ab^3}{2ab^2}$

105. $\dfrac{x^3 + x^2 - 3x + 9}{x + 3}$

106. $\dfrac{x^3 - 4x^2 - 9x + 20}{x - 5}$

107. $\dfrac{6x^3 - 11x^2 + 7x - 6}{2x - 3}$

108. $\dfrac{9x^3 - 9x^2 - x + 1}{3x - 1}$

109. $\dfrac{16x^3 + 8x^2 - 39x + 17}{4x - 3}$

110. $\dfrac{12x^3 - 56x^2 + 55x + 30}{2x - 5}$

111. $\dfrac{6x^4 + 13x^3 - 9x^2 - x + 6}{3x + 2}$

112. $\dfrac{25x^4 - 10x^3 + 11x^2 - 7x + 1}{5x - 1}$

113. $\dfrac{20x^4 + 12x^3 + 9x^2 + 10x + 5}{2x + 1}$

114. $\dfrac{25x^4 - 45x^3 - 26x^2 + 36x - 11}{5x - 2}$

115. $\dfrac{3x^4 + x^2 - 1}{x - 2}$

116. $\dfrac{x^4 + x - 3}{x + 3}$

117. $\dfrac{x^3 - 10}{x - 2}$

118. $\dfrac{x^3 + 15}{x + 3}$

119. $\dfrac{y^5 + 1}{y + 1}$

120. $\dfrac{y^6 + 1}{y + 1}$

121. $\dfrac{x^4 - 4x^3 + 6x^2 - 7x - 1}{x^2 - x + 2}$

122. $\dfrac{6x^4 + x^3 - 2x^2 + 2x + 4}{3x^2 - x + 1}$

123. $\dfrac{2x^3 - 7x^2 + 8x - 3}{x^2 - 2x + 1}$

124. $\dfrac{2x^4 + 3x^3 - 6x^2 - 4x + 3}{x^2 + x - 3}$

125. $\dfrac{x^4 + 4x^3 - 2x^2 - 4x + 1}{x^2 - 1}$

126. $\dfrac{x^4 + x - 1}{x^2 + 1}$

127. $\dfrac{x^3 + 6x^2y + 4xy^2 - y^3}{x + y}$

128. $\dfrac{2x^3 - 3x^2y + 4xy^2 - 3y^3}{x - y}$

129. $\dfrac{8a^3 - b^3}{2a - b}$

130. $\dfrac{a^3 + 27b^3}{a + 3b}$

131. Find the quotient of $10x^2 - 11x + 3$ and $2x - 1$.

132. Find the quotient of $12x^2 + x - 11$ and $3x - 2$.

ANSWERS

1. $-x^{2}-x+1$
3. $-y^{3}-3y^{2}+y+5$
5. $x^{4}-5x^{3}-x^{2}+6x+2$
7. Trinomial; degree 2
9. Trinomial; degree 4
11. Binomial; degree 4
13. Quadratic, −9
15. Linear, 2
17. Quadratic, 5
19. Quadratic, −2
21. $7x^{2}-9x+5$
23. $2x^{2}+5x$
25. $9x^{2}+9x-3$
27. $\frac{1}{2}x^{2}-\frac{7}{4}x-\frac{1}{2}$
29. $-x^{2}y^{2}+2xy-1$
31. $2a^{2}b^{2}+12ab-8$
33. $9x^{2}y+6xy^{2}$
35. $7x^{2}y^{2}-3xy-6$
37. $-\frac{3}{2}a^{2}-\frac{5}{8}ab-\frac{1}{20}b^{2}$
39. $3x^{2n}+2x^{n}-3$
41. $y^{2}+3y-7$
43. $SA=4\pi h^{2}$
45. $-16x^{3}$

47. $10x^2 - 2x$

49. $14x^3 - 42x^2$

51. $-5y^6 + 10y^5 - 15y^4$

53. $2x^3y - 14x^2y^2 + 2xy^3$

55. $x^{n+2} + x^{n+1} + x^n$

57. $x^2 - x - 20$

59. $6x^2 - 11x + 3$

61. $3x^4 - 16x^2y^2 + 5y^4$

63. $9x^2 - 25$

65. $a^4 - b^4$

67. $12x^3 - 15x^2y^2 - 4xy + 5y^3$

69. $x^3 - 8x^2 + 23x - 40$

71. $2x^4 + 11x^3 - 24x^2 - 4x + 1$

73. $x^2 + 16x + 64$

75. $4x^2 - 20x + 25$

77. $a^2 - 6ab + 9b^2$

79. $x^4 + 4x^2y^2 + 4y^4$

81. $a^4 - 2a^3 + 11a - 10a + 25$

83. $x^3 - 9x^2 + 27x - 27$

85. $27x^3 + 27x^2 + 9x + 1$

87. $x^4 + 8x^3 + 24x^2 + 32x + 16$

89. $16x^4 - 32x^3 + 24x^2 - 8x + 1$

91. $x^{4n} - 25$

93. $x^{4n} - 2x^{2n} + 1$

95. $3x^3 - 17x^2 + 4x + 4$

97. $9x^6$ square units

99. $5x$

101. $5x^2 - 3x + 1$

103. $3a + 7b - 1$

105. $x^2 - 2x + 3$

107. $3x^2 - x + 2$

109. $4x^2 + 5x - 6 - \frac{1}{4x-3}$

111. $2x^3 + 3x^2 - 5x + 3$

113. $10x^3 + x^2 + 4x + 3 + \frac{2}{2x+1}$

115. $3x^3 + 6x^2 + 13x + 26 + \frac{51}{x-2}$

117. $x^2 + 2x + 4 - \frac{2}{x-2}$

119. $y^4 - y^3 + y^2 - y + 1$

121. $x^2 - 3x + 1 - \dfrac{3}{x^2 - x + 2}$

123. $2x - 3$

125. $x^2 + 4x - 1$

127. $x^2 + 5xy - y^2$

129. $4a^2 + 2ab + b^2$

131. $5x - 3$

1.7 Solving Linear Equations

LEARNING OBJECTIVES

1. Use the properties of equality to solve basic linear equations.
2. Identify and solve conditional linear equations, identities, and contradictions.
3. Clear fractions from equations.
4. Set up and solve linear applications.

Solving Basic Linear Equations

An **equation**[129] is a statement indicating that two algebraic expressions are equal. A **linear equation with one variable**[130], x, is an equation that can be written in the standard form $ax + b = 0$ where a and b are real numbers and $a \neq 0$. For example,

$$3x - 12 = 0$$

A **solution**[131] to a linear equation is any value that can replace the variable to produce a true statement. The variable in the linear equation $3x - 12 = 0$ is x and the solution is $x = 4$. To verify this, substitute the value 4 in for x and check that you obtain a true statement.

$$\begin{aligned} 3x - 12 &= 0 \\ 3(4) - 12 &= 0 \\ 12 - 12 &= 0 \\ 0 &= 0 \quad ✓ \end{aligned}$$

Alternatively, when an equation is equal to a constant, we may verify a solution by substituting the value in for the variable and showing that the result is equal to that constant. In this sense, we say that solutions "satisfy the equation."

129. Statement indicating that two algebraic expressions are equal.

130. An equation that can be written in the standard form $ax + b = 0$, where a and b are real numbers and $a \neq 0$.

131. Any value that can replace the variable in an equation to produce a true statement.

Example 1

Is $a = -\frac{1}{2}$a solution to $-10a + 5 = 25$?

Solution:

Recall that when evaluating expressions, it is a good practice to first replace all variables with parentheses, and then substitute the appropriate values. By making use of parentheses, we avoid some common errors when working the order of operations.

$$-10a + 5 = -10\left(-\frac{1}{2}\right) + 5 = 5 + 5 = 10 \neq 25 \quad ✗$$

Answer: No, $a = -\frac{1}{2}$does not satisfy the equation.

Developing techniques for solving various algebraic equations is one of our main goals in algebra. This section reviews the basic techniques used for solving linear equations with one variable. We begin by defining **equivalent equations**[132] as equations with the same solution set.

$$\left.\begin{aligned} 3x - 5 &= 16 \\ 3x &= 21 \\ x &= 7 \end{aligned}\right\} \quad \textit{Equivalent equations}$$

Here we can see that the three linear equations are equivalent because they share the same solution set, namely, {7}. To obtain equivalent equations, use the following **properties of equality**[133]. Given algebraic expressions A and B, where c is a nonzero number:

132. Equations with the same solution set.

133. Properties that allow us to obtain equivalent equations by adding, subtracting, multiplying, and dividing both sides of an equation by nonzero real numbers.

Addition property of equality:	If $A = B$, then $A + c = B + c$
Subtraction property of equality:	If $A = B$, then $A - c = B - c$
Multiplication property of equality:	If $A = B$, then $cA = cB$
Division property of equality:	If $A = B$, then $\frac{A}{c} = \frac{B}{c}$

Note: Multiplying or dividing both sides of an equation by 0 is carefully avoided. Dividing by 0 is undefined and multiplying both sides by 0 results in the equation 0 = 0.

We solve algebraic equations by isolating the variable with a coefficient of 1. If given a linear equation of the form $ax + b = c$, then we can solve it in two steps. First, use the appropriate equality property of addition or subtraction to isolate the variable term. Next, isolate the variable using the equality property of multiplication or division. Checking the solution in the following examples is left to the reader.

Example 2

Solve: $7x - 2 = 19$.

Solution:

$$\begin{aligned} 7x - 2 &= 19 \\ 7x - 2 + 2 &= 19 + 2 && \textit{Add 2 to both sides.} \\ 7x &= 21 \\ \frac{7x}{7} &= \frac{21}{7} && \textit{Divide both sides by 7.} \\ x &= 3 \end{aligned}$$

Answer: The solution is 3.

Example 3

Solve: $56 = 8 + 12y$.

Solution:

When no sign precedes the term, it is understood to be positive. In other words, think of this as $56 = +8 + 12y$. Therefore, we begin by subtracting 8 on both sides of the equal sign.

$$\begin{aligned} 56 - 8 &= 8 + 12y - 8 \\ 48 &= 12y \\ \frac{48}{12} &= \frac{12y}{12} \\ 4 &= y \end{aligned}$$

It does not matter on which side we choose to isolate the variable because the **symmetric property**[134] states that $4 = y$ is equivalent to $y = 4$.

Answer: The solution is 4.

134. Allows you to solve for the variable on either side of the equal sign, because $x = 5$ is equivalent to $5 = x$.

Example 4

Solve: $\frac{5}{3}x + 2 = -8$.

Solution:

Isolate the variable term using the addition property of equality, and then multiply both sides of the equation by the reciprocal of the coefficient $\frac{5}{3}$.

$$
\begin{aligned}
\frac{5}{3}x + 2 &= -8 \\
\frac{5}{3}x + 2 - 2 &= -8 - 2 && \textit{Subtract 2 on both sides.} \\
\frac{5}{3}x &= -10 \\
\frac{3}{5} \cdot \frac{5}{3}x &= \frac{3}{\cancel{5}} \cdot \left(\overset{-2}{\cancel{-10}}\right) && \textit{Multiply both sides by } \frac{3}{5}. \\
1x &= 3 \cdot (-2) \\
x &= -6
\end{aligned}
$$

Answer: The solution is -6.

In summary, to retain equivalent equations, we must perform the same operation on both sides of the equation.

Try this! Solve: $\frac{2}{3}x + \frac{1}{2} = -\frac{5}{6}$.

Answer: $x = -2$

(click to see video)

General Guidelines for Solving Linear Equations

Typically linear equations are not given in standard form, and so solving them requires additional steps. When solving linear equations, the goal is to determine what value, if any, will produce a true statement when substituted in the original equation. Do this by isolating the variable using the following steps:

- **Step 1:** Simplify both sides of the equation using the order of operations and combine all like terms on the same side of the equal sign.
- **Step 2:** Use the appropriate properties of equality to combine like terms on opposite sides of the equal sign. The goal is to obtain the variable term on one side of the equation and the constant term on the other.
- **Step 3:** Divide or multiply as needed to isolate the variable.
- **Step 4:** Check to see if the answer solves the original equation.

We will often encounter linear equations where the expressions on each side of the equal sign can be simplified. If this is the case, then it is best to simplify each side first before solving. Normally this involves combining same-side like terms.

Note: At this point in our study of algebra the use of the properties of equality should seem routine. Therefore, displaying these steps in this text, usually in blue, becomes optional.

Example 5

Solve: $-4a + 2 - a = 1$.

Solution:

First combine the like terms on the left side of the equal sign.

$$\begin{aligned} -4a + 2 - a &= 1 && \textit{Combine same-side like terms.} \\ -5a + 2 &= 1 && \textit{Subtract 2 on both sides.} \\ -5a &= -1 && \textit{Divide both sides by } -5. \\ a &= \frac{-1}{-5} = \frac{1}{5} \end{aligned}$$

Always use the original equation to check to see if the solution is correct.

$$\begin{aligned} -4a + 2 - a &= -4\left(\frac{1}{5}\right) + 2 - \frac{1}{5} \\ &= -\frac{4}{5} + \frac{2}{1} \cdot \frac{5}{5} - \frac{1}{5} \\ &= \frac{-4 + 10 + 1}{5} \\ &= \frac{5}{5} = 1 \quad \checkmark \end{aligned}$$

Answer: The solution is $\frac{1}{5}$.

Given a linear equation in the form $ax + b = cx + d$, we begin the solving process by combining like terms on opposite sides of the equal sign. To do this, use the addition or subtraction property of equality to place like terms on the same side so that they can be combined. In the examples that remain, the check is left to the reader.

Example 6

Solve: $-2y - 3 = 5y + 11$.

Solution:

Subtract $5y$ on both sides so that we can combine the terms involving y on the left side.

$$\begin{aligned} -2y - 3 - 5y &= 5y + 11 - 5y \\ -7y - 3 &= 11 \end{aligned}$$

From here, solve using the techniques developed previously.

$$\begin{aligned} -7y - 3 &= 11 && \textit{Add 3 to both sides.} \\ -7y &= 14 \\ y &= \frac{14}{-7} && \textit{Divide both sides by } -7. \\ y &= -2 \end{aligned}$$

Answer: The solution is −2.

Solving will often require the application of the distributive property.

Example 7

Solve: $-\frac{1}{2}(10x - 2) + 3 = 7(1 - 2x)$.

Solution:

Simplify the linear expressions on either side of the equal sign first.

$$\begin{aligned} -\frac{1}{2}(10x - 2) + 3 &= 7(1 - 2x) && \textit{Distribute.} \\ -5x + 1 + 3 &= 7 - 14x && \textit{Combine same-side like terms.} \\ -5x + 4 &= 7 - 14x && \textit{Combine opposite-side like terms.} \\ 9x &= 3 && \textit{Solve.} \\ x &= \frac{3}{9} = \frac{1}{3} \end{aligned}$$

Answer: The solution is $\frac{1}{3}$.

Example 8

Solve: $5\left(3-a\right)-2\left(5-2a\right)=3$.

Solution:

Begin by applying the distributive property.

$$\begin{aligned} 5\left(3-a\right)-2\left(5-2a\right)&=3\\ 15-5a-10+4a&=3\\ 5-a&=3\\ -a&=-2 \end{aligned}$$

Here we point out that $-a$ is equivalent to $-1a$; therefore, we choose to divide both sides of the equation by -1.

$$\begin{aligned} -a&=-2\\ \frac{-1a}{-1}&=\frac{-2}{-1}\\ a&=2 \end{aligned}$$

Alternatively, we can multiply both sides of $-a=-2$ by negative one and achieve the same result.

$$\begin{aligned} -a&=-2\\ \left(-1\right)\left(-a\right)&=\left(-1\right)\left(-2\right)\\ a&=2 \end{aligned}$$

Answer: The solution is 2.

Try this! Solve: $6 - 3(4x - 1) = 4x - 7$.

Answer: $x = 1$

(click to see video)

There are three different types of equations. Up to this point, we have been solving **conditional equations**[135]. These are equations that are true for particular values. An **identity**[136] is an equation that is true for all possible values of the variable. For example,

$$x = x \quad \textit{Identity}$$

has a solution set consisting of all real numbers, $\mathbb{R}$. A **contradiction**[137] is an equation that is never true and thus has no solutions. For example,

$$x + 1 = x \qquad \textit{Contradiction}$$

has no solution. We use the empty set, $\varnothing$, to indicate that there are no solutions.

If the end result of solving an equation is a true statement, like 0 = 0, then the equation is an identity and any real number is a solution. If solving results in a false statement, like 0 = 1, then the equation is a contradiction and there is no solution.

135. Equations that are true for particular values.

136. An equation that is true for all possible values.

137. An equation that is never true and has no solution.

Example 9

Solve: $4\left(x+5\right)+6=2\left(2x+3\right)$.

Solution:

$$\begin{aligned} 4(x+5)+6 &= 2(2x+3) \\ 4x+20+6 &= 4x+6 \\ 4x+26 &= 4x+6 \\ 26 &= 6 \quad \text{✗} \end{aligned}$$

Solving leads to a false statement; therefore, the equation is a contradiction and there is no solution.

Answer: Ø

Example 10

Solve: $3\left(3y+5\right)+5=10\left(y+2\right)-y$.

Solution:

$$\begin{aligned} 3(3y+5)+5 &= 10(y+2)-y \\ 9y+15+5 &= 10y+20-y \\ 9y+20 &= 9y+20 \\ 9y &= 9y \\ 0 &= 0 \quad \checkmark \end{aligned}$$

Solving leads to a true statement; therefore, the equation is an identity and any real number is a solution.

Answer: $\mathbb{R}$

The coefficients of linear equations may be any real number, even decimals and fractions. When this is the case it is possible to use the multiplication property of equality to clear the fractional coefficients and obtain integer coefficients in a single step. If given fractional coefficients, then multiply both sides of the equation by the least common multiple of the denominators (LCD).

Example 11

Solve: $\frac{1}{3}x + \frac{1}{5} = \frac{1}{5}x - 1$.

Solution:

Clear the fractions by multiplying both sides by the least common multiple of the given denominators. In this case, it is the $LCD\left(3, 5\right) = 15$.

$$15 \cdot \left(\frac{1}{3}x + \frac{1}{5}\right) = 15 \cdot \left(\frac{1}{5}x - 1\right) \qquad \textit{Multiply both sides by 15.}$$

$$15 \cdot \frac{1}{3}x + 15 \cdot \frac{1}{5} = 15 \cdot \frac{1}{5}x - 15 \cdot 1 \qquad \textit{Simplify.}$$

$$5x + 3 = 3x - 15 \qquad \textit{Solve.}$$

$$2x = -18$$

$$x = \frac{-18}{2} = -9$$

Answer: The solution is −9.

It is important to know that this technique only works for equations. *Do not try to clear fractions when simplifying expressions.* As a reminder:

Expression	Equation
$\frac{1}{2}x + \frac{5}{3}$	$\frac{1}{2}x + \frac{5}{3} = 0$

We simplify expressions and solve equations. If you multiply an expression by 6, you will change the problem. However, if you multiply both sides of an equation by 6, you obtain an equivalent equation.

Incorrect	Correct
$\frac{1}{2}x + \frac{5}{3}$ $\neq 6 \cdot \left(\frac{1}{2}x + \frac{5}{3}\right)$ $= 3x + 10$ ✗	$\frac{1}{2}x + \frac{5}{3} = 0$ $6 \cdot \left(\frac{1}{2}x + \frac{5}{3}\right) = 6 \cdot 0$ $3x + 10 = 0$ ✓

Applications Involving Linear Equations

Algebra simplifies the process of solving real-world problems. This is done by using letters to represent unknowns, restating problems in the form of equations, and by offering systematic techniques for solving those equations. To solve problems using algebra, first translate the wording of the problem into mathematical statements that describe the relationships between the given information and the unknowns. Usually, this translation to mathematical statements is the difficult step in the process. The key to the translation is to carefully read the problem and identify certain key words and phrases.

Key Words	Translation
Sum, increased by, more than, plus, added to, total	+
Difference, decreased by, subtracted from, less, minus	−

Key Words	Translation
Product, multiplied by, of, times, twice	$\cdot$
Quotient, divided by, ratio, per	$\div$
Is, total, result	$=$

When translating sentences into mathematical statements, be sure to read the sentence several times and parse out the key words and phrases. It is important to first identify the variable, "*let x represent…*" and state in words what the unknown quantity is. This step not only makes our work more readable, but also forces us to think about what we are looking for.

Example 12

When 6 is subtracted from twice the sum of a number and 8 the result is 5. Find the number.

Solution:

Let n represent the unknown number.

$$\overbrace{2\,\cdot}^{\text{"twice"}}\;\overbrace{(n+8)}^{\text{"the sum of a number and 8"}}\;\overbrace{-\;6}^{\text{"6 is subtracted from"}}\;\overbrace{=}^{\text{"result is"}}\;5$$

To understand why we included the parentheses in the set up, you must study the structure of the following two sentences and their translations:

"*twice the sum of a number and 8*"	$2\,(n+8)$
"*the sum of* ***twice a number*** *and 8*"	$2n+8$

The key was to focus on the phrase "*twice the sum*," this prompted us to group the sum within parentheses and then multiply by 2. After translating the sentence into a mathematical statement we then solve.

$$\begin{aligned} 2\,(n+8)-6 &= 5 \\ 2n+16-6 &= 5 \\ 2n+10 &= 5 \\ 2n &= -5 \\ n &= \frac{-5}{2} \end{aligned}$$

Check.

$$\begin{aligned} 2(n+8)-6 &= 2\left(-\frac{5}{2}+8\right)-6 \\ &= 2\left(\frac{11}{2}\right)-6 \\ &= 11-6 \\ &= 5 \quad \checkmark \end{aligned}$$

Answer: The number is $-\frac{5}{2}$.

General guidelines for setting up and solving word problems follow.

- **Step 1:** Read the problem several times, identify the key words and phrases, and organize the given information.
- **Step 2:** Identify the variables by assigning a letter or expression to the unknown quantities.
- **Step 3:** Translate and set up an algebraic equation that models the problem.
- **Step 4:** Solve the resulting algebraic equation.
- **Step 5:** Finally, answer the question in sentence form and make sure it makes sense (check it).

For now, set up all of your equations using only one variable. Avoid two variables by looking for a relationship between the unknowns.

Example 13

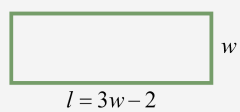
A rectangle has a perimeter measuring 92 meters. The length is 2 meters less than 3 times the width. Find the dimensions of the rectangle.

Solution:

The sentence "*The length is 2 meters less than 3 times the width*," gives us the relationship between the two variables.

Let *w* represent the width of the rectangle.

Let $3w - 2$ represent the length.



The sentence "*A rectangle has a perimeter measuring 92 meters*" suggests an algebraic set up. Substitute 92 for the perimeter and the expression $3w - 2$ for the length into the appropriate formula as follows:

$$\begin{array}{ccc} P = & 2l & +2w \\ \downarrow & \downarrow & \\ 92 = & 2(3w - 2) & +2w \end{array}$$

Once you have set up an algebraic equation with one variable, solve for the width, *w*.

$$\begin{aligned} 92 &= 2(3w - 2) + 2w & &\textit{Distribute.} \\ 92 &= 6w - 4 + 2w & &\textit{Combine like terms.} \\ 92 &= 8w - 4 & &\textit{Solve for } w. \\ 96 &= 8w \\ 12 &= w \end{aligned}$$

Use $3w - 2$ to find the length.

$$l = 3w - 2 = 3(12) - 2 = 36 - 2 = 34$$

To check, make sure the perimeter is 92 meters.

$$\begin{aligned} P &= 2l \quad + \quad 2w \\ &= 2(34) + 2(12) \\ &= 68 + 24 \\ &= 92 \end{aligned}$$

Answer: The rectangle measures 12 meters by 34 meters.

Example 14

Given a $4\,\frac{3}{8}\,\%$annual interest rate, how long will it take $2,500 to yield $437.50 in simple interest?

Solution:

Let t represent the time needed to earn $437.50 at $4\,\frac{3}{8}\,\%$.Organize the information needed to use the formula for simple interest, $I = prt$.

Given interest for the time period:	$I = \$437.50$
Given principal:	$p = \$2{,}500$
Given rate:	$r = 4\,\frac{3}{8}\,\% = 4.375\% = 0.04375$

Next, substitute all of the known quantities into the formula and then solve for the only unknown, t.

$$\begin{aligned} I &= prt \\ 437.50 &= 2500\,(0.04375)\,t \\ 437.50 &= 109.375t \\ \frac{437.50}{109.375} &= \frac{109.375t}{109.375} \\ 4 &= t \end{aligned}$$

Answer: It takes 4 years for $2,500 invested at $4\frac{3}{8}\%$to earn $437.50 in simple interest.

Example 15

Susan invested her total savings of $12,500 in two accounts earning simple interest. Her mutual fund account earned 7% last year and her CD earned 4.5%. If her total interest for the year was $670, how much was in each account?

Solution:

The relationship between the two unknowns is that they total $12,500. When a total is involved, a common technique used to avoid two variables is to represent the second unknown as the difference of the total and the first unknown.

Let x represent the amount invested in the mutual fund.

Let 12,500 – x represent the remaining amount invested in the CD.

Organize the data.

Interest earned in the mutual fund:	$\begin{aligned} I &= prt \\ &= x \cdot 0.07 \cdot 1 \\ &= 0.07x \end{aligned}$
Interest earned in the CD:	$\begin{aligned} I &= prt \\ &= (12,500 - x) \cdot 0.045 \cdot 1 \\ &= 0.045(12,500 - x) \end{aligned}$
Total interest:	$\$670$

The total interest is the sum of the interest earned from each account.

$$\begin{array}{ccccc} \textit{mutual fund interest} & + & \textit{CD interest} & = & \textit{total interest} \\ 0.07x & + & 0.045\left(12{,}500 - x\right) & = & 670 \end{array}$$

This equation models the problem with one variable. Solve for x.

$$\begin{aligned} 0.07x + 0.045\left(12{,}500 - x\right) &= 670 \\ 0.07x + 562.5 - 0.045x &= 670 \\ 0.025x + 562.5 &= 670 \\ 0.025x &= 107.5 \\ x &= \frac{107.5}{0.025} \\ x &= 4{,}300 \end{aligned}$$

Use $12,500 - x$ to find the amount in the CD.

$$12{,}500 - x = 12{,}500 - 4{,}300 = 8{,}200$$

Answer: Susan invested $4,300 at 7% in a mutual fund and $8,200 at 4.5% in a CD.

KEY TAKEAWAYS

- Solving general linear equations involves isolating the variable, with coefficient 1, on one side of the equal sign. To do this, first use the appropriate equality property of addition or subtraction to isolate the variable term on one side of the equal sign. Next, isolate the variable using the equality property of multiplication or division. Finally, check to verify that your solution solves the original equation.
- If solving a linear equation leads to a true statement like $0 = 0$, then the equation is an identity and the solution set consists of all real numbers, $\mathbb{R}$.
- If solving a linear equation leads to a false statement like $0 = 5$, then the equation is a contradiction and there is no solution, $\emptyset$.
- Clear fractions by multiplying both sides of an equation by the least common multiple of all the denominators. Distribute and multiply all terms by the LCD to obtain an equivalent equation with integer coefficients.
- Simplify the process of solving real-world problems by creating mathematical models that describe the relationship between unknowns. Use algebra to solve the resulting equations.

TOPIC EXERCISES

PART A: SOLVING BASIC LINEAR EQUATIONS

Determine whether or not the given value is a solution.

1. $-5x + 4 = -1;\ x = -1$
2. $4x - 3 = -7;\ x = -1$
3. $3y - 4 = 5;\ y = \frac{9}{3}$
4. $-2y + 7 = 12;\ y = -\frac{5}{2}$
5. $3a - 6 = 18 - a;\ a = -3$
6. $5(2t - 1) = 2 - t;\ t = 2$
7. $ax - b = 0;\ x = \frac{b}{a}$
8. $ax + b = 2b;\ x = \frac{b}{a}$

Solve.

9. $5x - 3 = 27$
10. $6x - 7 = 47$
11. $4x + 13 = 35$
12. $6x - 9 = 18$
13. $9a + 10 = 10$
14. $5 - 3a = 5$
15. $-8t + 5 = 15$
16. $-9t + 12 = 33$
17. $\frac{2}{3}x + \frac{1}{2} = 1$
18. $\frac{3}{8}x + \frac{5}{4} = \frac{3}{2}$
19. $\frac{1 - 3y}{5} = 2$

20. $\frac{2-5y}{6} = -8$

21. $7 - y = 22$

22. $6 - y = 12$

23. Solve for x: $ax - b = c$

24. Solve for x: $ax + b = 0$

PART B: SOLVING LINEAR EQUATIONS

Solve.

25. $6x - 5 + 2x = 19$

26. $7 - 2x + 9 = 24$

27. $12x - 2 - 9x = 5x + 8$

28. $16 - 3x - 22 = 8 - 4x$

29. $5y - 6 - 9y = 3 - 2y + 8$

30. $7 - 9y + 12 = 3y + 11 - 11y$

31. $3 + 3a - 11 = 5a - 8 - 2a$

32. $2 - 3a = 5a + 7 - 8a$

33. $\frac{1}{3}x - \frac{3}{2} + \frac{5}{2}x = \frac{5}{6}x + \frac{1}{4}$

34. $\frac{5}{8} + \frac{1}{5}x - \frac{3}{4} = \frac{3}{10}x - \frac{1}{4}$

35. $1.2x - 0.5 - 2.6x = 2 - 2.4x$

36. $1.59 - 3.87x = 3.48 - 4.1x - 0.51$

37. $5 - 10x = 2x + 8 - 12x$

38. $8x - 3 - 3x = 5x - 3$

39. $5\left(y + 2\right) = 3\left(2y - 1\right) + 10$

40. $7\left(y - 3\right) = 4\left(2y + 1\right) - 21$

41. $7 - 5\,(3t - 9) = 22$

42. $10 - 5(3t + 7) = 20$
43. $5 - 2x = 4 - 2(x - 4)$
44. $2\left(4x - 5\right) + 7x = 5(3x - 2)$
45. $4(4a - 1) = 5(a - 3) + 2(a - 2)$
46. $6\left(2b - 1\right) + 24b = 8\left(3b - 1\right)$
47. $\frac{2}{3}(x + 18) + 2 = \frac{1}{3}x - 13$
48. $\frac{2}{5}x - \frac{1}{2}\left(6x - 3\right) = \frac{4}{3}$
49. $1.2(2x + 1) + 0.6x = 4x$
50. $6 + 0.5\left(7x - 5\right) = 2.5x + 0.3$
51. $5\left(y + 3\right) = 15\left(y + 1\right) - 10y$
52. $3\left(4 - y\right) - 2\left(y + 7\right) = -5y$
53. $\frac{1}{5}(2a + 3) - \frac{1}{2} = \frac{1}{3}a + \frac{1}{10}$
54. $\frac{3}{2}a = \frac{3}{4}(1 + 2a) - \frac{1}{5}\left(a + 5\right)$
55. $6 - 3(7x + 1) = 7(4 - 3x)$
56. $6\left(x - 6\right) - 3(2x - 9) = -9$
57. $\frac{3}{4}\left(y - 2\right) + \frac{2}{3}\left(2y + 3\right) = 3$
58. $\frac{5}{4} - \frac{1}{2}\left(4y - 3\right) = \frac{2}{5}\left(y - 1\right)$
59. $-2(3x + 1) - (x - 3) = -7x + 1$
60. $6(2x + 1) - (10x + 9) = 0$
61. Solve for *w*: $P = 2l + 2w$
62. Solve for *a*: $P = a + b + c$
63. Solve for *t*: $D = rt$
64. Solve for *w*: $V = lwh$
65. Solve for *b*: $A = \frac{1}{2}bh$

66. Solve for *a*: $s = \frac{1}{2}at^2$
67. Solve for *a*: $A = \frac{1}{2}h\left(a + b\right)$
68. Solve for *h*: $V = \frac{1}{3}\pi r^2 h$
69. Solve for *F*: $C = \frac{5}{9}(F - 32)$
70. Solve for *x*: $ax + b = c$

PART C: APPLICATIONS

Set up an algebraic equation then solve.

Number Problems

71. When 3 is subtracted from the sum of a number and 10 the result is 2. Find the number.
72. The sum of 3 times a number and 12 is equal to 3. Find the number.
73. Three times the sum of a number and 6 is equal to 5 times the number. Find the number.
74. Twice the sum of a number and 4 is equal to 3 times the sum of the number and 1. Find the number.
75. A larger integer is 1 more than 3 times another integer. If the sum of the integers is 57, find the integers.
76. A larger integer is 5 more than twice another integer. If the sum of the integers is 83, find the integers.
77. One integer is 3 less than twice another integer. Find the integers if their sum is 135.
78. One integer is 10 less than 4 times another integer. Find the integers if their sum is 100.
79. The sum of three consecutive integers is 339. Find the integers.
80. The sum of four consecutive integers is 130. Find the integers.
81. The sum of three consecutive even integers is 174. Find the integers.
82. The sum of four consecutive even integers is 116. Find the integers.

83. The sum of three consecutive odd integers is 81. Find the integers.

84. The sum of four consecutive odd integers is 176. Find the integers.

Geometry Problems

85. The length of a rectangle is 5 centimeters less than twice its width. If the perimeter is 134 centimeters, find the length and width.

86. The length of a rectangle is 4 centimeters more than 3 times its width. If the perimeter is 64 centimeters, find the length and width.

87. The width of a rectangle is one-half that of its length. If the perimeter measures 36 inches, find the dimensions of the rectangle.

88. The width of a rectangle is 4 inches less than its length. If the perimeter measures 72 inches, find the dimensions of the rectangle.

89. The perimeter of a square is 48 inches. Find the length of each side.

90. The perimeter of an equilateral triangle is 96 inches. Find the length of each side.

91. The circumference of a circle measures 80π units. Find the radius.

92. The circumference of a circle measures 25 centimeters. Find the radius rounded off to the nearest hundredth.

Simple Interest Problems

93. For how many years must $1,000 be invested at $5\frac{1}{2}$% to earn $165 in simple interest?

94. For how many years must $20,000 be invested at $6\frac{1}{4}$% to earn $3,125 in simple interest?

95. At what annual interest rate must $6500 be invested for 2 years to yield $1,040 in simple interest?

96. At what annual interest rate must $5,750 be invested for 1 year to yield $333.50 in simple interest?

97. If the simple interest earned for 5 years was $1,860 and the annual interest rate was 6%, what was the principal?

98. If the simple interest earned for 2 years was $543.75 and the annual interest rate was $3\frac{3}{4}\%$, what was the principal?

99. How many years will it take $600 to double earning simple interest at a 5% annual rate? (Hint: To double, the investment must earn $600 in simple interest.)

100. How many years will it take $10,000 to double earning simple interest at a 5% annual rate? (Hint: To double, the investment must earn $10,000 in simple interest.)

101. Jim invested $4,200 in two accounts. One account earns 3% simple interest and the other earns 6%. If the interest after 1 year was $159, how much did he invest in each account?

102. Jane has her $6,500 savings invested in two accounts. She has part of it in a CD at 5% annual interest and the rest in a savings account that earns 4% annual interest. If the simple interest earned from both accounts is $303 for the year, then how much does she have in each account?

103. Jose put last year's bonus of $8,400 into two accounts. He invested part in a CD with 2.5% annual interest and the rest in a money market fund with 1.5% annual interest. His total interest for the year was $198. How much did he invest in each account?

104. Mary invested her total savings of $3,300 in two accounts. Her mutual fund account earned 6.2% last year and her CD earned 2.4%. If her total interest for the year was $124.80, how much was in each account?

105. Alice invests money into two accounts, one with 3% annual interest and another with 5% annual interest. She invests 3 times as much in the higher yielding account as she does in the lower yielding account. If her total interest for the year is $126, how much did she invest in each account?

106. James invested an inheritance in two separate banks. One bank offered $5\frac{1}{2}\%$ annual interest rate and the other $6\frac{1}{4}\%$. He invested twice as much in the higher yielding bank account than he did in the other. If his total simple interest for 1 year was $5,760, then what was the amount of his inheritance?

Uniform Motion Problems

107. If it takes Jim $1\frac{1}{4}$ hours to drive the 40 miles to work, then what is Jim's average speed?

108. It took Jill $3\frac{1}{2}$ hours to drive the 189 miles home from college. What was her average speed?

109. At what speed should Jim drive if he wishes to travel 176 miles in $2\frac{3}{4}$ hours?

110. James and Martin were able to drive the 1,140 miles from Los Angeles to Seattle. If the total trip took 19 hours, then what was their average speed?

PART D: DISCUSSION BOARD

111. What is regarded as the main business of algebra? Explain.

112. What is the origin of the word *algebra*?

113. Create an identity or contradiction of your own and share it on the discussion board. Provide a solution and explain how you found it.

114. Post something you found particularly useful or interesting in this section. Explain why.

115. Conduct a web search for “solving linear equations.” Share a link to website or video tutorial that you think is helpful.

ANSWERS

1. No
3. Yes
5. No
7. Yes
9. 6
11. $\frac{11}{2}$
13. 0
15. $-\frac{5}{4}$
17. $\frac{3}{4}$
19. −3
21. −15
23. $x = \frac{b+c}{a}$
25. 3
27. −5
29. $-\frac{17}{2}$
31. $\mathbb{R}$
33. $\frac{7}{8}$
35. 2.5
37. Ø
39. 3
41. 2
43. Ø
45. $-\frac{5}{3}$

47. −81

49. 1.2

51. $\mathbb{R}$

53. 0

55. Ø

57. $\frac{6}{5}$

59. $\mathbb{R}$

61. $w = \frac{P - 2l}{2}$

63. $t = \frac{D}{r}$

65. $b = \frac{2A}{h}$

67. $a = \frac{2A}{h} - b$

69. $F = \frac{9}{5} C + 32$

71. −5

73. 9

75. 14, 43

77. 46, 89

79. 112, 113, 114

81. 56, 58, 60

83. 25, 27, 29

85. Width: 24 centimeters; length: 43 centimeters

87. Width: 6 inches; length: 12 inches

89. 12 inches

91. 40 units

93. 3 years

95. 8%

97. \$6,200

99. 20 years

101. He invested \$3,100 at 3% and \$1,100 at 6%.

103. Jose invested \$7,200 in the CD and \$1,200 in the money market fund.

105. Alice invested \$700 at 3% and \$2,100 at 5%.

107. 32 miles per hour

109. 64 miles per hour

111. Answer may vary

113. Answer may vary

115. Answer may vary

1.8 Solving Linear Inequalities with One Variable

LEARNING OBJECTIVES

1. Identify linear inequalities and check solutions.
2. Solve linear inequalities and express the solutions graphically on a number line and in interval notation.
3. Solve compound linear inequalities and express the solutions graphically on a number line and in interval notation.
4. Solve applications involving linear inequalities and interpret the results.

Linear Inequalities

A **linear inequality**[138] is a mathematical statement that relates a linear expression as either less than or greater than another. The following are some examples of linear inequalities, all of which are solved in this section:

$5x + 7 < 22$	$-2(x + 8) + 6 \geq 20$	$-2\left(4x - 5\right) < 9 - 2(x - 2)$

A **solution to a linear inequality**[139] is a real number that will produce a true statement when substituted for the variable. Linear inequalities have either infinitely many solutions or no solution. If there are infinitely many solutions, graph the solution set on a number line and/or express the solution using interval notation.

138. Linear expressions related with the symbols $\leq$, <, $\geq$, and >.

139. A real number that produces a true statement when its value is substituted for the variable.

Example 1

Are $x = -4$ and $x = 6$ solutions to $5x + 7 < 22$?

Solution:

Substitute the values in for *x*, simplify, and check to see if we obtain a true statement.

Check $x = -4$	Check $x = 6$
$5(-4) + 7 < 22$ $-20 + 7 < 22$ $-13 < 22$ ✓	$5(6) + 7 < 22$ $30 + 7 < 22$ $37 < 22$ ✗

Answer: $x = -4$ is a solution and $x = 6$ is not.

All but one of the techniques learned for solving linear equations apply to solving linear inequalities. You may add or subtract any real number to both sides of an inequality, and you may multiply or divide both sides by any *positive* real number to create equivalent inequalities. For example:

$$\begin{aligned} 10 &> -5 & \\ 10 - 7 &> -5 - 7 & \textit{Subtract 7 on both sides.} \\ 3 &> -12 \quad ✓ & \textit{True} \end{aligned}$$

$$\begin{aligned} 10 &> -5 \\ \frac{10}{5} &> \frac{-5}{5} && \textit{Divide both sides by } 5. \\ 2 &> -1 && ✓\ \textit{True} \end{aligned}$$

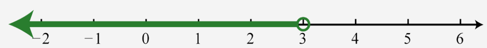
Subtracting 7 from each side and dividing each side by positive 5 results in an inequality that is true.

Example 2

Solve and graph the solution set: $5x + 7 < 22$.

Solution:

$$\begin{aligned} 5x + 7 &< 22 \\ 5x + 7 - 7 &< 22 - 7 \\ 5x &< 15 \\ \frac{5x}{5} &< \frac{15}{5} \\ x &< 3 \end{aligned}$$



It is helpful to take a minute and choose a few values in and out of the solution set, substitute them into the original inequality, and then verify the results. As indicated, you should expect $x = 0$ to solve the original inequality and that $x = 5$ should not.

Check $x = 0$	*Check* $x = 5$
$5(0) + 7 < 22$ $7 < 22$ ✓	$5(5) + 7 < 22$ $25 + 7 < 22$ $32 < 22$ ✗

Checking in this manner gives us a good indication that we have solved the inequality correctly.

We can express this solution in two ways: using set notation and interval notation.

$$\begin{array}{rl} \{x|x<3\} & \textit{Set notation} \\ (-\infty,3) & \textit{Interval notation} \end{array}$$

In this text we will choose to present answers using interval notation.

Answer: $(-\infty,\ 3)$

When working with linear inequalities, a different rule applies when multiplying or dividing by a negative number. To illustrate the problem, consider the true statement $10 > -5$ and divide both sides by -5.

$$\begin{array}{rl} 10 > -5 & \\ \frac{10}{-5} > \frac{-5}{-5} & \textit{Divide both sides by } -5. \\ -2 > 1 & ✗ \ \textit{False} \end{array}$$

Dividing by -5 results in a false statement. To retain a true statement, the inequality must be reversed.

$$10 > -5$$
$$\frac{10}{-5} < \frac{-5}{-5} \quad \textit{Reverse the inequality.}$$
$$-2 < 1 \quad ✓ \textit{ True}$$

The same problem occurs when multiplying by a negative number. This leads to the following new rule: *when multiplying or dividing by a negative number, reverse the inequality*. It is easy to forget to do this so take special care to watch for negative coefficients. In general, given algebraic expressions *A* and *B*, where *c* is a positive nonzero real number, we have the following **properties of inequalities**[140]:

Addition property of inequalities:	If $A < B$ then, $A + c < B + c$
Subtraction property of inequalities:	If $A < B$, then $A - c < B - c$
Multiplication property of inequalities:	If $A < B$, then $cA < cB$ If $A < B$, then $-cA > -cB$
Division property of inequalities:	If $A < B$, then $\frac{A}{c} < \frac{B}{c}$ If $A < B$, then $\frac{A}{-c} > \frac{B}{-c}$

We use these properties to obtain an **equivalent inequality**[141], one with the same solution set, where the variable is isolated. The process is similar to solving linear equations.

140. Properties used to obtain equivalent inequalities and used as a means to solve them.

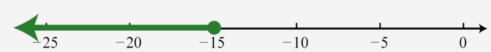
141. Inequalities that share the same solution set.

Example 3

Solve and graph the solution set: $-2(x+8)+6 \geq 20$.

Solution:

$$\begin{aligned} -2(x+8)+6 &\geq 20 && \textit{Distribute.} \\ -2x-16+6 &\geq 20 && \textit{Combine like terms.} \\ -2x-10 &\geq 20 && \textit{Solve for } x. \\ -2x &\geq 30 && \textit{Divide both sides by } -2. \\ \frac{-2x}{-2} &\leq \frac{30}{-2} && \textit{Reverse the inequality.} \\ x &\leq -15 \end{aligned}$$



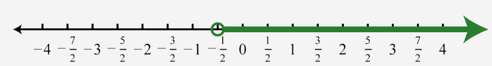
Answer: Interval notation $(-\infty, -15]$

Example 4

Solve and graph the solution set: $-2\left(4x-5\right) < 9-2\left(x-2\right)$.

Solution:

$$\begin{aligned} -2\left(4x-5\right) &< 9-2\left(x-2\right) \\ -8x+10 &< 9-2x+4 \\ -8x+10 &< 13-2x \\ -6x &< 3 \\ \frac{-6x}{-6} &> \frac{3}{-6} \qquad \textit{Reverse the inequality.} \\ x &> -\frac{1}{2} \end{aligned}$$



Answer: Interval notation $\left(-\frac{1}{2}, \infty\right)$

Example 5

Solve and graph the solution set: $\frac{1}{2}x - 2 \geq \frac{1}{2}\left(\frac{7}{4}x - 9\right) + 1.$

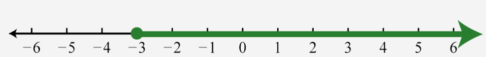
Solution:

$$
\begin{aligned}
\frac{1}{2}x - 2 &\geq \frac{1}{2}\left(\frac{7}{4}x - 9\right) + 1 \\
\frac{1}{2}x - 2 &\geq \frac{7}{8}x - \frac{9}{2} + 1 \\
\frac{1}{2}x - \frac{7}{8}x &\geq -\frac{7}{2} + 2 \\
-\frac{3}{8}x &\geq -\frac{3}{2} \\
\left(-\frac{8}{3}\right)\left(-\frac{3}{8}x\right) &\leq \left(-\frac{8}{3}\right)\left(-\frac{3}{2}\right) \qquad \textit{Reverse the inequality.} \\
x &\leq 4
\end{aligned}
$$

Answer: Interval notation: $(-\infty, \ 4]$

Try this! Solve and graph the solution set: $10 - 5(2x + 3) \leq 25.$

Answer: $[-3, \ \infty)$;



[(click to see video)]

Compound Inequalities

Following are some examples of compound linear inequalities:

$-13 < 3x - 7 < 17$	$4x + 5 \leq -15 \quad \text{or} \quad 6x - 11 > 7$

These **compound inequalities**[142] are actually two inequalities in one statement joined by the word *and* or by the word *or*. For example,

$$-13 < 3x - 7 < 17$$

is a compound inequality because it can be decomposed as follows:

$$-13 < 3x - 7 \text{ and } 3x - 7 < 17$$

We can solve each inequality individually; the intersection of the two solution sets solves the original compound inequality. While this method works, there is another method that usually requires fewer steps. Apply the properties of this section to all three parts of the compound inequality with the goal of *isolating the variable in the middle* of the statement to determine the bounds of the solution set.

142. Two or more inequalities in one statement joined by the word "and" or by the word "or."

Example 6

Solve and graph the solution set: $-13 < 3x - 7 < 17$.

Solution:

$$-13 < 3x - 7 < 17$$
$$-13 + 7 < 3x - 7 + 7 < 17 + 7$$
$$-6 < 3x < 24$$
$$\frac{-6}{3} < \frac{3x}{3} < \frac{24}{3}$$
$$-2 < x < 8$$

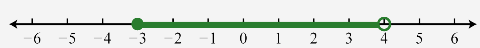
Answer: Interval notation: $(-2, 8)$

Example 7

Solve and graph the solution set: $\frac{5}{6} \leq \frac{1}{3}\left(\frac{1}{2}x + 4\right) < 2$.

Solution:

$$\frac{5}{6} \leq \frac{1}{3}\left(\frac{1}{2}x + 4\right) < 2$$
$$\frac{5}{6} \leq \frac{1}{6}x + \frac{4}{3} < 2$$
$$6 \cdot \left(\frac{5}{6}\right) \leq 6 \cdot \left(\frac{1}{6}x + \frac{4}{3}\right) < 6 \cdot (2)$$
$$5 \leq x + 8 < 12$$
$$5 - 8 \leq x + 8 - 8 < 12 - 8$$
$$-3 \leq x < 4$$



Answer: Interval notation $[-3, 4)$

It is important to note that when multiplying or dividing all three parts of a compound inequality by a negative number, you must reverse all of the inequalities in the statement. For example:

$$-10 < -2x < 20$$
$$\frac{-10}{-2} > \frac{-2x}{-2} > \frac{20}{-2}$$
$$5 > x > -10$$

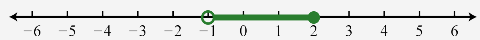
The answer above can be written in an equivalent form, where smaller numbers lie to the left and the larger numbers lie to the right, as they appear on a number line.

$$-10 < x < 5$$

Using interval notation, write: $(-10,\ 5)$.

Try this! Solve and graph the solution set: $-3 \leq -3\,(2x - 3) < 15$.

Answer: $(-1, 2]$;



(click to see video)

For compound inequalities with the word "*or*" you work both inequalities separately and then consider the union of the solution sets. Values in this union solve either inequality.

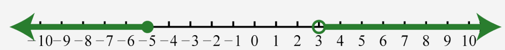
Example 8

Solve and graph the solution set: $4x + 5 \leq -15$ or $6x - 11 > 7$.

Solution:

Solve each inequality and form the union by combining the solution sets.

$4x + 5 \leq -15$ $4x \leq -20$ $x \leq -5$	or	$6x - 11 > 7$ $6x > 18$ $x > 3$



Answer: Interval notation $\left(-\infty, -5\right] \cup \left(3, \infty\right)$

Try this! Solve and graph the solution set:
$5\,(x - 3) < -20$ *or* $2\left(5 - 3x\right) < 1$.

Answer: $\left(-\infty, -1\right) \cup \left(\frac{3}{2}, \infty\right)$;

(click to see video)

Applications of Linear Inequalities

Some of the key words and phrases that indicate inequalities are summarized below:

Key Phrases	Translation
A number is **at least** 5.	$x \geq 5$
A number is 5 **or more inclusive**.	
A number is **at most** 3.	$x \leq 3$
A number is 3 **or less inclusive**.	
A number is **strictly less than** 4.	$x < 4$
A number is **less than** 4, **noninclusive**.	
A number is **greater than** 7.	$x > 7$
A number is **more than** 7, **noninclusive**.	
A number is **in between** 2 and 10.	$2 < x < 10$
A number is at least 5 **and** at most 15.	$5 \leq x \leq 15$

Key Phrases	Translation
A number may **range** from 5 to 15.	

As with all applications, carefully read the problem several times and look for key words and phrases. Identify the unknowns and assign variables. Next, translate the wording into a mathematical inequality. Finally, use the properties you have learned to solve the inequality and express the solution graphically or in interval notation.

Example 9

Seven less than 3 times the sum of a number and 5 is at most 11. Find all numbers that satisfy this condition.

Solution:

First, choose a variable for the unknown number and identify the key words and phrases.

Let n represent the unknown indicated by "*a number*."

$$\underbrace{3(n+5)}_{\text{three times the sum of a number and 5}} \quad \underbrace{-\quad 7}_{\text{seven less than}} \quad \underbrace{\leq}_{\text{is at most}} \quad 11$$

Solve for n.

$$\begin{aligned} 3\left(n+5\right)-7 &\leq 11 \\ 3n+15-7 &\leq 11 \\ 3n+8 &\leq 11 \\ 3n &\leq 3 \\ n &\leq 1 \end{aligned}$$

Answer: Any number less than or equal to 1 will satisfy the statement.

Example 10

To earn a B in a mathematics course the test average must be at least 80% and less than 90%. If a student earned 92%, 96%, 79%, and 83% on the first four tests, what must she score on the fifth test to earn a B?

Solution:

Set up a compound inequality where the test average is between 80% and 90%. In this case, include the lower bound, 80.

Let x represent the score on the fifth test.

$$\begin{aligned} 80 &\leq \textit{test average} < 90 \\ 80 &\leq \frac{92+96+79+83+x}{5} < 90 \\ 5\cdot 80 &\leq 5\cdot\frac{350+x}{5} < 5\cdot 90 \\ 400 &\leq 350+x < 450 \\ 50 &\leq x < 100 \end{aligned}$$

Answer: She must earn a score of at least 50% and less than 100%.

In the previous example, the upper bound 100% was not part of the solution set. What would happen if she did earn a 100% on the fifth test?

$$\begin{aligned} \textit{average} &= \frac{92+96+79+83+100}{5} \\ &= \frac{450}{5} \\ &= 90 \end{aligned}$$

As we can see, her average would be 90%, which would earn her an A.

KEY TAKEAWAYS

- Inequalities typically have infinitely many solutions. The solutions are presented graphically on a number line or using interval notation or both.
- All but one of the rules for solving linear inequalities are the same as solving linear equations. If you divide or multiply an inequality by a negative number, reverse the inequality to obtain an equivalent inequality.
- Compound inequalities involving the word "or" require us to solve each inequality and form the union of each solution set. These are the values that solve at least one of the given inequalities.
- Compound inequalities involving the word "and" require the intersection of the solution sets for each inequality. These are the values that solve both or all of the given inequalities.
- The general guidelines for solving word problems apply to applications involving inequalities. Be aware of a new list of key words and phrases that indicate a mathematical setup involving inequalities.

TOPIC EXERCISES

PART A: LINEAR INEQUALITIES

Determine whether or not the given value is a solution.

1. $5x - 1 < -2;\ x = -1$
2. $-3x + 1 > -10;\ x = 1$
3. $2x - 3 < -5;\ x = 1$
4. $5x - 7 < 0;\ x = 2$
5. $9y - 4 \geq 5;\ y = 1$
6. $-6y + 1 \leq 3;\ y = -1$
7. $12a + 3 \leq -2;\ a = -\frac{1}{3}$
8. $25a - 2 \leq -22;\ a = -\frac{4}{5}$
9. $-10 < 2x - 5 < -5;\ x = -\frac{1}{2}$
10. $3x + 8 < -2\ or\ 4x - 2 > 5;\ x = 2$

Graph all solutions on a number line and provide the corresponding interval notation.

11. $3x + 5 > -4$
12. $2x + 1 > -1$
13. $5 - 6y < -1$
14. $7 - 9y > 43$
15. $6 - a \leq 6$
16. $-2a + 5 > 5$
17. $\frac{5x+6}{3} \leq 7$
18. $\frac{4x+11}{6} \leq \frac{1}{2}$

19. $\frac{1}{2}y + \frac{5}{4} \geq \frac{1}{4}$
20. $\frac{1}{12}y + \frac{2}{3} \leq \frac{5}{6}$
21. $2(3x + 14) < -2$
22. $5(2y + 9) > -15$
23. $5 - 2(4 + 3y) \leq 45$
24. $-12 + 5(5 - 2x) < 83$
25. $6(7 - 2a) + 6a \leq 12$
26. $2a + 10(4 - a) \geq 8$
27. $9(2t - 3) - 3(3t + 2) < 30$
28. $-3(t - 3) - (4 - t) > 1$
29. $\frac{1}{2}(5x + 4) + \frac{5}{6}x > -\frac{4}{3}$
30. $\frac{2}{5} + \frac{1}{6}(2x - 3) \geq \frac{1}{15}$
31. $5x - 2(x - 3) < 3(2x - 1)$
32. $3(2x - 1) - 10 > 4(3x - 2) - 5x$
33. $-3y \geq 3(y + 8) + 6(y - 1)$
34. $12 \leq 4(y - 1) + 2(2y + 1)$
35. $-2(5t - 3) - 4 > 5(-2t + 3)$
36. $-7(3t - 4) > 2(3 - 10t) - t$
37. $\frac{1}{2}(x + 5) - \frac{1}{3}(2x + 3) > \frac{7}{6}x + \frac{3}{2}$
38. $-\frac{1}{3}(2x - 3) + \frac{1}{4}(x - 6) \geq \frac{1}{12}x - \frac{3}{4}$
39. $4(3x + 4) \geq 3(6x + 5) - 6x$
40. $1 - 4(3x + 7) < -3(x + 9) - 9x$
41. $6 - 3(2a - 1) \leq 4(3 - a) + 1$

42. $12 - 5\left(2a + 6\right) \geq 2\left(5 - 4a\right) - a$

PART B: COMPOUND INEQUALITIES

Graph all solutions on a number line and provide the corresponding interval notation.

43. $-1 < 2x + 1 < 9$
44. $-4 < 5x + 11 < 16$
45. $-7 \leq 6y - 7 \leq 17$
46. $-7 \leq 3y + 5 \leq 2$
47. $-7 < \frac{3x+1}{2} \leq 8$
48. $-1 \leq \frac{2x+7}{3} < 1$
49. $-4 \leq 11 - 5t < 31$
50. $15 < 12 - t \leq 16$
51. $-\frac{1}{3} \leq \frac{1}{6}a + \frac{1}{3} \leq \frac{1}{2}$
52. $-\frac{1}{6} < \frac{1}{3}a + \frac{5}{6} < \frac{3}{2}$
53. $5x + 2 < -3$ or $7x - 6 > 15$
54. $4x + 15 \leq -1$ or $3x - 8 \geq -11$
55. $8x - 3 \leq 1$ or $6x - 7 \geq 8$
56. $6x + 1 < -3$ or $9x - 20 > -5$
57. $8x - 7 < 1$ or $4x + 11 > 3$
58. $10x - 21 < 9$ or $7x + 9 \geq 30$
59. $7 + 2y < 5$ or $20 - 3y > 5$
60. $5 - y < 5$ or $7 - 8y \leq 23$
61. $15 + 2x < -15$ or $10 - 3x > 40$
62. $10 - \frac{1}{3}x \leq 5$ or $5 - \frac{1}{2}x \leq 15$

63. $9 - 2x \leq 15$ and $5x - 3 \leq 7$

64. $5 - 4x > 1$ and $15 + 2x \geq 5$

65. $7y - 18 < 17$ and $2y - 15 < 25$

66. $13y + 20 \geq 7$ and $8 + 15y > 8$

67. $5 - 4x \leq 9$ and $3x + 13 \leq 1$

68. $17 - 5x \geq 7$ and $4x - 7 > 1$

69. $9y + 20 \leq 2$ and $7y + 15 \geq 1$

70. $21 - 6y \leq 3$ and $-7 + 2y \leq -1$

71. $-21 < 6(x - 3) < -9$

72. $0 \leq 2\left(2x + 5\right) < 8$

73. $-15 \leq 5 + 4\left(2y - 3\right) < 17$

74. $5 < 8 - 3\left(3 - 2y\right) \leq 29$

75. $5 < 5 - 3(4 + t) < 17$

76. $-3 \leq 3 - 2\left(5 + 2t\right) \leq 21$

77. $-40 < 2\left(x + 5\right) - \left(5 - x\right) \leq -10$

78. $-60 \leq 5(x - 4) - 2\left(x + 5\right) \leq 15$

79. $-\frac{1}{2} < \frac{1}{30}(x - 10) < \frac{1}{3}$

80. $-\frac{1}{5} \leq \frac{1}{15}(x - 7) \leq \frac{1}{3}$

81. $-1 \leq \frac{a + 2(a - 2)}{5} \leq 0$

82. $0 < \frac{5 + 2(a - 1)}{6} < 2$

PART C: APPLICATIONS

Find all numbers that satisfy the given condition.

83. Three less than twice the sum of a number and 6 is at most 13.
84. Five less than 3 times the sum of a number and 4 is at most 10.
85. Five times the sum of a number and 3 is at least 5.
86. Three times the difference between a number and 2 is at least 12.
87. The sum of 3 times a number and 8 is between 2 and 20.
88. Eight less than twice a number is between −20 and −8.
89. Four subtracted from three times some number is between −4 and 14.
90. Nine subtracted from 5 times some number is between 1 and 11.

Set up an algebraic inequality and then solve.

91. With a golf club membership, costing $120 per month, each round of golf costs only $35.00. How many rounds of golf can a member play if he wishes to keep his costs $270 per month at most?
92. A rental truck costs $95 per day plus $0.65 per mile driven. How many miles can be driven on a one-day rental to keep the cost at most $120?
93. Mark earned 6, 7, and 10 points out of 10 on the first three quizzes. What must he score on the fourth quiz to average at least 8?
94. Joe earned scores of 78, 82, 88 and 70 on his first four algebra exams. What must he score on the fifth exam to average at least 80?
95. A gymnast scored 13.2, 13.0, 14.3, 13.8, and 14.6 on the first five events. What must he score on the sixth event to average at least 14.0?
96. A dancer scored 7.5 and 8.2 from the first two judges. What must her score from the third judge come in as if she is to average 8.4 or higher?
97. If two times an angle is between 180 degrees and 270 degrees, then what are the bounds of the original angle?
98. The perimeter of a square must be between 120 inches and 460 inches. Find the length of all possible sides that satisfy this condition.
99. A computer is set to shut down if the temperature exceeds 45°C. Give an equivalent statement using degrees Fahrenheit. Hint: $C = \frac{5}{9}(F - 32)$.
100. A certain antifreeze is effective for a temperature range of −35°C to 120°C. Find the equivalent range in degrees Fahrenheit.

PART D: DISCUSSION BOARD

101. Often students reverse the inequality when solving $5x + 2 < -18$? Why do you think this is a common error? Explain to a beginning algebra student why we do not.

102. Conduct a web search for “solving linear inequalities.” Share a link to website or video tutorial that you think is helpful.

103. Write your own 5 key takeaways for this entire chapter. What did you find to be review and what did you find to be new? Share your thoughts on the discussion board.

ANSWERS

1. Yes

3. No

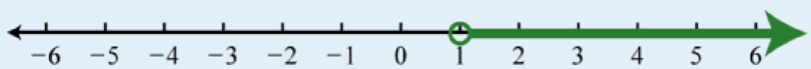
5. Yes

7. No

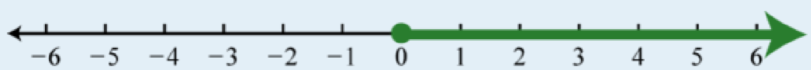
9. Yes

11. $(-3, \infty)$;

13. $(1, \infty)$;

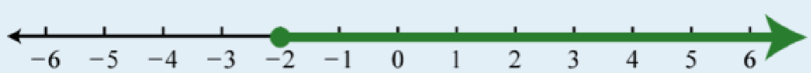


15. $[0, \infty)$;

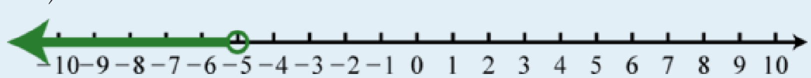


17. $(-\infty, 3]$;

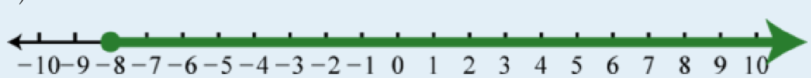
19. $[-2, \infty)$;



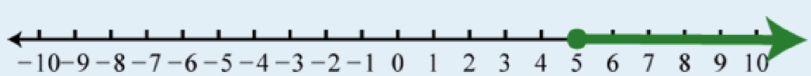
21. $(-\infty, -5)$;



23. $[-8, \infty)$;



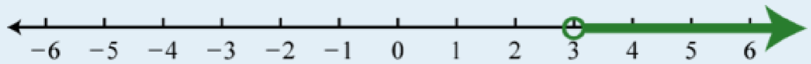
25. $[5, \infty)$;



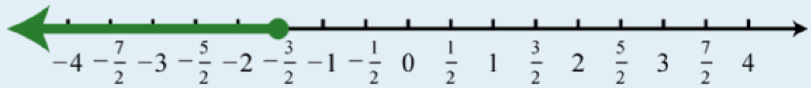
27. $(-\infty, 7)$;

29. $(-1, \infty)$;

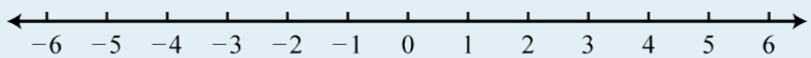
31. $\left(3, \infty\right)$;



33. $\left(-\infty, -\frac{3}{2}\right]$;

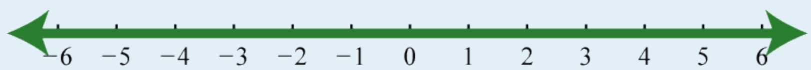


35. Ø;

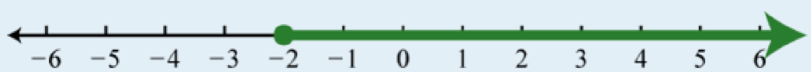


37. $\left(-\infty, 0\right)$;

39. $\mathbb{R}$;

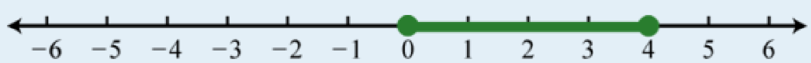


41. $\left[-2, \infty\right)$;



43. $(-1, 4)$;

45. $\left[0, 4\right]$;

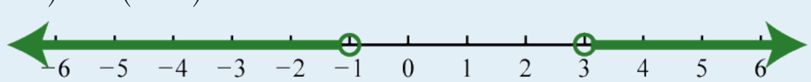


47. $\left(-5, 5\right]$;

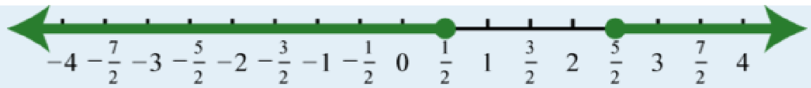
49. $(-4, 3\right]$;

51. $\left[-4, 1\right]$;

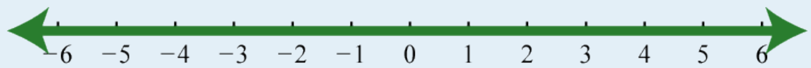
53. $\left(-\infty, -1\right) \cup \left(3, \infty\right)$;



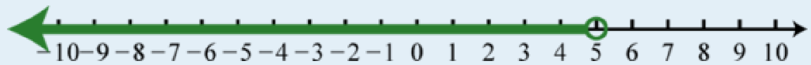
55. $\left(-\infty, \frac{1}{2}\right] \cup \left[\frac{5}{2}, \infty\right)$;



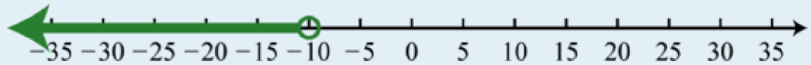
57. $\mathbb{R}$;



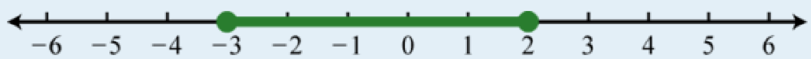
59. $(-\infty, 5)$;



61. $(-\infty, -10)$;

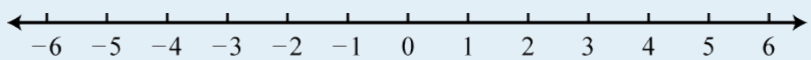


63. $[-3, 2]$;

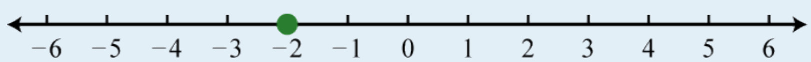


65. $(-\infty, 5)$;

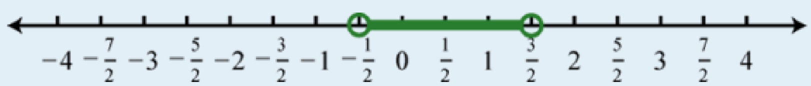
67. Ø;



69. -2;

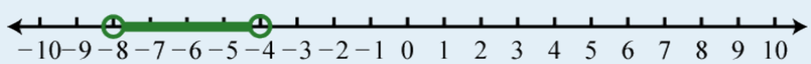


71. $\left(-\frac{1}{2}, \frac{3}{2}\right)$;

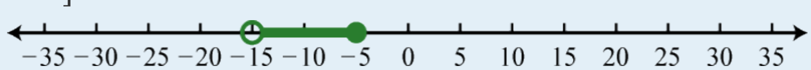


73. $[-1, 3)$;

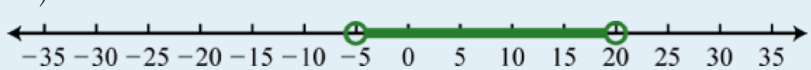
75. $(-8, -4)$;



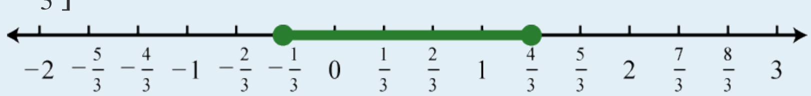
77. $(-15, -5]$;



79. $(-5, 20)$;



81. $\left[-\frac{1}{3}, \frac{4}{3}\right]$;



83. $(-\infty, 2]$

85. $[-2, \infty)$

87. $(-2, 4)$

89. $(0, 6)$

91. Members may play 4 rounds or fewer.

93. Mark must earn at least 9 points on the fourth quiz.

95. He must score a 15.1 on the sixth event.

97. The angle is between 90 degrees and 135 degrees.

99. The computer will shut down when the temperature exceeds 113°F.

101. Answer may vary

103. Answer may vary

1.9 Review Exercises and Sample Exam

REVIEW EXERCISES

REVIEW OF REAL NUMBERS AND ABSOLUTE VALUE

Reduce to lowest terms.

1. $\frac{56}{120}$
2. $\frac{54}{60}$
3. $\frac{155}{90}$
4. $\frac{315}{120}$

Simplify.

5. $-\left(-\frac{1}{2}\right)$
6. $-\left(-\left(-\frac{5}{8}\right)\right)$
7. $-(-(-a))$
8. $-(-(-(-a)))$

Graph the solution set and give the interval notation equivalent.

9. $x \geq -10$
10. $x < 0$
11. $-8 \leq x < 0$
12. $-10 < x \leq 4$
13. $x < 3$ and $x \geq -1$
14. $x < 0$ and $x > 1$
15. $x < -2$ or $x > -6$
16. $x \leq -1$ or $x > 3$

Determine the inequality that corresponds to the set expressed using interval notation.

17. $[-8, \infty)$
18. $(-\infty, -7)$
19. $[12, 32]$
20. $[-10, 0)$
21. $(-\infty, 1] \cup (5, \infty)$
22. $(-\infty, -10) \cup (-5, \infty)$
23. $(-4, \infty)$
24. $(-\infty, 0)$

Simplify.

25. $-\left|-\frac{3}{4}\right|$
26. $-\left|-\left(-\frac{2}{3}\right)\right|$
27. $-(-|-4|)$
28. $-(-(-|-3|))$

Determine the values represented by *a*.

29. $|a| = 6$
30. $|a| = 1$
31. $|a| = -5$
32. $|a| = a$

OPERATIONS WITH REAL NUMBERS

Perform the operations.

33. $\frac{1}{4} - \frac{1}{5} + \frac{3}{20}$

34. $\frac{2}{3} - \left(-\frac{3}{4}\right) - \frac{5}{12}$

35. $\frac{5}{3}\left(-\frac{6}{7}\right) \div \left(\frac{5}{14}\right)$

36. $\left(-\frac{8}{9}\right) \div \frac{16}{27}\left(\frac{2}{15}\right)$

37. $\left(-\frac{2}{3}\right)^3$

38. $\left(-\frac{3}{4}\right)^2$

39. $(-7)^2 - 8^2$

40. $-4^2 + (-4)^3$

41. $10 - 8\left((3-5)^2 - 2\right)$

42. $4 + 5\left(3 - (2-3)^2\right)$

43. $-3^2 - \left(7 - (-4+2)^3\right)$

44. $(-4+1)^2 - (3-6)^3$

45. $\dfrac{10 - 3(-2)^3}{3^2 - (-4)^2}$

46. $\dfrac{6\left[(-5)^2 - (-3)^2\right]}{4 - 6(-2)^2}$

47. $7 - 3\left|6 - (-3-2)^2\right|$

48. $-6^2 + 5\left|3 - 2(-2)^2\right|$

49. $\dfrac{12 - \left|6 - 2(-4)^2\right|}{3 - |-4|}$

50. $\dfrac{-(5 - 2|-3|)^3}{\left|4 - (-3)^2\right| - 3^2}$

SQUARE AND CUBE ROOTS OF REAL NUMBERS

Simplify.

51. $3\sqrt{8}$

52. $5\sqrt{18}$

53. $6\sqrt{0}$

54. $\sqrt{-6}$

55. $\sqrt{\frac{75}{16}}$

56. $\sqrt{\frac{80}{49}}$

57. $\sqrt[3]{40}$

58. $\sqrt[3]{81}$

59. $\sqrt[3]{-81}$

60. $\sqrt[3]{-32}$

61. $\sqrt[3]{\frac{250}{27}}$

62. $\sqrt[3]{\frac{1}{125}}$

Use a calculator to approximate the following to the nearest thousandth.

63. $\sqrt{12}$

64. $3\sqrt{14}$

65. $\sqrt[3]{18}$

66. $7\sqrt[3]{25}$

67. Find the length of the diagonal of a square with sides measuring 8 centimeters.

68. Find the length of the diagonal of a rectangle with sides measuring 6 centimeters and 12 centimeters.

ALGEBRAIC EXPRESSIONS AND FORMULAS

Multiply.

69. $\frac{2}{3}\left(9x^2 + 3x - 6\right)$
70. $-5\left(\frac{1}{5}y^2 - \frac{3}{5}y + \frac{1}{2}\right)$
71. $\left(a^2 - 5ab - 2b^2\right)(-3)$
72. $\left(2m^2 - 3mn + n^2\right) \cdot 6$

Combine like terms.

73. $5x^2y - 3xy^2 - 4x^2y - 7xy^2$
74. $9x^2y^2 + 8xy + 3 - 5x^2y^2 - 8xy - 2$
75. $a^2b^2 - 7ab + 6 - a^2b^2 + 12ab - 5$
76. $5m^2n - 3mn + 2mn^2 - 2nm - 4m^2n + mn^2$

Simplify.

77. $5x^2 + 4x - 3\left(2x^2 - 4x - 1\right)$
78. $\left(6x^2y^2 + 3xy - 1\right) - \left(7x^2y^2 - 3xy + 2\right)$
79. $a^2 - b^2 - \left(2a^2 + ab - 3b^2\right)$
80. $m^2 + mn - 6\left(m^2 - 3n^2\right)$

Evaluate.

81. $x^2 - 3x + 1$ where $x = -\frac{1}{2}$
82. $x^2 - x - 1$ where $x = -\frac{2}{3}$
83. $a^4 - b^4$ where $a = -3$ and $b = -1$
84. $a^2 - 3ab + 5b^2$ where $a = 4$ and $b = -2$
85. $(2x + 1)(x - 3)$ where $x = -3$
86. $(3x + 1)\left(x + 5\right)$ where $x = -5$

87. $\sqrt{b^2 - 4ac}$ where $a = 2, b = -4$, and $c = -1$

88. $\sqrt{b^2 - 4ac}$ where $a = 3, b = -6$, and $c = -2$

89. $\pi r^2 h$ where $r = 2\sqrt{3}$ and $h = 5$

90. $\frac{4}{3}\pi r^3$ where $r = 2\sqrt[3]{6}$

91. What is the simple interest earned on a 4 year investment of \$4,500 at an annual interest rate of $4\frac{3}{4}$%?

92. James traveled at an average speed of 48 miles per hour for $2\frac{1}{4}$ hours. How far did he travel?

93. The period of a pendulum T in seconds is given by the formula $T = 2\pi\sqrt{\frac{L}{32}}$ where L represents its length in feet. Approximate the period of a pendulum with length 2 feet. Round off to the nearest tenth of a foot.

94. The average distance d, in miles, a person can see an object is given by the formula $d = \frac{\sqrt{6h}}{2}$ where h represents the person's height above the ground, measured in feet. What average distance can a person see an object from a height of 10 feet? Round off to the nearest tenth of a mile.

RULES OF EXPONENTS AND SCIENTIFIC NOTATION

Multiply.

95. $\frac{x^{10} \cdot x^2}{x^5}$

96. $\frac{x^6 \left(x^2\right)^4}{x^3}$

97. $-7x^2yz^3 \cdot 3x^4y^2z$

98. $3a^2b^3c\left(-4a^2bc^4\right)^2$

99. $\frac{-10a^5b^0c^{-4}}{25a^{-2}b^2c^{-3}}$

100. $\frac{-12x^{-6}y^{-2}z}{36x^{-3}y^4z^6}$

101. $\left(-2x^{-5}y^{-3}z\right)^{-4}$

102. $\left(3x^{6}y^{-3}z^{0}\right)^{-3}$

103. $\left(\frac{-5a^{2}b^{3}}{c^{5}}\right)^{2}$

104. $\left(\frac{-3m^{5}}{5n^{2}}\right)^{3}$

105. $\left(\frac{-2a^{-2}b^{3}c}{3ab^{-2}c^{0}}\right)^{-3}$

106. $\left(\frac{6a^{3}b^{-3}c}{2a^{7}b^{0}c^{-4}}\right)^{-2}$

Perform the operations.

107. $\left(4.3 \times 10^{22}\right)\left(3.1 \times 10^{-8}\right)$

108. $\left(6.8 \times 10^{-33}\right)\left(1.6 \times 10^{7}\right)$

109. $\frac{1.4\times10^{-32}}{2\times10^{-10}}$

110. $\frac{1.15\times10^{26}}{2.3\times10^{-7}}$

111. The value of a new tablet computer in dollars can be estimated using the formula $v = 450(t + 1)^{-1}$ where t represents the number of years after it is purchased. Use the formula to estimate the value of the tablet computer $2\frac{1}{2}$ years after it was purchased.

112. The speed of light is approximately 6.7×10^{8} miles per hour. Express this speed in miles per minute and determine the distance light travels in 4 minutes.

POLYNOMIALS AND THEIR OPERATIONS

Simplify.

113. $\left(x^{2} + 3x - 5\right) - \left(2x^{2} + 5x - 7\right)$

114. $\left(6x^2 - 3x + 5\right) + \left(9x^2 + 3x - 4\right)$

115. $\left(a^2b^2 - ab + 6\right) - \left(ab + 9\right) + \left(a^2b^2 - 10\right)$

116. $\left(x^2 - 2y^2\right) - \left(x^2 + 3xy - y^2\right) - \left(3xy + y^2\right)$

117. $-\frac{3}{4}\left(16x^2 + 8x - 4\right)$

118. $6\left(\frac{4}{3}x^2 - \frac{3}{2}x + \frac{5}{6}\right)$

119. $\left(2x + 5\right)(x - 4)$

120. $(3x - 2)\left(x^2 - 5x + 2\right)$

121. $\left(x^2 - 2x + 5\right)\left(2x^2 - x + 4\right)$

122. $\left(a^2 + b^2\right)\left(a^2 - b^2\right)$

123. $\left(2a + b\right)\left(4a^2 - 2ab + b^2\right)$

124. $(2x - 3)^2$

125. $(3x - 1)^3$

126. $(2x + 3)^4$

127. $\left(x^2 - y^2\right)^2$

128. $\left(x^2y^2 + 1\right)^2$

129. $\frac{27a^2b - 9ab + 81ab^2}{3ab}$

130. $\frac{125x^3y^3 - 25x^2y^2 + 5xy^2}{5xy^2}$

131. $\frac{2x^3 - 7x^2 + 7x - 2}{2x - 1}$

132. $\frac{12x^3 + 5x^2 - 7x - 3}{4x + 3}$

133. $\frac{5x^3 - 21x^2 + 6x - 3}{x - 4}$

134. $\frac{x^4 + x^3 - 3x^2 + 10x - 1}{x + 3}$

135. $\frac{a^4 - a^3 + 4a^2 - 2a + 4}{a^2 + 2}$

136. $\frac{8a^4 - 10}{a^2 - 2}$

SOLVING LINEAR EQUATIONS

Solve.

137. $6x - 8 = 2$

138. $12x - 5 = 3$

139. $\frac{5}{4}x - 3 = \frac{1}{2}$

140. $\frac{5}{6}x - \frac{1}{4} = \frac{3}{2}$

141. $\frac{9x+2}{3} = \frac{5}{6}$

142. $\frac{3x-8}{10} = \frac{5}{2}$

143. $3a - 5 - 2a = 4a - 6$

144. $8 - 5y + 2 = 4 - 7y$

145. $5x - 6 - 8x = 1 - 3x$

146. $17 - 6x - 10 = 5x + 7 - 11x$

147. $5(3x + 3) - (10x - 4) = 4$

148. $6 - 2(3x - 1) = -4(1 - 3x)$

149. $9 - 3(2x + 3) + 6x = 0$

150. $-5(x + 2) - (4 - 5x) = 1$

151. $\frac{5}{9}(6y + 27) = 2 - \frac{1}{3}(2y + 3)$

152. $4 - \frac{4}{5}(3a + 10) = \frac{1}{10}(4 - 2a)$

153. Solve for s: $A = \pi r^2 + \pi rs$

154. Solve for x: $y = mx + b$

155. A larger integer is 3 more than twice another. If their sum divided by 2 is 9, find the integers.

156. The sum of three consecutive odd integers is 171. Find the integers.

157. The length of a rectangle is 3 meters less than twice its width. If the perimeter measures 66 meters, find the length and width.

158. How long will it take $500 to earn $124 in simple interest earning 6.2% annual interest?

159. It took Sally $3\frac{1}{2}$ hours to drive the 147 miles home from her grandmother's house. What was her average speed?

160. Jeannine invested her bonus of $8,300 in two accounts. One account earned $3\frac{1}{2}$% simple interest and the other earned $4\frac{3}{4}$% simple interest. If her total interest for one year was $341.75, how much did she invest in each account?

SOLVING LINEAR INEQUALITIES WITH ONE VARIABLE

Solve. Graph all solutions on a number line and provide the corresponding interval notation.

161. $5x - 7 < 18$

162. $2x - 1 > 2$

163. $9 - x \leq 3$

164. $3 - 7x \geq 10$

165. $61 - 3(x + 3) > 13$

166. $7 - 3(2x - 1) \geq 6$

167. $\frac{1}{3}(9x + 15) - \frac{1}{2}(6x - 1) < 0$

168. $\frac{2}{3}(12x - 1) + \frac{1}{4}(1 - 32x) < 0$

169. $20 + 4(2a - 3) \geq \frac{1}{2}a + 2$

170. $\frac{1}{3}\left(2x + \frac{3}{2}\right) - \frac{1}{4}x < \frac{1}{2}\left(1 - \frac{1}{2}x\right)$

171. $-4 \leq 3x + 5 < 11$
172. $5 < 2x + 15 \leq 13$
173. $-1 < 4(x + 1) - 1 < 9$
174. $0 \leq 3(2x - 3) + 1 \leq 10$
175. $-1 < \dfrac{2x - 5}{4} < 1$
176. $-2 \leq \dfrac{3 - x}{3} < 1$
177. $2x + 3 < 13$ and $4x - 1 > 10$
178. $3x - 1 \leq 8$ and $2x + 5 \geq 23$
179. $5x - 3 < -2$ or $5x - 3 > 2$
180. $1 - 3x \leq -1$ or $1 - 3x \geq 1$
181. $5x + 6 < 6$ or $9x - 2 > -11$
182. $2(3x - 1) < -16$ or $3(1 - 2x) < -15$
183. Jerry scored 90, 85, 92, and 76 on the first four algebra exams. What must he score on the fifth exam so that his average is at least 80?
184. If 6 degrees less than 3 times an angle is between 90 degrees and 180 degrees, then what are the bounds of the original angle?

ANSWERS

1. $\frac{7}{15}$

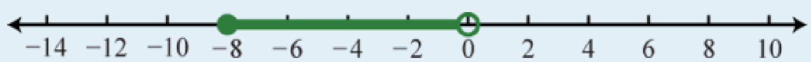
3. $\frac{31}{18}$

5. $\frac{1}{2}$

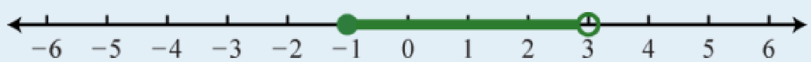
7. $-a$

9. $[-10, \infty)$;

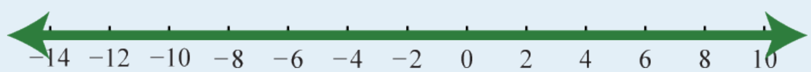
11. $[-8, 0)$;



13. $[-1, 3)$;



15. $\mathbb{R}$;



17. $x \geq -8$

19. $12 \leq x \leq 32$

21. $x \leq 1 \ \ or \ \ x > 5$

23. $x > -4$

25. $-\frac{3}{4}$

27. 4

29. $a = \pm 6$

31. $\emptyset$

33. $\frac{1}{5}$

35. −4

37. $-\frac{8}{27}$

39. −15

41. −6

43. −24

45. $-\frac{34}{7}$

47. −50

49. 14

51. $6\sqrt{2}$

53. 0

55. $\frac{5\sqrt{3}}{4}$

57. $2\sqrt[3]{5}$

59. $-3\sqrt[3]{3}$

61. $\frac{5\sqrt[3]{2}}{3}$

63. 3.464

65. 2.621

67. $8\sqrt{2}$ centimeters

69. $6x^2 + 2x - 4$

71. $-3a^2 + 15ab + 6b^2$

73. $x^2y - 10xy^2$

75. $5ab + 1$

77. $-x^2 + 16x + 3$

79. $-a^2 - ab + 2b^2$

81. $\frac{11}{4}$

83. 80

85. 30

87. $2\sqrt{6}$

89. 60π

91. $855

93. 1.6 seconds

95. x^7

97. $-21x^6y^3z^4$

99. $-\frac{2a^7}{5b^2c}$

101. $\frac{x^{20}y^{12}}{16z^4}$

103. $\frac{25a^4b^6}{c^{10}}$

105. $-\frac{27a^9}{8b^{15}c^3}$

107. 1.333×10^{15}

109. 7×10^{-23}

111. $128.57

113. $-x^2 - 2x + 2$

115. $2a^2b^2 - 2ab - 13$

117. $-12x^2 - 6x + 3$

119. $2x^2 - 3x - 20$

121. $2x^4 - 5x^3 + 16x^2 - 13x + 20$

123. $8a^3 + b^3$

125. $27x^3 - 27x^2 + 9x - 1$

127. $x^4 - 2x^2y^2 + y^4$

129. $9a + 27b - 3$

131. $x^2 - 3x + 2$

133. $5x^2 - x + 2 + \frac{5}{x-4}$

135. $a^2 - a + 2$

137. $\frac{5}{3}$

139. $\frac{14}{5}$

141. $\frac{1}{18}$

143. $\frac{1}{3}$

145. Ø

147. -3

149. $\mathbb{R}$

151. $-\frac{7}{2}$

153. $s = \frac{A - \pi r^2}{\pi r}$

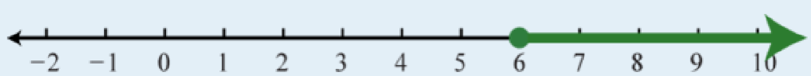
155. 5, 13

157. Length: 21 meters; Width: 12 meters

159. 42 miles per hour

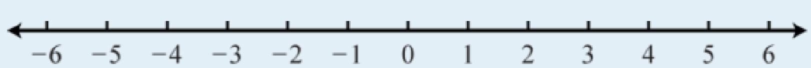
161. $(-\infty, 5)$;

163. $[6, \infty)$;

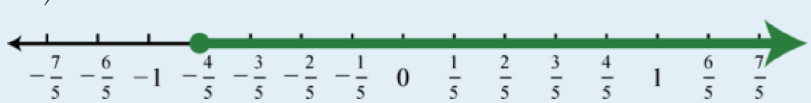


165. $(-\infty, 13)$;

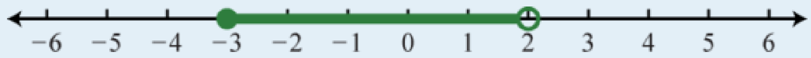
167. Ø;



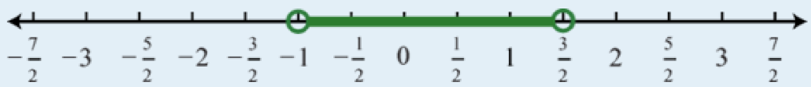
169. $\left[-\frac{4}{5}, \infty\right)$;



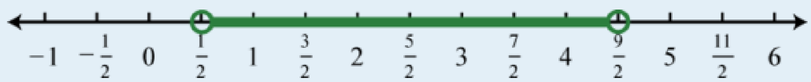
171. $[-3, 2)$;



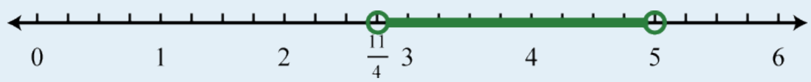
173. $\left(-1, \frac{3}{2}\right)$;



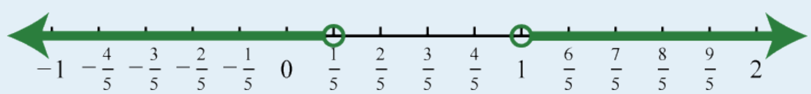
175. $\left(\frac{1}{2}, \frac{9}{2}\right)$;



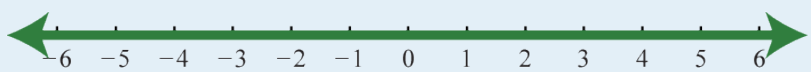
177. $\left(\frac{11}{4}, 5\right)$;



179. $\left(-\infty, \frac{1}{5}\right) \cup (1, \infty)$;



181. $\mathbb{R}$;



183. Jerry must score at least 57 on the fifth exam.

SAMPLE EXAM

Simplify.

1. $5 - 3\left(12 - \left|2 - 5^2\right|\right)$
2. $\left(-\frac{1}{2}\right)^2 - \left(3 - 2\left|-\frac{3}{4}\right|\right)^3$
3. $-7\sqrt{60}$
4. $5\sqrt[3]{-32}$
5. Find the diagonal of a square with sides measuring 6 centimeters.

Simplify.

6. $-5x^2yz^{-1}\left(3x^3y^{-2}z\right)$
7. $\left(\frac{-2a^{-4}b^2c}{a^{-3}b^0c^2}\right)^{-3}$
8. $2\left(3a^2b^2 + 2ab - 1\right) - a^2b^2 + 2ab - 1$
9. $\left(x^2 - 6x + 9\right) - \left(3x^2 - 7x + 2\right)$
10. $(2x - 3)^3$
11. $\left(3a - b\right)\left(9a^2 + 3ab + b^2\right)$
12. $\frac{6x^4 - 17x^3 + 16x^2 - 18x + 13}{2x - 3}$

Solve.

13. $\frac{4}{5}x - \frac{2}{15} = 2$
14. $\frac{3}{4}(8x - 12) - \frac{1}{2}(2x - 10) = 16$
15. $12 - 5(3x - 1) = 2(4x + 3)$
16. $\frac{1}{2}(12x - 2) + 5 = 4\left(\frac{3}{2}x - 8\right)$
17. Solve for y: $ax + by = c$

Solve. Graph the solutions on a number line and give the corresponding interval notation.

18. $2\left(3x - 5\right) - (7x - 3) \geq 0$
19. $2\left(4x - 1\right) - 4\left(5 + 2x\right) < -10$
20. $-6 \leq \frac{1}{4}\left(2x - 8\right) < 4$
21. $3x - 7 > 14$ or $3x - 7 < -14$

Use algebra to solve the following.

22. Degrees Fahrenheit F is given by the formula $F = \frac{9}{5}C + 32$ where C represents degrees Celsius. What is the Fahrenheit equivalent to 35° Celsius?
23. The length of a rectangle is 5 inches less than its width. If the perimeter is 134 inches, find the length and width of the rectangle.
24. Melanie invested 4,500 in two separate accounts. She invested part in a CD that earned 3.2% simple interest and the rest in a savings account that earned 2.8% simple interest. If the total simple interest for one year was $138.80, how much did she invest in each account?
25. A rental car costs $45.00 per day plus $0.48 per mile driven. If the total cost of a one-day rental is to be at most $105, how many miles can be driven?

ANSWERS

1. 38

3. $-14\sqrt{15}$

5. $6\sqrt{2}$ centimeters

7. $-\dfrac{a^3c^3}{8b^6}$

9. $-2x^2 + x + 7$

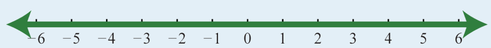
11. $27a^3 - b^3$

13. $\dfrac{8}{3}$

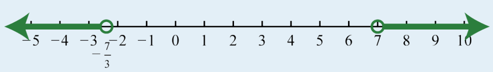
15. $\frac{11}{23}$

17. $y = \frac{c-ax}{b}$

19. $\mathbb{R}$;



21. $\left(-\infty, -\frac{7}{3}\right) \cup (7, \infty)$;



23. Length: 31 inches; width: 36 inches

25. The car can be driven at most 125 miles.

Chapter 2

Graphing Functions and Inequalities

2.1 Relations, Graphs, and Functions

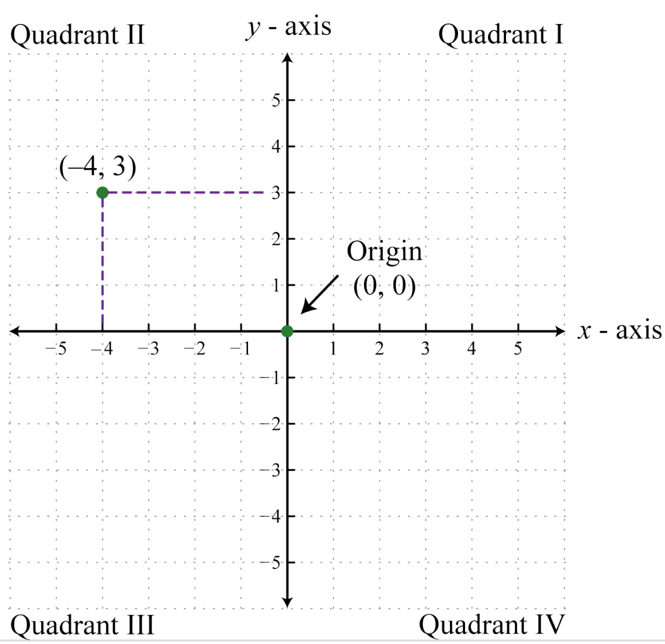
LEARNING OBJECTIVES

1. State the domain and range of a relation.
2. Identify a function.
3. Use function notation.

Graphs, Relations, Domain, and Range

The **rectangular coordinate system**[1] consists of two real number lines that intersect at a right angle. The horizontal number line is called the ***x*-axis**[2], and the vertical number line is called the ***y*-axis**[3]. These two number lines define a flat surface called a **plane**[4], and each point on this plane is associated with an **ordered pair**[5] of real numbers (x, y). The first number is called the x-coordinate, and the second number is called the y-coordinate. The intersection of the two axes is known as the **origin**[6], which corresponds to the point (0, 0).

The x- and y-axes break the plane into four regions called **quadrants**[7], named using roman numerals I, II, III, and IV, as pictured. The ordered pair (x, y) represents the position of points relative to the origin. For example, the ordered pair (−4, 3) represents the position 4 units to the left of the origin, and 3 units above in the second quadrant.



1. A system with two number lines at right angles specifying points in a plane using ordered pairs (x, y).
2. The horizontal number line used as reference in a rectangular coordinate system.
3. The vertical number line used as reference in a rectangular coordinate system.
4. The flat surface defined by x- and y-axes.
5. Pairs (x, y) that identify position relative to the origin on a rectangular coordinate plane.
6. The point where the x- and y-axes cross, denoted by (0, 0).
7. The four regions of a rectangular coordinate plane partly bounded by the x- and y-axes and numbered using the Roman numerals I, II, III, and IV.

This system is often called the **Cartesian coordinate system**[8], named after the French mathematician René Descartes (1596–1650).

Figure 2.1

Rene Descartes Wikipedia

Next, we define a **relation**[9] as any set of ordered pairs. In the context of algebra, the relations of interest are sets of ordered pairs (x, y) in the rectangular coordinate plane. Typically, the coordinates are related by a rule expressed using an algebraic equation. For example, both the algebraic equations $y = |x| - 2$ and $x = |y| + 1$ define relationsips between x and y. Following are some integers that satisfy both equations:

8. Term used in honor of René Descartes when referring to the rectangular coordinate system.

9. Any set of ordered pairs.

$y = |x| - 2$

x	y
-3	1
-2	0
-1	-1
0	-2
1	-1
2	0
3	1

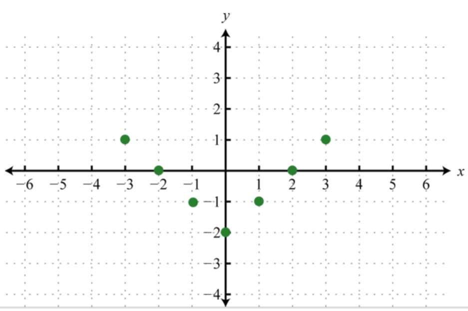
$x = |y| + 1$

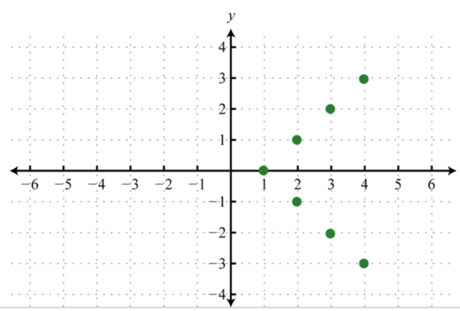
x	y
4	-3
3	-2
2	-1
1	0
2	1
3	2
4	3

Here two relations consisting of seven ordered pair solutions are obtained:

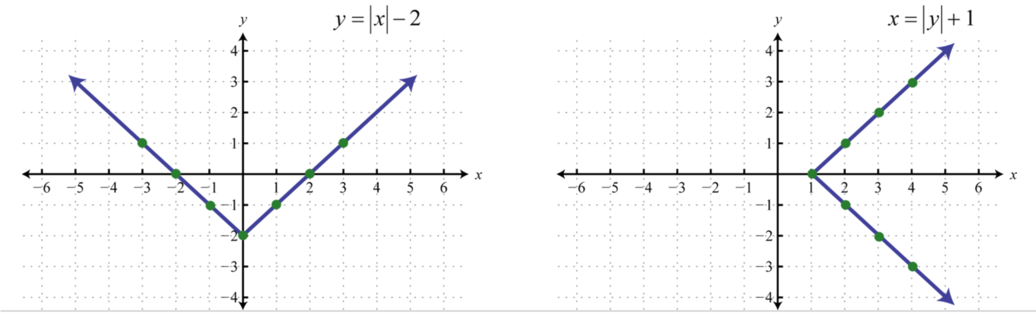
$$y = |x| - 2 \;\; \textit{has solutions} \;\; \{(-3, 1), (-2, 0), (-1, -1), (0, -2), (1, -1), (2, 0), (3$$
$$\textit{and}$$
$$x = |y| + 1 \;\; \textit{has solutions} \;\; \{(4, -3), (3, -2), (2, -1), (1, 0), (2, 1), (3, 2), (4, 3)\}$$

We can visually display any relation of this type on a coordinate plane by plotting the points.



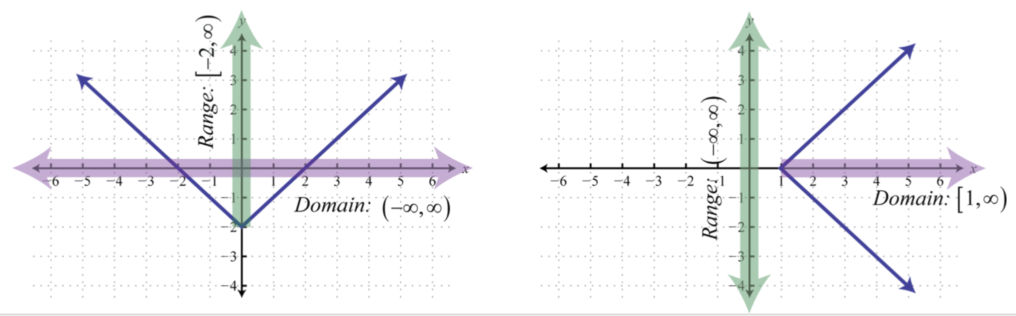


The solution sets of each equation will form a relation consisting of infinitely many ordered pairs. We can use the given ordered pair solutions to estimate all of the other ordered pairs by drawing a line through the given points. Here we put an arrow on the ends of our lines to indicate that this set of ordered pairs continues without bounds.



The representation of a relation on a rectangular coordinate plane, as illustrated above, is called a **graph**[10]. Any curve graphed on a rectangular coordinate plane represents a set of ordered pairs and thus defines a relation.

The set consisting of all of the first components of a relation, in this case the x-values, is called the **domain**[11]. And the set consisting of all second components of a relation, in this case the y-values, is called the **range**[12] (or **codomain**[13]). Often, we can determine the domain and range of a relation if we are given its graph.



Here we can see that the graph of $y = |x| - 2$ has a domain consisting of all real numbers, $\mathbb{R} = (-\infty, \infty)$, and a range of all y-values greater than or equal to -2, $[-2, \infty)$. The domain of the graph of $x = |y| + 1$ consists of all x-values greater than or equal to 1, $[1, \infty)$, and the range consists of all real numbers, $\mathbb{R} = (-\infty, \infty)$.

10. A visual representation of a relation on a rectangular coordinate plane.

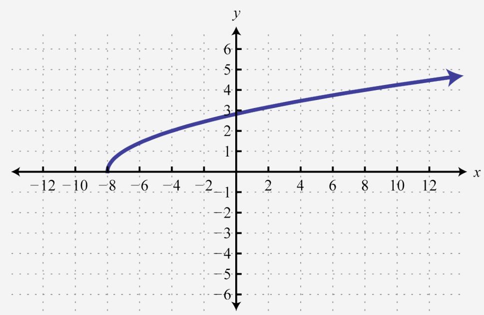
11. The set consisting of all of the first components of a relation. For relations consisting of points in the plane, the domain is the set of all x-values.

12. The set consisting of all of the second components of a relation. For relations consisting of points in the plane, the range is the set of all y-values.

13. Used when referencing the range.

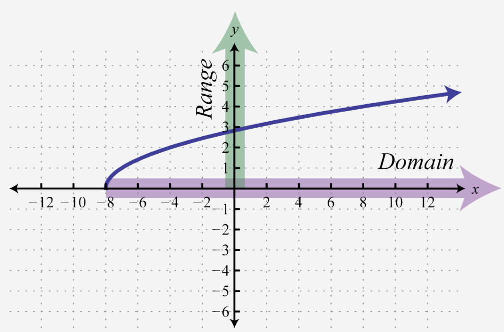
Example 1

Determine the domain and range of the following relation:



Solution:

The minimum *x*-value represented on the graph is -8 all others are larger. Therefore, the domain consists of all *x*-values in the interval $[-8, \infty)$. The minimum *y*-value represented on the graph is 0; thus, the range is $[0, \infty)$.



Answer: Domain: $[-8, \infty)$; range: $[0, \infty)$

Functions

Of special interest are relations where every *x*-value corresponds to exactly one *y*-value. A relation with this property is called a **function**[14].

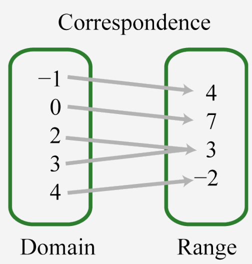
14. A relation where each element in the domain corresponds to exactly one element in the range.

Example 2

Determine the domain and range of the following relation and state whether it is a function or not: {(−1, 4), (0, 7), (2, 3), (3, 3), (4, −2)}

Solution:

Here we separate the domain (*x-values*), and the range (*y-values*), and depict the correspondence between the values with arrows.



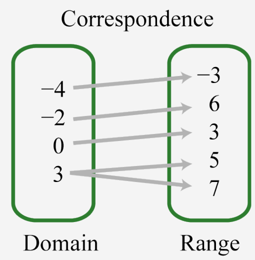
The relation is a function because each *x*-value corresponds to exactly one *y*-value.

Answer: The domain is {−1, 0, 2, 3, 4} and the range is {−2, 3, 4, 7}. The relation is a function.

Example 3

Determine the domain and range of the following relation and state whether it is a function or not: {(−4, −3), (−2, 6), (0, 3), (3, 5), (3, 7)}

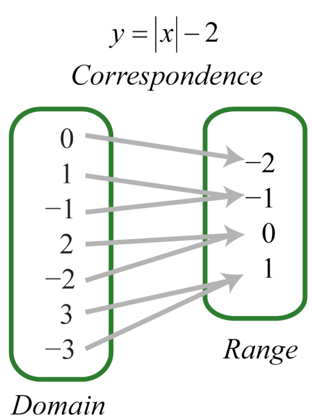
Solution:



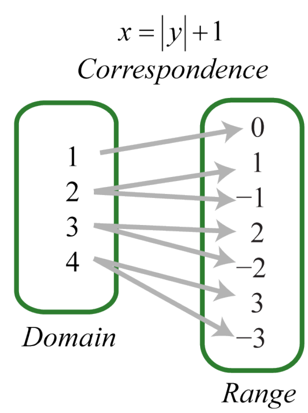
The given relation is not a function because the *x*-value 3 corresponds to two *y*-values. We can also recognize functions as relations where no *x*-values are repeated.

Answer: The domain is {−4, −2, 0, 3} and the range is {−3, 3, 5, 6, 7}. This relation is not a function.

Consider the relations consisting of the seven ordered pair solutions to $y = |x| - 2$ and $x = |y| + 1$. The correspondence between the domain and range of each can be pictured as follows:



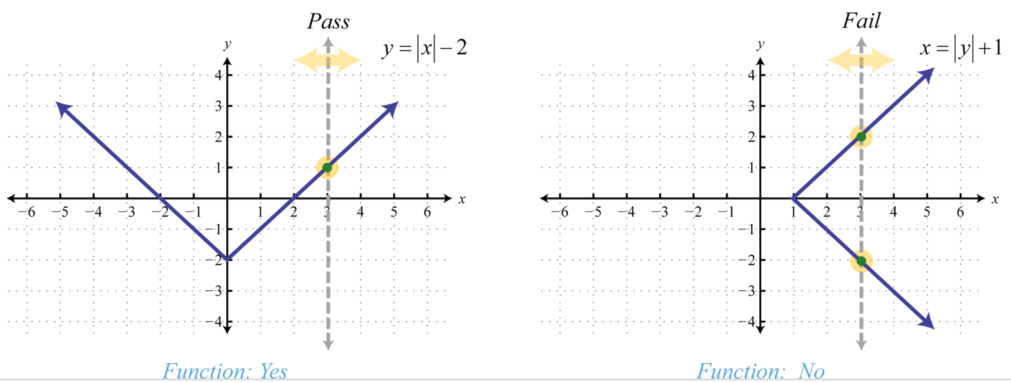
Function: Yes



Function: No

Notice that every element in the domain of the solution set of $y = |x| - 2$ corresponds to only one element in the range; it is a function. The solutions to $x = |y| + 1$, on the other hand, have values in the domain that correspond to two elements in the range. In particular, the x-value 4 corresponds to two y-values −3 and 3. Therefore, $x = |y| + 1$ does not define a function.

We can visually identify functions by their graphs using the **vertical line test**[15]. If any vertical line intersects the graph more than once, then the graph does not represent a function.

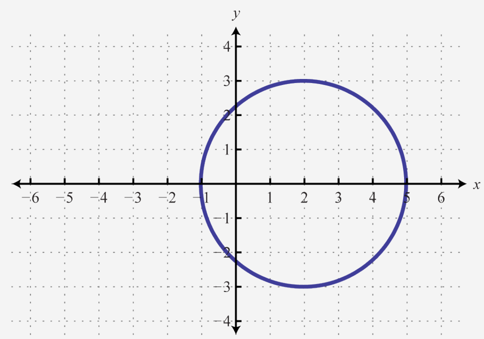


The vertical line represents a value in the domain, and the number of intersections with the graph represent the number of values to which it corresponds. As we can see, any vertical line will intersect the graph of $y = |x| - 2$ only once; therefore, it is a function. A vertical line can cross the graph of $x = |y| + 1$ more than once; therefore, it is not a function. As pictured, the x-value 3 corresponds to more than one y-value.

15. If any vertical line intersects the graph more than once, then the graph does not represent a function.

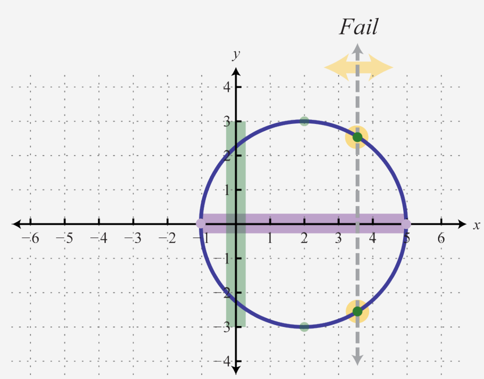
Example 4

Given the graph, state the domain and range and determine whether or not it represents a function:



Solution:

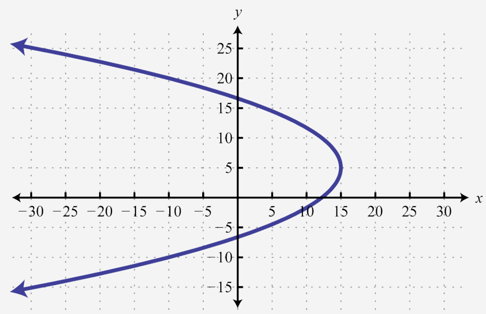
From the graph we can see that the minimum *x*-value is −1 and the maximum *x*-value is 5. Hence, the domain consists of all the real numbers in the set from $[-1, 5]$. The maximum *y*-value is 3 and the minimum is −3; hence, the range consists of *y*-values in the interval $[-3, 3]$.



In addition, since we can find a vertical line that intersects the graph more than once, we conclude that the graph is not a function. There are many *x*-values in the domain that correspond to two *y*-values.

Answer: Domain: $[-1, 5]$; range: $[-3, 3]$; function: no

Try this! Given the graph, determine the domain and range and state whether or not it is a function:



Answer: Domain: $(-\infty, 15]$; range:$\mathbb{R}$; function: no

(click to see video)

Function Notation

With the definition of a function comes special notation. If we consider each *x*-value to be the input that produces exactly one output, then we can use **function notation**[16]:

$$f(x) = y$$

The notation $f(x)$ reads, "*f of x*" and should not be confused with multiplication. Algebra frequently involves functions, and so the notation becomes useful when performing common tasks. Here *f* is the function name, and $f(x)$ denotes the value in the range associated with the value *x* in the domain. Functions are often named with different letters; some common names for functions are *f*, *g*, *h*, *C*, and *R*. We have determined that the set of solutions to $y = |x| - 2$ is a function; therefore, using function notation we can write:

16. The notation $f(x) = y$, which reads "*f* of *x* is equal to *y*." Given a function, *y* and $f(x)$ can be used interchangeably.

$$y = |x| - 2$$
$$\downarrow$$
$$f(x) = |x| - 2$$

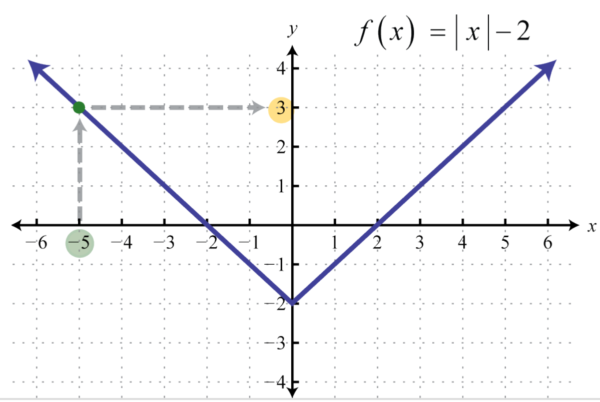
It is important to note that y and $f\,(x)$ are used interchangeably. This notation is used as follows:

$$\begin{aligned} f\,(x) \quad &= |x| - 2 \\ \downarrow \qquad &\quad\ \downarrow \\ f\,(-5) &= |-5| - 2 = 5 - 2 = 3 \end{aligned}$$

Here the compact notation $f(-5) = 3$ indicates that where $x = -5$ (*the input*), the function results in $y = 3$ (*the output*). In other words, replace the variable with the value given inside the parentheses.

$$\underset{\substack{\uparrow \\ \text{Input}}}{} \; f\,(\overset{\substack{\text{Input} \\ \downarrow}}{-5}) = |-5| - 2 = \underset{\substack{\uparrow \\ \text{Output}}}{3}$$

Functions are compactly defined by an algebraic equation, such as $f\,(x) = |x| - 2$. Given values for x in the domain, we can quickly calculate the corresponding values in the range. As we have seen, functions are also expressed using graphs. In this case, we interpret $f(-5) = 3$ as follows:



Function notation streamlines the task of evaluating. For example, use the function h defined by $h(x) = \frac{1}{2}x - 3$to evaluate for x-values in the set {-2, 0, 7}.

$$
\begin{aligned}
h(-2) &= \frac{1}{2}(-2) - 3 = -1 - 3 = -4 \\
h(0) &= \frac{1}{2}(0) - 3 = 0 - 3 = -3 \\
h(7) &= \frac{1}{2}(7) - 3 = \frac{7}{2} - 3 = \frac{1}{2}
\end{aligned}
$$

Given any function defined by $h(x) = y$, the value x is called the **argument of the function**[17]. The argument can be any algebraic expression. For example:

$$
\begin{aligned}
h\left(4a^3\right) &= \frac{1}{2}\left(4a^3\right) - 3 = 2a^3 - 3 \\
h(2x - 1) &= \frac{1}{2}(2x - 1) - 3 = x - \frac{1}{2} - 3 = x - \frac{7}{2}
\end{aligned}
$$

17. The value or algebraic expression used as input when using function notation.

Example 5

Given $g(x) = x^2$, find $g(-2)$, $g\left(\frac{1}{2}\right)$, and $g\left(x+h\right)$.

Solution:

Recall that when evaluating, it is a best practice to begin by replacing the variables with parentheses and then substitute the appropriate values. This helps with the order of operations when simplifying expressions.

$$\begin{aligned} g(-2) &= (-2)^2 = 4 \\ g\left(\frac{1}{2}\right) &= \left(\frac{1}{2}\right)^2 = \frac{1}{4} \\ g\left(x+h\right) &= \left(x+h\right)^2 = x^2 + 2xh + h^2 \end{aligned}$$

Answer: $g(-2) = 4$, $g\left(\frac{1}{2}\right) = \frac{1}{4} g\left(x+h\right) = x^2 + 2xh + h^2$

At this point, it is important to note that, in general, $f\left(x+h\right) \neq f(x) + f\left(h\right)$. The previous example, where $g(x) = x^2$, illustrates this nicely.

$$\begin{aligned} g\left(x+h\right) &\neq g(x) + g\left(h\right) \\ \left(x+h\right)^2 &\neq x^2 + h^2 \end{aligned}$$

Example 6

Given $f(x) = \sqrt{2x+4}$, find $f(-2)$, $f(0)$, and $f\left(\frac{1}{2}a^2 - 2\right)$.

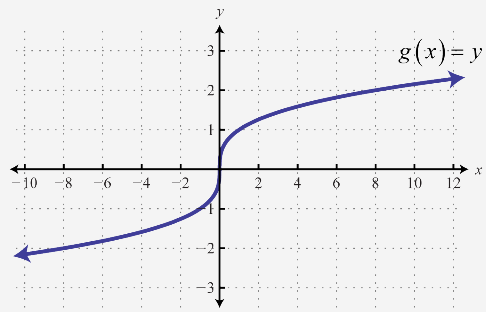
Solution:

$$f(-2) = \sqrt{2(-2)+4} = \sqrt{-4+4} = \sqrt{0} = 0$$
$$f(0) = \sqrt{2(0)+4} = \sqrt{0+4} = \sqrt{4} = 2$$
$$f\left(\frac{1}{2}a^2 - 2\right) = \sqrt{2\left(\frac{1}{2}a^2 - 2\right) + 4} = \sqrt{a^2 - 4 + 4} = \sqrt{a^2} = |a|$$

Answer: $f(-2) = 0$, $f(0) = 2$, $f\left(\frac{1}{2}a^2 - 2\right) = |a|$

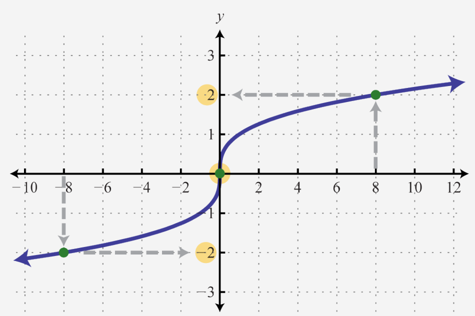
Example 7

Given the graph of $g(x)$, find $g(-8)$, $g(0)$, and $g(8)$.



Solution:

Use the graph to find the corresponding y-values where x = −8, 0, and 8.



Answer: $g(-8) = -2, g(0) = 0, g(8) = 2$

Sometimes the output is given and we are asked to find the input.

Example 8

Given $f(x) = 5x + 7$, find x where $f(x) = 27$.

Solution:

In this example, the output is given and we are asked to find the input. Substitute $f(x)$ with 27 and solve.

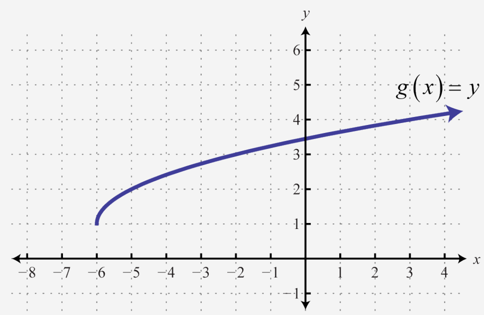
$$\begin{aligned} f(x) &= 5x + 7 \\ &\downarrow \\ 27 &= 5x + 7 \\ 20 &= 5x \\ 4 &= x \end{aligned}$$

Therefore, $f(4) = 27$. As a check, we can evaluate $f(4) = 5(4) + 7 = 27$.

Answer: $x = 4$

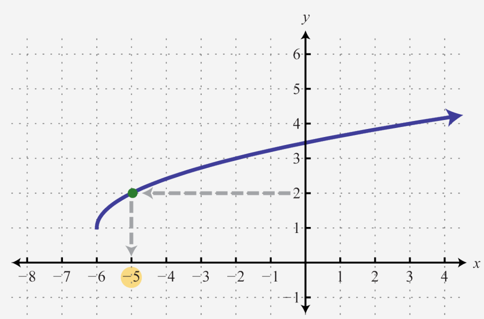
Example 9

Given the graph of g, find x where $g(x) = 2$.



Solution:

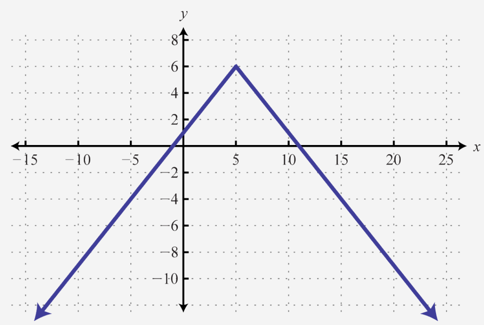
Here we are asked to find the x-value given a particular y-value. We begin with 2 on the y-axis and then read the corresponding x-value.



We can see that $g(x) = 2$ where $x = -5$; in other words, $g(-5) = 2$.

Answer: $x = -5$

Try this! Given the graph of h, find x where $h(x) = -4$.



Answer: $x = -5$ and $x = 15$

(click to see video)

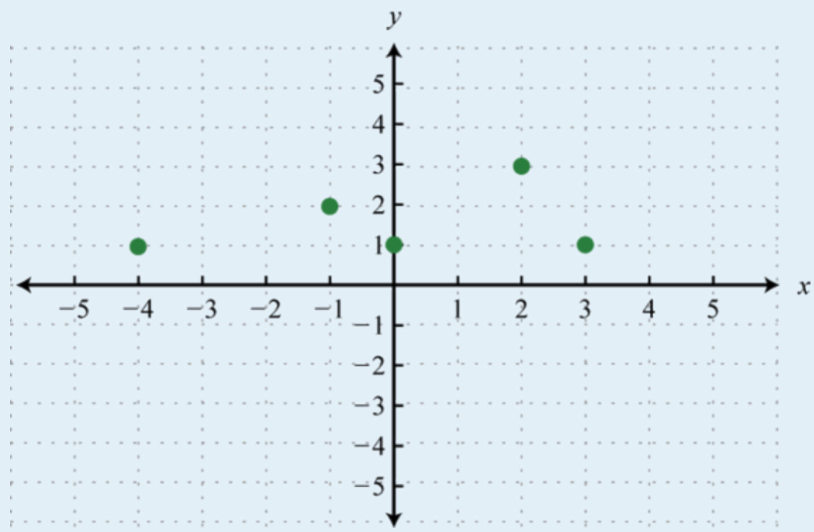
KEY TAKEAWAYS

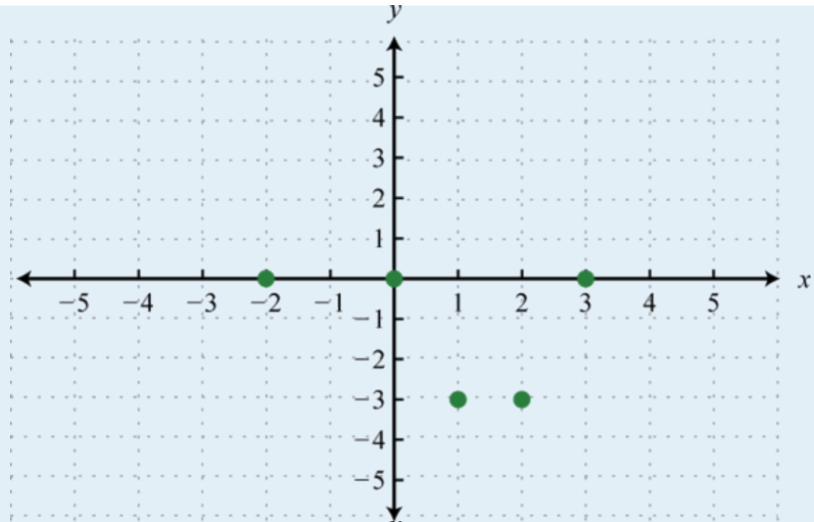
- A relation is any set of ordered pairs. However, in this course, we will be working with sets of ordered pairs (x, y) in the rectangular coordinate system. The set of x-values defines the domain and the set of y-values defines the range.
- Special relations where every x-value (input) corresponds to exactly one y-value (output) are called functions.
- We can easily determine whether or not an equation represents a function by performing the vertical line test on its graph. If any vertical line intersects the graph more than once, then the graph does not represent a function.
- If an algebraic equation defines a function, then we can use the notation $f(x) = y$. The notation $f(x)$ is read "f of x" and should not be confused with multiplication. When working with functions, it is important to remember that y and $f(x)$ are used interchangeably.
- If asked to find $f(a)$, we substitute the argument a in for the variable and then simplify. The argument could be an algebraic expression.
- If asked to find x where $f(x) = a$, we set the function equal to a and then solve for x.

TOPIC EXERCISES

PART A: RELATIONS AND FUNCTIONS

Determine the domain and range and state whether the relation is a function or not.

1. {(3, 1), (5, 2), (7, 3), (9, 4), (12, 4)}
2. {(2, 0), (4, 3), (6, 6), (8, 6), (10, 9)}
3. {(7, 5), (8, 6), (10, 7), (10, 8), (15, 9)}
4. {(1, 1), (2, 1), (3, 1), (4, 1), (5, 1)}
5. {(5, 0), (5, 2), (5, 4), (5, 6), (5, 8)}
6. {(−3, 1), (−2, 2), (−1, 3), (0, 4), (0, 5)}
7. 

8. 
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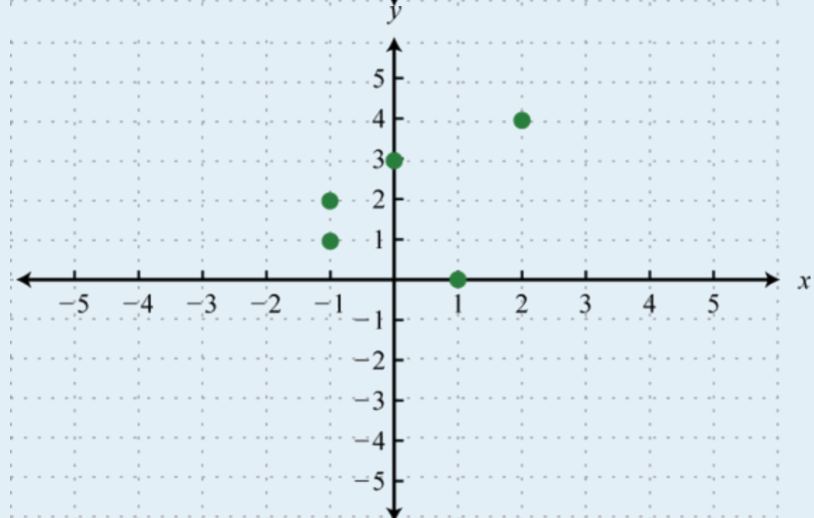
9. 
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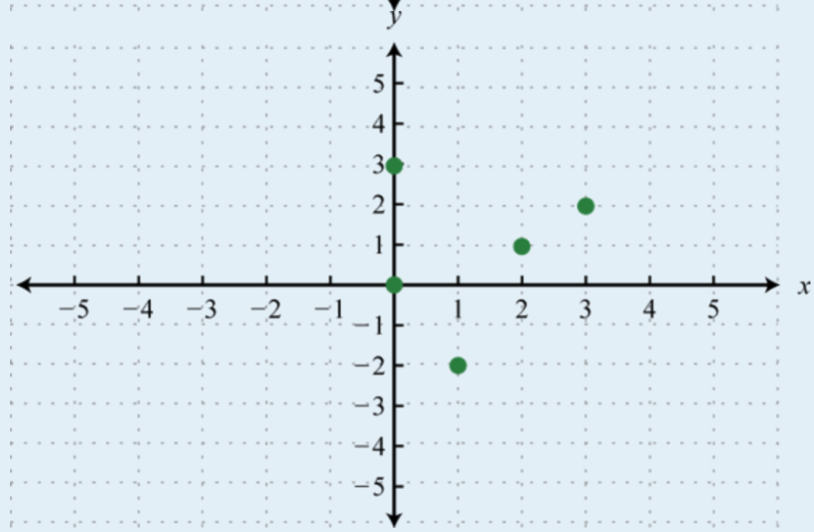
10. 
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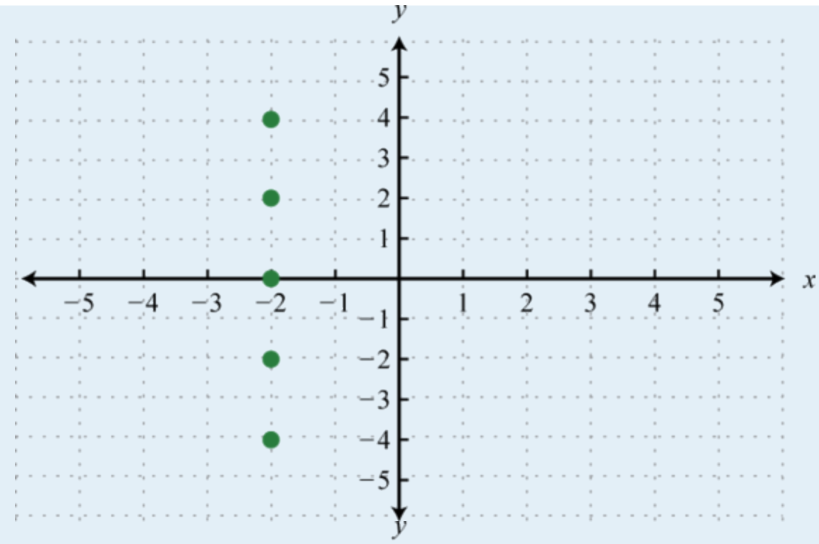
11. 
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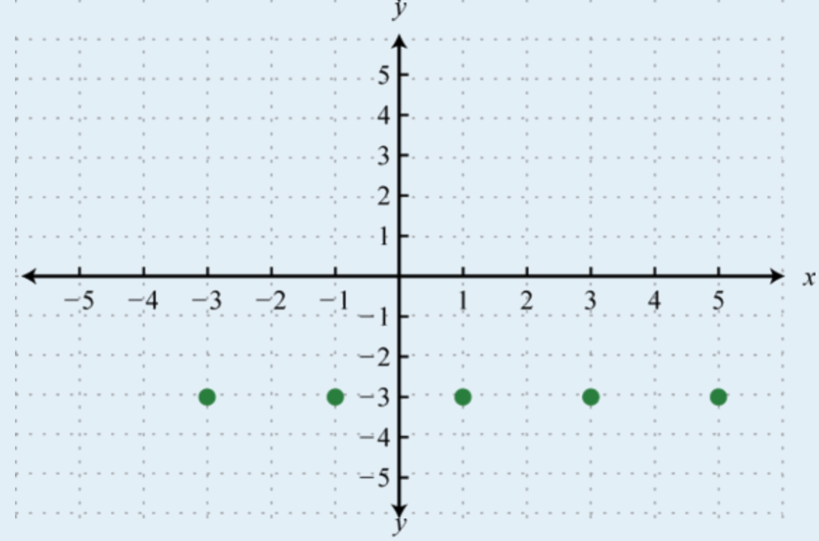
12. 
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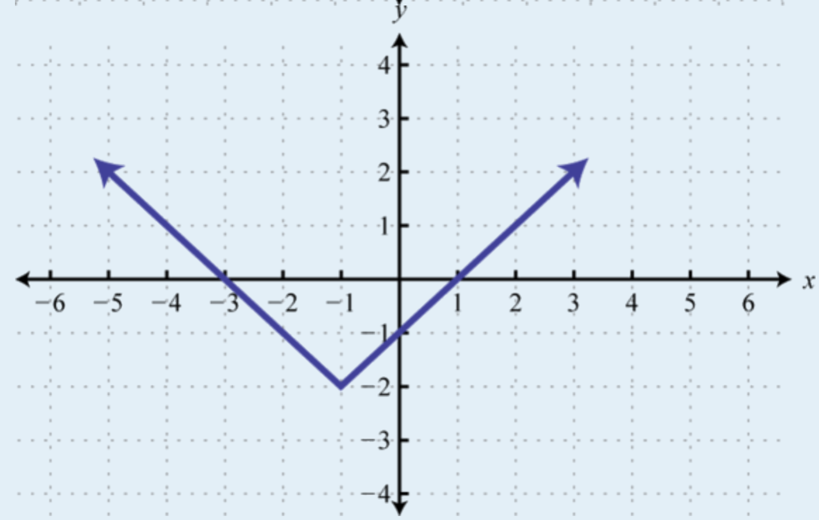
13. 
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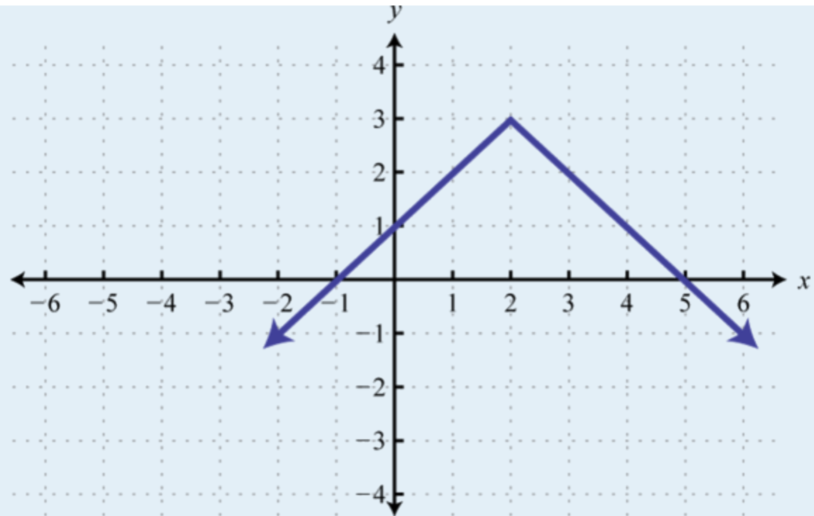
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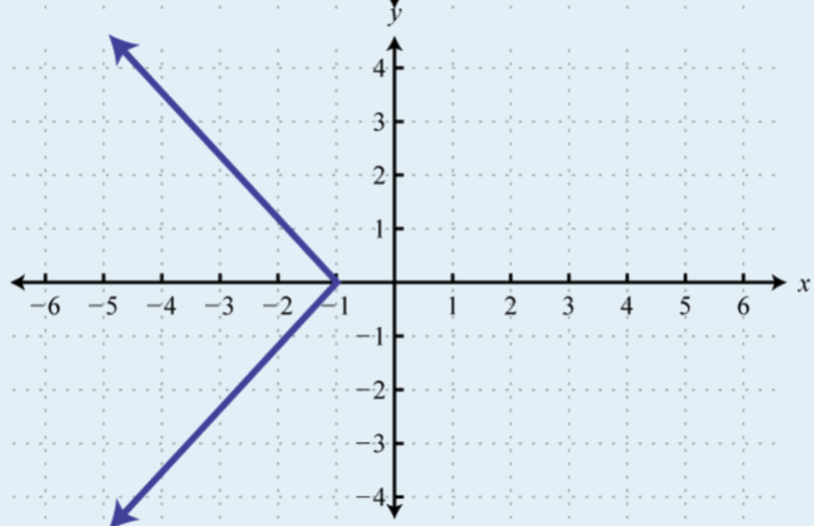
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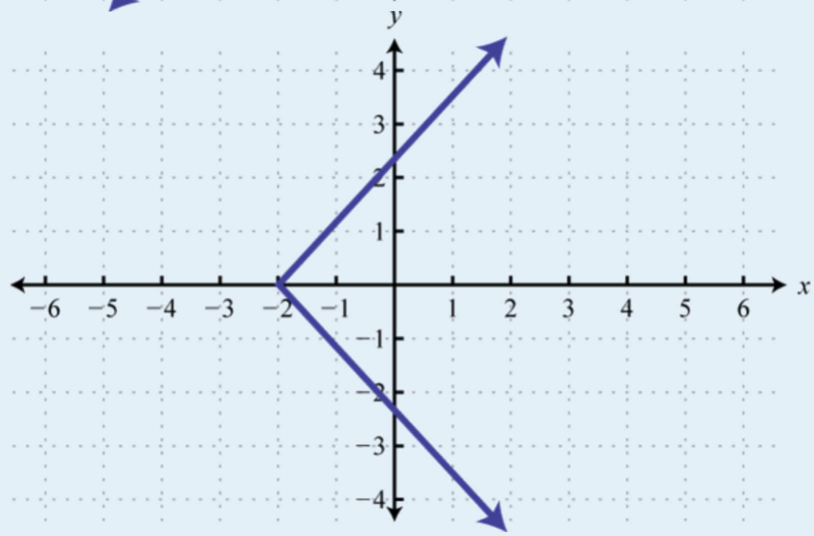
16.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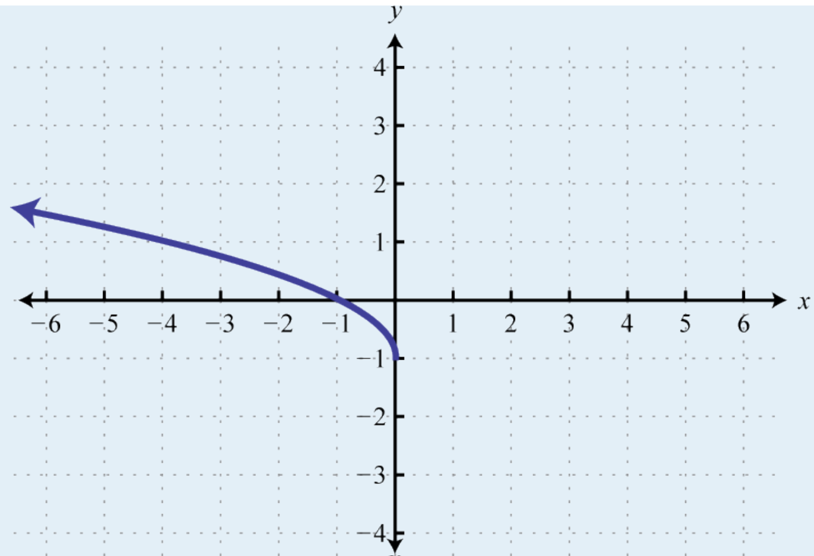
17. 
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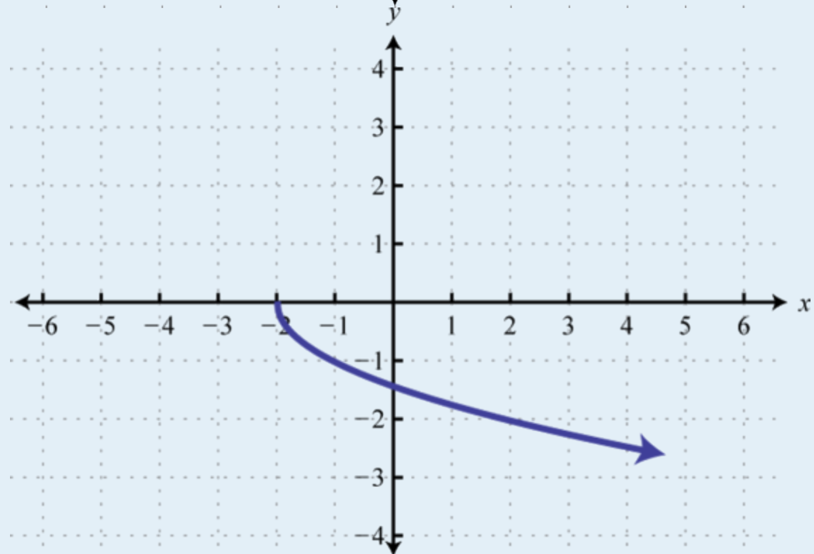
18. 
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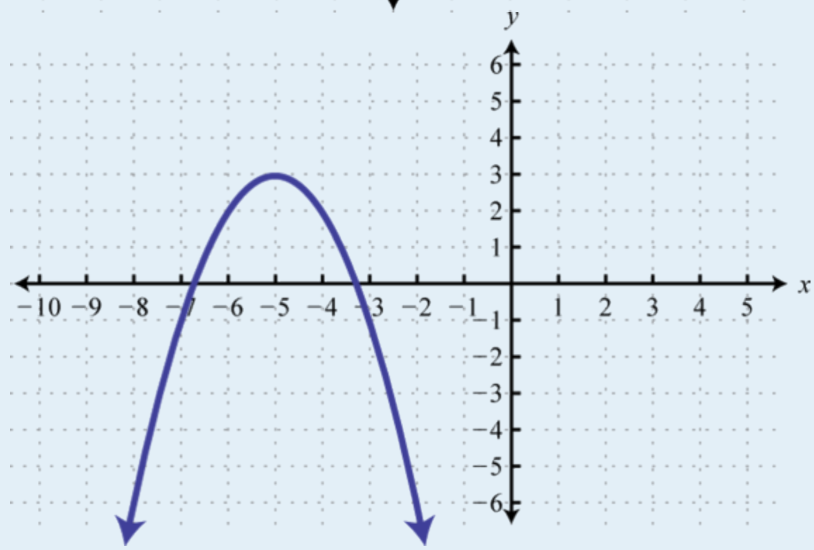
19. 
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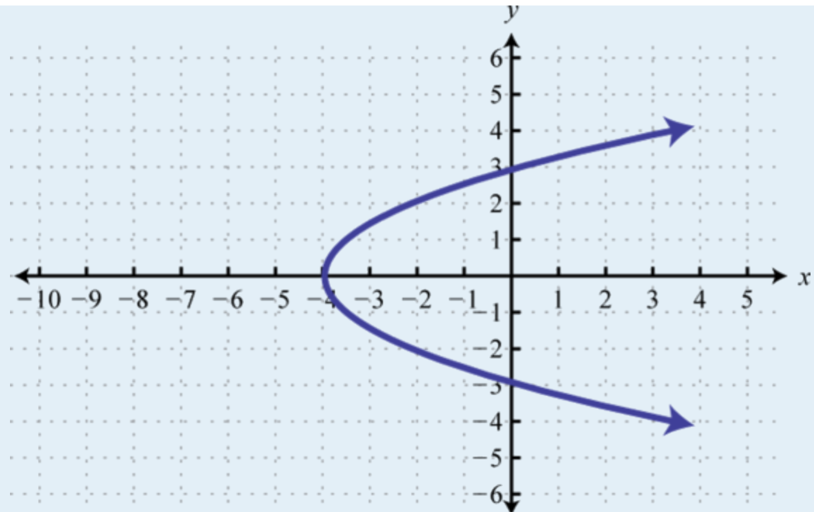
20. 
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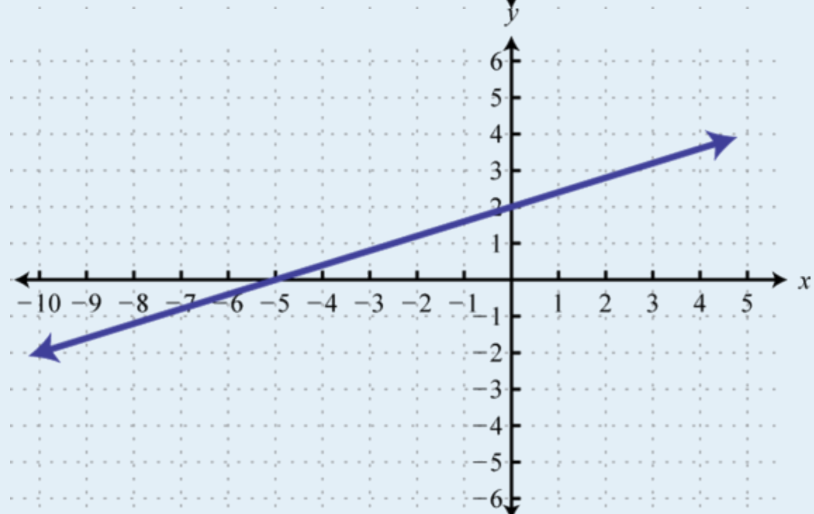
21. 
y
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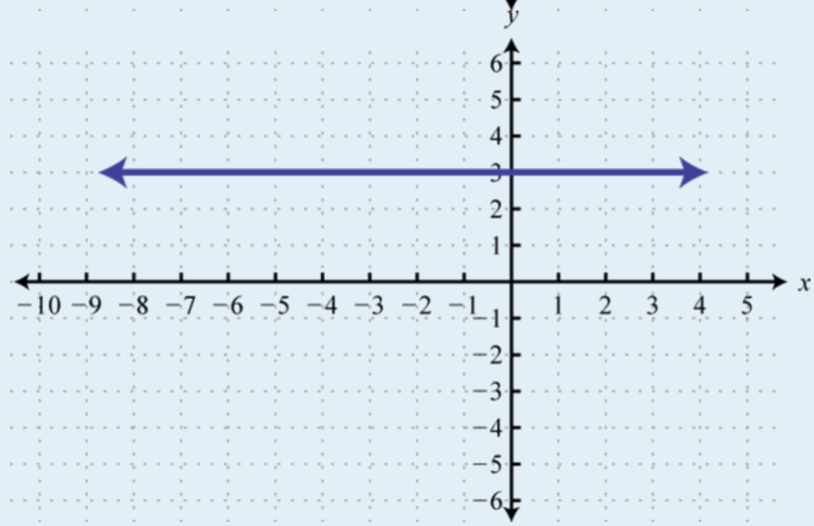
22. 
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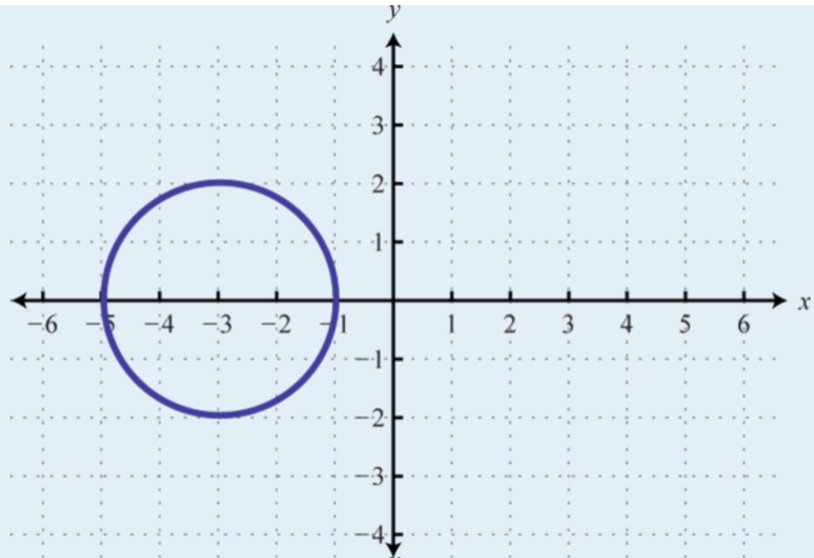
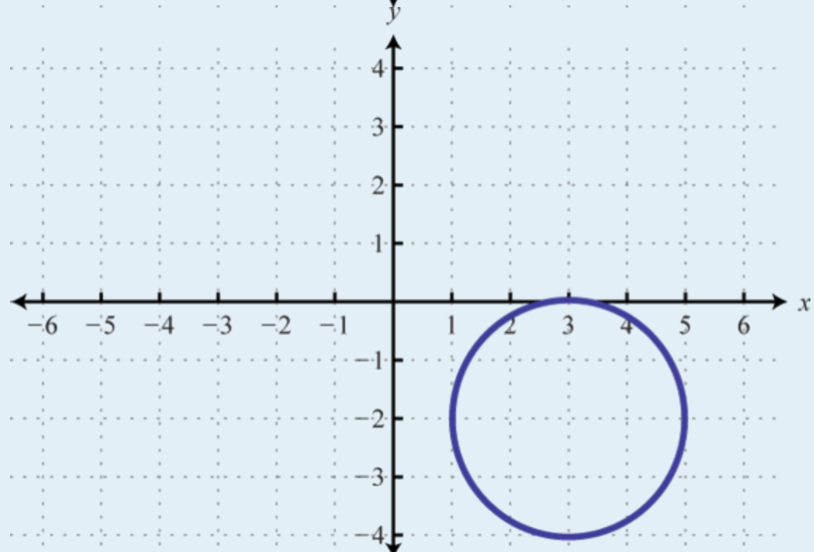
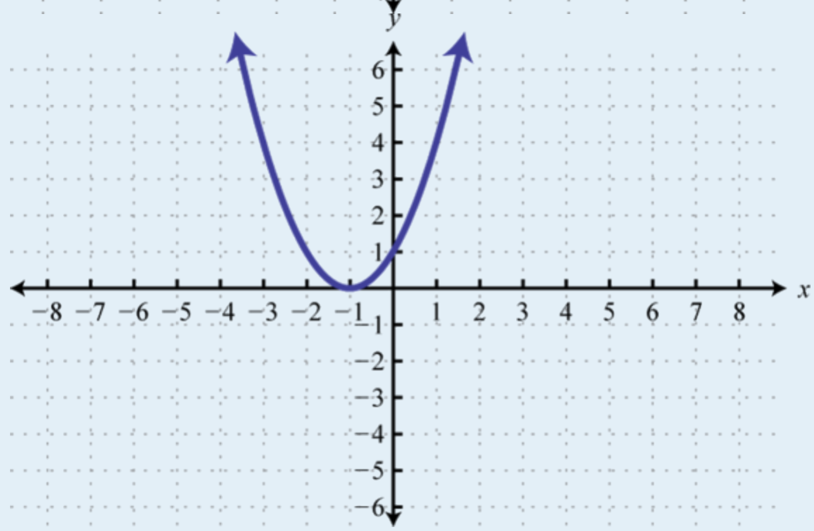
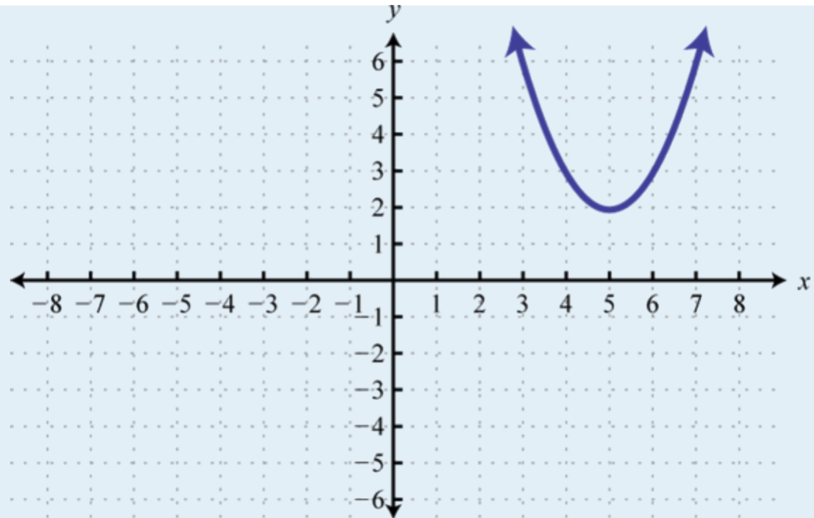
6 5 4 3 2 1 −1 −2 −3 −4 −5 −6

23. 
y
x
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6
4 3 2 1 −1 −2 −3 −4

24. 
y
x
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6
4 3 2 1 −1 −2 −3 −4

25. 
y
x
−8 −7 −6 −5 −4 −3 −2 −1 1 2 3 4 5 6 7 8
6 5 4 3 2 1 −1 −2 −3 −4 −5 −6



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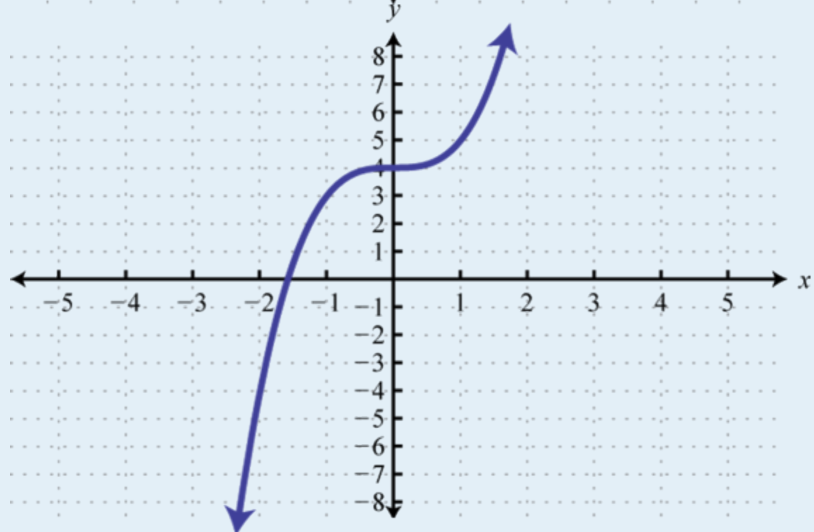

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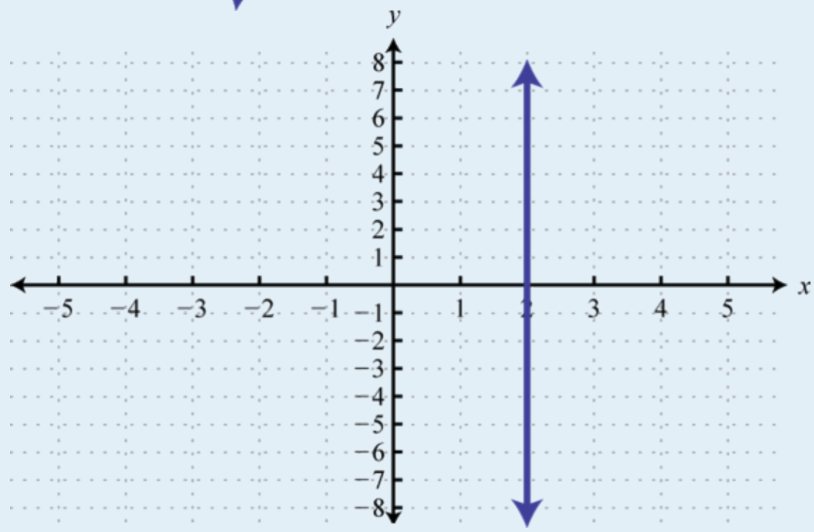
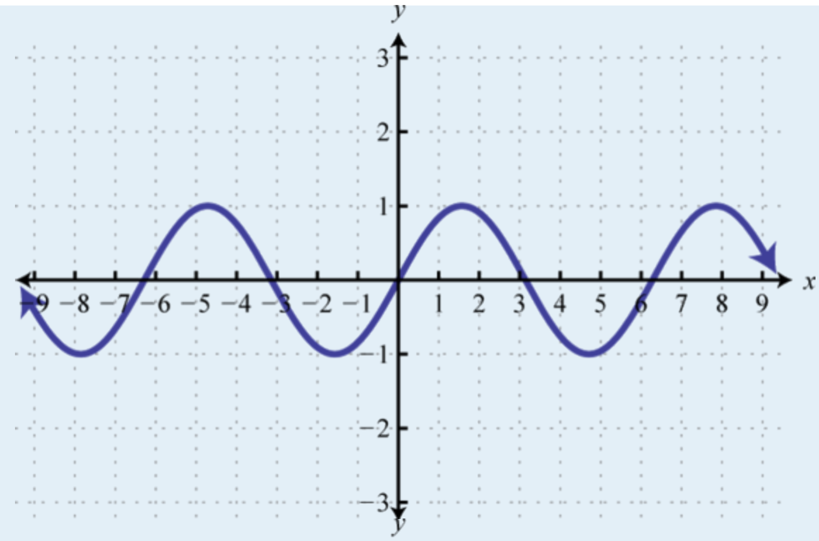


28.

29.



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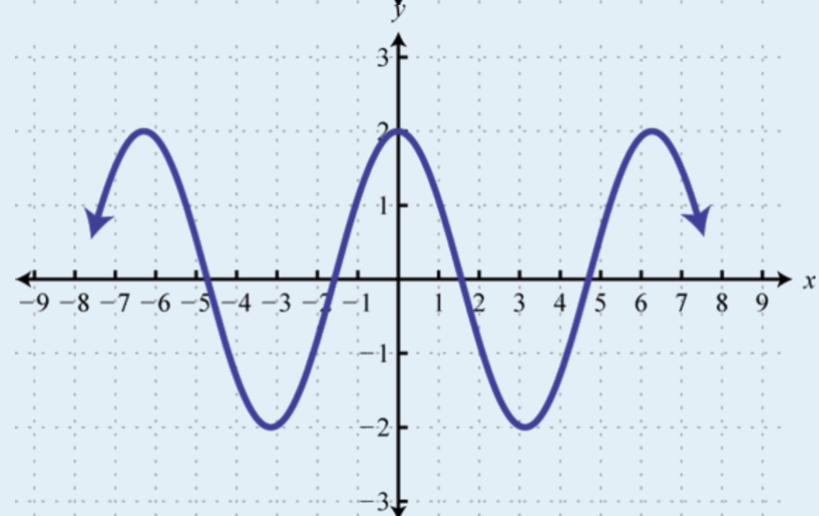

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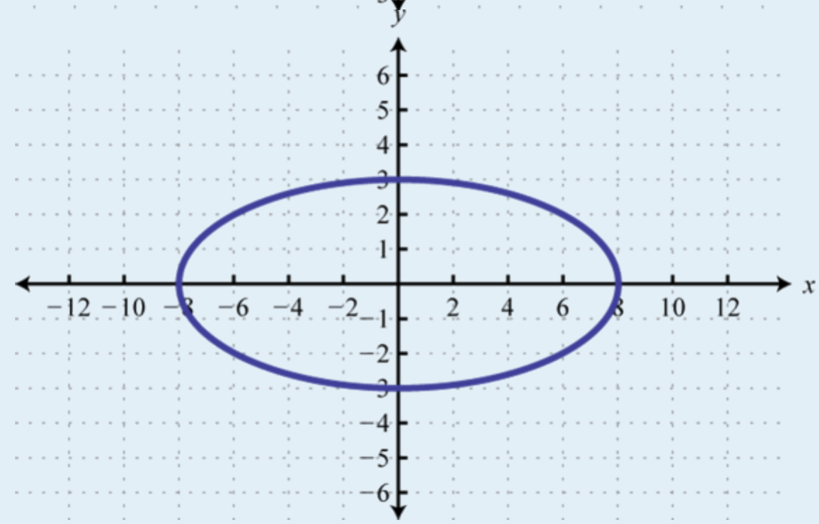
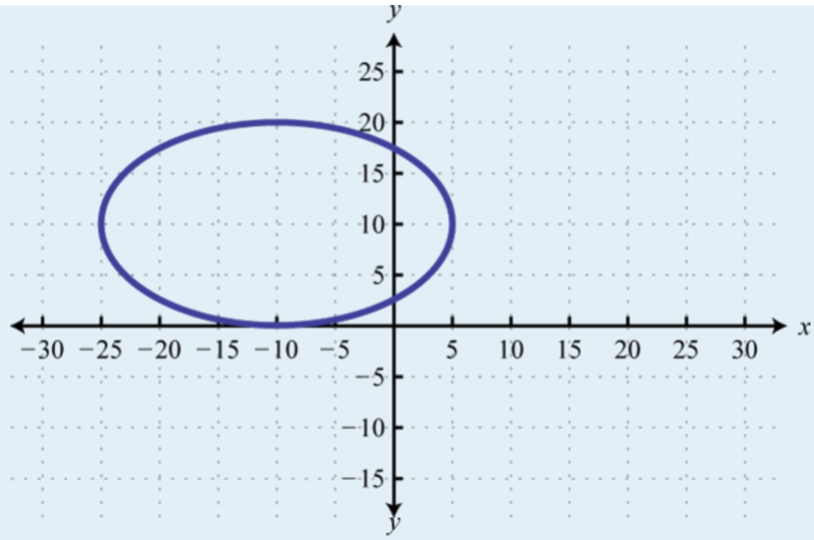
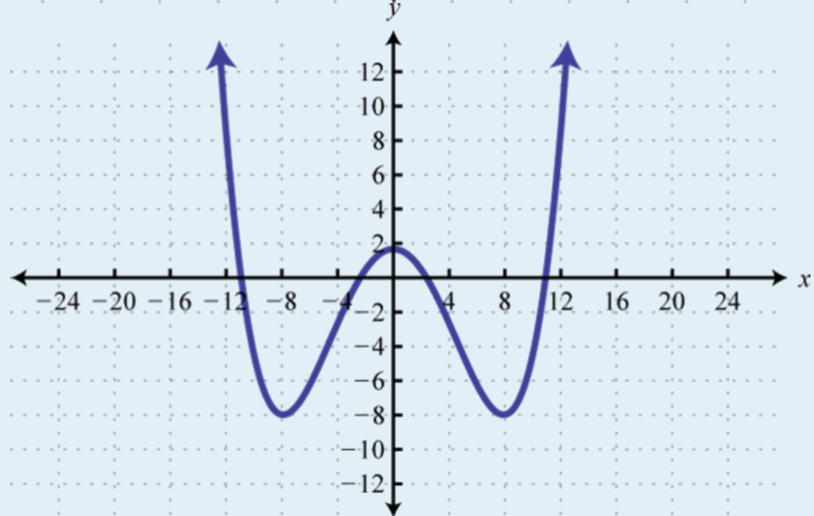
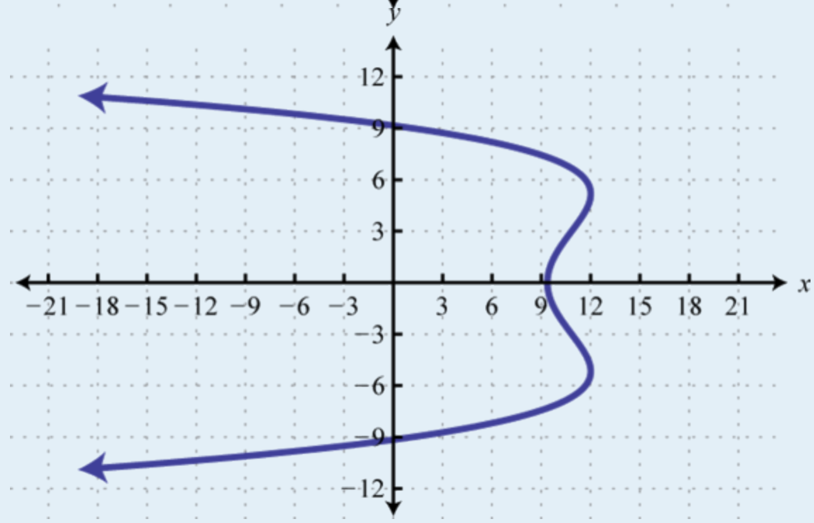


32. 
y
x
25
20
15
10
5
−5
−10
−15
−30 −25 −20 −15 −10 −5 5 10 15 20 25 30

33. 
y
x
12
10
8
6
4
2
−2
−4
−6
−8
−10
−12
−24 −20 −16 −12 −8 −4 4 8 12 16 20 24

34. 
y
x
12
9
6
3
−3
−6
−9
−12
−21 −18 −15 −12 −9 −6 −3 3 6 9 12 15 18 21

PART B: FUNCTION NOTATION

Evaluate.

35. $g(x) = |x - 5|$ find $g(-5)$, $g(0)$, and $g(5)$.
36. $g(x) = |x| - 5$; find $g(-5)$, $g(0)$, and $g(5)$.
37. $g(x) = |2x - 3|$; find $g(-1)$, $g(0)$, and $g\left(\frac{3}{2}\right)$.
38. $g(x) = 3 - |2x|$; find $g(-3)$, $g(0)$, and $g(3)$.
39. $f(x) = 2x - 3$; find $f(-2)$, $f(0)$, and $f(x - 3)$.
40. $f(x) = 5x - 1$; find $f(-2)$, $f(0)$, and $f(x + 1)$.
41. $g(x) = \frac{2}{3}x + 1$; find $g(-3)$, $g(0)$, and $f(9x + 6)$.
42. $g(x) = -\frac{3}{4}x - \frac{1}{2}$; find $g(-4)$, $g(0)$, and $g(6x - 2)$.
43. $g(x) = x^2$; find $g(-5)$, $g\left(\sqrt{3}\right)$, and $g(x - 5)$.
44. $g(x) = x^2 + 1$; find $g(-1)$, $g\left(\sqrt{6}\right)$, and $g(2x - 1)$.
45. $f(x) = x^2 - x - 2$; find $f(0)$, $f(2)$, and $f(x + 2)$.
46. $f(x) = -2x^2 + x - 4$; find $f(-2)$, $f\left(\frac{1}{2}\right)$, and $f(x - 3)$.
47. $h(t) = -16t^2 + 32$; find $h\left(\frac{1}{4}\right)$, $h\left(\frac{1}{2}\right)$, and $h(2a - 1)$.
48. $h(t) = -16t^2 + 32$; find $h(0)$, $h\left(\sqrt{2}\right)$, $h(2a + 1)$.
49. $f(x) = \sqrt{x + 1} - 2$ find $f(-1)$, $f(0)$, $f(x - 1)$.
50. $f(x) = \sqrt{x - 3} + 1$; find $f(12)$, $f(3)$, $f(x + 3)$.
51. $g(x) = \sqrt{x + 8}$; find $g(0)$, $g(-8)$, and $g(x - 8)$.
52. $g(x) = \sqrt{3x - 1}$; find $g\left(\frac{1}{3}\right)$, $g\left(\frac{5}{3}\right)$, and $g\left(\frac{1}{3}a^2 + \frac{1}{3}\right)$.
53. $f(x) = x^3 + 1$; find $f(-1)$, $f(0)$, $f\left(a^2\right)$.

54. $f(x) = x^3 - 8$; find $f(2), f(0), f\left(a^3\right)$.

Given the function find $f\left(x + h\right)$.

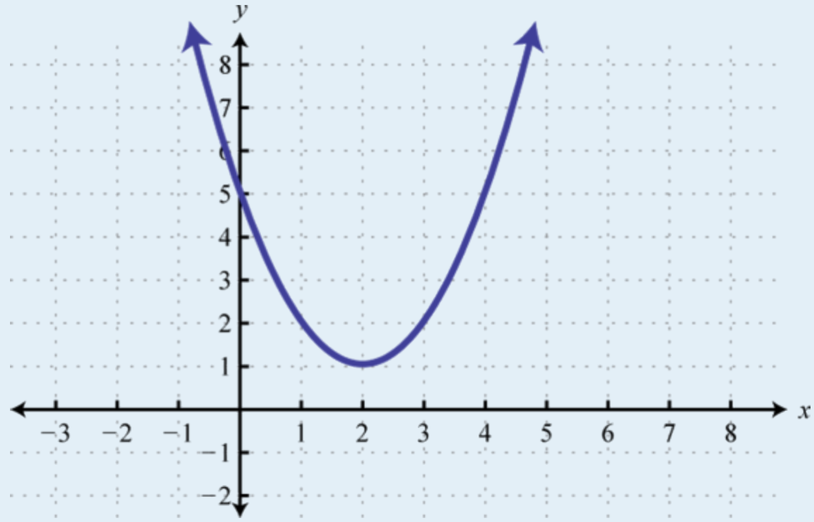
55. $f(x) = 3x - 1$
56. $f(x) = -5x + 2$
57. $f(x) = x^2 + x + 1$
58. $f(x) = 2x^2 - x - 1$
59. $f(x) = x^3$
60. $f(x) = 2x^3 - 1$

Find *x* given the function.

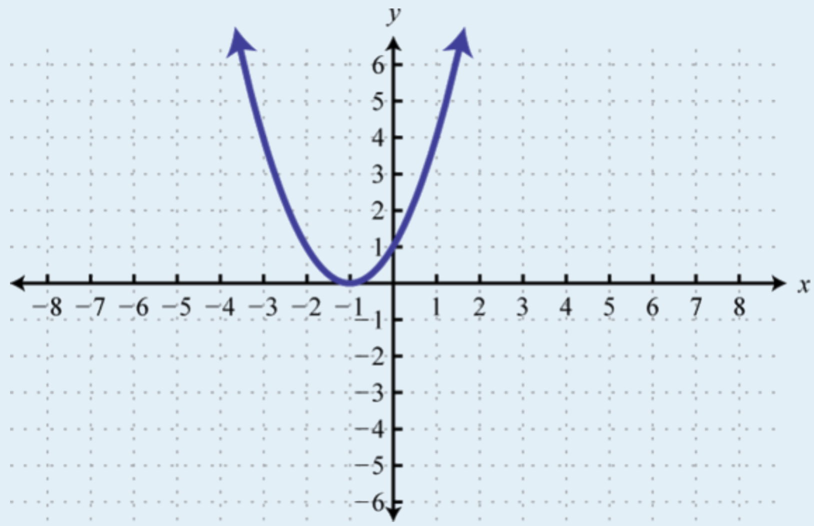
61. $f(x) = 2x - 3$; find *x* where $f(x) = 25$.
62. $f(x) = 7 - 3x$; find *x* where $f(x) = -27$.
63. $f(x) = 2x + 5$; find *x* where $f(x) = 0$
64. $f(x) = -2x + 1$; find *x* where $f(x) = 0$
65. $g(x) = 6x + 2$; find *x* where $g(x) = 5$.
66. $g(x) = 4x + 5$; find *x* where $g(x) = 2$.
67. $h(x) = \frac{2}{3}x - \frac{1}{2}$; find *x* where $h(x) = \frac{1}{6}$.
68. $h(x) = \frac{5}{4}x + \frac{1}{3}$; find *x* where $h(x) = \frac{1}{2}$.
69. The value of a new car in dollars is given by the function $V(t) = -1{,}800t + 22{,}000$ where *t* represents the age of the car in years. Use the function to determine the value of the car when it is 4 years old. What was the value of the car new?
70. The monthly income in dollars of a commissioned car salesperson is given by the function $I(n) = 350n + 1{,}450$ where *n* represents the number of cars sold in the month. Use the function to determine the salesperson's income if he sells 3 cars this month. What is his income if he does not sell any cars in one month?

Given the graph of the function f, find the function values.

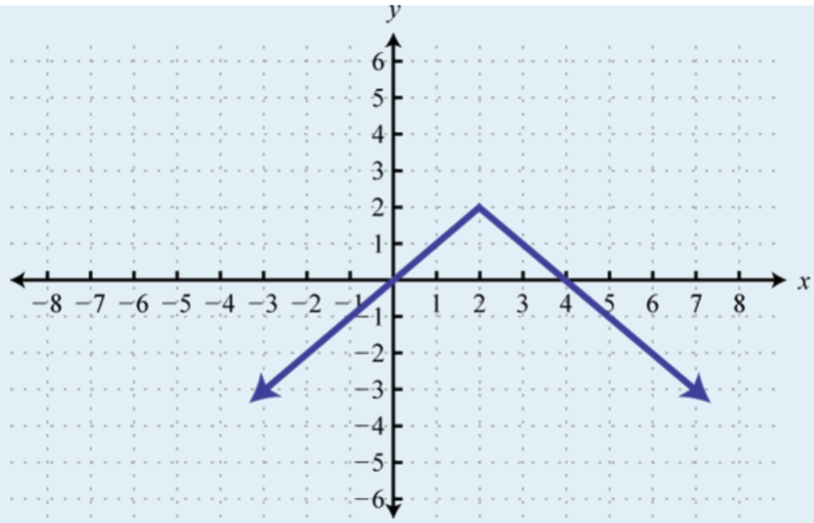
71. Find $f(0)$, $f(2)$, and $f(4)$.



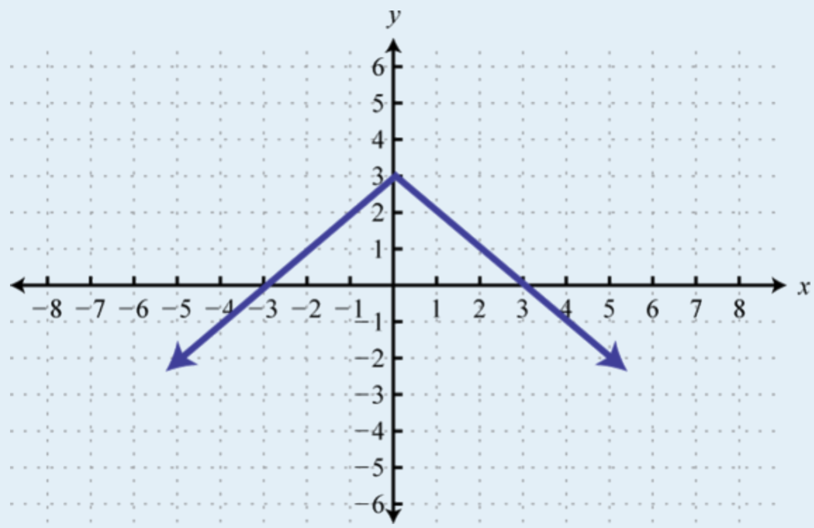
72. Find $f(-1)$, $f(0)$, and $f(1)$.



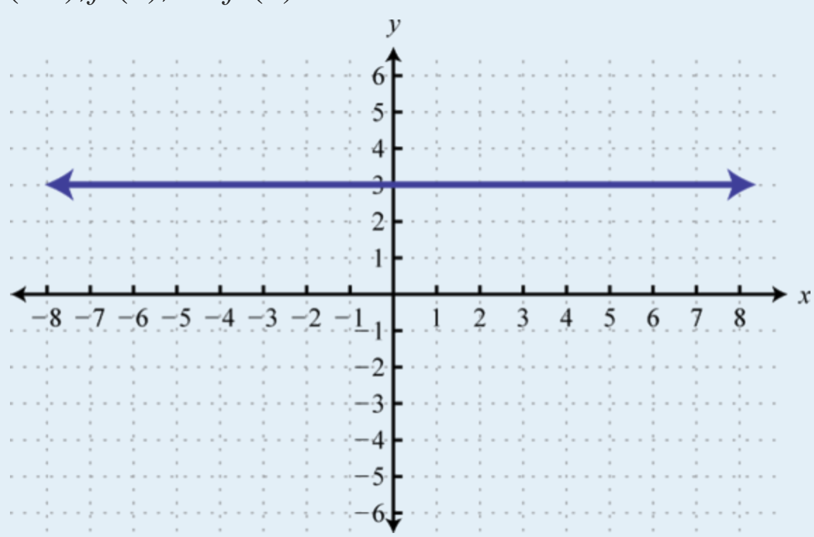
73. Find $f(0)$, $f(2)$, and $f(4)$.



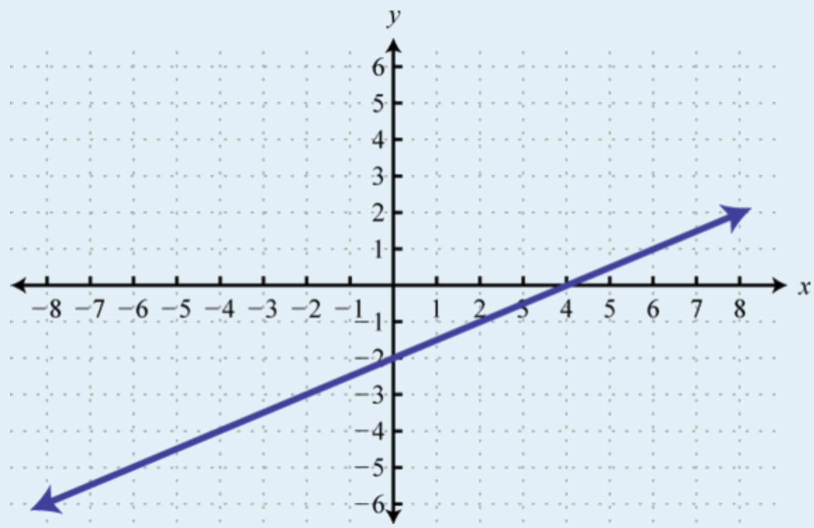
74. Find $f\ (-3), f\ (0)$, and $f\ (3)$.



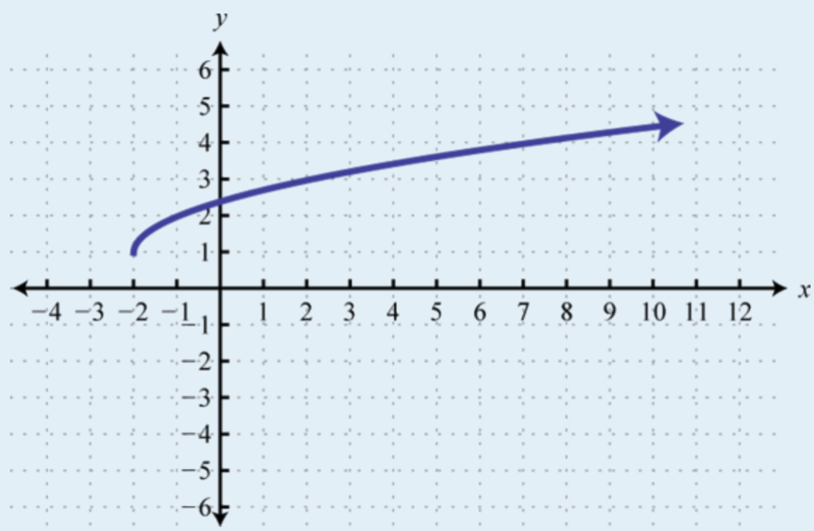
75. Find $f\ (-4), f\ (0)$, and $f\ (2)$.



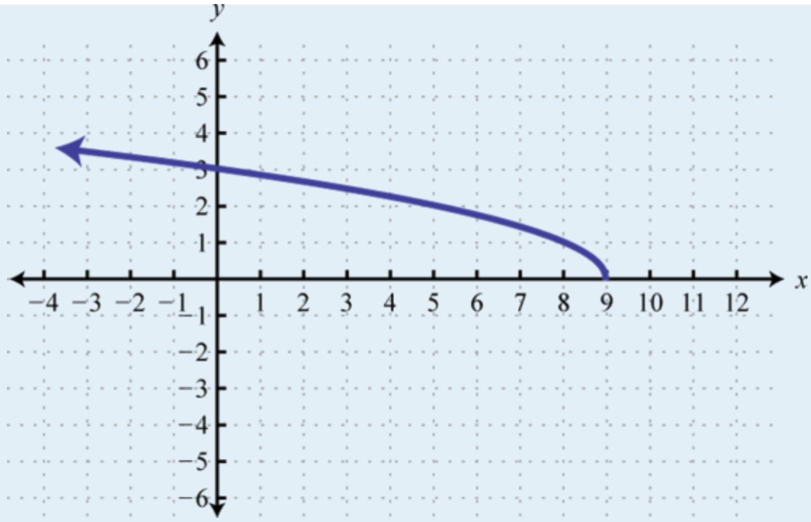
76. Find $f\left(-6\right)$, $f\,(0)$, and $f\left(6\right)$.



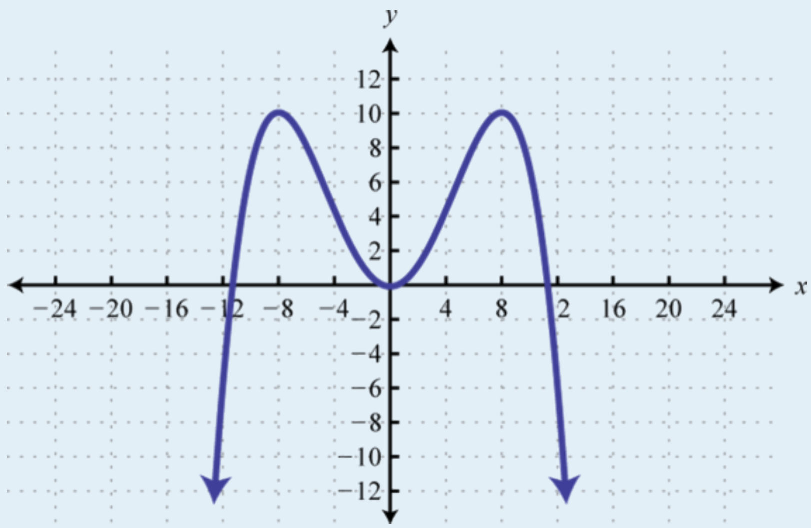
77. Find $f\,(-2)$, $f\,(2)$, and $f\,(7)$.



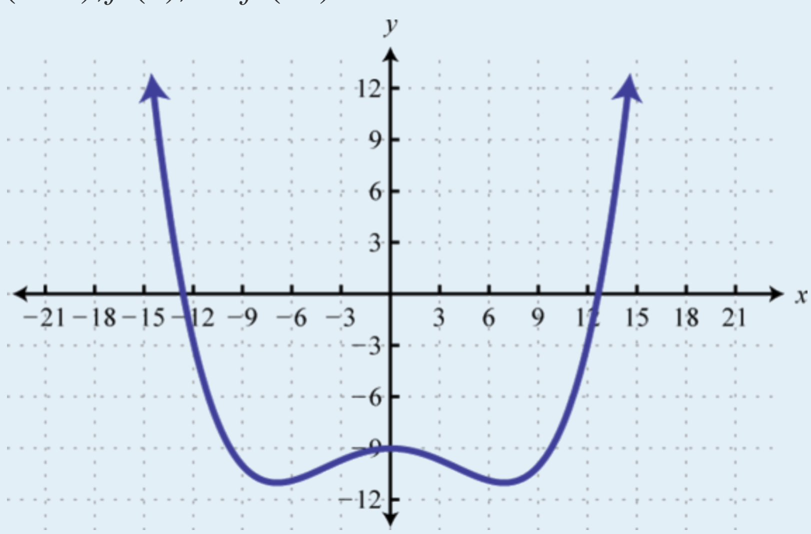
78. Find $f\,(0)$, $f\left(5\right)$, and $f\,(9)$.



79. Find $f\,(-8), f\,(0)$, and $f\,(8)$.

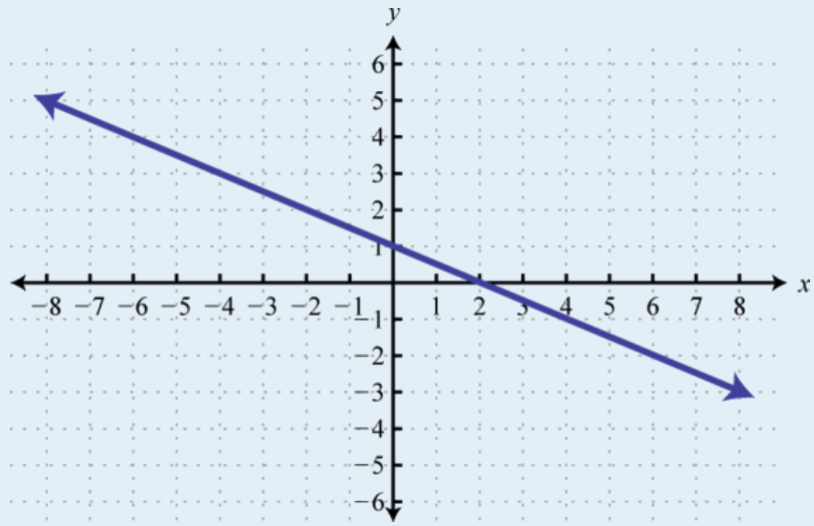


80. Find $f\,(-12), f\,(0)$, and $f\,(12)$.

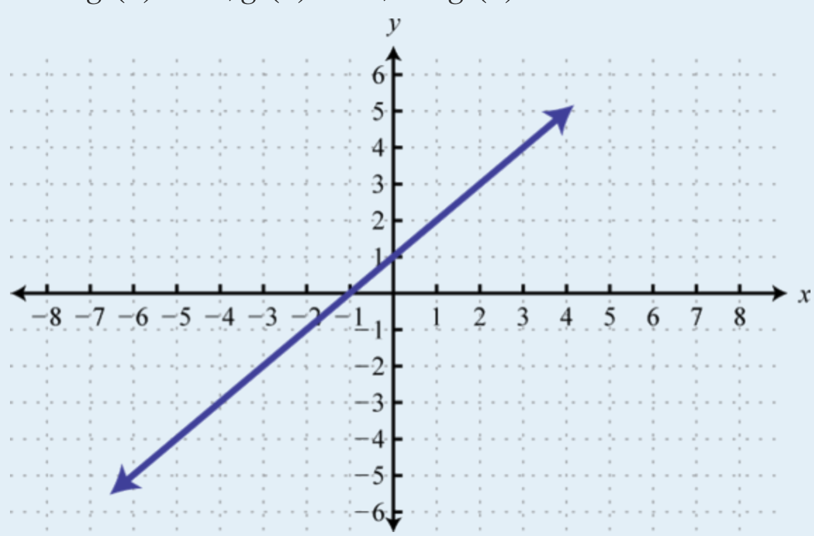


Given the graph of a function g, find the *x*-values.

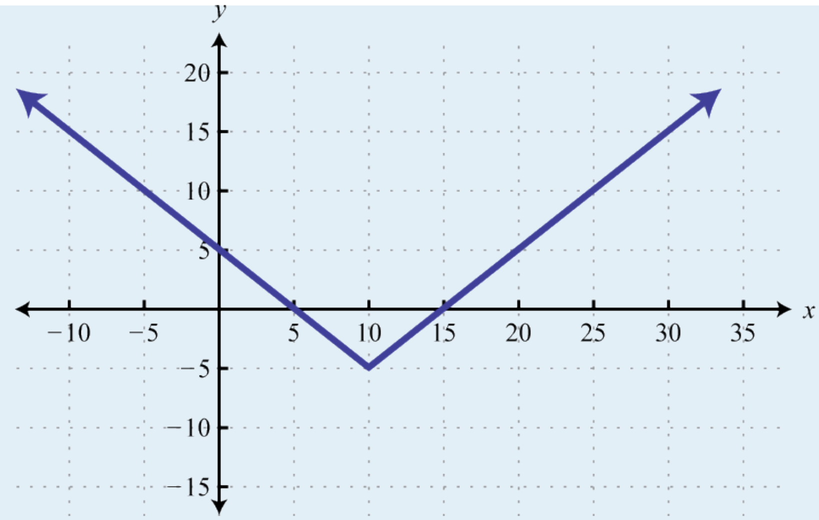
81. Find *x* where $g(x) = 3$, $g(x) = 0$, and $g(x) = -2$.



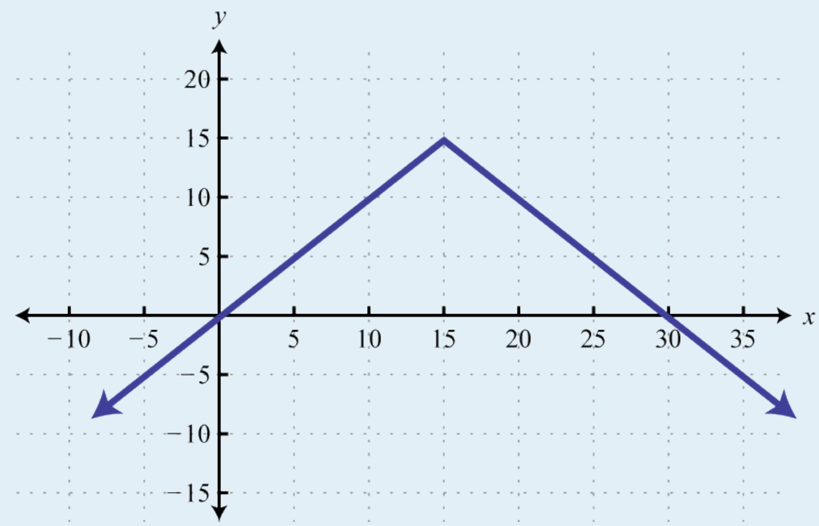
82. Find *x* where $g(x) = 0$, $g(x) = 1$, and $g(x) = 4$.



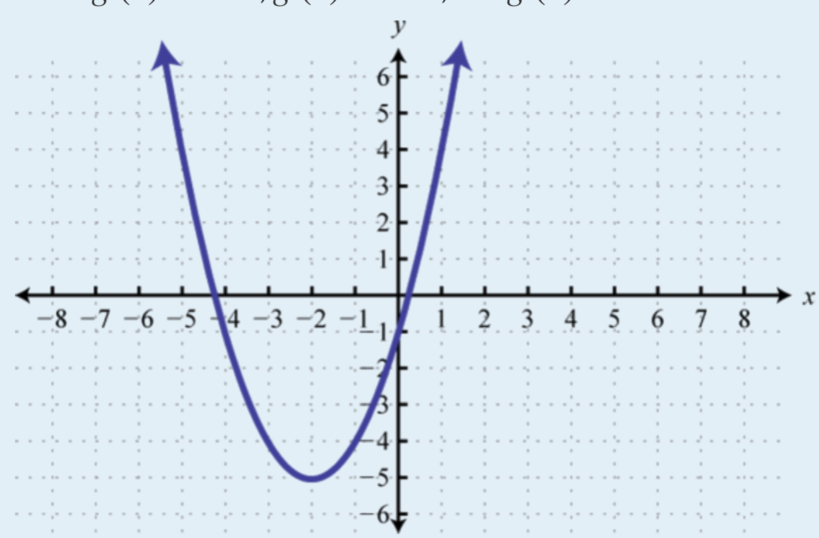
83. Find *x* where $g(x) = -5$, $g(x) = 0$, and $g(x) = 10$.



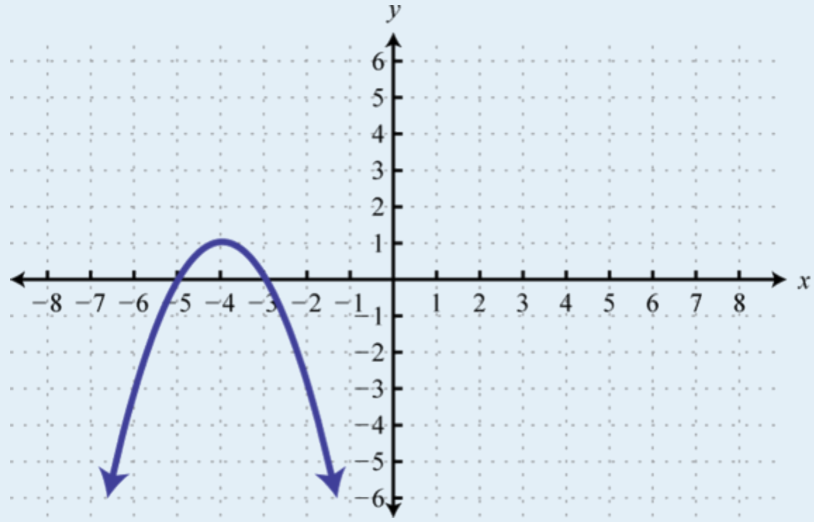
84. Find x where $g(x) = 0$, $g(x) = 10$, and $g(x) = 15$.



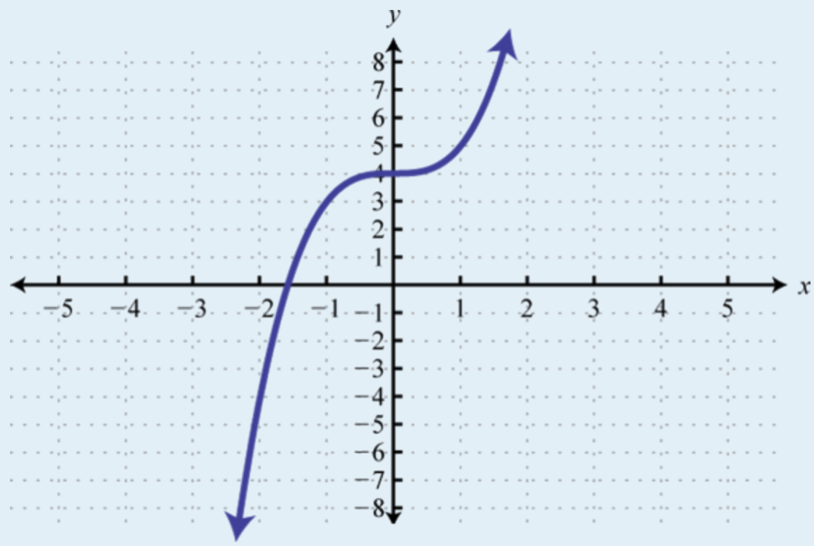
85. Find x where $g(x) = -5$, $g(x) = -4$, and $g(x) = 4$.



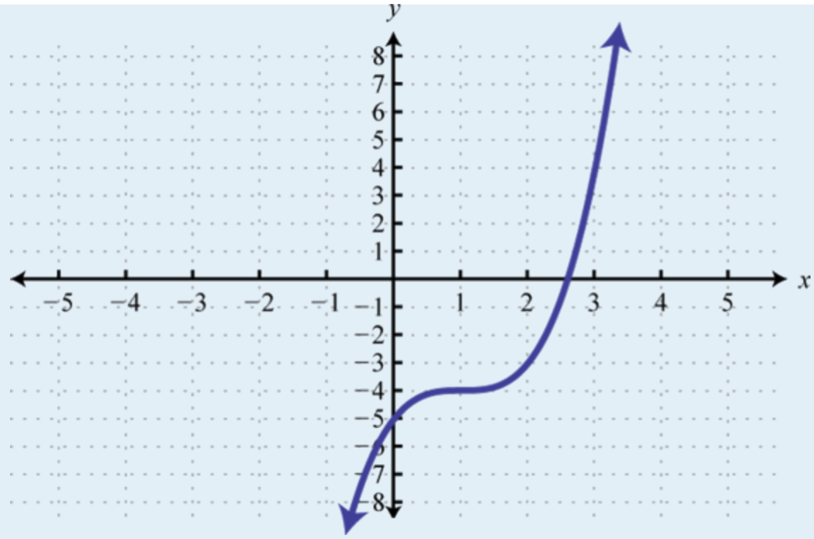
86. Find x where $g(x) = 1$, $g(x) = 0$, and $g(x) = -3$.



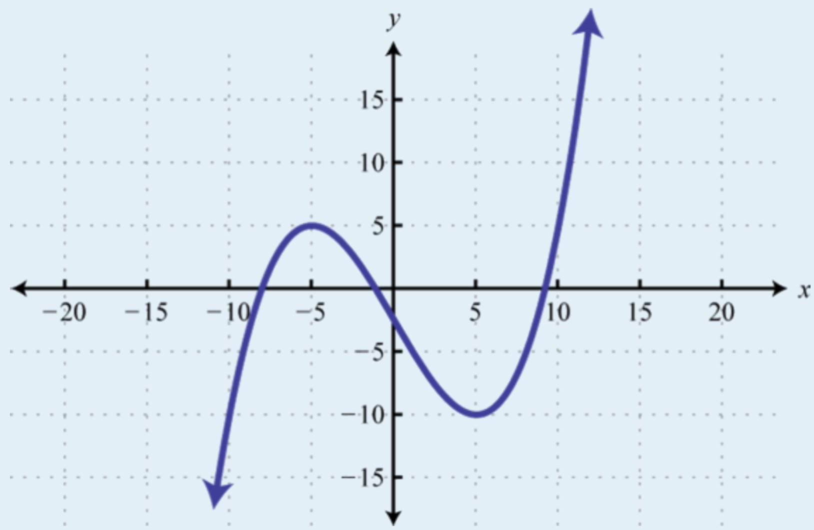
87. Find x where $g(x) = -4$, $g(x) = 3$, and $g(x) = 4$.



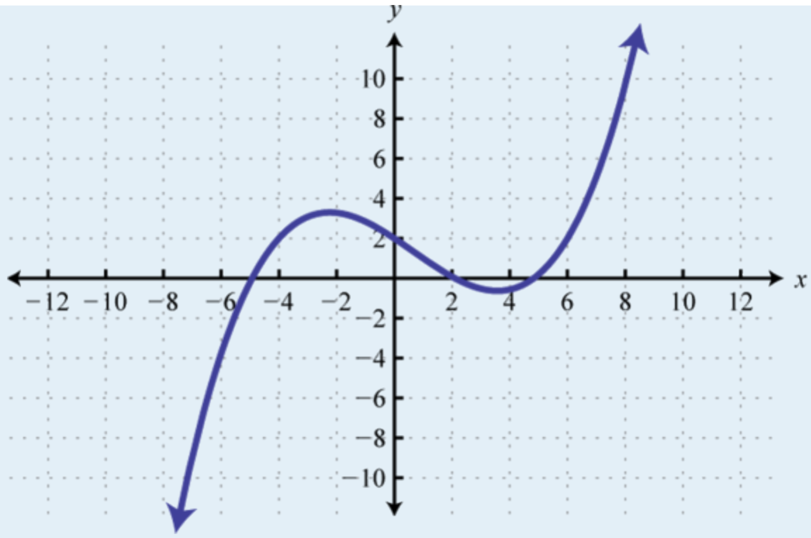
88. Find x where $g(x) = -5$, $g(x) = -4$, and $g(x) = 4$.



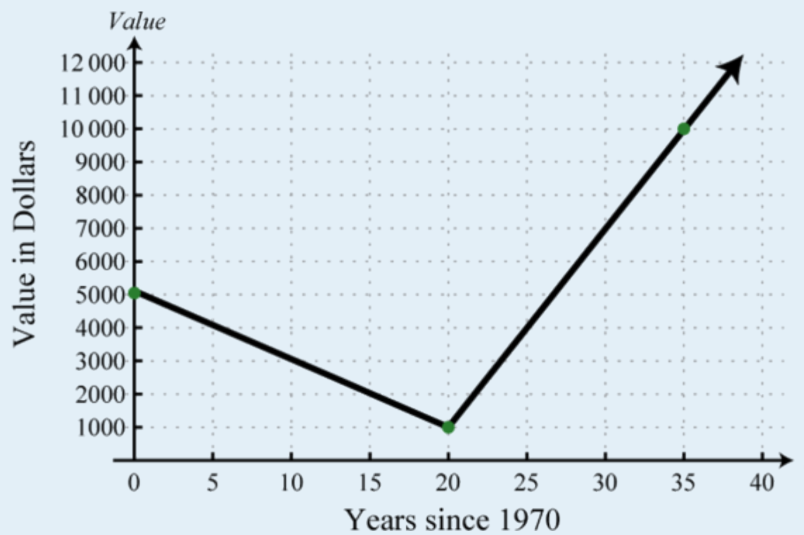
89. Find x where $g(x) = -10$ and $g(x) = 5$.



90. Find x where $g(x) = 2$.



The value of a certain automobile in dollars depends on the number of years since it was purchased in 1970 according to the following function:



91. What was the value of the car when it was new in 1970?

92. In what year was the value of the car at a minimum?

93. What was the value of the car in 2005?

94. In what years was the car valued at $4,000?

Given the linear function defined by $f(x) = 2x - 5$, simplify the following.

95. $f(5) - f(3)$

96. $f(0) - f(7)$

97. $f(x+2) - f(2)$

98. $f(x+7) - f(7)$

99. $f\left(x+h\right) - f(x)$

100. $\dfrac{f\left(x+h\right) - f(x)}{h}$

101. Simplify $\frac{c(x+h)-c(x)}{h}$ given $c(x) = 3x + 1$.

102. Simplify $\frac{p(x+h)-p(x)}{h}$ given $p(x) = 7x - 3$.

103. Simplify $\frac{g(x+h)-g(x)}{h}$ given $g(x) = mx + b$.

104. Simplify $\frac{q(x+h)-q(x)}{h}$ given $q(x) = ax$.

PART C: DISCUSSION BOARD

105. Who is credited with the introduction of the notation $y = f(x)$? Provide a brief summary of his life and accomplishments.

106. Explain to a beginning algebra student what the vertical line test is and why it works.

107. Research and discuss the life and contributions of René Descartes.

108. Conduct an Internet search for the vertical line test, functions, and evaluating functions. Share a link to a page that you think others may find useful.

ANSWERS

1. Domain: {3, 5, 7, 9, 12}; range: {1, 2, 3, 4}; function: yes
3. Domain: {7, 8, 10, 15}; range: {5, 6, 7, 8, 9}; function: no
5. Domain: {5}; range: {0, 2, 4, 6, 8}; function: no
7. Domain: {-4, -1, 0, 2, 3}; range: {1, 2, 3}; function: yes
9. Domain: {-1, 0, 1, 2}; range: {0, 1, 2, 3, 4}; function: no
11. Domain: {-2}; range: {-4, -2, 0, 2, 4}; function: no
13. Domain: $\mathbb{R}$; range: $\left[-2, \infty\right)$; function: yes
15. Domain: $\left(-\infty, -1\right]$; range: $\mathbb{R}$; function: no
17. Domain: $\left(-\infty, 0\right]$; range: $\left[-1, \infty\right)$; function: yes
19. Domain: $\mathbb{R}$; range: $\left(-\infty, 3\right]$; function: yes
21. Domain: $\mathbb{R}$; range: $\mathbb{R}$; function: yes
23. Domain: $\left[-5, -1\right]$; range: $\left[-2, 2\right]$; function: no
25. Domain: $\mathbb{R}$; range: $\left[0, \infty\right]$; function: yes
27. Domain: $\mathbb{R}$; range: $\mathbb{R}$; function: yes
29. Domain: $\mathbb{R}$; range: $\left[-1, 1\right]$; function: yes
31. Domain: $\left[-8, 8\right]$; range: $\left[-3, 3\right]$; function: no
33. Domain: $\mathbb{R}$; range: $\left[-8, \infty\right]$; function: yes
35. $g\left(-5\right) = 10, g\left(0\right) = 5, g\left(5\right) = 0$
37. $g\left(-1\right) = 5, g\left(0\right) = 3, g\left(\frac{3}{2}\right) = 0$
39. $f\left(-2\right) = -7, f\left(0\right) = -3, f\left(x - 3\right) = 2x - 9$
41. $g\left(-3\right) = -1, g\left(0\right) = 1, g\left(9x + 6\right) = 6x + 5$
43. $g\left(-5\right) = 25, g\left(\sqrt{3}\right) = 3, g\left(x - 5\right) = x^2 - 10x + 25$

45. $f(0) = -2, f(2) = 0, f(x+2) = x^2 + 3x$

47. $h\left(\frac{1}{4}\right) = 31, h\left(\frac{1}{2}\right) = 28, h(2a-1) = -64a^2 + 64a + 16$

49. $f(-1) = -2, f(0) = -1, f(x-1) = \sqrt{x} - 2$

51. $g(0) = 2\sqrt{2}, g(-8) = 0, g\left(a^2 - 8\right) = |a|$

53. $f(-1) = 0, f(0) = 1, f\left(a^2\right) = a^6 + 1$

55. $f\left(x+h\right) = 3x + 3h - 1$

57. $f\left(x+h\right) = x^2 + 2xh + h^2 + x + h + 1$

59. $f\left(x+h\right) = x^3 + 3hx^2 + 3h^2x + h^3$

61. $x = 14$

63. $x = -\dfrac{5}{2}$

65. $x = \dfrac{1}{2}$

67. $x = 1$

69. New: \$22,000; 4 yrs old: \$14,800

71. $f(0) = 5, f(2) = 1, f(4) = 5$

73. $f(0) = 0, f(2) = 2, f(4) = 0$

75. $f(-4) = 3, f(0) = 3, f(2) = 3$

77. $f(-2) = 1, f(2) = 3, f(7) = 4$

79. $f(-8) = 10, f(0) = 0, f(8) = 10$

81. $g(-4) = 3, g(2) = 0,$ and $g\left(6\right) = -2.$

83. $g(10) = -5, g\left(5\right) = 0 \ \ and \ \ g\left(15\right) = 0,$
$g\left(-5\right) = 10 \ \ and \ \ g\left(25\right) = 10$

85. $g(-2) = -5, g(-3) = -4 \ \ and \ \ g(-1) = -4,$
$g\left(-5\right) = 4 \ \ and \ \ g(1) = 4$

87. $g(-2) = -4, g(-1) = 3, g(0) = 4$

89. $g(-10) = -10 \ \ and \ \ g\left(5\right) = -10;$
$g\left(-5\right) = 5 \ \ and \ \ g(10) = 5$

91. \$5,000
93. \$10,000
95. 4
97. $2x$
99. $2h$
101. 3
103. m
105. Answer may vary
107. Answer may vary

2.2 Linear Functions and Their Graphs

LEARNING OBJECTIVES

1. Graph a line by plotting points.
2. Determine the slope of a line.
3. Identify and graph a linear function using the slope and y-intercept.
4. Interpret solutions to linear equations and inequalities graphically.

A Review of Graphing Lines

Recall that the set of all solutions to a linear equation can be represented on a rectangular coordinate plane using a straight line through at least two points; this line is called its graph. For example, to graph the linear equation $8x + 4y = 12$ we would first solve for y.

$$\begin{aligned} 8x + 4y &= 12 && \textit{Subtract } 8x \textit{ on both sides.} \\ 4y &= -8x + 12 && \textit{Divide both sides by } 4. \\ y &= \frac{-8x + 12}{4} && \textit{Simplify } y. \\ y &= \frac{-8x}{4} + \frac{12}{4} \\ y &= -2x + 3 \end{aligned}$$

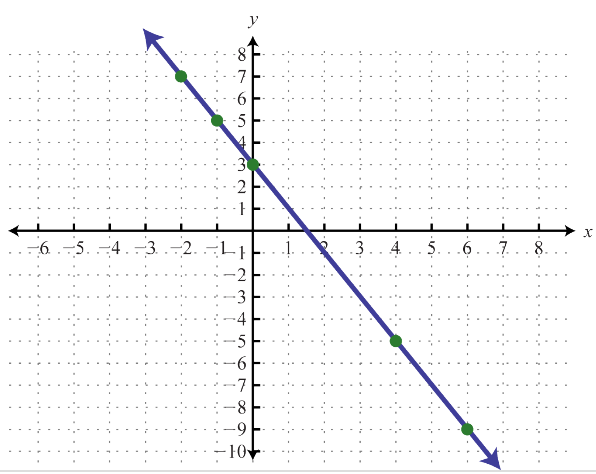
Written in this form, we can see that y depends on x; in other words, x is the **independent variable**[18] and y is the **dependent variable**[19]. Choose at least two x-values and find the corresponding y-values. It is a good practice to choose zero, some negative numbers, as well as some positive numbers. Here we will choose five x values, determine the corresponding y-values, and then form a representative set of ordered pair solutions.

18. The variable that determines the values of other variables. Usually we think of the x-value of an ordered pair (x, y) as the independent variable.

19. The variable whose value is determined by the value of the independent variable. Usually we think of the y-value of an ordered pair (x, y) as the dependent variable.

x	y	$y = -2x + 3$	*Solutions*
-2	7	$y = -2(-2) + 3 = 4 + 3 = 7$	(-2, 7)
-1	5	$y = -2(-1) + 3 = 2 + 3 = 5$	(-1, 5)
0	3	$y = -2(0) + 3 = 0 + 3 = 3$	(0, 3)
4	-5	$y = -2(4) + 3 = -8 + 3 = -5$	(4, -5)
6	-9	$y = -2\left(6\right) + 3 = -12 + 3 = -9$	(6, -9)

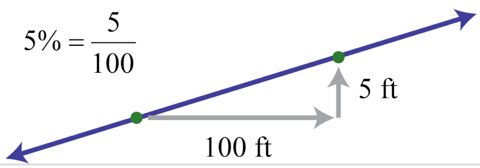
Plot the points and draw a line through the points with a straightedge. Be sure to add arrows on either end to indicate that the graph extends indefinitely.



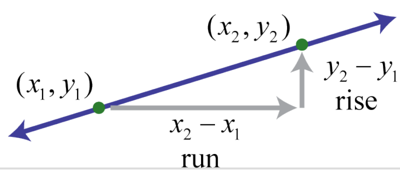
The resulting line represents all solutions to $8x + 4y = 12$, of which there are infinitely many. The above process describes the technique for graphing known as

plotting points[20]. This technique will be used to graph more complicated functions as we progress in this course.

The steepness of any incline can be measured as the ratio of the vertical change to the horizontal change. For example, a 5% incline can be written as $\frac{5}{100}$, which means that for every 100 feet forward, the height increases 5 feet.



In mathematics, we call the incline of a line the **slope**[21], denoted by the letter *m*. The vertical change is called the **rise**[22] and the horizontal change is called the **run**[23]. Given any two points (x_1, y_1) and (x_2, y_2), we can obtain the rise and run by subtracting the corresponding coordinates.



This leads us to the **slope formula**[24]. Given any two points (x_1, y_1) and (x_2, y_2), the slope is given by:

$$\text{Slope} \quad m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} \quad \begin{array}{l} \leftarrow \text{Change in } y \\ \leftarrow \text{Change in } x \end{array}$$

The Greek letter delta (Δ) is often used to describe the change in a quantity. Therefore, the slope is sometimes described using the notation $\frac{\Delta y}{\Delta x}$, which represents the change in *y* divided by the change in *x*.

20. A way of determining a graph using a finite number of representative ordered pair solutions.
21. The incline of a line measured as the ratio of the vertical change to the horizontal change, often referred to as "rise over run."
22. The vertical change between any two points on a line.
23. The horizontal change between any two points on a line.
24. The slope of the line through the points (x_1, y_1) and (x_2, y_2) is given by the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Example 1

Find the slope of the line passing through (−3, −5) and (2, 1).

Solution:

Given (−3, −5) and (2, 1), calculate the difference of the *y*-values divided by the difference of the *x*-values. Take care to be consistent when subtracting the coordinates:

$$\begin{array}{cc} (x_1, y_1) & (x_2,\ y_2) \\ (-3, -5) & (2, 1) \end{array}$$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1 - \ (-5)}{2 - \ (-3)} \\ &= \frac{1 + 5}{2 + 3} \\ &= \frac{6}{5} \end{aligned}$$

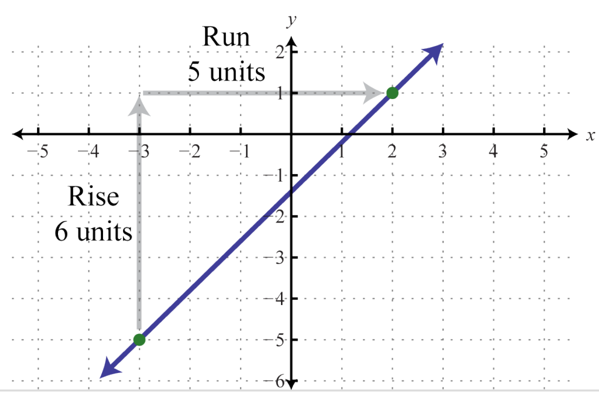
It does not matter which point you consider to be the first and second. However, because subtraction is not commutative, you must take care to subtract the coordinates of the first point from the coordinates of the second point in the same order. For example, we obtain the same result if we apply the slope formula with the points switched:

$$\begin{array}{cc} (x_1, y_1) & (x_2, y_2) \\ (2, 1) & (-3, -5) \end{array}$$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-5 - 1}{-3 - 2} \\ &= \frac{-6}{-5} \\ &= \frac{6}{5} \end{aligned}$$

Answer: $m = \frac{6}{5}$

Verify that the slope is $\frac{6}{5}$ by graphing the line described in the previous example.



Certainly the graph is optional; the beauty of the slope formula is that, given any two points, we can obtain the slope using only algebra.

Example 2

Find the y-value for which the slope of the line passing through $(6, -3)$ and $(-9, y)$ is $-\frac{2}{3}$.

Solution:

Substitute the given information into the slope formula.

$$\begin{array}{ccc} \textit{Slope} & (x_1, y_1) & (x_2, y_2) \\ m = -\dfrac{2}{3} & (6, -3) & (-9, y) \end{array}$$

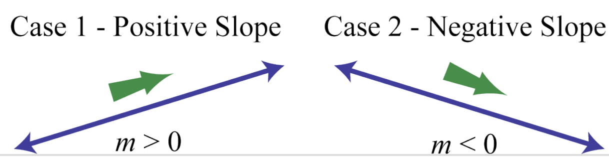
$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ -\frac{2}{3} &= \frac{y - (-3)}{-9 - 6} \\ -\frac{2}{3} &= \frac{y + 3}{-15} \end{aligned}$$

After substituting in the given information, the only variable left is y. Solve.

$$\begin{aligned} {\color{blue}-15}\left(-\frac{2}{3}\right) &= {\color{blue}-15}\left(-\frac{y + 3}{15}\right) \\ 10 &= y + 3 \\ 7 &= y \end{aligned}$$

Answer: $y = 7$

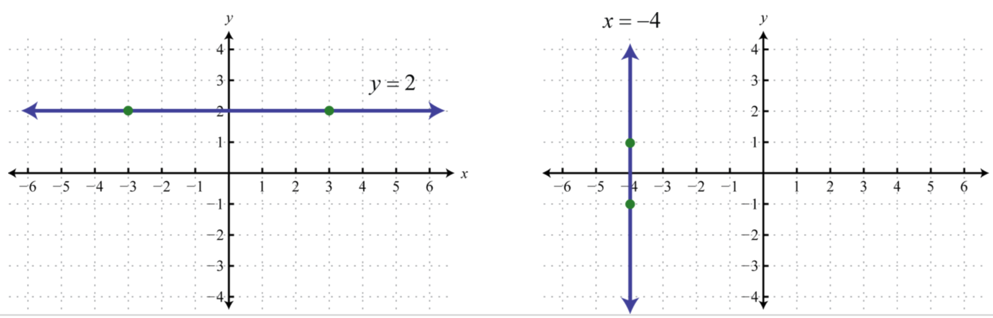
There are four geometric cases for the value of the slope.



Reading the graph from left to right, lines with an upward incline have positive slopes and lines with a downward incline have negative slopes. The other two cases involve horizontal and vertical lines. Recall that if k is a real number we have

$$y = k \quad \textit{Horizontal Line}$$
$$x = k \quad \textit{Vertical Line}$$

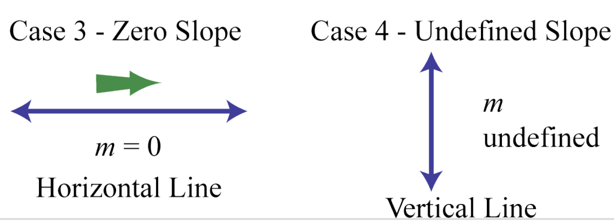
For example, if we graph $y = 2$ we obtain a horizontal line, and if we graph $x = -4$ we obtain a vertical line.



From the graphs we can determine two points and calculate the slope using the slope formula.

Horizontal Line	*Vertical Line*
$(x_1, y_1) \quad (x_2, y_2)$ $(-3, 2) \quad (3, 2)$ $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-(2)}{3-(-3)} = \frac{2-2}{3+3} = \frac{0}{6} = 0$	$(x_1, y_1) \quad (x_2, y_2)$ $(-4, -1) \quad (-4, 1)$ $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-(-1)}{-4-(-4)} = \frac{1+1}{-4+4} = \frac{2}{0}$ *Undefined*

Notice that the points on the horizontal line share the same y-values. Therefore, the rise is zero and hence the slope is zero. The points on the vertical line share the same x-values. Consequently, the run is zero, leading to an undefined slope. In general,



Linear Functions

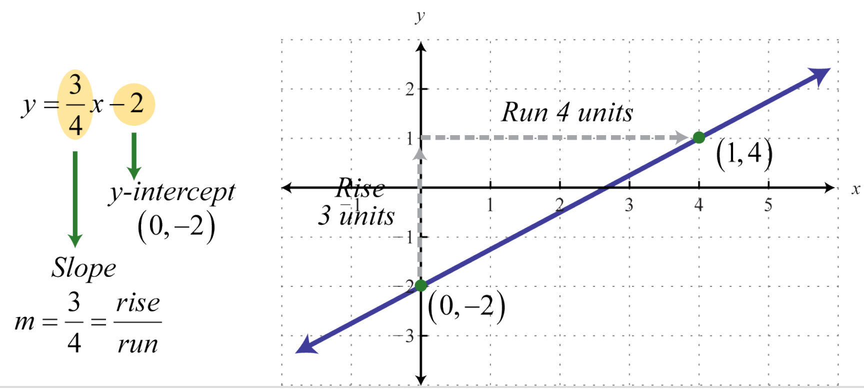
Given any linear equation in **standard form**[25], $ax + by = c$, we can solve for y to obtain **slope-intercept form**[26], $y = mx + b$. For example,

25. Any nonvertical line can be written in the standard form $ax + by = c$.

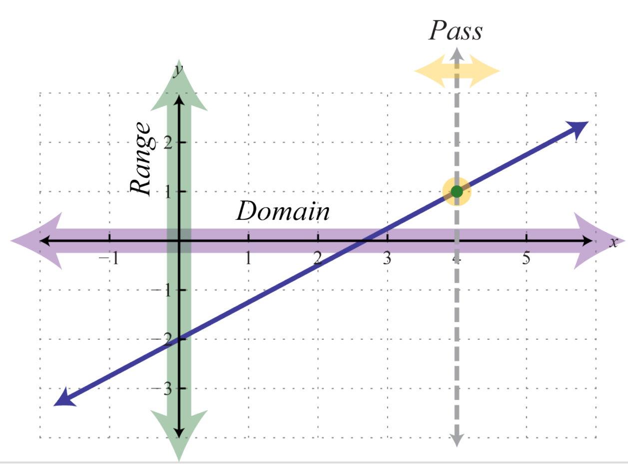
26. Any nonvertical line can be written in the form $y = mx + b$, where m is the slope and $(0, b)$ is the y-intercept.

$$\begin{aligned} 3x - 4y &= 8 && \leftarrow \textit{Standard Form} \\ -4y &= -3x + 8 \\ y &= \frac{-3x+8}{-4} \\ y &= \frac{-3x}{-4} + \frac{8}{-4} \\ y &= \frac{3}{4}x - 2 && \leftarrow \textit{Slope-Intercept Form} \end{aligned}$$

Where $x = 0$, we can see that $y = -2$ and thus $(0, -2)$ is an ordered pair solution. This is the point where the graph intersects the y-axis and is called the ***y*-intercept**[27]. We can use this point and the slope as a means to quickly graph a line. For example, to graph $y = \frac{3}{4}x - 2$, start at the y-intercept $(0, -2)$ and mark off the slope to find a second point. Then use these points to graph the line as follows:



The vertical line test indicates that this graph represents a function. Furthermore, the domain and range consists of all real numbers.



27. The point (or points) where a graph intersects the y-axis, expressed as an ordered pair $(0, y)$.

In general, a **linear function**[28] is a function that can be written in the form

$$f(x) = mx + b \textit{ Linear Function}$$

where the slope *m* and *b* represent any real numbers. Because $y = f(x)$, we can use y and $f(x)$ interchangeably, and ordered pair solutions on the graph (x, y) can be written in the form $(x, f(x))$.

$$(x, y) \quad \Leftrightarrow \quad (x, f(x))$$

We know that any *y*-intercept will have an *x*-value equal to zero. Therefore, the *y*-intercept can be expressed as the ordered pair $(0, f(0))$. For linear functions,

$$\begin{aligned} f(0) &= m(0) + b \\ &= b \end{aligned}$$

Hence, the *y*-intercept of any linear function is $(0, b)$. To find the ***x*-intercept**[29], the point where the function intersects the *x*-axis, we find *x* where $y = 0$ or $f(x) = 0$.

28. Any function that can be written in the form $f(x) = mx + b$

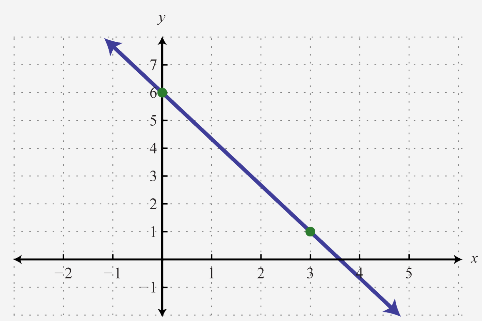
29. The point (or points) where a graph intersects the *x*-axis, expressed as an ordered pair $(x, 0)$.

Example 3

Graph the linear function $f(x) = -\frac{5}{3}x + 6$ and label the x-intercept.

Solution:

From the function, we see that $f(0) = 6$ (or $b = 6$) and thus the y-intercept is $(0, 6)$. Also, we can see that the slope $m = -\frac{5}{3} = \frac{-5}{3} = \frac{rise}{run}$ Starting from the y-intercept, mark a second point down 5 units and right 3 units. Draw the line passing through these two points with a straightedge.

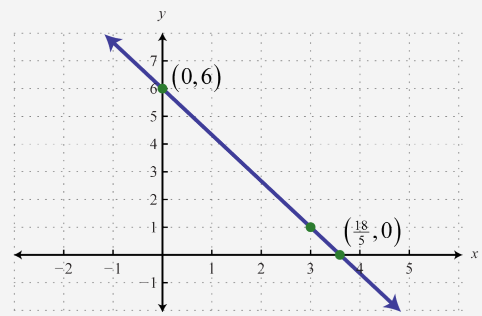


To determine the x-intercept, find the x-value where the function is equal to zero. In other words, determine x where $f(x) = 0$.

$$
\begin{aligned}
f(x) &= -\frac{5}{3}x + 6 \\
0 &= -\frac{5}{3}x + 6 \\
\frac{5}{3}x &= 6 \\
\left(\frac{3}{5}\right)\frac{5}{3}x &= \left(\frac{3}{5}\right)6 \\
x &= \frac{18}{5} = 3\frac{3}{5}
\end{aligned}
$$

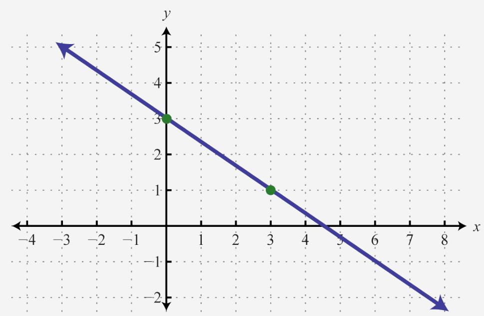
Therefore, the x-intercept is $\left(\frac{18}{5}, 0\right)$. The general rule is to label all important points that cannot be clearly read from the graph.

Answer:



Example 4

Determine a linear function that defines the given graph and find the *x*-intercept.



Solution:

We begin by reading the slope from the graph. In this case, two points are given and we can see that,

$$m = \frac{rise}{run} = \frac{-2}{3}$$

In addition, the *y*-intercept is (0, 3) and thus $b = 3$. We can substitute into the equation for any linear function.

$$g(x) = mx + b$$

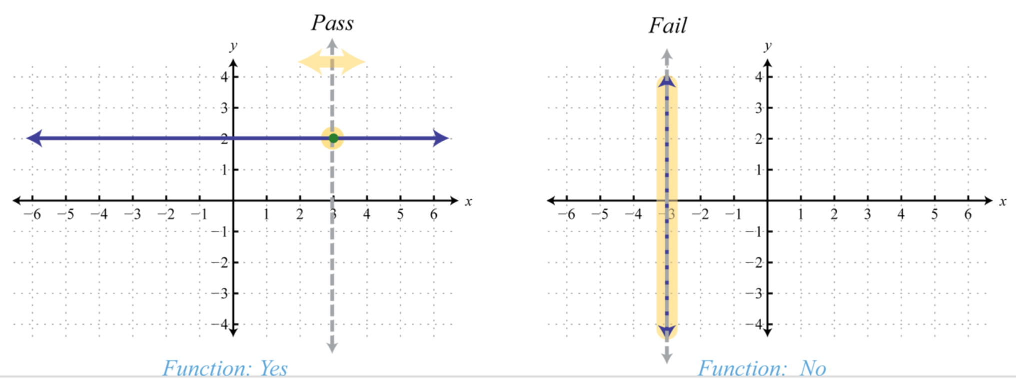
$$g(x) = -\frac{2}{3}x + 3$$

To find the *x*-intercept, we set $g(x) = 0$ and solve for *x*.

$$g(x) = -\frac{2}{3}x + 3$$
$$0 = -\frac{2}{3}x + 3$$
$$\frac{2}{3}x = 3$$
$$\left(\frac{3}{2}\right)\frac{2}{3}x = \left(\frac{3}{2}\right)3$$
$$x = \frac{9}{2} = 4\frac{1}{2}$$

Answer: $g(x) = -\frac{2}{3}x + 3$; x-intercept: $\left(\frac{9}{2}, 0\right)$

Next, consider horizontal and vertical lines. Use the vertical line test to see that any horizontal line represents a function, and that a vertical line does not.



Given any horizontal line, the vertical line test shows that every x-value in the domain corresponds to exactly one y-value in the range; it is a function. A vertical line, on the other hand, fails the vertical line test; it is not a function. A vertical line represents a set of ordered pairs where all of the elements in the domain are the same. This violates the requirement that functions must associate exactly one element in the range to each element in the domain. We summarize as follows:

	Horizontal Line	*Vertical Line*
Equation:	$y = 2$	$x = -3$
x-intercept:	None	$(-3, 0)$
y-intercept:	$(0, 2)$	None
Domain:	$(-\infty, \infty)$	$\{-3\}$
Range:	$\{2\}$	$(-\infty, \infty)$
Function:	Yes	No

A horizontal line is often called a **constant function**. Given any real number c,

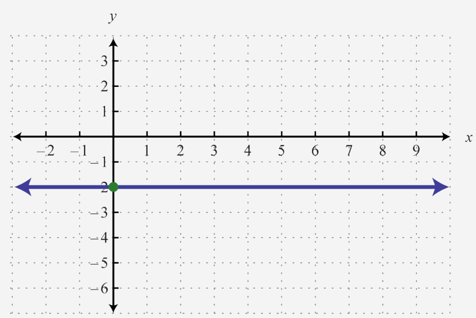
$$f(x) = c \quad \textit{Constant Function}$$

Example 5

Graph the constant function $g\,(x) = -2$ and state the domain and range.

Solution:

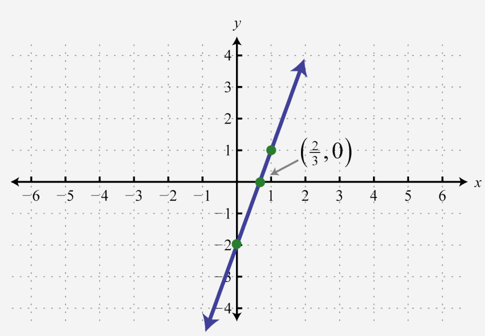
Here we are given a constant function that is equivalent to $y = -2$. This defines a horizontal line through $(0, -2)$.



Answer: Domain: $\mathbb{R}$; range: $\{-2\}$

Try this! Graph $f\,(x) = 3x - 2$ and label the x-intercept.

Answer:



(click to see video)

Linear Equations and Inequalities: A Graphical Interpretation

We can use the ideas in this section to develop a geometric understanding of what it means to solve equations of the form $f(x) = g(x)$, where f and g are linear functions. Using algebra, we can solve the linear equation $\frac{1}{2}x + 1 = 3$ as follows:

$$\begin{aligned} \frac{1}{2}x + 1 &= 3 \\ \frac{1}{2}x &= 2 \\ (2)\frac{1}{2}x &= (2)2 \\ x &= 4 \end{aligned}$$

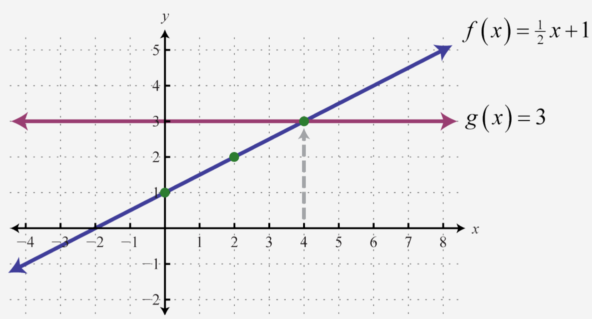
The solution to this equation is $x = 4$. Geometrically, this is the x-value of the intersection of the two graphs $f(x) = \frac{1}{2}x + 1$ and $g(x) = 3$. The idea is to graph the linear functions on either side of the equation and determine where the graphs coincide.

Example 6

Graph $f(x) = \frac{1}{2}x + 1$ and $g(x) = 3$ on the same set of axes and determine where $f(x) = g(x)$.

Solution:

Here f is a linear function with slope $\frac{1}{2}$ and y-intercept (0,1). The function g is a constant function and represents a horizontal line. Graph both of these functions on the same set of axes.



From the graph we can see that $f(x) = g(x)$ where $x = 4$. In other words, $\frac{1}{2}x + 1 = 3$ where $x = 4$.

Answer: $x = 4$

We can extend the geometric interpretation a bit further to solve inequalities. For example, we can solve the linear inequality $\frac{1}{2}x + 1 \geq 3$, using algebra, as follows:

$$\frac{1}{2}x + 1 \geq 3$$
$$\frac{1}{2}x \geq 2$$
$$(2)\frac{1}{2}x \geq (2)\,2$$
$$x \geq 4$$

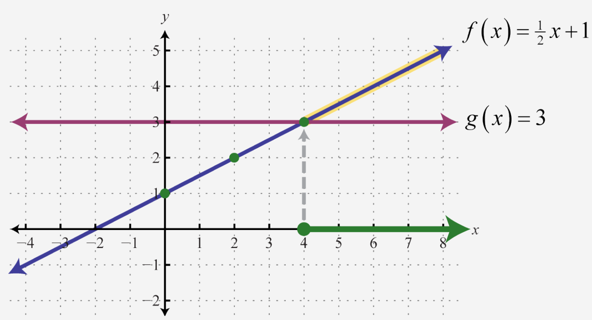
The solution set consists of all real numbers greater than or equal to 4. Geometrically, these are the *x*-values for which the graph $f(x) = \frac{1}{2}x + 1$ lies above the graph of $g(x) = 3$.

Example 7

Graph $f(x) = \frac{1}{2}x + 1$ and $g(x) = 3$ on the same set of axes and determine where $f(x) \geq g(x)$.

Solution:

On the graph we can see this shaded.



From the graph we can see that $f(x) \geq g(x)$ or $\frac{1}{2}x + 1 \geq 3$ where $x \geq 4$.

Answer: The *x*-values that solve the inequality, in interval notation, are $[4, \infty)$.

KEY TAKEAWAYS

- We can graph lines by plotting points. Choose a few values for x, find the corresponding y-values, and then plot the resulting ordered pair solutions. Draw a line through the points with a straightedge to complete the graph.
- Given any two points on a line, we can calculate the slope algebraically using the slope formula, $m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$.
- Use slope-intercept form $y = mx + b$ to quickly sketch the graph of a line. From the y-intercept $(0, b)$, mark off the slope to determine a second point. Since two points determine a line, draw a line through these two points with a straightedge to complete the graph.
- Linear functions have the form $f(x) = mx + b$, where the slope m and b are real numbers. To find the x-intercept, if one exists, set $f(x) = 0$ and solve for x.
- Since $y = f(x)$ we can use y and $f(x)$ interchangeably. Any point on the graph of a function can be expressed using function notation $(x, f(x))$.

TOPIC EXERCISES

PART A: GRAPHING LINES BY PLOTTING POINTS

Find five ordered pair solutions and graph.

1. $y = 3x - 6$
2. $y = 2x - 4$
3. $y = -5x + 15$
4. $y = -3x + 18$
5. $y = \frac{1}{2}x + 8$
6. $y = \frac{2}{3}x + 2$
7. $y = -\frac{3}{5}x + 1$
8. $y = -\frac{3}{2}x + 4$
9. $y = \frac{1}{4}x$
10. $y = -\frac{2}{5}x$
11. $y = 10$
12. $x = -1$
13. $6x + 3y = 18$
14. $8x - 2y = 16$
15. $-2x + 4y = 8$
16. $-x + 3y = 18$
17. $\frac{1}{2}x - \frac{1}{5}y = 1$
18. $\frac{1}{6}x - \frac{2}{3}y = 2$
19. $x + y = 0$
20. $-x + y = 0$

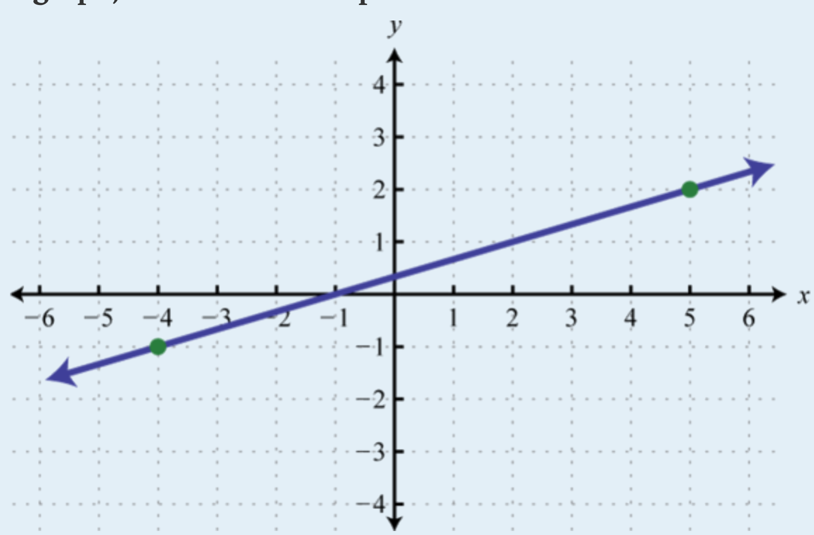
Find the slope of the line passing through the given points.

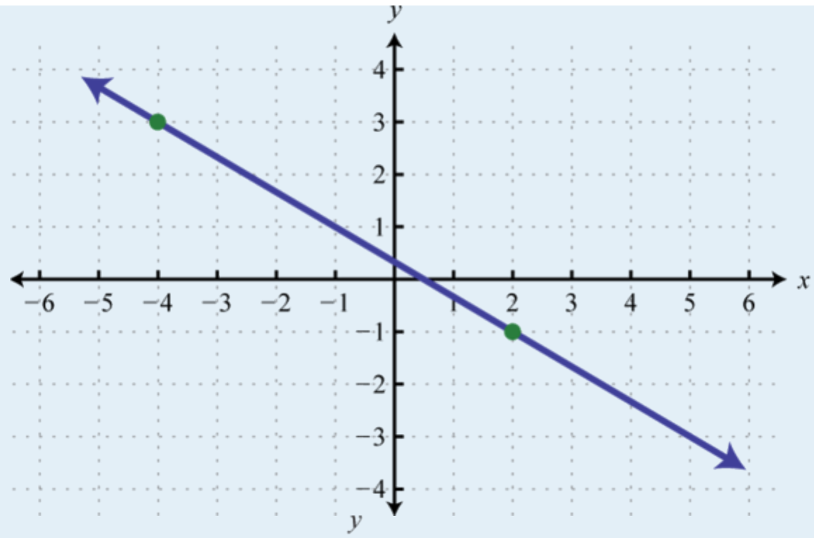
21. $(-2, -4)$ and $(1, -1)$
22. $(-3, 0)$ and $(3, -4)$
23. $\left(-\frac{5}{2}, \frac{1}{4}\right)$ and $\left(-\frac{1}{2}, \frac{5}{4}\right)$
24. $(-4, -3)$ and $(-2, -3)$
25. $(9, -5)$ and $(9, -6)$
26. $\left(\frac{1}{2}, -1\right)$ and $\left(-1, -\frac{3}{2}\right)$

Find the *y*-value for which the slope of the line passing through given points has the given slope.

27. $m = \frac{3}{2}$; $(6, 10), (-4, y)$
28. $m = -\frac{1}{3}$; $(-6, 4), (9, y)$
29. $m = -4$; $(-2, 5), (-1, y)$
30. $m = 3$; $(1, -2), (-2, y)$
31. $m = \frac{1}{5}$; $(1, y), \left(6, \frac{1}{5}\right)$
32. $m = -\frac{3}{4}$; $(-1, y), (-4, 5)$

Given the graph, determine the slope.

33. 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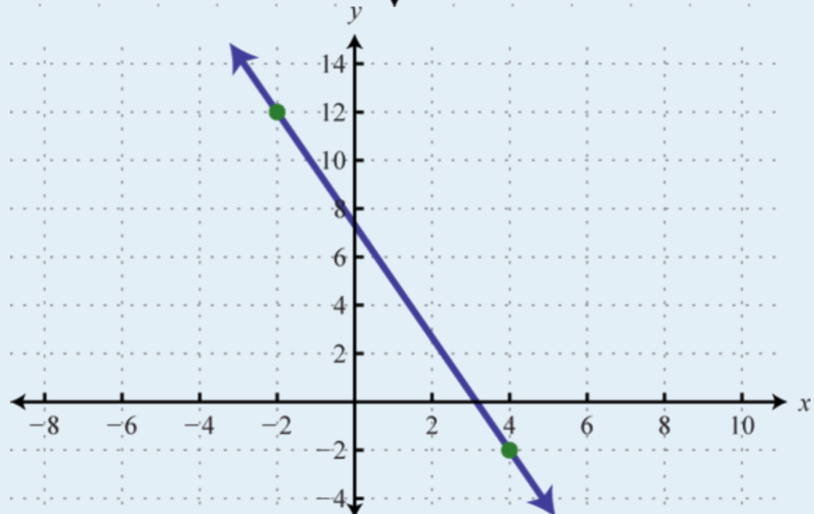
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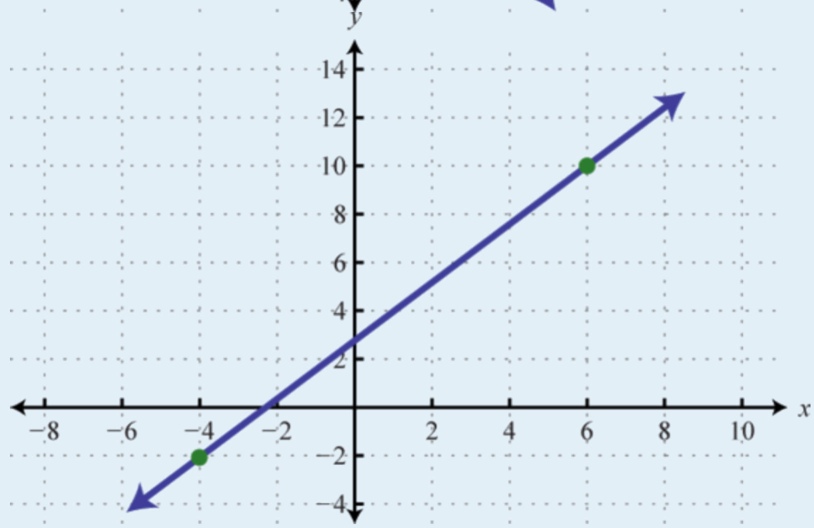
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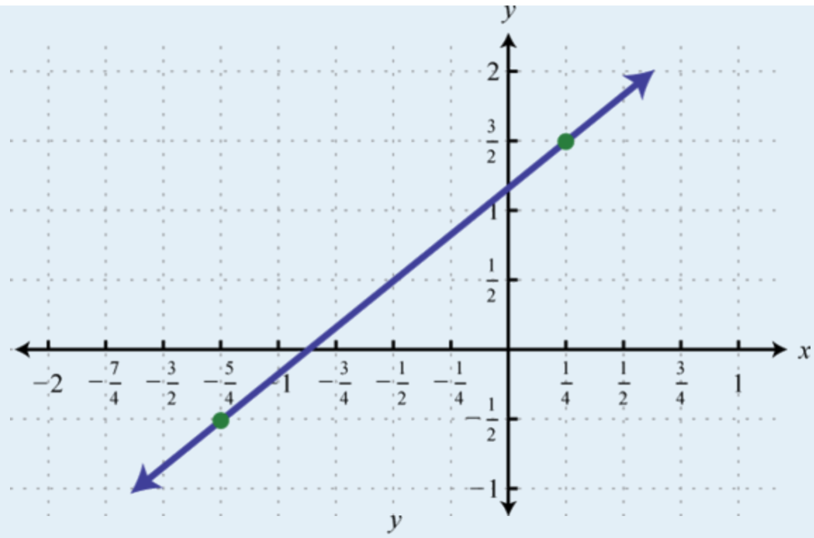
37. 
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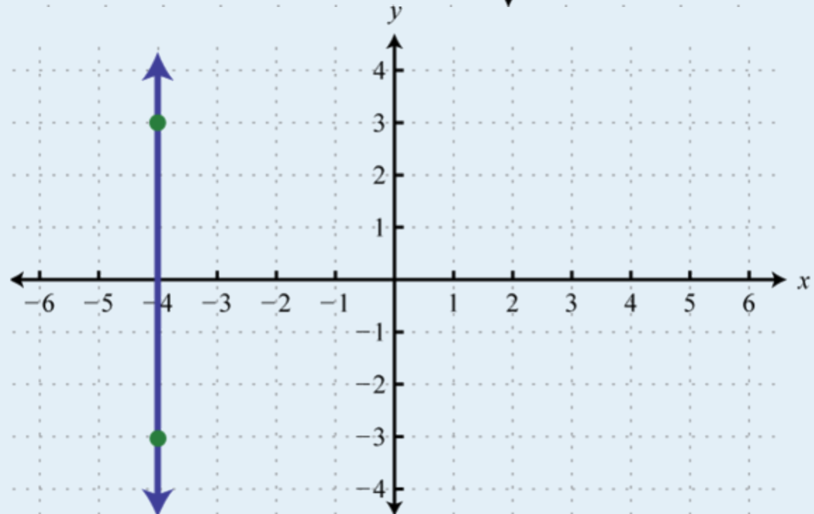
38. 
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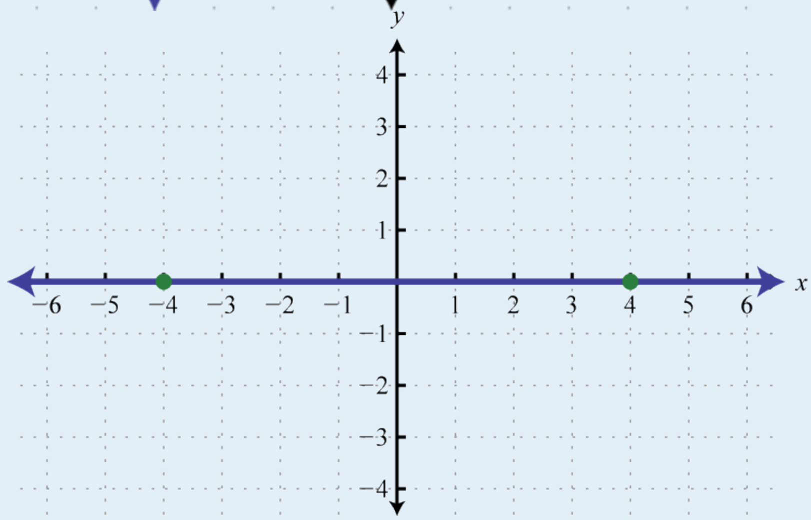
39. 
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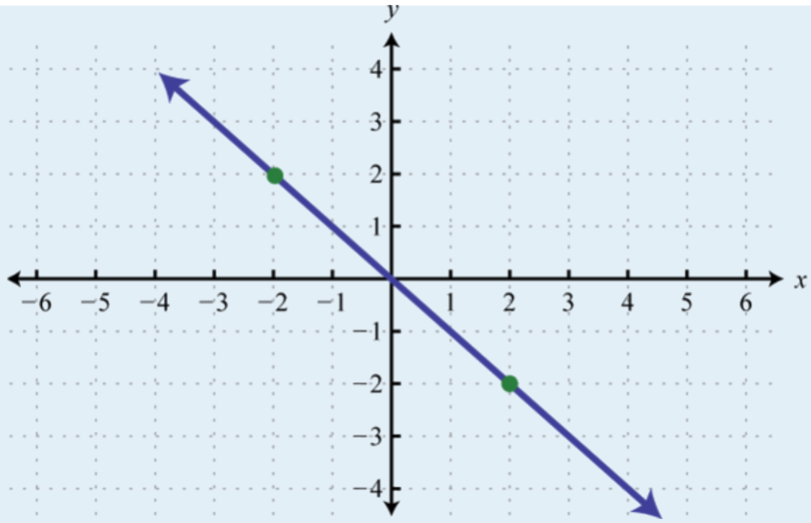
40. 

PART B: LINEAR FUNCTIONS

Find the *x*- and *y*-intercepts and use them to graph the following functions.

41. $6x - 3y = 18$
42. $8x - 2y = 8$
43. $-x + 12y = 6$
44. $-2x - 6y = 8$
45. $x - 2y = 5$
46. $-x + 3y = 1$
47. $2x + 3y = 2$
48. $5x - 4y = 2$
49. $9x - 4y = 30$
50. $-8x + 3y = 28$
51. $\frac{1}{3}x + \frac{1}{2}y = -3$
52. $\frac{1}{4}x - \frac{1}{3}y = 3$
53. $\frac{7}{9}x - \frac{2}{3}y = \frac{14}{3}$
54. $\frac{1}{8}x - \frac{1}{6}y = -\frac{3}{2}$

55. $-\frac{1}{6}x + \frac{2}{9}y = \frac{4}{3}$

56. $\frac{2}{15}x + \frac{1}{6}y = \frac{4}{3}$

57. $y = -\frac{1}{4}x + \frac{1}{2}$

58. $y = \frac{3}{8}x - \frac{3}{2}$

59. $y = \frac{2}{3}x + \frac{1}{2}$

60. $y = \frac{4}{5}x + 1$

Graph the linear function and label the *x*-intercept.

61. $f(x) = -5x + 15$

62. $f(x) = -2x + 6$

63. $f(x) = -x - 2$

64. $f(x) = x + 3$

65. $f(x) = \frac{1}{3}x + 2$

66. $f(x) = \frac{5}{2}x + 10$

67. $f(x) = \frac{5}{3}x + 2$

68. $f(x) = \frac{2}{5}x - 3$

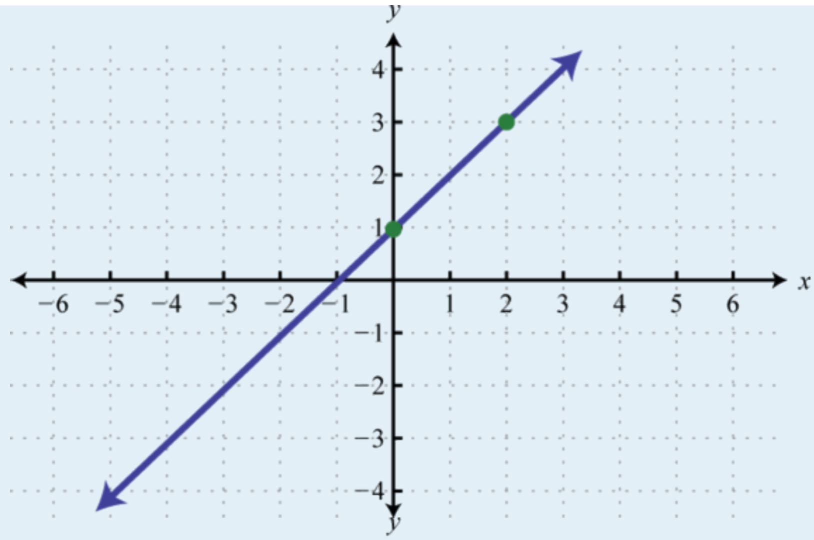
69. $f(x) = -\frac{5}{6}x + 2$

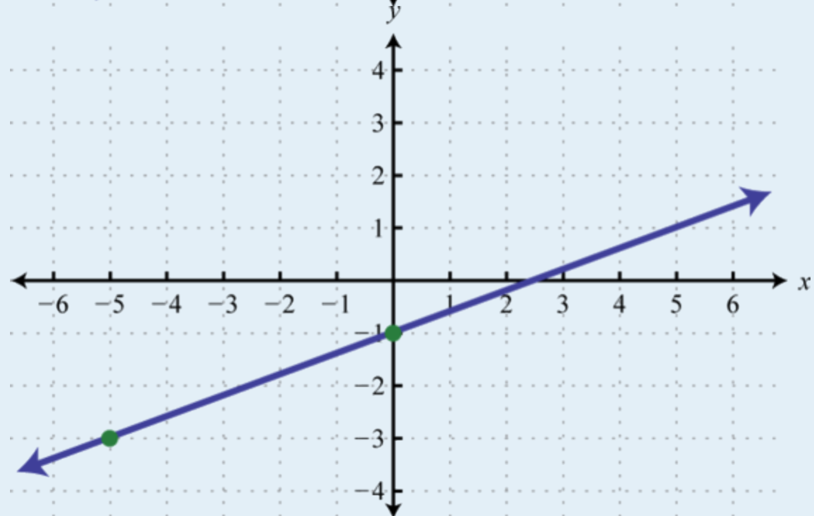
70. $f(x) = -\frac{4}{3}x + 3$

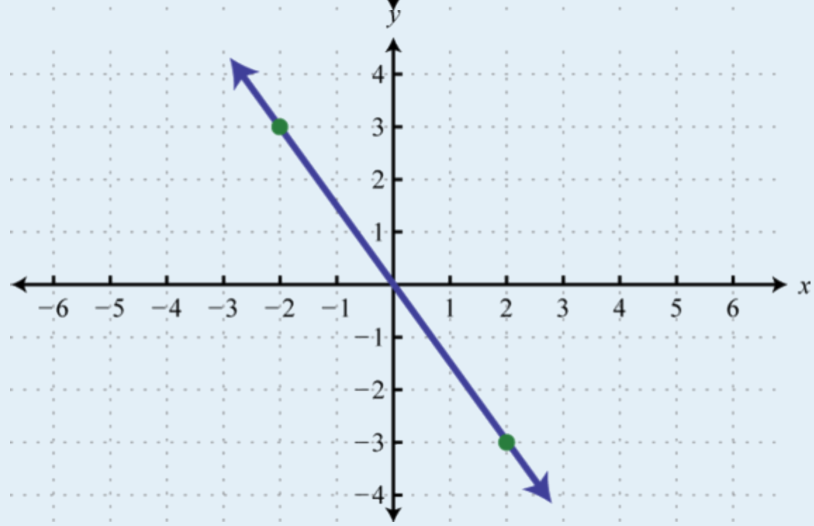
71. $f(x) = 2x$

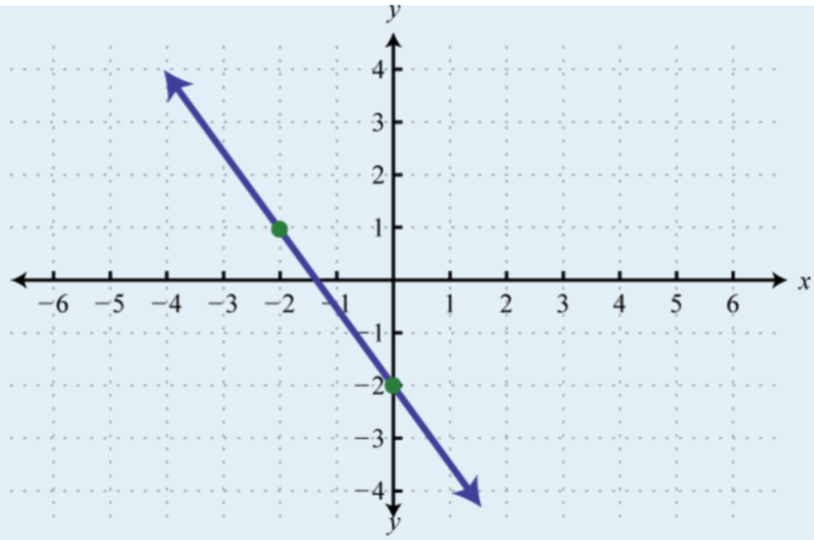
72. $f(x) = 3$

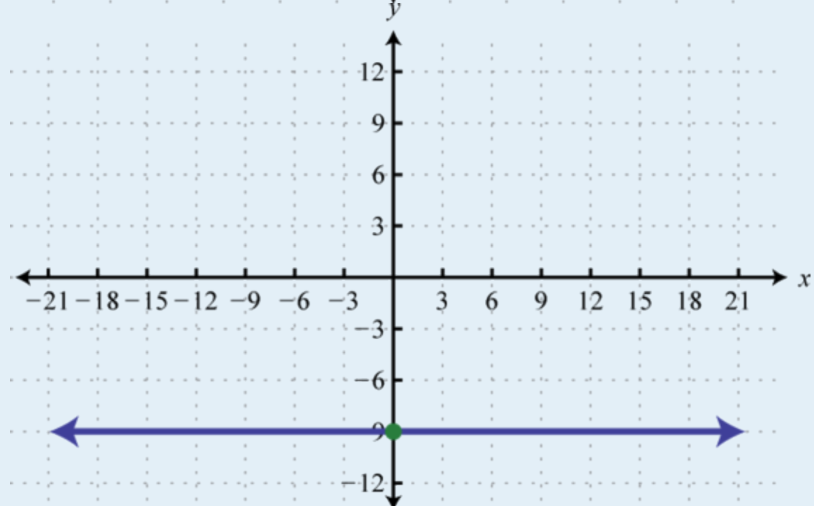
Determine the linear function that defines the given graph and find the *x*-intercept.

73. 
y
x
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6
4 3 2 1 −1 −2 −3 −4

74. 
y
x
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6
4 3 2 1 −1 −2 −3 −4

75. 
y
x
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6
4 3 2 1 −1 −2 −3 −4

76.

y
x
4
3
2
1
−1
−2
−3
−4
−6
−5
−4
−3
−2
1
2
3
4
5
6

77.

y
x
12
9
6
3
−3
−6
−12
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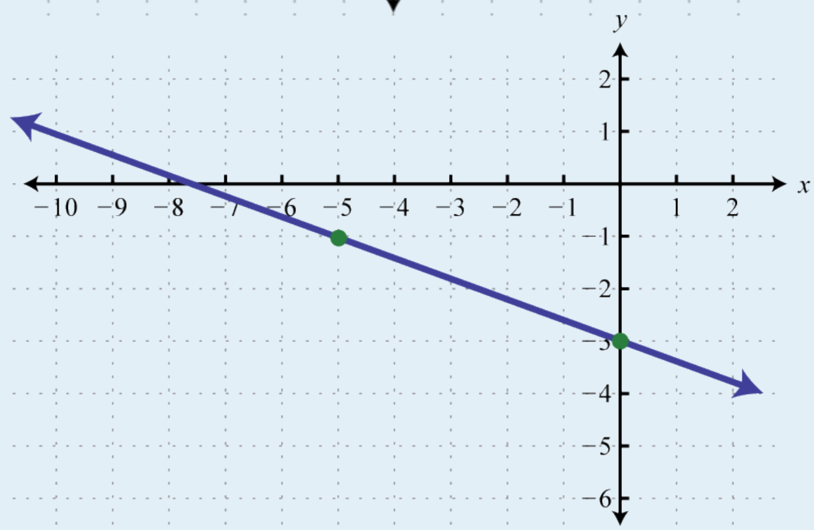
−18
−15
−12
−9
−6
−3
3
6
9
12
15
18
21

78.

y
x
2
1
−1
−2
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−5
−6
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−9
−8
−5
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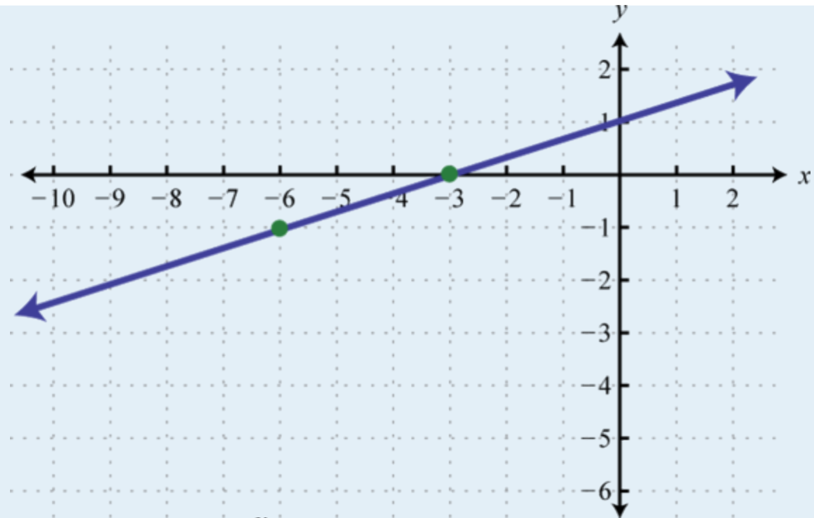
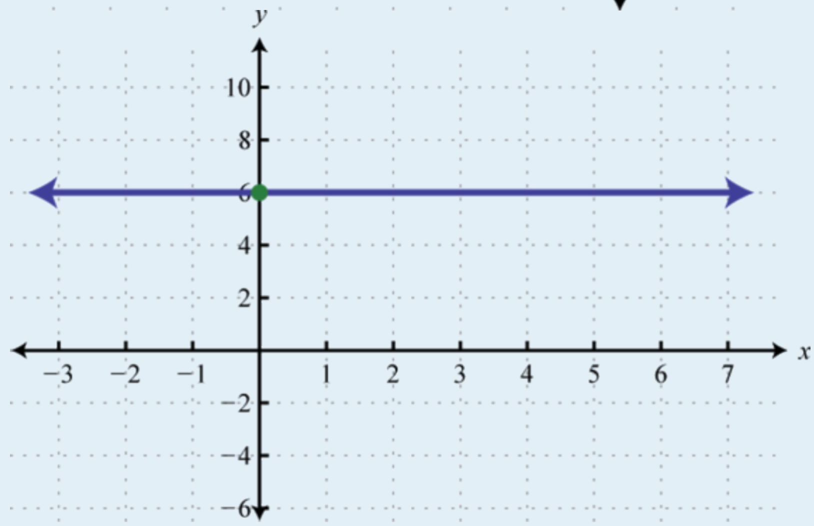
−3
−2
−1
1
2

79. 

80. 

PART C: A GRAPHICAL INTERPRETATION OF LINEAR EQUATIONS AND INEQUALITIES

Graph the functions f and g on the same set of axes and determine where $f(x) = g(x)$. Verify your answer algebraically.

81. $f(x) = \frac{1}{2}x - 3, g(x) = 1$

82. $f(x) = \frac{1}{3}x + 2, g(x) = -1$

83. $f(x) = 3x - 2, g(x) = -5$

84. $f(x) = x + 2, g(x) = -3$

85. $f(x) = -\frac{2}{3}x + 4, g(x) = 2$

86. $f(x) = -\frac{5}{2}x + 6, g(x) = 1$
87. $f(x) = 3x - 2, g(x) = -2x + 3$
88. $f(x) = -x + 6, g(x) = x + 2$
89. $f(x) = -\frac{1}{3}x, g(x) = -\frac{2}{3}x + 1$
90. $f(x) = \frac{2}{3}x - 1, g(x) = -\frac{4}{3}x - 3$

Graph the functions f and g on the same set of axes and determine where $f(x) \geq g(x)$. Verify your answer algebraically.

91. $f(x) = 3x + 7, g(x) = 1$
92. $f(x) = 5x - 3, g(x) = 2$
93. $f(x) = \frac{2}{3}x - 3, g(x) = -3$
94. $f(x) = \frac{3}{4}x + 2, g(x) = -1$
95. $f(x) = -x + 1, g(x) = -3$
96. $f(x) = -4x + 4, g(x) = 8$
97. $f(x) = x - 2, g(x) = -x + 4$
98. $f(x) = 4x - 5, g(x) = x + 1$

Graph the functions f and g on the same set of axes and determine where $f(x) < g(x)$. Verify your answer algebraically.

99. $f(x) = x + 5, g(x) = -1$
100. $f(x) = 3x - 3, g(x) = 6$
101. $f(x) = -\frac{4}{5}x, g(x) = -8$
102. $f(x) = -\frac{3}{2}x + 6, g(x) = -3$
103. $f(x) = \frac{1}{4}x + 1, g(x) = 0$
104. $f(x) = \frac{3}{5}x - 6, g(x) = 0$
105. $f(x) = \frac{1}{3}x + 2, g(x) = -\frac{1}{3}x$

106. $f(x) = \frac{3}{2}x + 3, g(x) = -\frac{3}{2}x - 3$

PART D: DISCUSSION BOARD

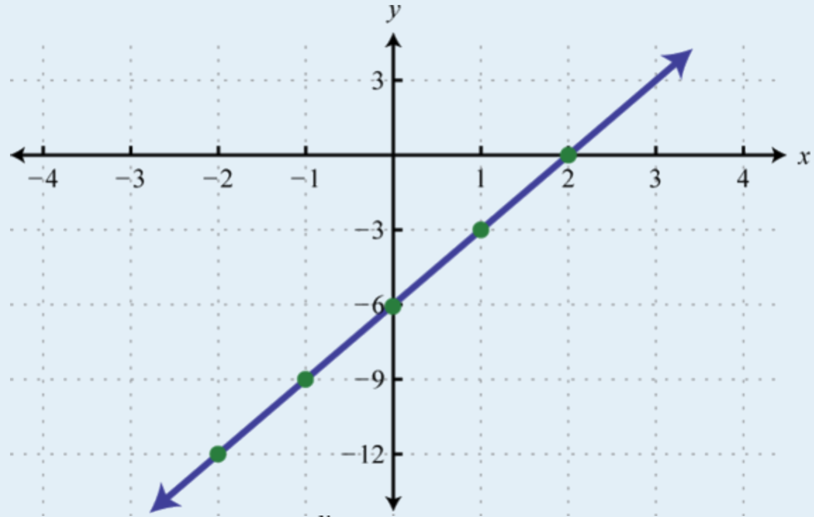
107. Do all linear functions have y-intercepts? Do all linear functions have x-intercepts? Explain.

108. Can a function have more than one y-intercept? Explain.

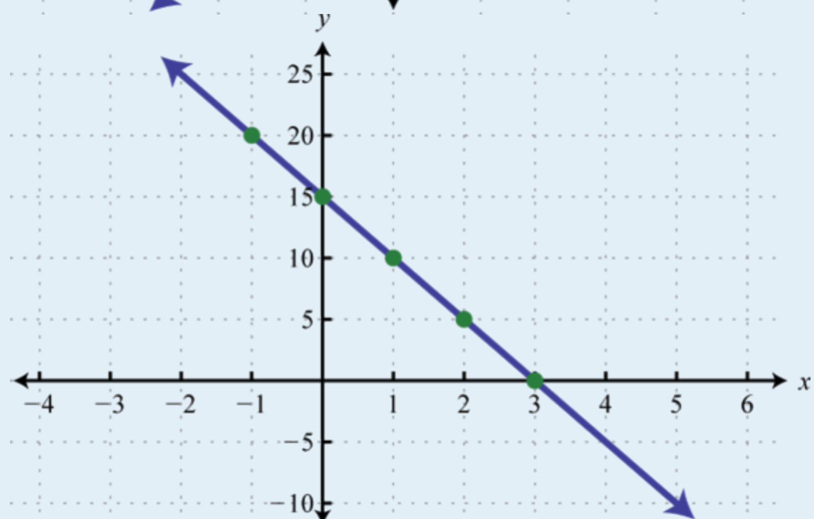
109. How does the vertical line test show that a vertical line is not a function?

ANSWERS

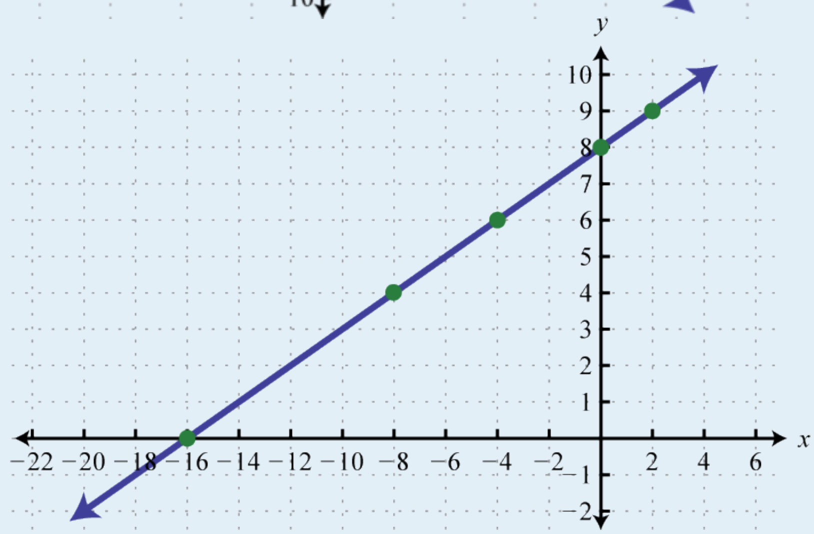
1.


y
x
3
−3
−6
−9
−12
−4 −3 −2 −1 1 2 3 4

3.


y
x
25
20
15
10
5
−5
−10
−4 −3 −2 −1 1 2 3 4 5 6

5.


y
x
10
9
8
7
6
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4
3
2
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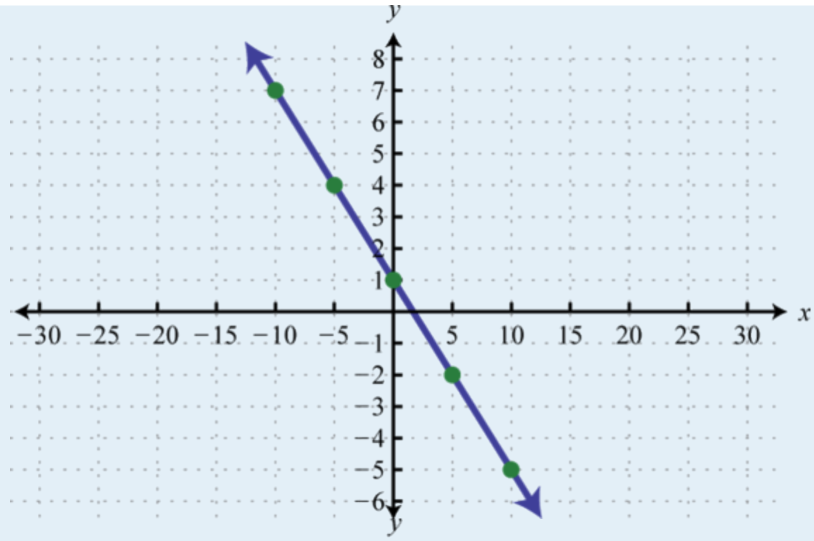
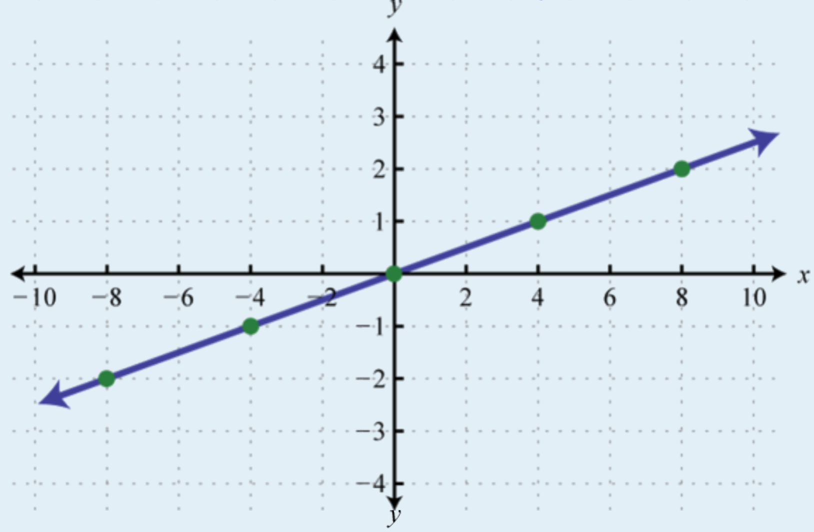
−2
−22 −20 −18 −16 −14 −12 −10 −8 −6 −4 −2 2 4 6

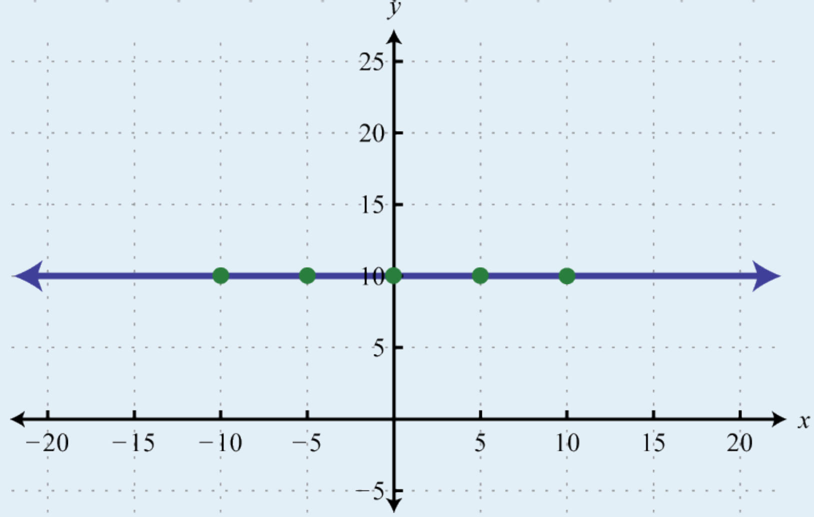
7.


y
8
7
6
5
4
3
2
1
x
−30 −25 −20 −15 −10 −5 5 10 15 20 25 30
−1
−2
−3
−4
−5
−6

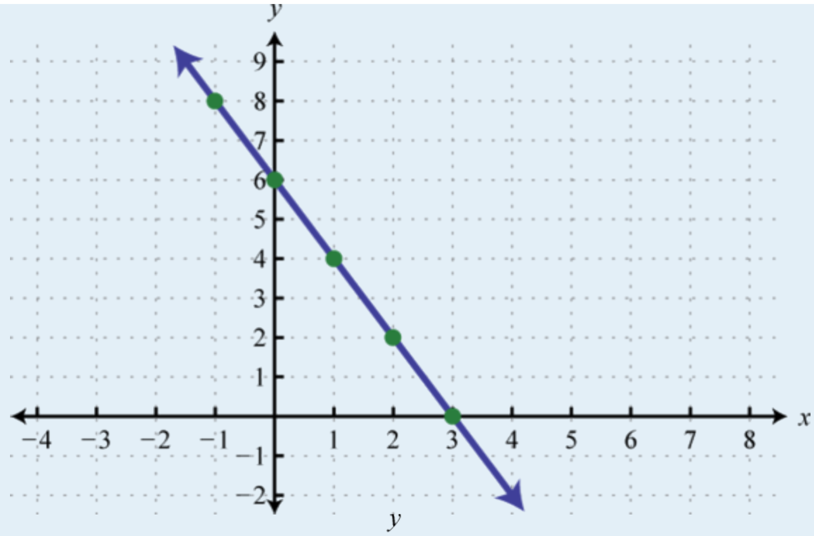
9.


y
4
3
2
1
x
−10 −8 −6 −4 −2 2 4 6 8 10
−1
−2
−3
−4

11.


y
25
20
15
10
5
x
−20 −15 −10 −5 5 10 15 20
−5

13.


y
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8
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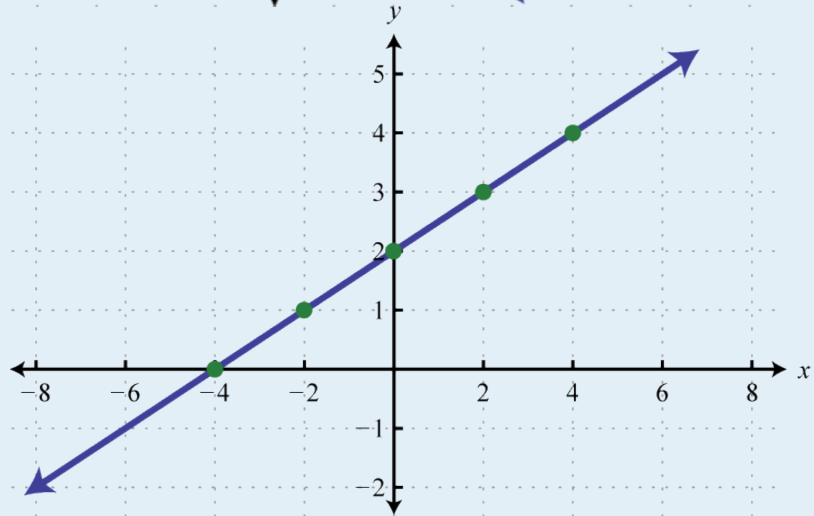
6
5
4
3
2
1
x
−4 −3 −2 −1 1 2 3 4 5 6 7 8
−1
−2

15.


y
5
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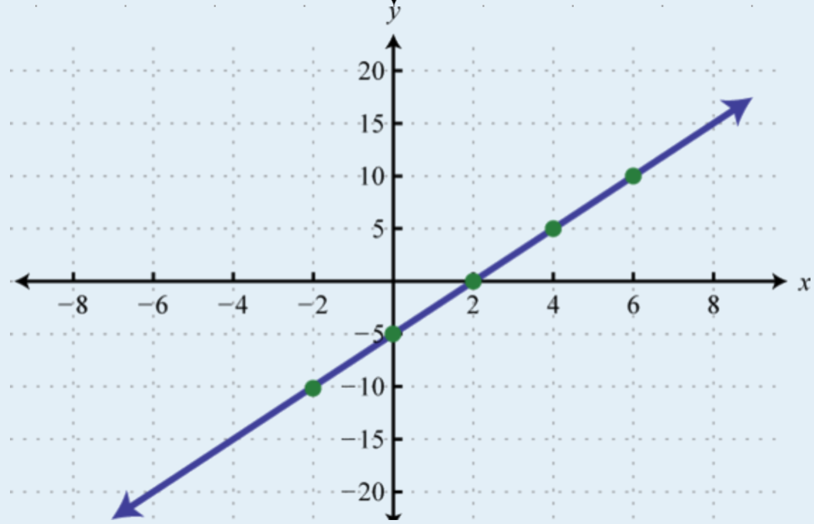
2
1
x
−8 −6 −4 −2 2 4 6 8
−1
−2

17.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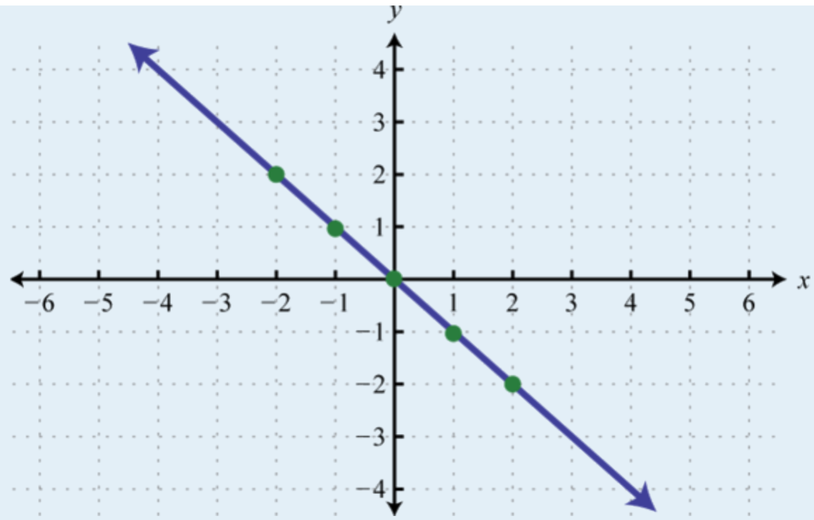
20
15
10
5
x
−8 −6 −4 −2 2 4 6 8
−5
−10
−15
−20

19. 

21. 1

23. $\frac{1}{2}$

25. Undefined

27. $y = -5$

29. $y = 1$

31. $y = -\frac{4}{5}$

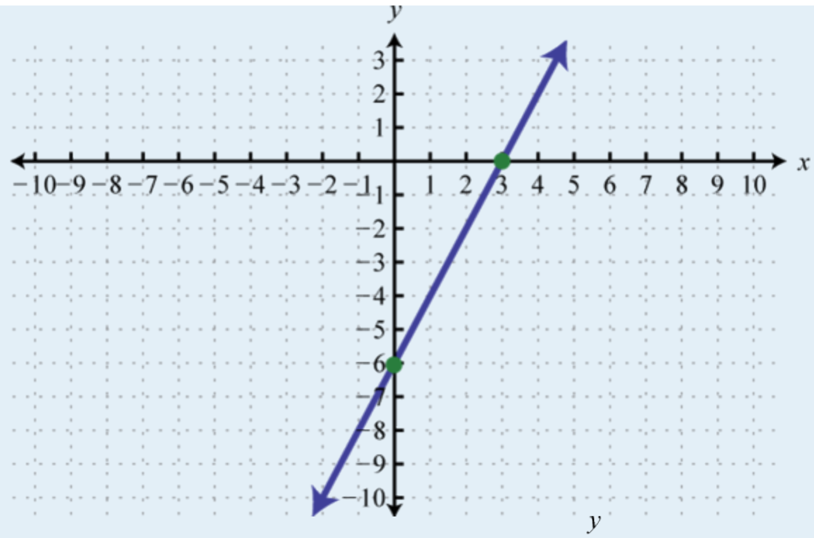
33. $m = \frac{1}{3}$

35. $m = -\frac{7}{3}$

37. $m = \frac{4}{3}$

39. $m = 0$

41.



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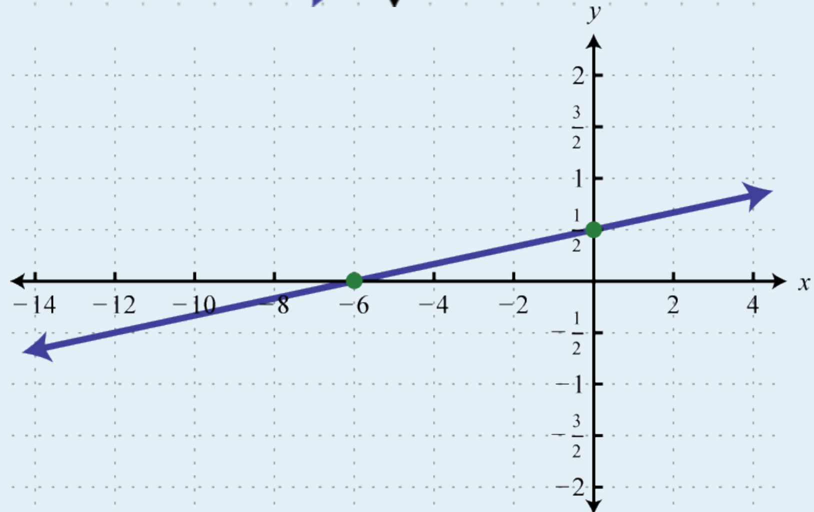

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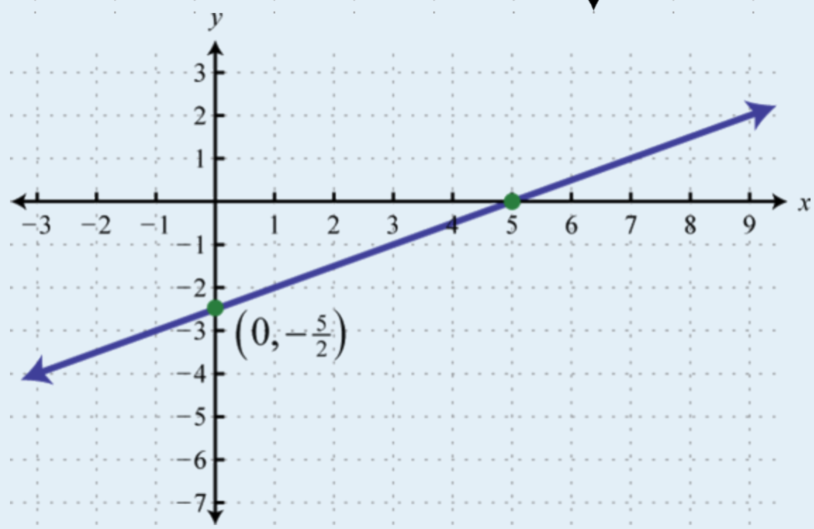
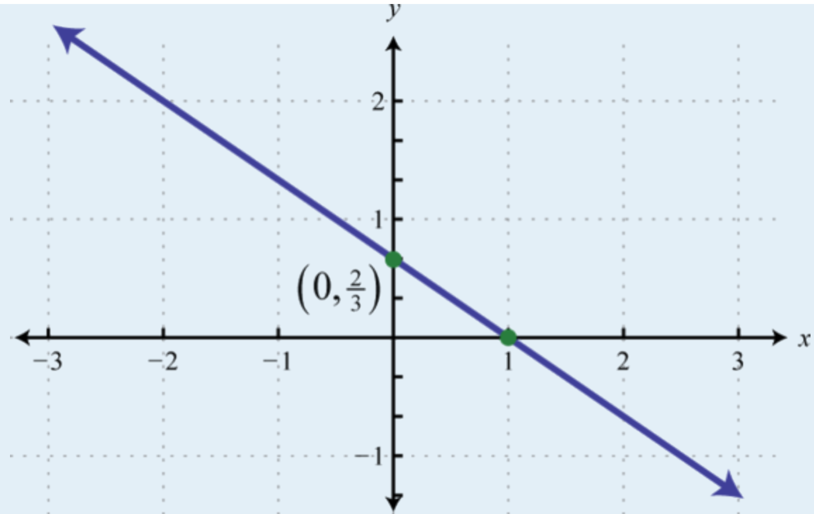
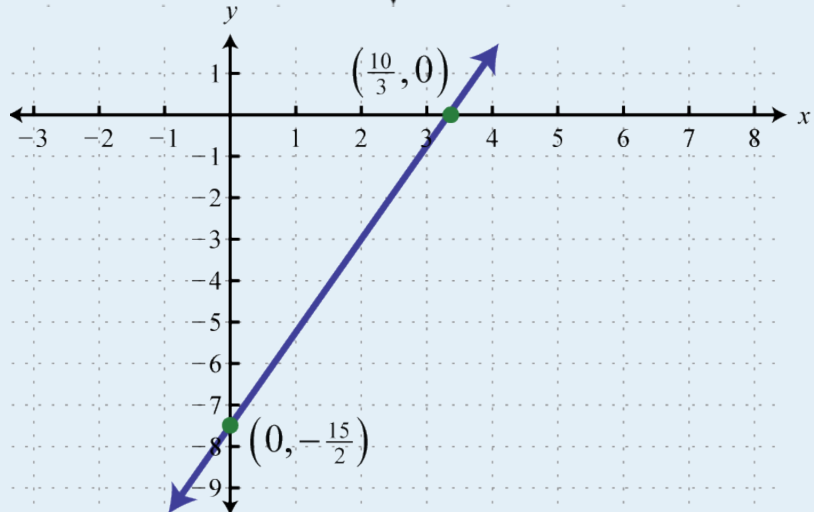
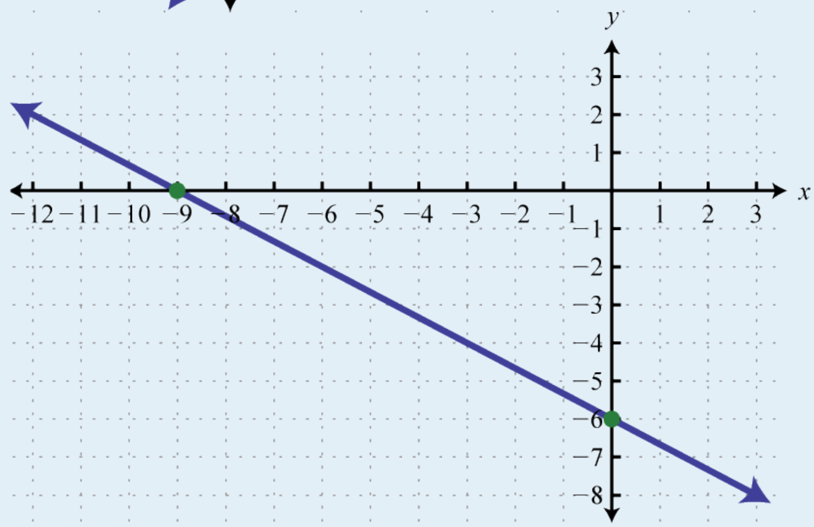


47. 

49. 

51. 

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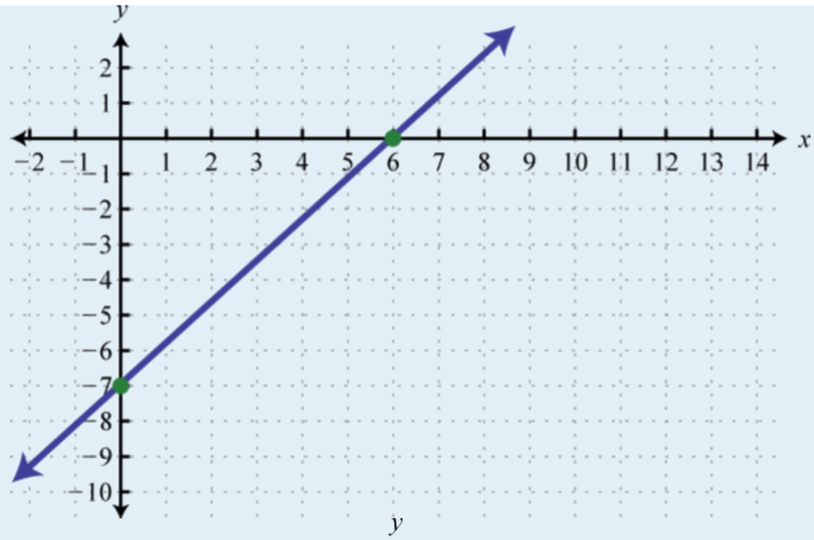

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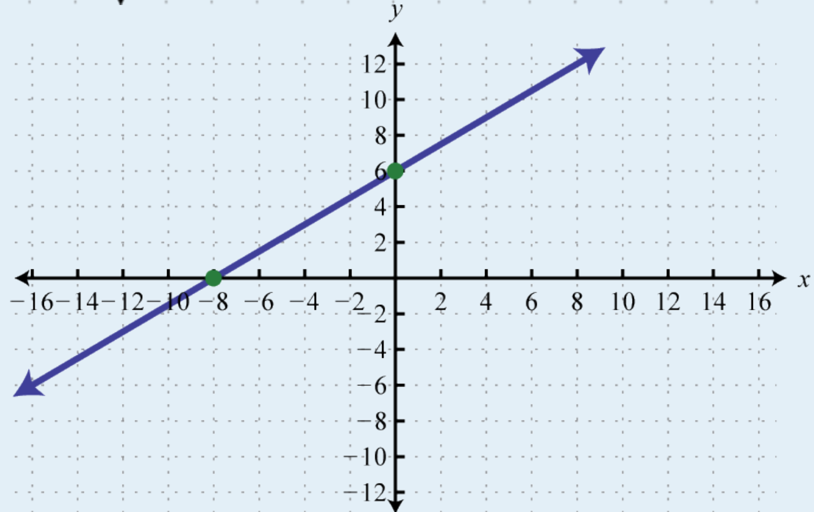

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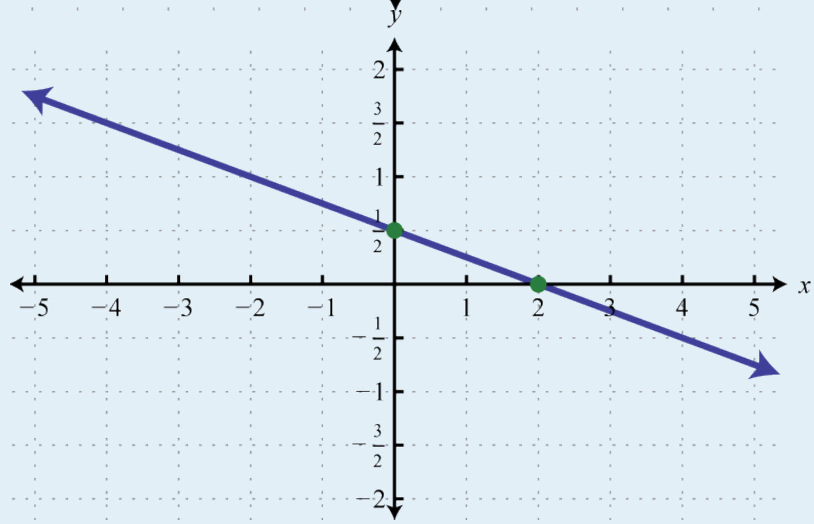

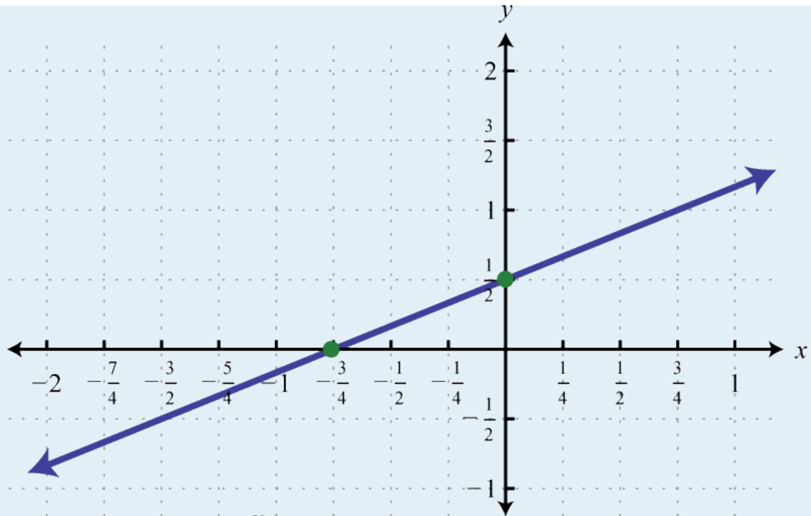

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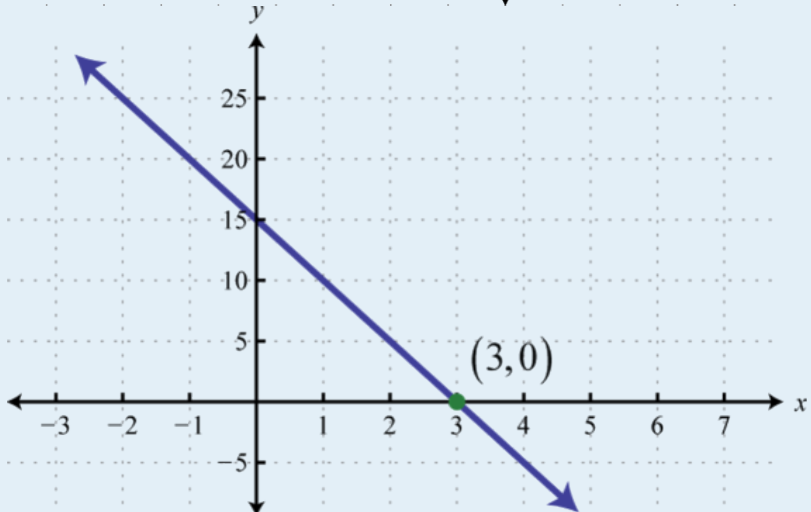

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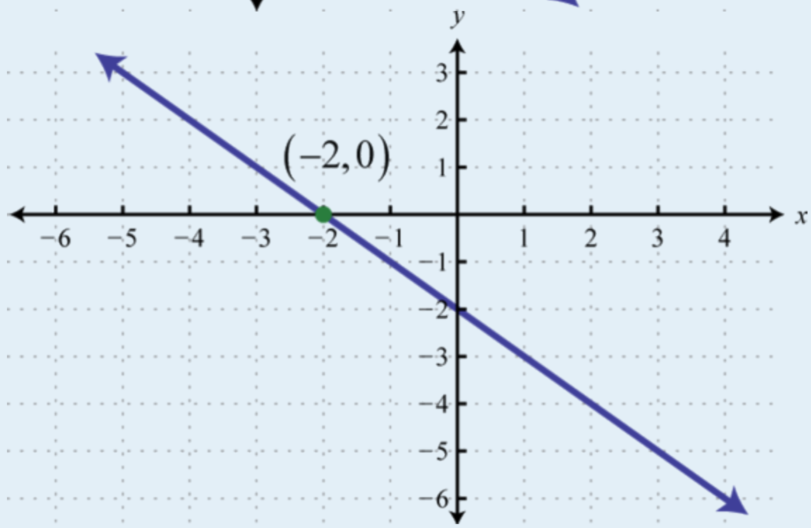

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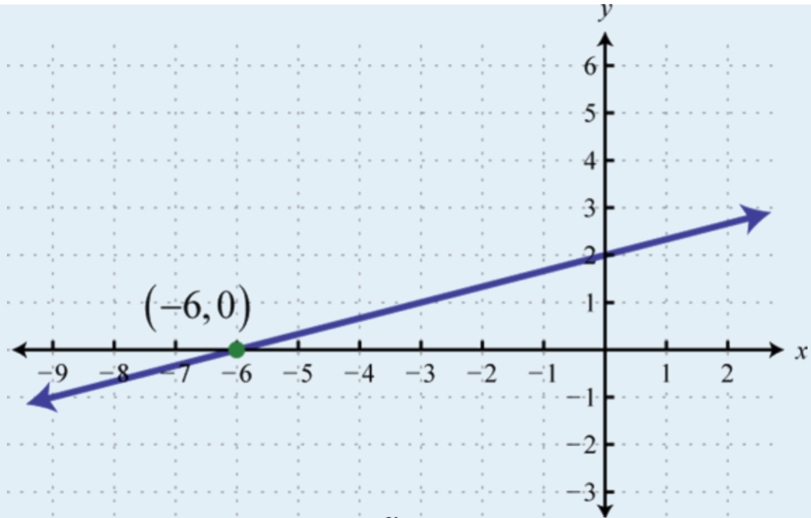

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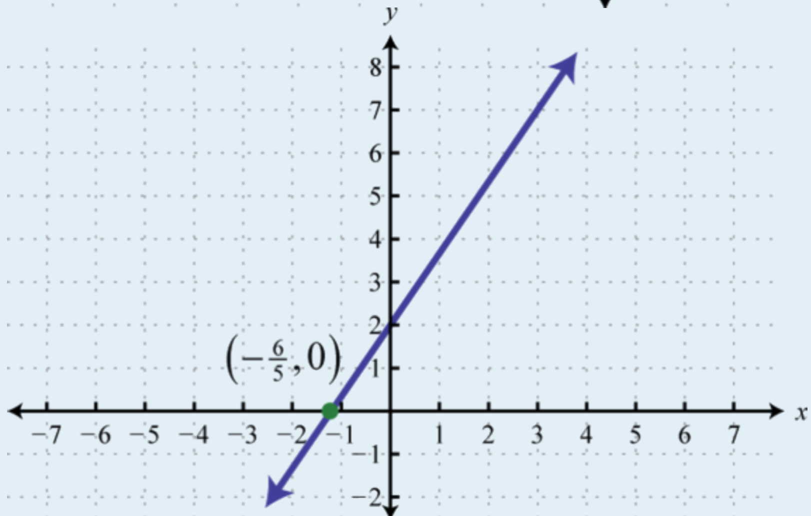

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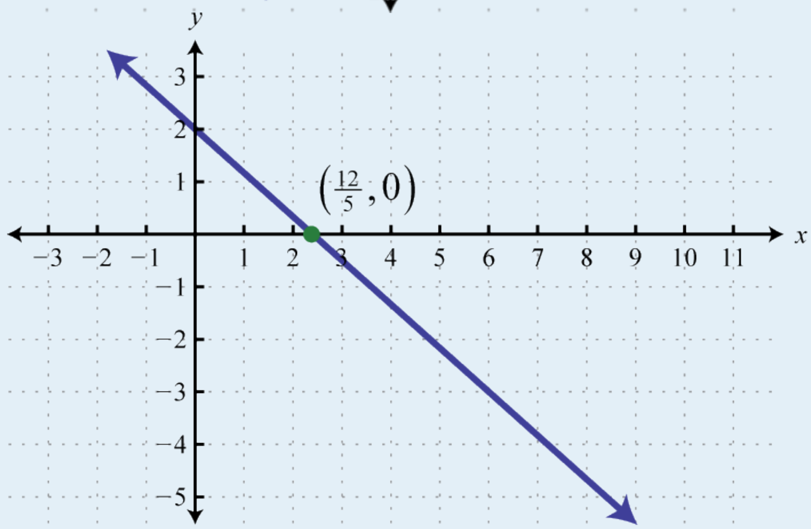

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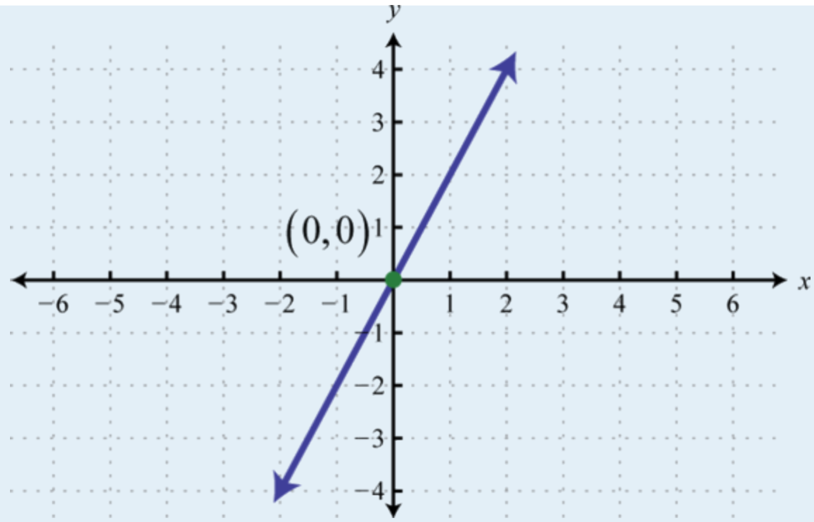
71. 

73. $f(x) = x + 1;\ (-1, 0)$

75. $f(x) = -\frac{3}{2}x;\ (0, 0)$

77. $f(x) = -9;$ none

79. $f(x) = \frac{1}{3}x + 1; (-3, 0)$

81. $x = 8$

83. $x = -1$

85. $x = 3$

87. $x = 1$

89. $x = 3$

91. $[-2, \infty)$

93. $[0, \infty)$

95. $(-\infty, 4]$

97. $[3, \infty)$

99. $(-\infty, -6)$

101. $(10, \infty)$

103. $(-\infty, -4)$

105. $(-\infty, -3)$

107. Answer may vary

109. Answer may vary

2.3 Modeling Linear Functions

LEARNING OBJECTIVES

1. Determine the equation of a line given two points.
2. Determine the equation of a line given the slope and y-intercept.
3. Find linear functions that model common applications.

Equations of Lines

Given the algebraic equation of a line, we can graph it in a number of ways. In this section, we will be given a geometric description of a line and find the algebraic equation. Finding the equation of a line can be accomplished in a number of ways. The following example makes use of slope-intercept form, $y = mx + b$, or using function notation, $f(x) = mx + b$. If we can determine the slope, m, and the y-intercept, $(0, b)$, we can then construct the equation.

Example 1

Find the equation of the line passing through $(-3, 6)$ and $(5, -4)$.

Solution:

We begin by finding the slope. Given two points, we can find the slope using the slope formula.

$$\begin{array}{ll} (x_1,\ y_1) & (x_2,\ y_2) \\ (-3, 6) & (5, -4) \end{array}$$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-4 - \ (6)}{5 - \ (-3)} \\ &= \frac{-4 - 6}{5 + 3} \\ &= \frac{-10}{8} \\ &= -\frac{5}{4} \end{aligned}$$

Here $m = -\frac{5}{4}$ and we have

$$\begin{aligned} f(x) &= mx + b \\ f(x) &= -\frac{5}{4}x + b \end{aligned}$$

To find b, substitute either one of the given points through which the line passes. Here we will use $(-3, 6)$, but $(5, -4)$ would work just as well:

$$\begin{aligned} f(x) &= -\frac{5}{4}x + b \qquad \textit{Use } (x, f(x)) = (-3, 6) \\ 6 &= -\frac{5}{4}(-3) + b \\ 6 &= \frac{15}{4} + b \\ \frac{6 \cdot 4}{1 \cdot 4} - \frac{15}{4} &= b \\ \frac{24 - 15}{4} &= b \\ \frac{9}{4} &= b \end{aligned}$$

Therefore, the equation of the line passing through the two given points is:

$$\begin{aligned} f(x) &= mx \quad + b \\ & \quad\ \downarrow \qquad \downarrow \\ f(x) &= -\frac{5}{4}x + \frac{9}{4} \end{aligned}$$

Answer: $f(x) = -\frac{5}{4}x + \frac{9}{4}$

Next, we outline an alternative method for finding equations of lines. Begin by applying the slope formula with a given point (x_1, y_1) and a variable point (x, y).

$$
\begin{aligned}
m &= \frac{y - y_1}{x - x_1} \\
\frac{m}{1} &= \frac{y - y_1}{x - x_1} && \textit{Cross multiply.} \\
m(x - x_1) &= y - y_1 && \textit{Apply the symmetric property.} \\
y - y_1 &= m(x - x_1)
\end{aligned}
$$

Therefore, the equation of a nonvertical line can be written in **point-slope form**[30]:

$$y - y_1 = m(x - x_1) \quad \textit{Point-slope form.}$$

Point-slope form is particularly useful for finding the equation of a line given the slope and any ordered pair solution. After finding the slope, $-\frac{5}{4}$ in the previous example, we could use this form to find the equation.

$$
\begin{array}{ll}
\textit{Point} & \textit{Slope} \\
(x_1, y_1) & \\
(-3, 6) & m = -\frac{5}{4}
\end{array}
$$

Substitute as follows.

30. Any nonvertical line can be written in the form $y - y_1 = m(x - x_1)$, where m is the slope and (x_1, y_1) is any point on the line.

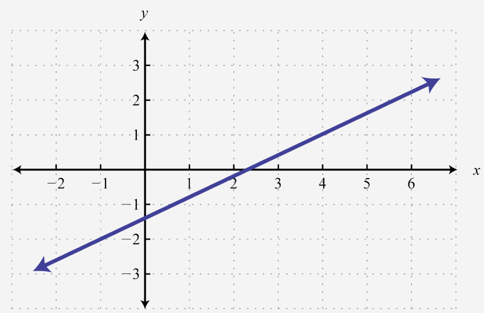
$$
\begin{aligned}
y - y_1 &= m\,(x - x_1) \\
y - (6) &= -\frac{5}{4}\,(x - \ (-3)) \qquad \textit{Solve for y.} \\
y - 6 &= -\frac{5}{4}\,(x + 3) \qquad \textit{Distribute.} \\
y - 6 &= -\frac{5}{4}x - \frac{15}{4} \\
y &= -\frac{5}{4}x - \frac{15}{4} + 6 \\
y &= -\frac{5}{4}x + \frac{9}{4}
\end{aligned}
$$

Notice that we obtain the same linear function $f\,(x) = -\frac{5}{4}\,x + \frac{9}{4}$.

Note: Sometimes a variable is not expressed explicitly in terms of another; however, it is still assumed that one variable is dependent on the other. For example, the equation $2x + 3y = 6$ implicitly represents the function $f\,(x) = -\frac{2}{3}\,x + 2$.You should become comfortable with working with functions in either form.

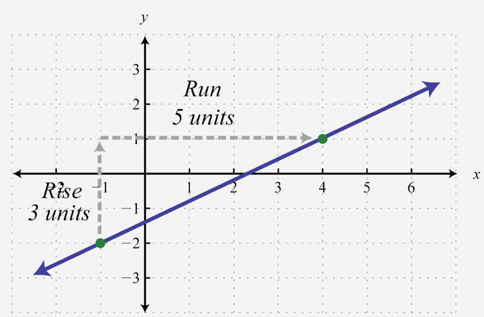
Example 2

Find the equation of the following linear function:



Solution:

From the graph we can determine two points (-1, -2) and (4, 1). Use these points to read the slope from the graph. The rise is 3 units and the run is 5 units.



Therefore, we have the slope and a point. (It does not matter which of the given points we use, the result will be the same.)

$$\begin{array}{cc} \textit{Point} & \textit{Slope} \\ (-1,-2) & m=\dfrac{3}{5} \end{array}$$

Use point-slope form to determine the equation of the line.

$$
\begin{aligned}
y - y_1 &= m\,(x - x_1) \\
y - (-2) &= \frac{3}{5}\,(x - \ (-1)) \qquad \textit{Solve for } y. \\
y + 2 &= \frac{3}{5}\,(x + 1) \\
y + 2 &= \frac{3}{5}x + \frac{3}{5} \\
y &= \frac{3}{5}x + \frac{3}{5} - 2 \\
y &= \frac{3}{5}x - \frac{7}{5}
\end{aligned}
$$

Answer: $f(x) = \frac{3}{5}x - \frac{7}{5}$

Recall that **parallel lines**[31] are lines in the same plane that never intersect. Two non-vertical lines in the same plane with slopes m_1 and m_2 are parallel if their slopes are the same, $m_1 = m_2$.

31. Lines in the same plane that do not intersect; their slopes are the same.

Example 3

Find the equation of the line passing through $(3, -2)$ and parallel to $x - 2y = -2$.

Solution:

To find the slope of the given line, solve for y.

$$\begin{aligned} x - 2y &= -2 \\ -2y &= -x - 2 \\ y &= \frac{-x - 2}{-2} \\ y &= \frac{-x}{-2} - \frac{2}{-2} \\ y &= \frac{1}{2}x + 1 \end{aligned}$$

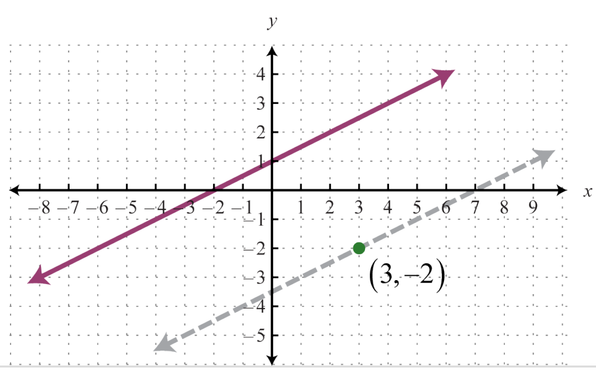
Here the given line has slope $m = \frac{1}{2}$ and thus the slope of a parallel line $m_{\|} = \frac{1}{2}$ .The notation $m_{\|}$ reads "m parallel." Since we are given a point and we now have the slope, we will choose to use point-slope form of a line to determine the equation.

Point	*Slope*
$(3, -2)$	$m_{\|} = \frac{1}{2}$

$$\begin{aligned} y - y_1 &= m(x - x_1) \quad \textit{Point-Slope form} \\ y - (-2) &= \frac{1}{2}(x - 3) \\ y + 2 &= \frac{1}{2}x - \frac{3}{2} \\ y + 2 - 2 &= \frac{1}{2}x - \frac{3}{2} - 2 \\ y &= \frac{1}{2}x - \frac{7}{2} \end{aligned}$$

Answer: $f(x) = \frac{1}{2}x - \frac{7}{2}$

It is important to have a geometric understanding of this question. We were asked to find the equation of a line parallel to another line passing through a certain point.



Through the point $(3, -2)$ we found a parallel line, $y = \frac{1}{2}x - \frac{7}{2}$, shown as a dashed line. Notice that the slope is the same as the given line, $y = \frac{1}{2}x + 1$, but the y-intercept is different.

Recall that **perpendicular lines**[32] are lines in the same plane that intersect at right angles (90 degrees). Two nonvertical lines, in the same plane with slopes m_1 and m_2, are perpendicular if the product of their slopes is -1, $m_1 \cdot m_2 = -1$. We can solve for m_1 and obtain $m_1 = -\frac{1}{m_2}$. In this form, we see that perpendicular lines have slopes that are **negative reciprocals**[33], or **opposite reciprocals**[34]. In general, given real numbers a and b,

32. Lines in the same plane that intersect at right angles; their slopes are opposite reciprocals.

33. Used when referring to opposite reciproacals.

34. Two real numbers whose product is -1. Given a real number $\frac{a}{b}$, the opposite reciprocal is $-\frac{b}{a}$.

$$\text{If } m = \frac{a}{b} \text{ then } m_{\perp} = -\frac{b}{a}$$

The mathematical notation $m_{\perp}$ reads "m perpendicular". For example, the opposite reciprocal of $m = -\frac{3}{5}$ is $m_{\perp} = \frac{5}{3}$ .We can verify that two slopes produce perpendicular lines if their product is -1.

$$m \cdot m_{\perp} = -\frac{3}{5} \cdot \frac{5}{3} = -\frac{15}{15} = -1 \checkmark$$

Example 4

Find the equation of the line passing through $(-5, -2)$ and perpendicular to $x + 4y = 4$.

Solution:

To find the slope of the given line, solve for y.

$$\begin{aligned} x + 4y &= 4 \\ 4y &= -x + 4 \\ y &= \frac{-x + 4}{4} \\ y &= \frac{-x}{4} + \frac{4}{4} \\ y &= -\frac{1}{4}x + 1 \end{aligned}$$

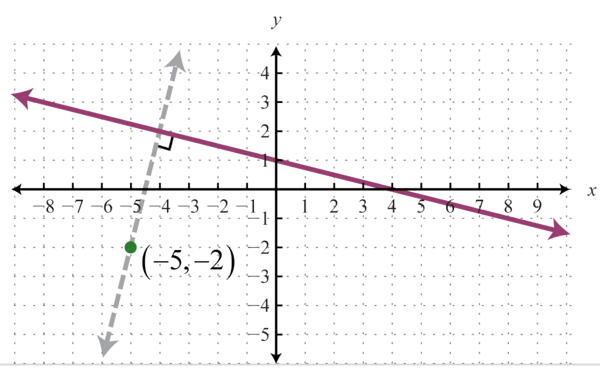
The given line has slope $m = -\frac{1}{4}$, and thus, $m_{\perp} = +\frac{4}{1} = 4$.Substitute this slope and the given point into point-slope form.

$$\begin{array}{cc} \textit{Point} & \textit{Slope} \\ (-5, -2) & m_{\perp} = 4 \end{array}$$

$$
\begin{aligned}
y - y_1 &= m\,(x - x_1)\\
y - \ (-2) &= 4\left(x - \ (-5)\right)\\
y + 2 &= 4\left(x + 5\right)\\
y + 5 &= 4x + 20\\
y &= 4x + 18
\end{aligned}
$$

Answer: $f\,(x) = 4x + 18$

Geometrically, we see that the line $y = 4x + 18$, shown as a dashed line in the graph, passes through $\left(-5, -2\right)$ and is perpendicular to the given line $y = -\frac{1}{4}x + 1.$



Try this! Find the equation of the line passing through $\left(-5, -2\right)$ and perpendicular to $\frac{1}{3}x - \frac{1}{2}y = -2.$

Answer: $y = -\frac{3}{2}x - \frac{19}{2}$

(click to see video)

Modeling Linear Applications

Data can be used to construct functions that model real-world applications. Once an equation that fits given data is determined, we can use the equation to make certain predictions; this is called **mathematical modeling**[35].

Example 5

The cost of a daily truck rental is $48.00, plus an additional $0.45 for every mile driven. Write a function that gives the cost of the daily truck rental and use it to determine the total cost of renting the truck for a day and driving it 60 miles.

Solution:

The total cost of the truck rental depends on the number of miles driven. If we let x represent the number of miles driven, then $0.45x$ represents the variable cost of renting the truck. Use this and the fixed cost, $48.00, to write a function that models the total cost,

$$C(x) = 0.45x + 48$$

Use this function to calculate the cost of the rental when $x = 60$ miles.

$$\begin{aligned} C(60) &= 0.45(60) + 48 \\ &= 27 + 48 \\ &= 75 \end{aligned}$$

Answer: The total cost of renting the truck for the day and driving it 60 miles would be $75.

35. Using data to find mathematical equations that describe, or model, real-world applications.

We can use the model $C(x) = 0.45x + 48$ to answer many more questions. For example, how many miles can be driven to keep the cost of the rental at most $66? To answer this question, set up an inequality that expresses the cost less than or equal to $66.

$$\begin{aligned} C(x) &\leq \$66 \\ 0.45x + 48 &\leq 66 \end{aligned}$$

Solve for *x* to determine the number of miles that can be driven.

$$\begin{aligned} 0.45x + 48 &\leq 66 \\ 0.45x &\leq 18 \\ x &\leq 40 \end{aligned}$$

To limit the rental cost to $66, the truck can be driven 40 miles or less.

Example 6

A company purchased a new piece of equipment for $12,000. Four years later it was valued at $9,000 dollars. Use this data to construct a linear function that models the value of the piece of equipment over time.

Solution:

The value of the item depends on the number of years after it was purchased. Therefore, the age of the piece of equipment is the independent variable. Use ordered pairs where the x-values represent the age and the y-values represent the corresponding value.

$$\left(age, value\right)$$

From the problem, we can determine two ordered pairs. Purchased new (age = 0), the item cost $12,000, and 4 years later the item was valued at $9,000. Therefore, we can write the following two $\left(age, value\right)$ordered pairs:

$$(0, 12{,}000) \quad and \quad (4, 9{,}000)$$

Use these two ordered pairs to construct a linear model. Begin by finding the slope m.

$$
\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{9{,}000 - 12{,}000}{4 - 0} \\
&= \frac{-3{,}000}{4} \\
&= -750
\end{aligned}
$$

Here we have $m = -750$. The ordered pair (0, 12,000) gives the y-intercept; therefore, $b = 12{,}000$.

$$
\begin{aligned}
y &= mx + b \\
y &= -750x + 12{,}000
\end{aligned}
$$

Lastly, write this model as a function which gives the value of the piece of equipment over time. Choose the function name V, for value, and the variable t instead of x to represent time in years.

$$V(t) = -750t + 12{,}000$$

Answer: $V(t) = -750t + 12{,}000$

The function $V(t) = -750t + 12{,}000$ called a **linear depreciation model**[36]. It uses a linear equation to expresses the declining value of an item over time. Using this function to determine the value of the item between the given data points is called **interpolation**[37]. For example, we can use the function to determine the value of the item where $t = 2$,

36. A linear function used to describe the declining value of an item over time.

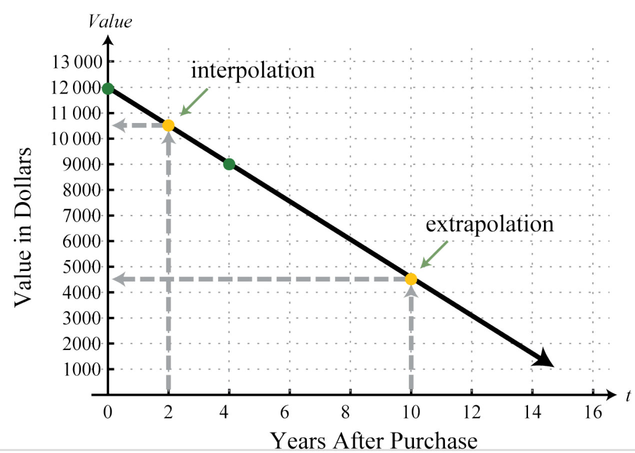
37. Using a linear function to estimate a value between given data points.

$$\begin{aligned} V(2) &= -750(2) + 12{,}000 \\ &= 10{,}500 \end{aligned}$$

The function shows that the item was worth \$10,500 two years after it was purchased. Using this model to predict the value outside the given data points is called **extrapolation**[38]. For example, we can use the function to determine the value of the item when $t = 10$:

$$\begin{aligned} V(10) &= -750(10) + 12{,}000 \\ &= -7{,}500 + 12{,}000 \\ &= 4{,}500 \end{aligned}$$

The model predicts that the piece of equipment will be worth \$4,500 ten years after it is purchased.



In a business application, revenue results from the sale of a number of items. For example, if an item can be sold for \$150 and we let n represent the number of units sold, then we can form the following **revenue function**[39]:

38. Using a linear function to estimate values that extend beyond the given data points.

39. A function that models income based on a number of units sold.

$$R(n) = 150n$$

Use this function to determine the revenue generated from selling $n = 100$ units,

$$R(100) = 150(100) = 15{,}000$$

The function shows that the revenue generated from selling 100 items is \$15,000. Typically, selling items does not represent the entire story. There are a number of costs associated with the generation of revenue. For example, if there is a one-time set up fee of \$5,280 and each item cost \$62 to produce, then we can form the following **cost function**[40]:

$$C(n) = 62n + 5{,}280$$

Here n represents the number of items produced. Use this function to determine the cost associated with producing $n = 100$ units:

$$C(100) = 62(100) + 5{,}280 = 11{,}480$$

The function shows that the cost associated with producing 100 items is \$11,480. Profit is revenue less costs:

$$\begin{aligned} \text{Profit} &= \text{Revenue} - \text{Cost} \\ &= 15{,}000 - 11{,}480 \\ &= 3{,}520 \end{aligned}$$

Therefore, the profit generated by producing and selling 100 items is \$3,520. In general, given a revenue function R and a cost function C, we can form a **profit function**[41] by subtracting as follows:

40. A function that models the cost of producing a number of units.

41. A function that models the profit as revenue less cost.

$$P(n) = R(n) - C(n)$$

Example 7

The cost in dollars of producing n items is given by the formula $C(n) = 62n + 5{,}280.$ The revenue in dollars is given by $R(n) = 150n$, where n represents the number items sold. Write a function that gives the profit generated by producing and selling n items. Use the function to determine how many items must be produced and sold in order to earn a profit of at least \$7,000.

Solution:

Obtain the profit function by subtracting the cost function from the revenue function.

$$\begin{aligned} P(n) &= R(n) - C(n) \\ &= 150n - \left(62n + 5{,}280\right) \\ &= 150n - 62n - 5{,}280 \\ &= 88n - 5{,}280 \end{aligned}$$

Therefore, $P(n) = 88n + 5{,}280$models the profit. To determine the number of items that must be produced and sold to profit at least \$7,000, solve the following:

$$\begin{aligned} P(n) &\geq 7{,}000 \\ 88n - 5{,}280 &\geq 7{,}000 \\ 88n &\geq 12{,}280 \\ n &\geq 139.5 \end{aligned}$$

Round up because the number of units produced and sold must be an integer. To see this, calculate the profit where n is 139 and 140 units.

$$\begin{aligned} P(139) &= 88(139) - 5{,}280 = 6{,}952 \\ P(140) &= 88(140) - 5{,}280 = 7{,}040 \end{aligned}$$

Answer: 140 or more items must be produced and sold in order to earn a profit of at least $7,000.

Sometimes the costs exceed the revenue, in which case, the profit will be negative. For example, use the profit function of the previous example, $P(n) = 88n - 5{,}280$, to calculate the profit generated where $n = 50$.

$$P(50) = 88(50) - 5{,}280 = -880$$

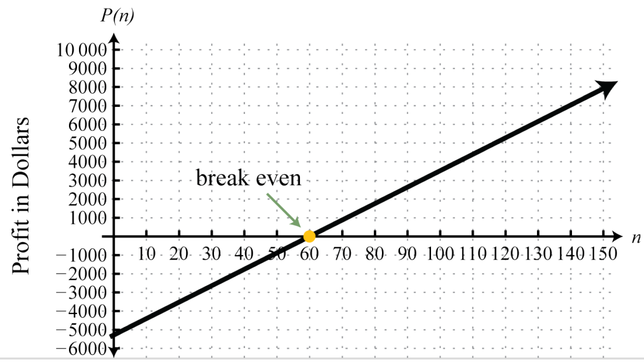
This indicates that when 50 units are produced and sold the corresponding profit is a loss of $880.

It is often important to determine how many items must be produced and sold to break even. To break even means to neither have a gain nor a loss; in this case, the profit will be equal to zero. To determine the **breakeven point**[42], set the profit function equal to zero and solve:

$$\begin{aligned} P(n) &= 88n - 5{,}280 \\ 0 &= 88n - 5{,}280 \\ 5{,}280 &= 88n \\ 60 &= n \end{aligned}$$

Therefore, 60 items must be produced and sold to break even.

42. The point at which profit is neither negative nor positive; profit is equal to zero.



Try this! Custom t-shirts can be sold for $6.50 each. In addition to an initial set-up fee of $120, each t-shirt cost $3.50 to produce. a. Write a function that models the revenue and a function that models the cost. b. Determine a function that models the profit and use it to determine the profit from producing and selling 150 t-shirts. c. Calculate the number of t-shirts that must be sold to break even.

Answer: a. Revenue: $R\left(x\right) = 6.50x$; cost: $C\left(x\right) = 3.50x + 120$; b. profit: $P\left(x\right) = 3x + 120$; $330 c. 40

(click to see video)

KEY TAKEAWAYS

- Given two points we can find the equation of a line.
- Parallel lines have the same slope.
- Perpendicular lines have slopes that are opposite reciprocals. In other words, if $m = \frac{a}{b}$, then $m_{\perp} = -\frac{b}{a}$.
- To find an equation of a line, first use the given information to determine the slope. Then use the slope and a point on the line to find the equation using point-slope form.
- To construct a linear function that models a real-world application, first identify the dependent and independent variables. Next, find two ordered pairs that describe the given situation. Use these two ordered pairs to construct a linear function by finding the slope and y-intercept.

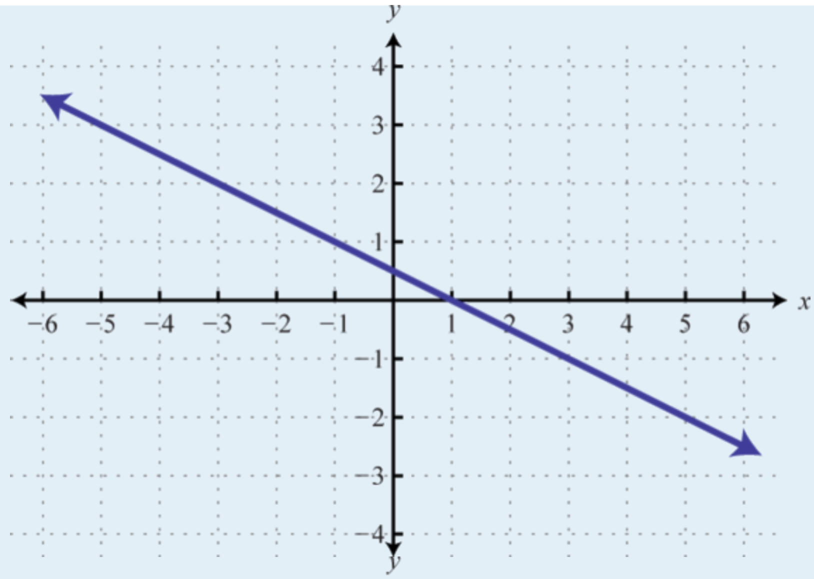
TOPIC EXERCISES

PART A: EQUATIONS OF LINES

Find the linear function *f* passing through the given points.

1. $(-1, 2)$ and $(3, -4)$
2. $(3, -2)$ and $(-1, -4)$
3. $(-5, -6)$ and $(-4, 2)$
4. $(2, -7)$ and $(3, -5)$
5. $(10, -15)$ and $(7, -6)$
6. $(-9, 13)$ and $(-8, 12)$
7. $(-12, 22)$ and $(6, -20)$
8. $(6, -12)$ and $(-4, 13)$
9. $\left(\frac{1}{3}, \frac{4}{5}\right)$ and $\left(\frac{1}{2}, 1\right)$
10. $\left(-\frac{3}{2}, -\frac{5}{2}\right)$ and $\left(1, \frac{5}{6}\right)$
11. $(-5, 10)$ and $(-1, 10)$
12. $(4, 0)$ and $(-7, 0)$

Find the equation of the given linear function.

13. 
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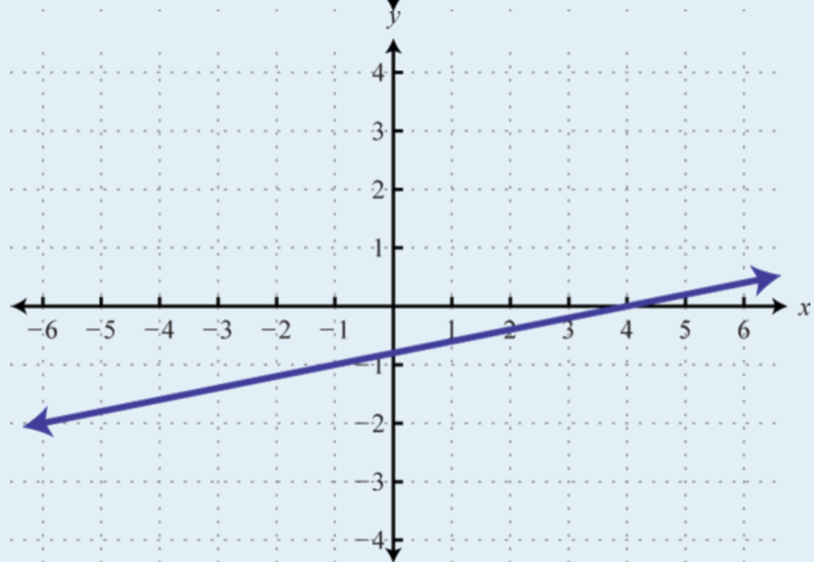
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6

14. 
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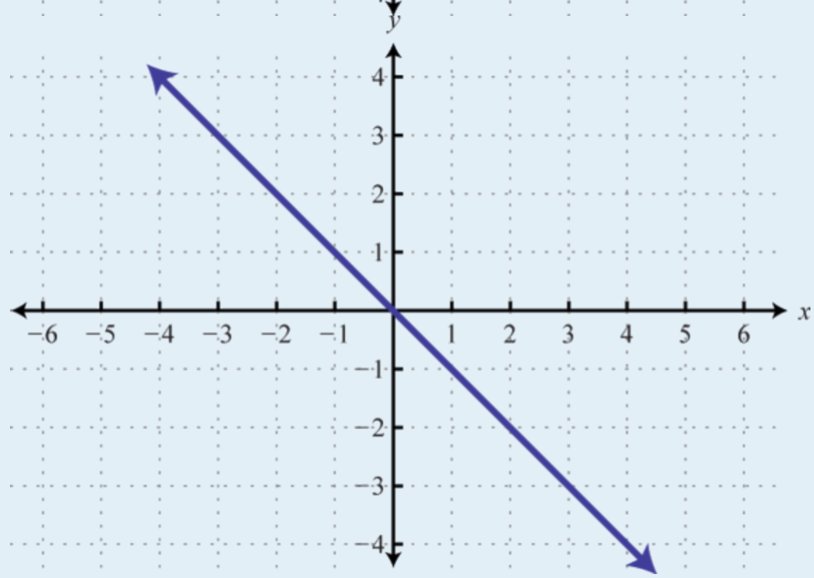
15. 
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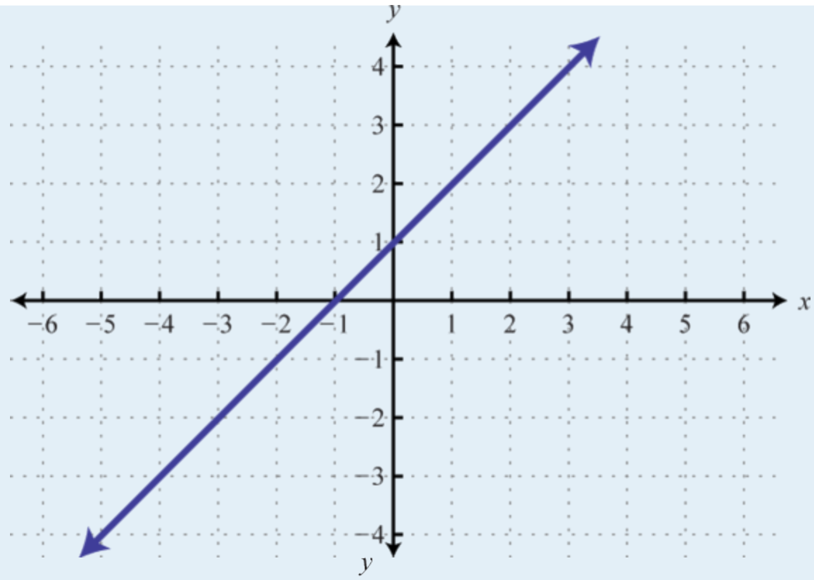
16. 
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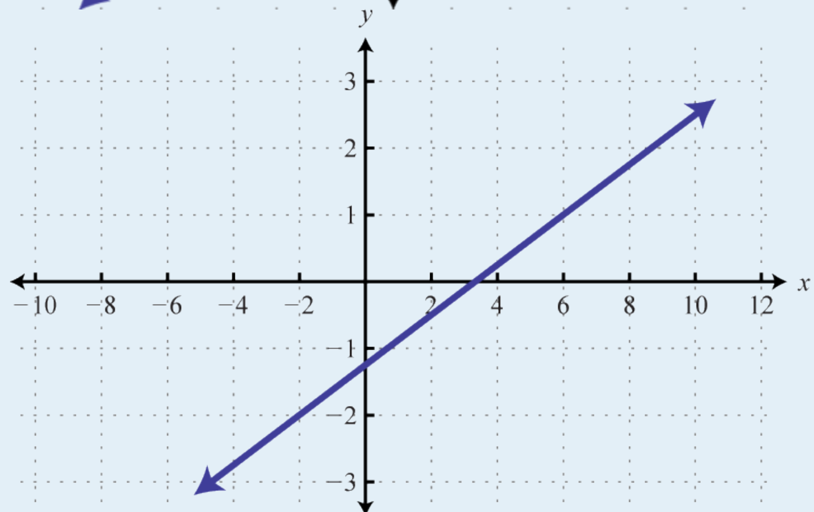
17. 
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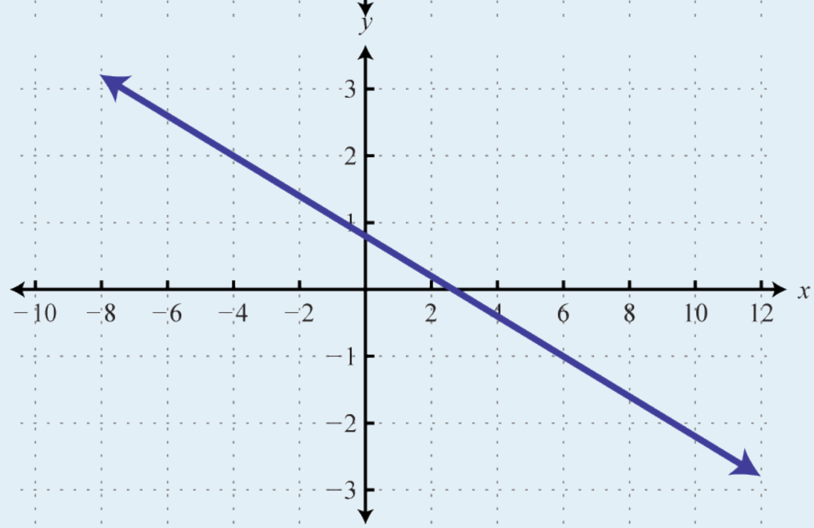
18. 
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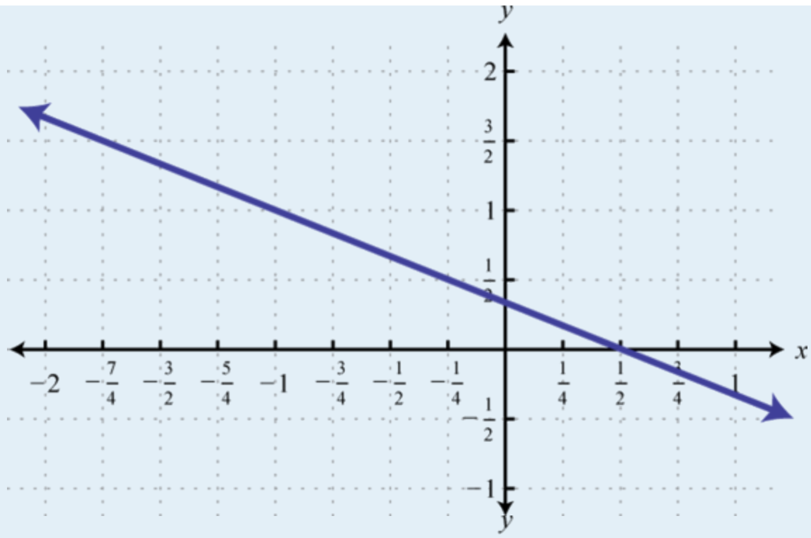
19. 
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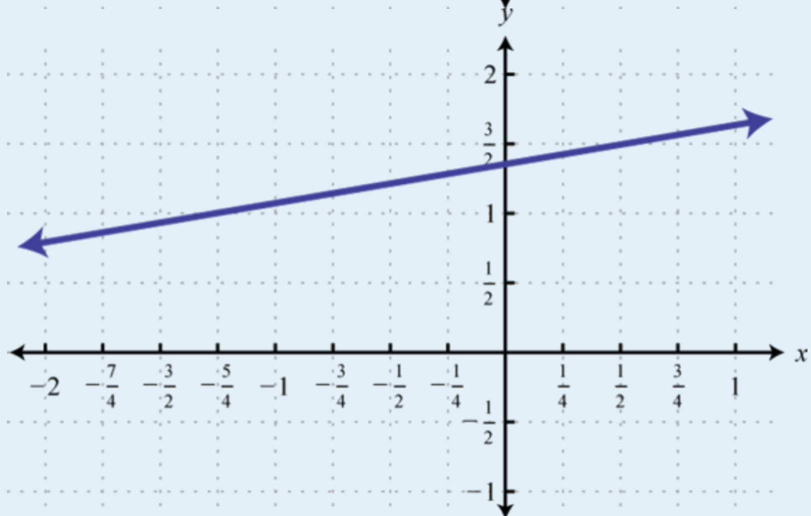
20. 
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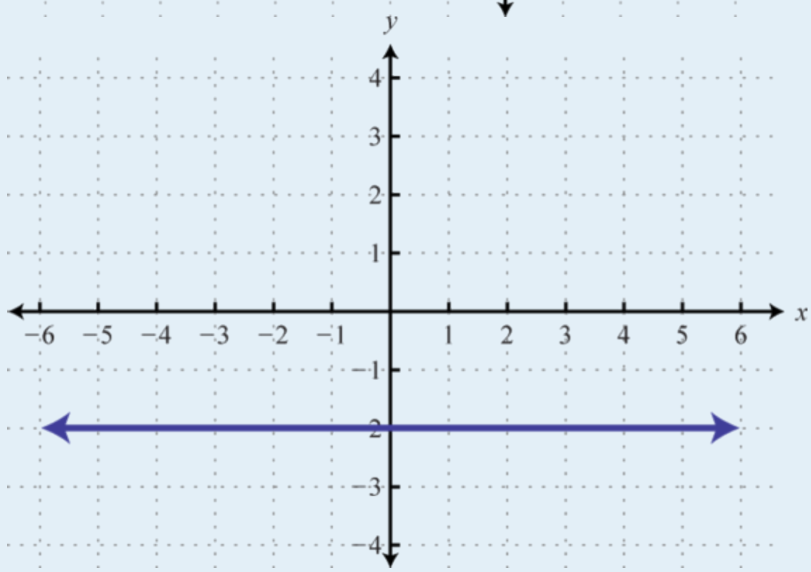
21. 
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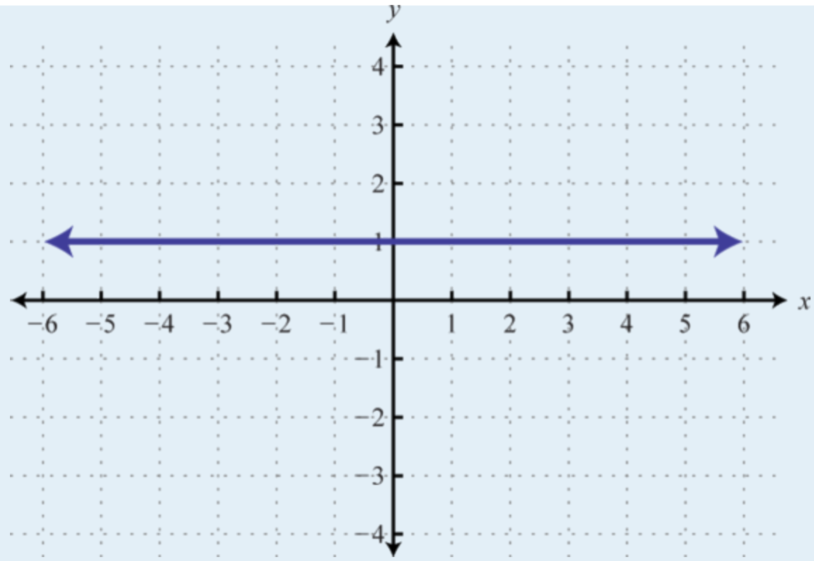
22. 

Find the equation of the line:

23. Parallel to $y = -\frac{5}{3}x + \frac{1}{2}$ and passing through $(-3, 4)$.

24. Parallel to $y = -\frac{3}{4}x - \frac{7}{3}$ and passing through $(-8, -1)$.

25. Parallel to $y = \frac{1}{3}x + 6$ and passing through $\left(2, -5\right)$.

26. Parallel to $y = \frac{1}{4}x + \frac{5}{3}$ and passing through $\left(5, 6\right)$.

27. Parallel to $4x - 5y = 15$ and passing through $(-1, -2)$.

28. Parallel to $3x - 4y = 2$ and passing through $\left(-6, 8\right)$.

29. Parallel to $2x + 12y = 9$ and passing through $(10, -9)$.

30. Parallel to $9x + 24y = 2$ and passing through $(-12, -4)$.

31. Parallel to $\frac{2}{15}x + \frac{1}{3}y = \frac{1}{10}$ and passing through $\left(-15, 4\right)$.

32. Parallel to $\frac{1}{3}x + \frac{2}{7}y = 1$ and passing through $(12, -11)$.

33. Perpendicular to $y = 5x + 2$ and passing through $\left(10, -5\right)$.

34. Perpendicular to $y = -2x + 1$ and passing through $(-8, -11)$.

35. Perpendicular to $y = \frac{3}{2}x - 5$ and passing through $\left(5, -3\right)$.

36. Perpendicular to $y = \frac{3}{4}x - \frac{1}{2}$ and passing through $\left(-6, -4\right)$.

37. Perpendicular to $12x + 15y = 3$ and passing through $(12, 15)$.
38. Perpendicular to $24x + 15y = 12$ and passing through $(2, -1)$.
39. Perpendicular to $14x - y = 3$ and passing through $(7, 3)$.
40. Perpendicular to $x - y = 4$ and passing through $(6, -2)$.
41. Perpendicular to $\frac{2}{15}x - \frac{3}{5}y = -1$ and passing through $(1, -1)$.
42. Perpendicular to $\frac{3}{4}x - \frac{2}{3}y = \frac{1}{2}$ and passing through $(-3, 6)$.
43. Give the equation of the line that coincides with the *x*-axis.
44. Give the equation of the line that coincides with the *y*-axis.
45. Given any line in standard form, $ax + by = c$, determine the slope of any perpendicular line.
46. Given any line in standard form, $ax + by = c$, determine the slope of any parallel line.

PART B: MODELING LINEAR APPLICATIONS

Use algebra to solve the following.

47. A company wishes to purchase pens stamped with the company logo. In addition to an initial set-up fee of $90, each pen cost $1.35 to produce. Write a function that gives the cost in terms of the number of pens produced. Use the function to determine the cost of producing 500 pens with the company logo stamped on it.
48. A rental car company charges a daily rate of $42.00 plus $0.51 per mile driven. Write a function that gives the cost of renting the car for a day in terms of the number of miles driven. Use the function to determine the cost of renting the car for a day and driving it 76 miles.
49. A certain cellular phone plan charges $16 per month and $0.15 per minute of usage. Write a function that gives the cost of the phone per month based on the number of minutes of usage. Use the function to determine the number of minutes of usage if the bill for the first month was $46.
50. A web-services company charges $2.50 a month plus $0.14 per gigabyte of storage on their system. Write a function that gives the cost of storage per

month in terms of the number of gigabytes stored. How many gigabytes are stored if the bill for this month was $6.00?

51. Mary has been keeping track of her cellular phone bills for the last two months. The bill for the first month was $45.00 for 150 minutes of usage. The bill for the second month was $25.00 for 50 minutes of usage. Find a linear function that gives the total monthly bill based on the minutes of usage.

52. A company in its first year of business produced 1,200 brochures for a total cost of $5,050. The following year, the company produced 500 more brochures at a cost of $2,250. Use this information to find a linear function that gives the total cost of producing brochures from the number of brochures produced.

53. A Webmaster has noticed that the number of registered users has been steadily increasing since beginning an advertising campaign. Before starting to advertise, he had 2,200 registered users, and after 4 months of advertising he now has 5,480 registered users. Use this data to write a linear function that gives the total number of registered users, given the number of months after starting to advertise. Use the function to predict the number of users 8 months into the advertising campaign.

54. A corn farmer in California was able to produce 154 bushels of corn per acre 2 years after starting his operation. Currently, after 7 years of operation, he has increased his yield to 164 bushels per acre. Use this information to write a linear function that gives the total yield per acre based on the number of years of operation, and use it to predict the yield for next year.

55. A commercial van was purchased new for $22,500 and is expected to be worthless in 12 years. Use this information to write a linear depreciation function for the value of the van. Use the function to determine the value of the van after 8 years of use.

56. The average lifespan of an industrial welding robot is 10 years, after which it is considered to have no value. If an industrial welding robot was purchased new for $58,000, write a function that gives the value of the robot in terms of the number of years of operation. Use the function to value the robot after 3 years of operation.

57. A business purchased a piece of equipment new for $2,400. After 5 years of use the equipment is valued at $1,650. Find a linear function that gives the value of the equipment in terms of years of usage. Use the function to determine the number of years after which the piece of equipment will have no value.

58. A salesman earns a base salary of $2,400 a month plus a 5% commission on all sales. Write a function that gives the salesman’s monthly salary in terms of

sales. Use the function to determine the monthly sales required to earn at least \$3,600 a month.

59. When a certain professor was hired in 2005, the enrollment at a college was 8,500 students. Five years later, in 2010, the enrollment grew to 11,200 students. Determine a linear growth function that models the student population in years since 2005. Use the model to predict the year in which enrollment will exceed 13,000 students.

60. In 1980, the population of California was about 24 million people. Twenty years later, in the year 2000, the population was about 34 million. Use this data to construct a linear function to model the population growth in years since 1980. Use the function to predict the year in which the population will reach 40 million.

61. A classic car is purchased for \$24,500 and is expected to increase in value each year by \$672. Write a linear function that models the appreciation of the car in terms of the number of years after purchase. Use the function to predict the value of the car in 7 years.

62. A company reported first and second quarter sales of \$52,000 and \$64,500, respectively.

    a. Write a linear function that models the sales for the year in terms of the quarter *n*.
    b. Use the model to predict the sales in the third and fourth quarters.

63. A particular search engine assigns a ranking to a webpage based on the number of links that direct users to the webpage. If no links are found, the webpage is assigned a ranking of 1. If 20 links are found directing users to the webpage, the search engine assigns a page ranking of 3.5. a. Find a linear function that gives the webpage ranking based on the number of links that direct users to it. b. How many links will be needed to obtain a page ranking of 5?

64. Online sales of a particular product are related to the number of clicks on its advertisement. It was found that 1,520 clicks in a month results in \$2,748 of online sales, and that 1,840 clicks results in \$2,956 of online sales. Write a linear function that models the online sales of the product based on the number of clicks on its advertisement. How many clicks would we need to expect \$3,385 in monthly online sales from this particular product?

65. A bicycle manufacturing business can produce x bicycles at a cost, in dollars, given by the formula $C(x) = 85x + 2{,}400.$ The company sells each bicycle at a wholesale price of \$145. The revenue, in dollars, is given by $R(x) = 145x$, where x represents the number of bicycles sold. Write a

function that gives profit in terms of the number of bicycles produced and sold. Use the function to determine the number of bicycles that need to be produced and sold to break even.

66. The cost, in dollars, of producing n custom lamps is given by the formula $C(n) = 28n + 360.$ Each lamp can be sold online for $79. The revenue in dollars, is given by $R(n) = 79n$, where n represents the number of lamps sold. Write a function that gives the profit from producing and selling n custom lamps. Use the function to determine how many lamps must be produced and sold to earn at least $1,000 in profit.

67. A manufacturer can produce a board game at a cost of $12 per unit after an initial fixed retooling investment of $12,500. The games can be sold for $22 each to retailers.

    a. Write a function that gives the manufacturing costs when n games are produced.
    b. Write a function that gives the revenue from selling n games to retailers.
    c. Write a function that gives the profit from producing and selling n units.
    d. How many units must be sold to earn a profit of at least $37,500?

68. A vending machine can be leased at a cost of $90 per month. The items used to stock the machine can be purchased for $0.50 each and sold for $1.25 each.

    a. Write a function that gives the monthly cost of leasing and stocking the vending machine with n items.
    b. Write a function that gives the revenue generated by selling n items.
    c. Write a function that gives the profit from stocking and selling n items per month.
    d. How many items must be sold each month to break even?

PART C: DISCUSSION BOARD

69. Research and discuss linear depreciation. In a linear depreciation model, what do the slope and y-intercept represent?

70. Write down your own steps for finding the equation of a line. Post your steps on the discussion board.

ANSWERS

1. $f(x) = -\frac{3}{2}x + \frac{1}{2}$
3. $f(x) = 8x + 34$
5. $f(x) = -3x + 15$
7. $f(x) = -\frac{7}{3}x - 6$
9. $f(x) = \frac{6}{5}x + \frac{2}{5}$
11. $f(x) = 10$
13. $f(x) = -\frac{1}{2}x + \frac{1}{2}$
15. $f(x) = -x$
17. $f(x) = \frac{3}{8}x - \frac{5}{4}$
19. $f(x) = -\frac{2}{3}x + \frac{1}{3}$
21. $f(x) = -2$
23. $y = -\frac{5}{3}x - 1$
25. $y = \frac{1}{3}x - \frac{17}{3}$
27. $y = \frac{4}{5}x - \frac{6}{5}$
29. $y = -\frac{1}{6}x - \frac{22}{3}$
31. $y = -\frac{2}{5}x - 2$
33. $y = -\frac{1}{5}x - 3$
35. $y = -\frac{2}{3}x + \frac{1}{3}$
37. $y = \frac{5}{4}x$
39. $y = -\frac{1}{14}x + \frac{7}{2}$
41. $$y = -\frac{9}{2}x + \frac{7}{2}$$

43. $y = 0$

45. $m_{\perp} = \frac{b}{a}$

47. $C(x) = 1.35x + 90$; \$765

49. $C(x) = 0.15x + 16$; 200 minutes

51. $C(x) = 0.20x + 15$

53. $U(x) = 820x + 2{,}200$; 8,760 users

55. $V(t) = -1,875t + 22,500$; \$7,500

57. $V(t) = -150t + 2{,}400$; 16 years

59. $P(x) = 540x + 8{,}500$; 2013

61. $V(t) = 672t + 24{,}500$; \$29,204

63.
   a. $r(n) = 0.125n + 1$;
   b. 32 links

65. $P(x) = 60x - 2{,}400$; 40 bicycles

67.
   a. $C(n) = 12n + 12{,}500$;
   b. $R(n) = 22n$;
   c. $P(n) = 10n - 12{,}500$;
   d. at least 5,000 units

69. Answer may vary

2.4 Graphing the Basic Functions

LEARNING OBJECTIVES

1. Define and graph seven basic functions.
2. Define and graph piecewise functions.
3. Evaluate piecewise defined functions.
4. Define the greatest integer function.

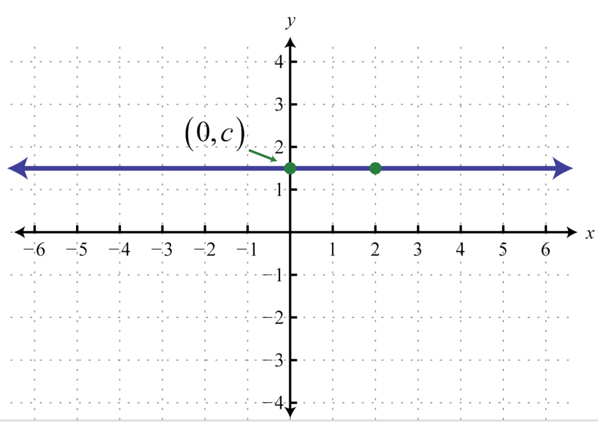
Basic Functions

In this section we graph seven basic functions that will be used throughout this course. Each function is graphed by plotting points. Remember that $f(x) = y$ and thus $f(x)$ and y can be used interchangeably.

Any function of the form $f(x) = c$, where c is any real number, is called a **constant function**[43]. Constant functions are linear and can be written $f(x) = 0x + c$. In this form, it is clear that the slope is 0 and the y-intercept is $(0, c)$. Evaluating any value for x, such as $x = 2$, will result in c.

$f(x) = c$

x	$f(x)$
0	c
2	c



The graph of a constant function is a horizontal line. The domain consists of all real numbers $\mathbb{R}$ and the range consists of the single value $\{c\}$.

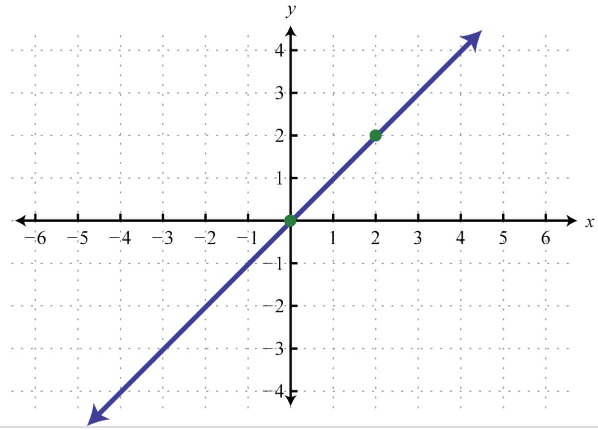
We next define the **identity function**[44] $f(x) = x$. Evaluating any value for x will result in that same value. For example, $f(0) = 0$ and $f(2) = 2$. The identity function is linear, $f(x) = 1x + 0$, with slope $m = 1$ and y-intercept $(0, 0)$.

43. Any function of the form $f(x) = c$ where c is a real number.

44. The linear function defined by $f(x) = x$.

$f(x) = x$

x	$f(x)$
0	0
2	2

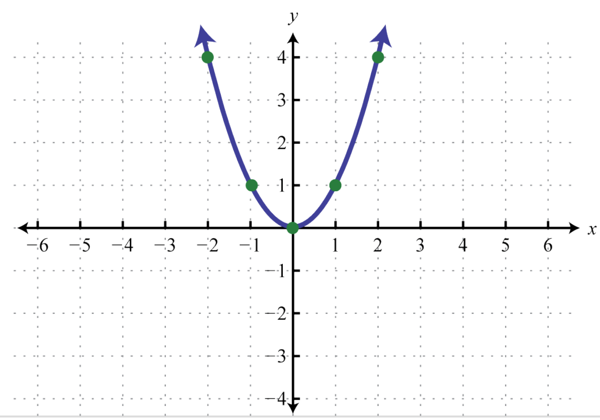


The domain and range both consist of all real numbers.

The **squaring function**[45], defined by $f(x) = x^2$, is the function obtained by squaring the values in the domain. For example, $f(2) = (2)^2 = 4$ and $f(-2) = (-2)^2 = 4$. The result of squaring nonzero values in the domain will always be positive.

$f(x) = x^2$

x	$f(x)$
−2	4
−1	1
0	0
1	1
2	4



The resulting curved graph is called a **parabola**[46]. The domain consists of all real numbers $\mathbb{R}$ and the range consists of all y-values greater than or equal to zero $[0, \infty)$.

The **cubing function**[47], defined by $f(x) = x^3$, raises all of the values in the domain to the third power. The results can be either positive, zero, or negative. For example, $f(1) = (1)^3 = 1$, $f(0) = (0)^3 = 0$, and $f(-1) = (-1)^3 = -1$.

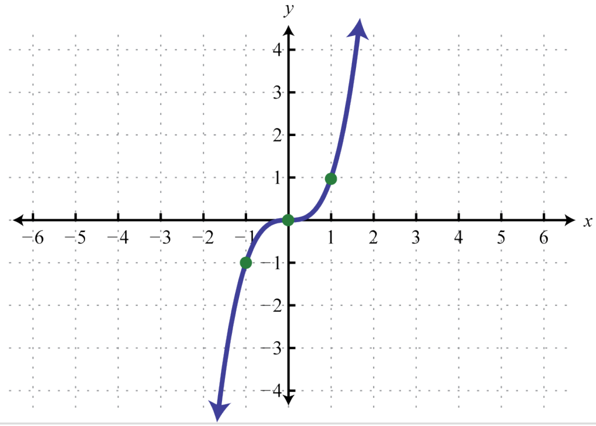
45. The quadratic function defined by $f(x) = x^2$.

46. The curved graph formed by the squaring function.

47. The cubic function defined by $f(x) = x^3$.

$f(x) = x^3$

x	$f(x)$
-2	-8
-1	-1
0	0
1	1
2	8



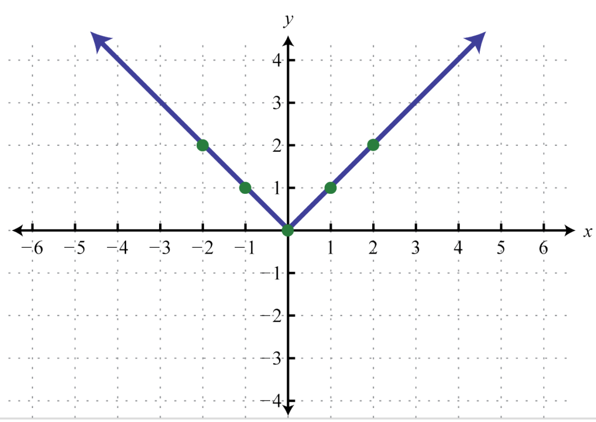
The domain and range both consist of all real numbers $\mathbb{R}$.

Note that the constant, identity, squaring, and cubing functions are all examples of basic polynomial functions. The next three basic functions are not polynomials.

The **absolute value function**[48], defined by $f(x) = |x|$, is a function where the output represents the distance to the origin on a number line. The result of evaluating the absolute value function for any nonzero value of x will always be positive. For example, $f(-2) = |-2| = 2$ and $f(2) = |2| = 2$.

$f(x) = |x|$

x	$f(x)$
-2	2
-1	1
0	0
1	1
2	2



The domain of the absolute value function consists of all real numbers $\mathbb{R}$ and the range consists of all y-values greater than or equal to zero $[0, \infty)$.

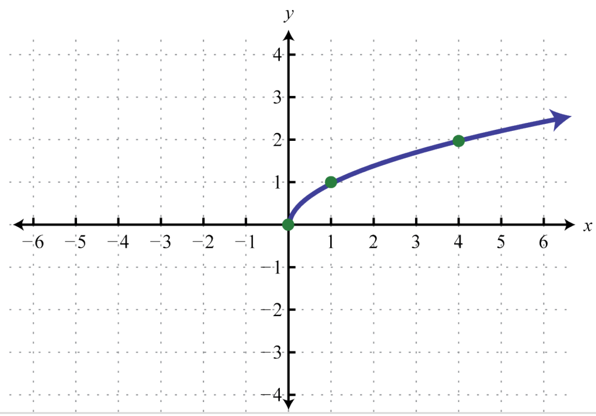
The **square root function**[49], defined by $f(x) = \sqrt{x}$, is not defined to be a real number if the x-values are negative. Therefore, the smallest value in the domain is zero. For example, $f(0) = \sqrt{0} = 0$ and $f(4) = \sqrt{4} = 2$.

48. The function defined by $f(x) = |x|$.

49. The function defined by $f(x) = \sqrt{x}$.

$f(x) = \sqrt{x}$

x	$f(x)$
0	0
1	1
4	2
9	3



The domain and range both consist of real numbers greater than or equal to zero $[0, \infty)$.

The **reciprocal function**[50], defined by $f(x) = \frac{1}{x}$, is a rational function with one restriction on the domain, namely $x \neq 0$. The reciprocal of an x-value very close to zero is very large. For example,

$$f(1/10) = \frac{1}{\left(\frac{1}{10}\right)} = 1 \cdot \frac{10}{1} = 10$$

$$f(1/100) = \frac{1}{\left(\frac{1}{100}\right)} = 1 \cdot \frac{100}{1} = 100$$

$$f(1/1{,}000) = \frac{1}{\left(\frac{1}{1{,}000}\right)} = 1 \cdot \frac{1{,}000}{1} = 1{,}000$$

In other words, as the x-values approach zero their reciprocals will tend toward either positive or negative infinity. This describes a **vertical asymptote**[51] at the y-axis. Furthermore, where the x-values are very large the result of the reciprocal function is very small.

50. The function defined by $f(x) = \frac{1}{x}$.

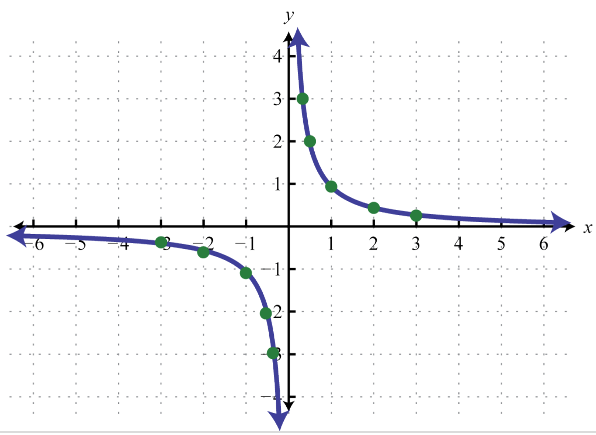
51. A vertical line to which a graph becomes infinitely close.

$$f(10) = \frac{1}{10} = 0.1$$
$$f(100) = \frac{1}{100} = 0.01$$
$$f(1000) = \frac{1}{1{,}000} = 0.001$$

In other words, as the *x*-values become very large the resulting *y*-values tend toward zero. This describes a **horizontal asymptote**[52] at the *x*-axis. After plotting a number of points the general shape of the reciprocal function can be determined.

$$f(x) = \frac{1}{x}$$

x	$f(x)$
-3	$-1/3$
-2	$-1/2$
-1	-1
$-1/2$	-2
$-1/3$	-3

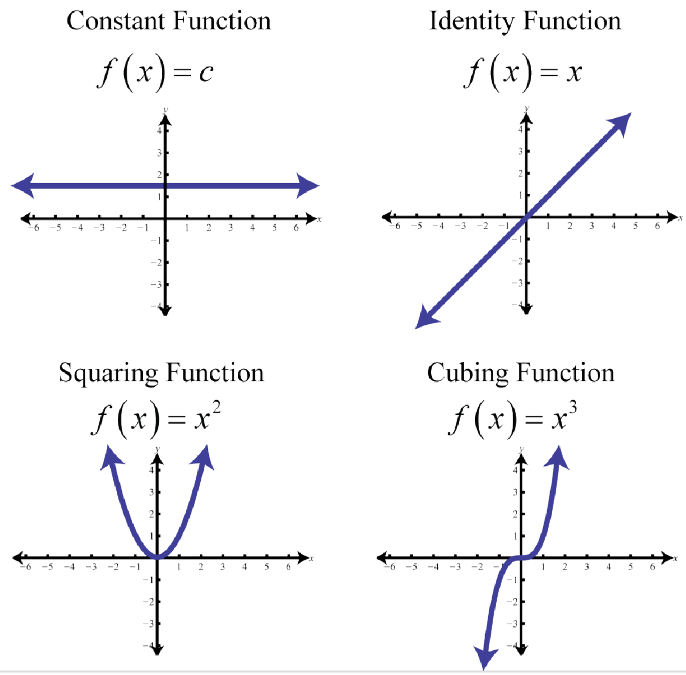


x	$f(x)$
$1/3$	3
$1/2$	2
1	1
2	$1/2$
3	$1/3$

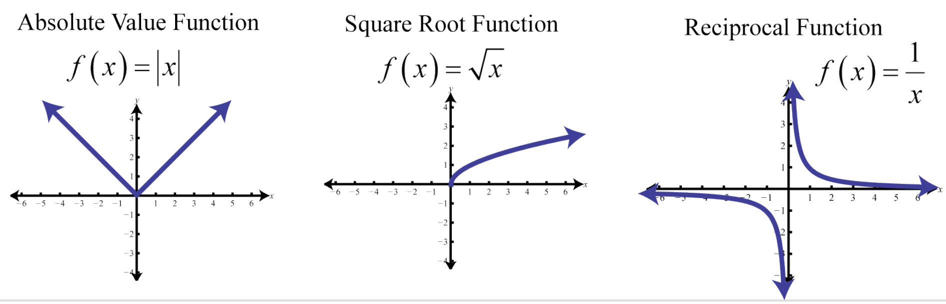
Both the domain and range of the reciprocal function consists of all real numbers except 0, which can be expressed using interval notation as follows: $(-\infty, 0) \cup (0, \infty)$.

In summary, the basic polynomial functions are:

52. A horizontal line to which a graph becomes infinitely close where the *x*-values tend toward ±∞.



The basic nonpolynomial functions are:



Piecewise Defined Functions

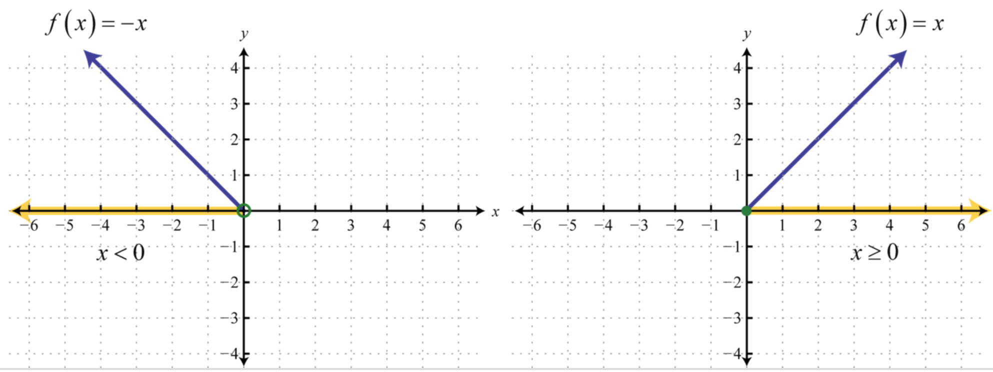
A **piecewise function**[53], or **split function**[54], is a function whose definition changes depending on the value in the domain. For example, we can write the absolute value function $f(x) = |x|$ as a piecewise function:

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

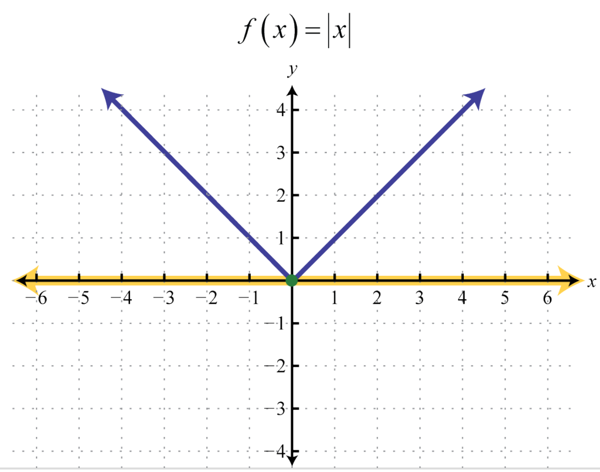
53. A function whose definition changes depending on the values in the domain.

54. A term used when referring to a piecewise function.

In this case, the definition used depends on the sign of the *x*-value. If the *x*-value is positive, $x \geq 0$, then the function is defined by $f(x) = x$. And if the *x*-value is negative, $x < 0$, then the function is defined by $f(x) = -x$.



Following is the graph of the two pieces on the same rectangular coordinate plane:

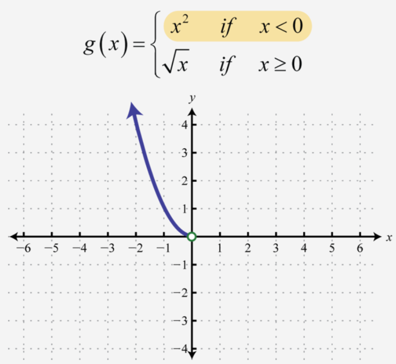


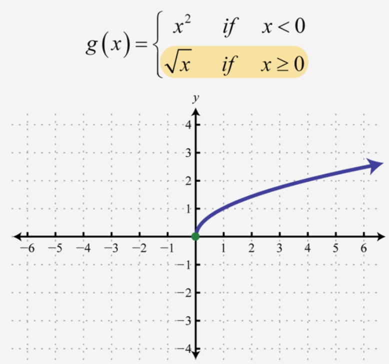
Example 1

Graph: $g(x) = \begin{cases} x^2 & \text{if} \quad x < 0 \\ \sqrt{x} & \text{if} \quad x \geq 0 \end{cases}$.

Solution:

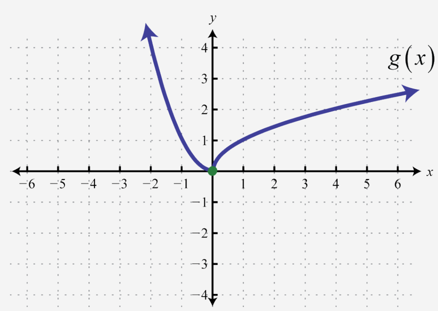
In this case, we graph the squaring function over negative *x*-values and the square root function over positive *x*-values.





Notice the open dot used at the origin for the squaring function and the closed dot used for the square root function. This was determined by the inequality that defines the domain of each piece of the function. The entire function consists of each piece graphed on the same coordinate plane.

Answer:



When evaluating, the value in the domain determines the appropriate definition to use.

Example 2

Given the function h, find $h(-5)$, $h(0)$, and $h(3)$.

$$h(t) = \begin{cases} 7t + 3 & \text{if} t < 0 \\ -16t^2 + 32t \text{if} t \geq 0 \end{cases}$$

Solution:

Use $h(t) = 7t + 3$where t is negative, as indicated by $t < 0$.

$$\begin{aligned} h(t) &= 7t + 5 \\ h(-5) &= 7(-5) + 3 \\ &= -35 + 3 \\ &= -32 \end{aligned}$$

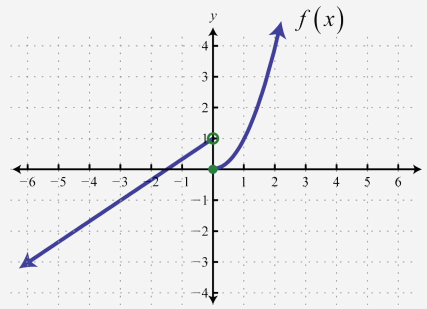
Where t is greater than or equal to zero, use $h(t) = -16t^2 + 32t$.

$$\begin{aligned} h(0) &= -16(0) + 32(0) h(3) = 16(3)^2 + 32(3) \\ &= 0 + 0 \qquad\qquad\qquad = -144 + 96 \\ &= 0 \qquad\qquad\qquad\qquad = -48 \end{aligned}$$

Answer: $h(-5) = -32$, $h(0) = 0$, and $h(3) = -48$

Try this! Graph: $f(x) = \begin{cases} \frac{2}{3}x + 1 & \text{if} \quad x < 0 \\ x^2 & \text{if} \quad x \geq 0 \end{cases}.$

Answer:



(click to see video)

The definition of a function may be different over multiple intervals in the domain.

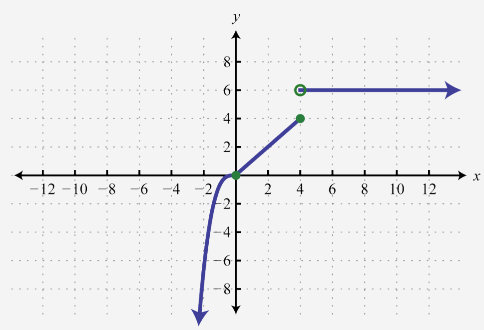
Example 3

Graph: $f(x) = \begin{cases} x^3 & \text{if} \quad x < 0 \\ x & \text{if} \quad 0 \le x \le 4 \\ 6 & \text{if} \quad x > 4 \end{cases}.$

Solution:

In this case, graph the cubing function over the interval $(-\infty, 0)$. Graph the identity function over the interval $[0, 4]$. Finally, graph the constant function $f(x) = 6$ over the interval $(4, \infty)$. And because $f(x) = 6$ where $x > 4$, we use an open dot at the point $(4, 6)$. Where $x = 4$, we use $f(x) = x$ and thus $(4, 4)$ is a point on the graph as indicated by a closed dot.

Answer:



The **greatest integer function**[55], denoted $f(x) = [\![x]\!]$, assigns the greatest integer less than or equal to any real number in its domain. For example,

$$\begin{aligned} f(2.7) &= [\![2.7]\!] = 2 \\ f(\pi) &= [\![\pi]\!] = 3 \\ f(0.23) &= [\![0.23]\!] = 0 \\ f(-3.5) &= [\![-3.5]\!] = -4 \end{aligned}$$

55. The function that assigns any real number x to the greatest integer less than or equal to x denoted $f(x) = [\![x]\!]$.

This function associates any real number with the greatest integer less than or equal to it and should not be confused with rounding off.

Example 4

Graph: $f(x) = [\![x]\!]$.

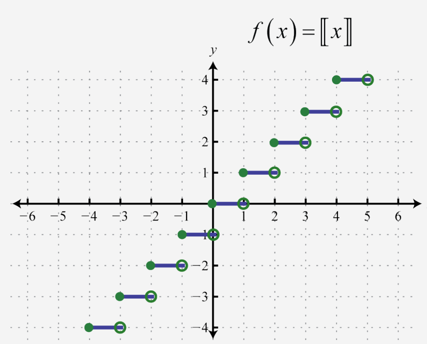
Solution:

If x is any real number, then $y = [\![x]\!]$ is the greatest integer less than or equal to x.

$$\begin{aligned} &\vdots \\ -1 \le x < 0 &\Rightarrow y = [\![x]\!] = -1 \\ 0 \le x < 1 &\Rightarrow y = [\![x]\!] = 0 \\ 1 \le x < 2 &\Rightarrow y = [\![x]\!] = 1 \\ &\vdots \end{aligned}$$

Using this, we obtain the following graph.

Answer:



The domain of the greatest integer function consists of all real numbers $\mathbb{R}$ and the range consists of the set of integers $\mathbb{Z}$. This function is often called the **floor function**[56] and has many applications in computer science.

KEY TAKEAWAYS

- Plot points to determine the general shape of the basic functions. The shape, as well as the domain and range, of each should be memorized.
- The basic polynomial functions are: $f(x) = c$, $f(x) = x$, $f(x) = x^2$, and $f(x) = x^3$.
- The basic nonpolynomial functions are: $f(x) = |x|$, $f(x) = \sqrt{x}$, and $f(x) = \frac{1}{x}$.
- A function whose definition changes depending on the value in the domain is called a piecewise function. The value in the domain determines the appropriate definition to use.

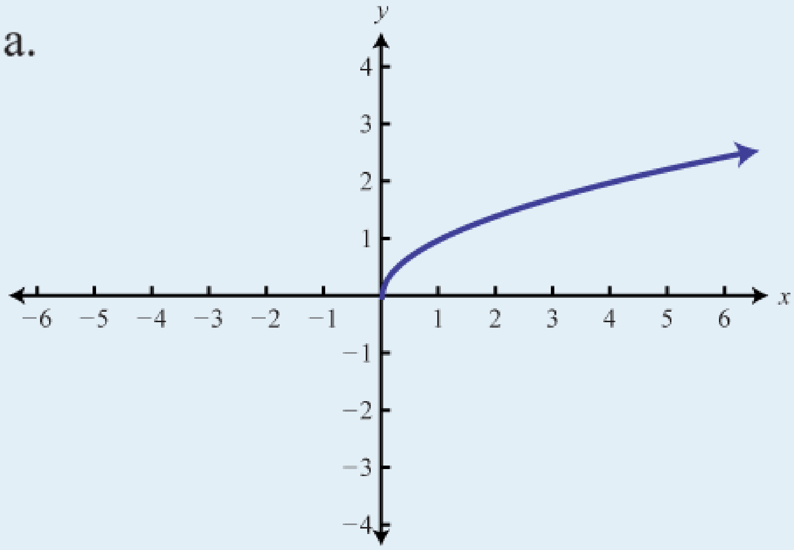
56. A term used when referring to the greatest integer function.

TOPIC EXERCISES

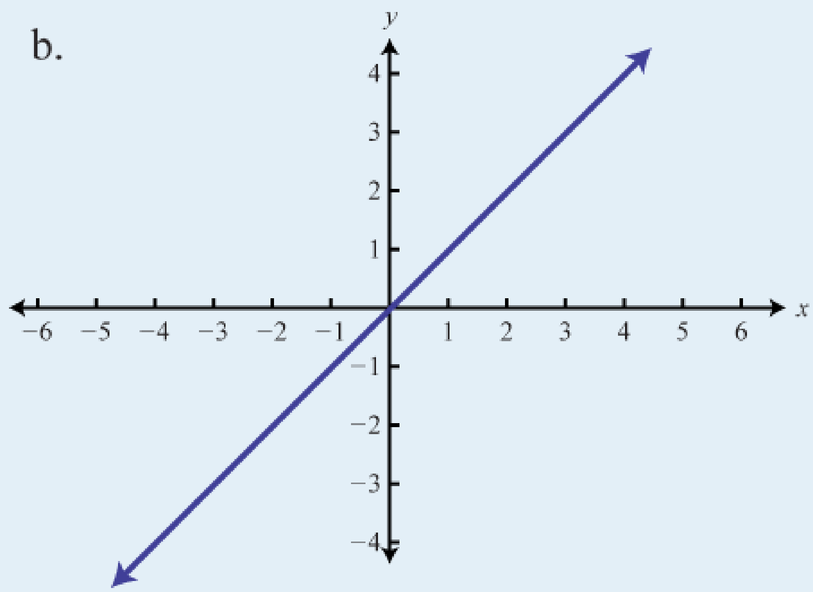
PART A: BASIC FUNCTIONS

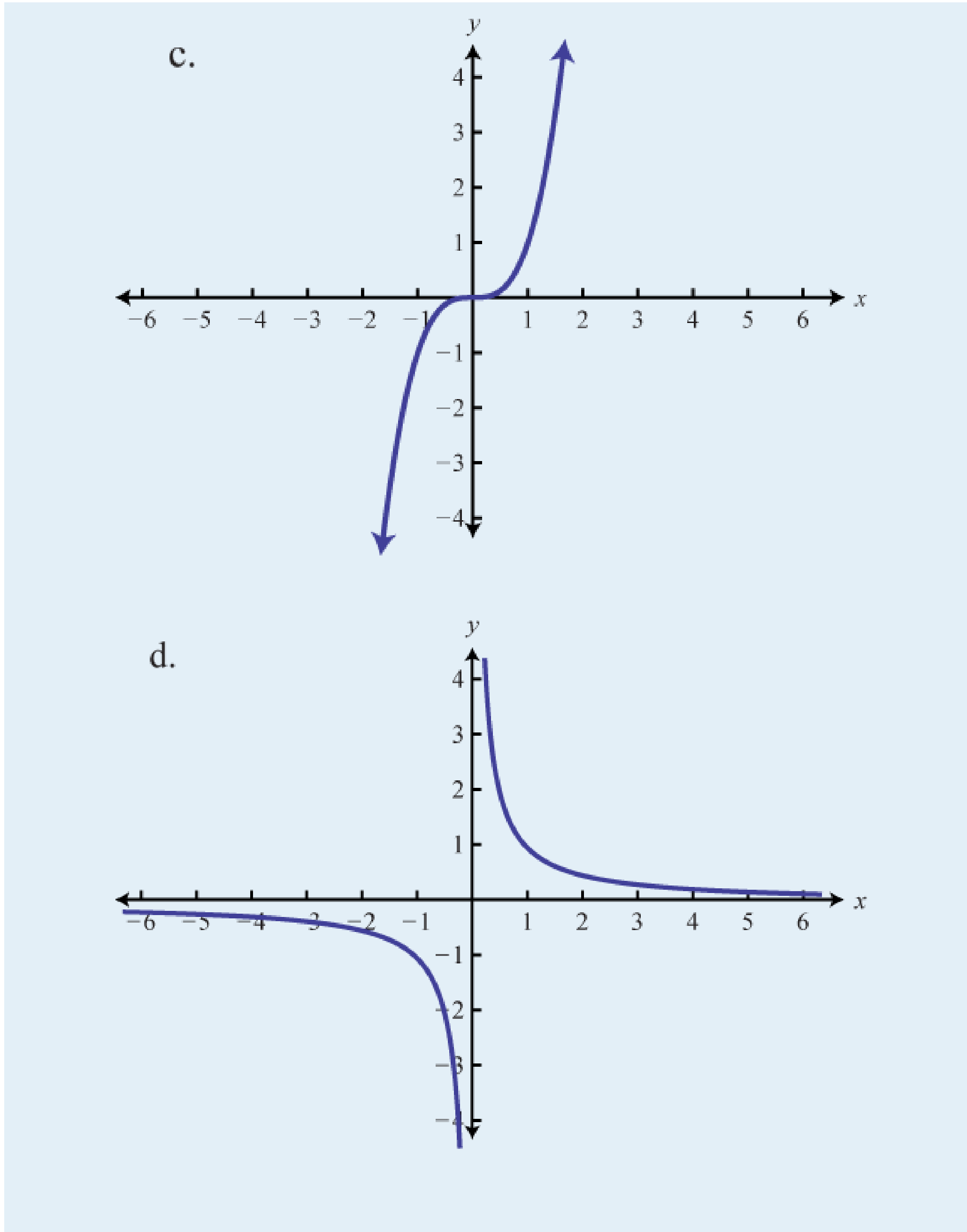
Match the graph to the function definition.

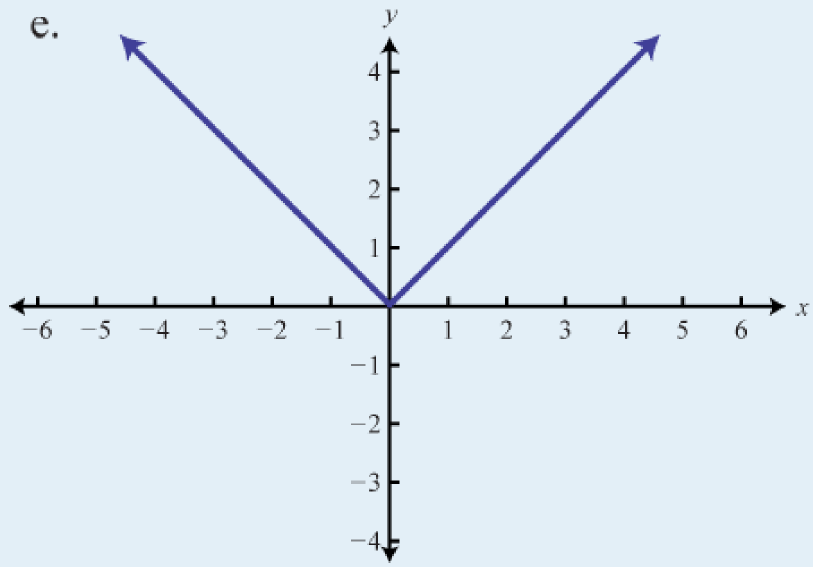
a.

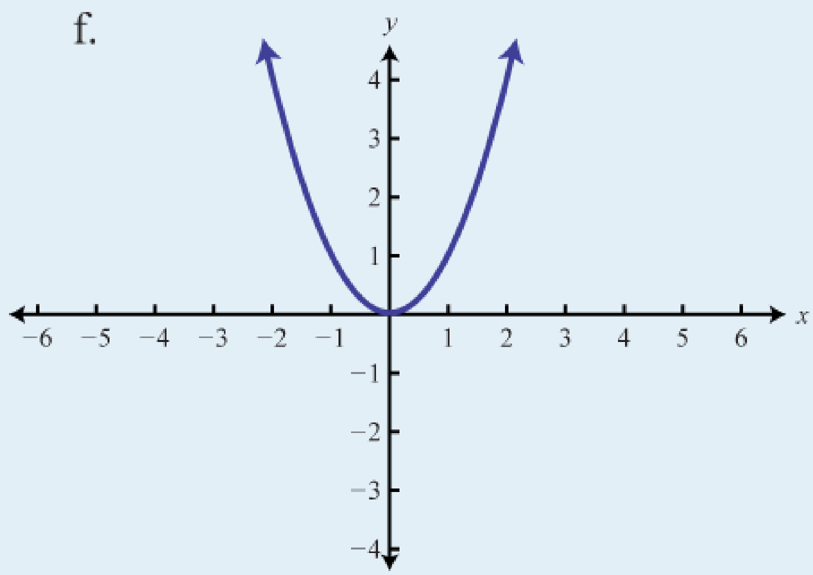


b.




c.
d.





1. $f(x) = x$
2. $f(x) = x^2$
3. $f(x) = x^3$
4. $f(x) = |x|$
5. $f(x) = \sqrt{x}$

6. $f(x) = \frac{1}{x}$

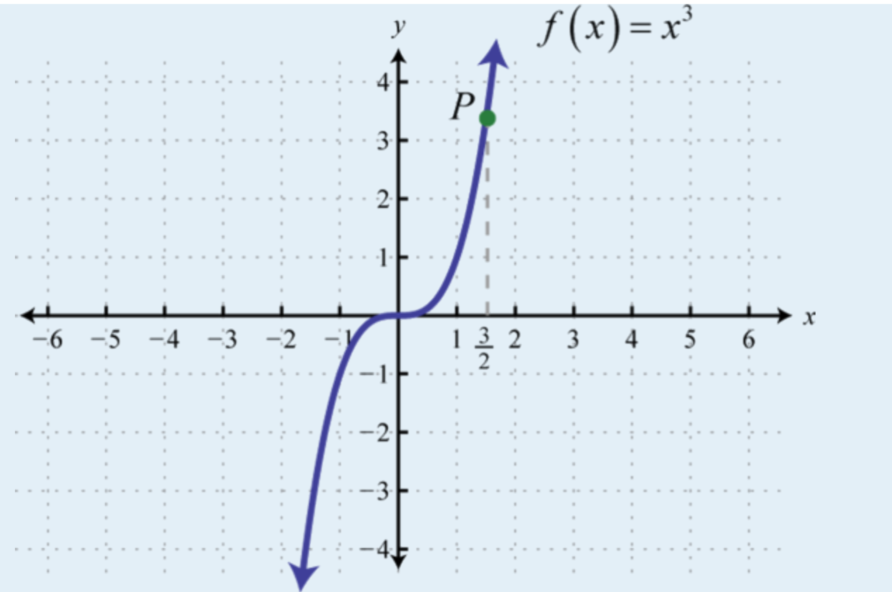
Evaluate.

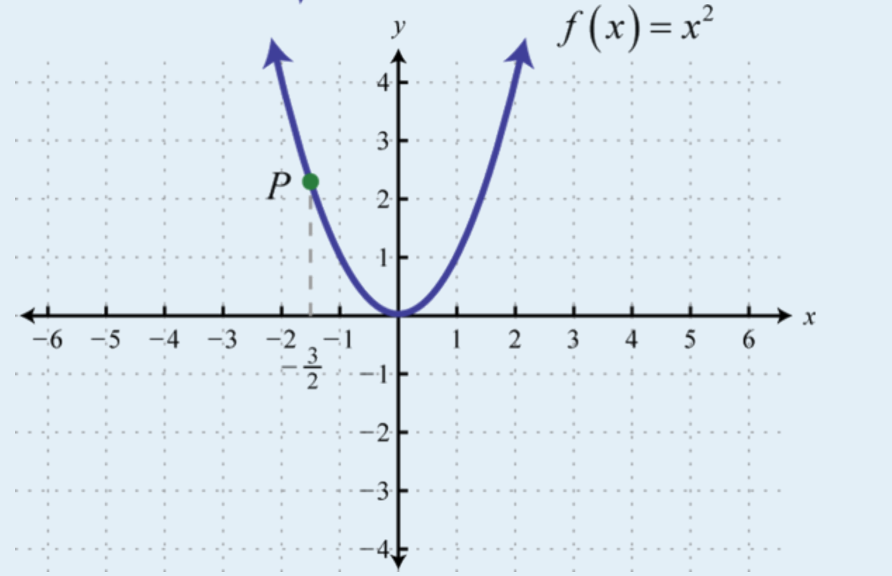
7. $f(x) = x$; find $f(-10)$, $f(0)$, and $f(a)$.
8. $f(x) = x^2$; find $f(-10)$, $f(0)$, and $f(a)$.
9. $f(x) = x^3$; find $f(-10)$, $f(0)$, and $f(a)$.
10. $f(x) = |x|$; find $f(-10)$, $f(0)$, and $f(a)$.
11. $f(x) = \sqrt{x}$; find $f(25)$, $f(0)$, and $f(a)$ where $a \geq 0$.
12. $f(x) = \frac{1}{x}$; find $f(-10)$, $f\left(\frac{1}{5}\right)$, and $f(a)$ where $a \neq 0$.
13. $f(x) = 5$; find $f(-10)$, $f(0)$, and $f(a)$.
14. $f(x) = -12$; find $f(-12)$, $f(0)$, and $f(a)$.
15. Graph $f(x) = 5$ and state its domain and range.
16. Graph $f(x) = -9$ and state its domain and range.

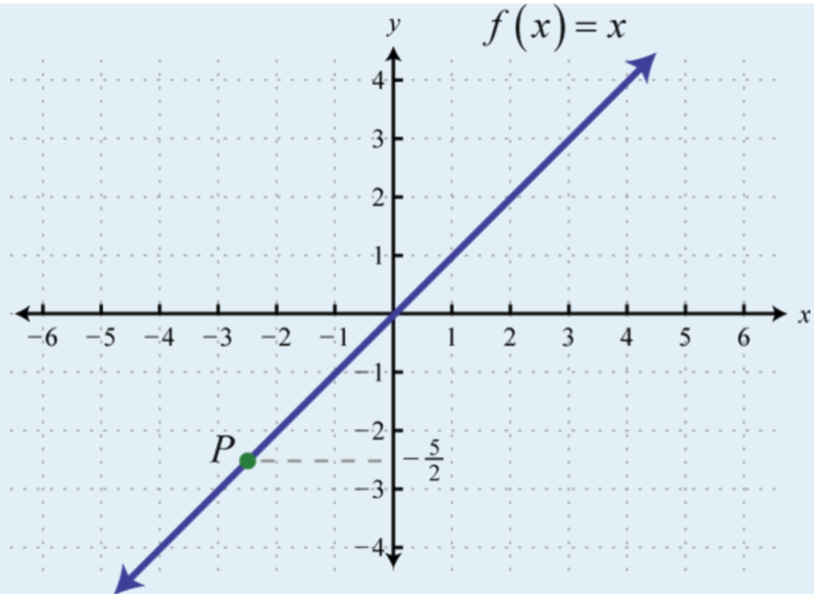
Cube root function.

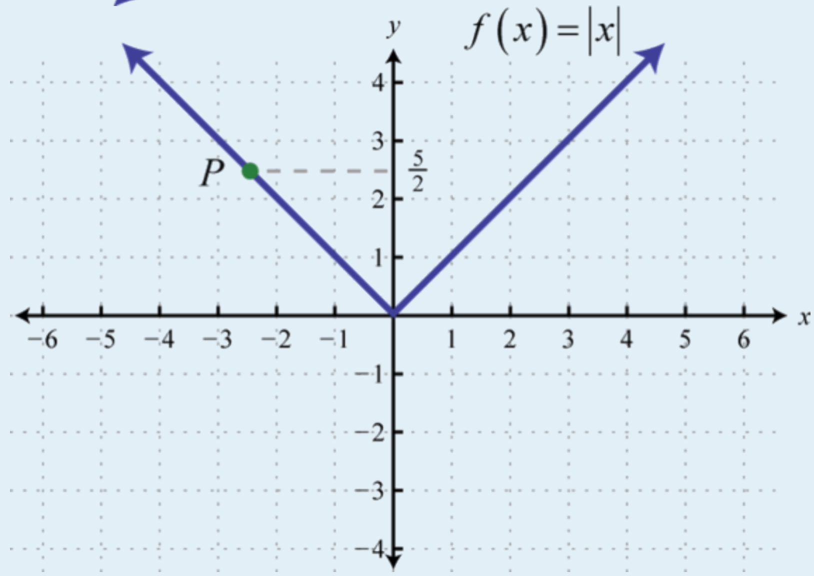
17. Find points on the graph of the function defined by $f(x) = \sqrt[3]{x}$ with x-values in the set {−8, −1, 0, 1, 8}.
18. Find points on the graph of the function defined by $f(x) = \sqrt[3]{x}$ with x-values in the set {−3, −2, 1, 2, 3}. Use a calculator and round off to the nearest tenth.
19. Graph the cube root function defined by $f(x) = \sqrt[3]{x}$ by plotting the points found in the previous two exercises.
20. Determine the domain and range of the cube root function.

Find the ordered pair that specifies the point *P*.

21. 
$f(x) = x^3$
y
x
P
4
3
2
1
−1
−2
−3
−4
−6
−5
−4
−3
−2
−1
1
$\frac{3}{2}$
2
3
4
5
6

22. 
$f(x) = x^2$
y
x
P
4
3
2
1
−1
−2
−3
−4
−6
−5
−4
−3
−2
−1
$-\frac{3}{2}$
1
2
3
4
5
6

23. 

24. 

PART B: PIECEWISE FUNCTIONS

Graph the piecewise functions.

25. $g(x) = \begin{cases} 2 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$

26. $g(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 3 & \text{if } x \geq 0 \end{cases}$

27. $h(x) = \begin{cases} x & \text{if} x < 0 \\ \sqrt{x} & \text{if} x \geq 0 \end{cases}$

28. $h(x) = \begin{cases} |x| & \text{if} x < 0 \\ x^3 & \text{if} x \geq 0 \end{cases}$

29. $f(x) = \begin{cases} |x| & \text{if} x < 2 \\ 4 & \text{if} x \geq 2 \end{cases}$

30. $f(x) = \begin{cases} x & \text{if} x < 1 \\ \sqrt{x} & \text{if} x \geq 1 \end{cases}$

31. $g(x) = \begin{cases} x^2 & \text{if} x \leq -1 \\ x & \text{if} x > -1 \end{cases}$

32. $g(x) = \begin{cases} -3 & \text{if} x \leq -1 \\ x^3 & \text{if} x > -1 \end{cases}$

33. $h(x) = \begin{cases} 0 & \text{if} x \leq 0 \\ \frac{1}{x} & \text{if} x > 0 \end{cases}$

34. $h(x) = \begin{cases} \frac{1}{x} & \text{if} x < 0 \\ x^2 & \text{if} x \geq 0 \end{cases}$

35. $f(x) = \begin{cases} x^2 & \text{if} x < 0 \\ x & \text{if} 0 \leq x < 2 \\ -2 & \text{if} x \geq 2 \end{cases}$

36. $f(x) = \begin{cases} x & \text{if} x < -1 \\ x^3 & \text{if} -1 \leq x < 1 \\ 3 & \text{if} x \geq 1 \end{cases}$

37. $g(x) = \begin{cases} 5 & \text{if} x < -2 \\ x^2 & \text{if} -2 \leq x < 2 \\ x & \text{if} x \geq 2 \end{cases}$

38. $g(x) = \begin{cases} x & \text{if} x < -3 \\ |x| & \text{if} -3 \leq x < 1 \\ \sqrt{x} & \text{if} x \geq 1 \end{cases}$

39. $h(x) = \begin{cases} \frac{1}{x} & \text{if} x < 0 \\ x^2 & \text{if} 0 \le x < 2 \\ 4 & \text{if} x \ge 2 \end{cases}$

40. $h(x) = \begin{cases} 0 & \text{if} x < 0 \\ x^3 & \text{if} 0 < x \le 2 \\ 8 & \text{if} x > 2 \end{cases}$

41. $f(x) = [\![x + 0.5]\!]$

42. $f(x) = [\![x]\!] + 1$

43. $f(x) = [\![0.5x]\!]$

44. $f(x) = 2[\![x]\!]$

Evaluate.

45. $f(x) = \begin{cases} x^2 & \text{if} x \le 0 \\ x + 2 & \text{if} x > 0 \end{cases}$

Find $f(-5), f(0)$, and $f(3)$.

46. $f(x) = \begin{cases} x^3 & \text{if} x < 0 \\ 2x - 1 & \text{if} x \ge 0 \end{cases}$

Find $f(-3), f(0)$, and $f(2)$.

47. $g(x) = \begin{cases} 5x - 2 & \text{if} x < 1 \\ \sqrt{x} & \text{if} x \ge 1 \end{cases}$

Find $g(-1)$, $g(1)$, and $g(4)$.

48. $g(x) = \begin{cases} x^3 & \text{if} x \le -2 \\ |x| & \text{if} x > -2 \end{cases}$

Find $g(-3)$, $g(-2)$, and $g(-1)$.

49. $h(x) = \begin{cases} -5 & \text{if} x < 0 \\ 2x - 3 & \text{if} 0 \le x < 2 \\ x^2 & \text{if} x \ge 2 \end{cases}$

Find $h(-2)$, $h(0)$, and $h(4)$.

50. $$h(x) = \begin{cases} -3x & \text{if } x \leq 0 \\ x^3 & \text{if } 0 < x \leq 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$$

Find $h(-5)$, $h(4)$, and $h(25)$.

51.

$$f(x) = [\![x - 0.5]\!]$$

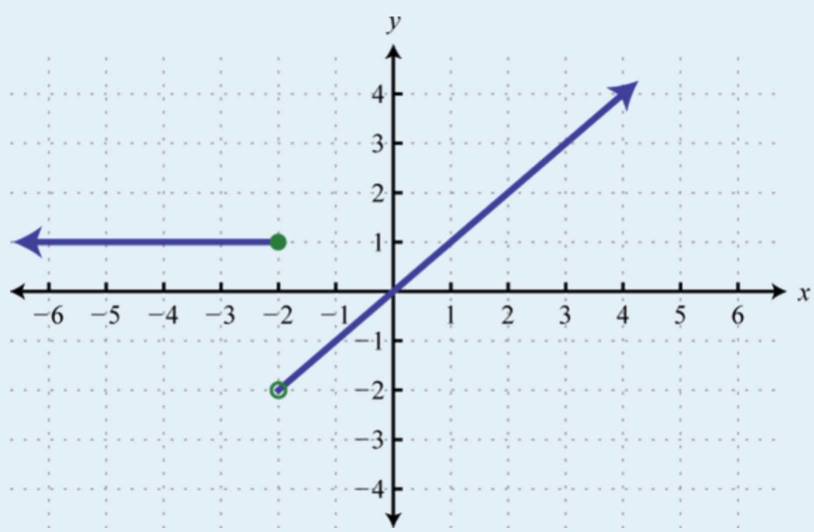
Find $f(-2)$, $f(0)$, and $f(3)$.

52.

$$f(x) = [\![2x]\!] + 1$$

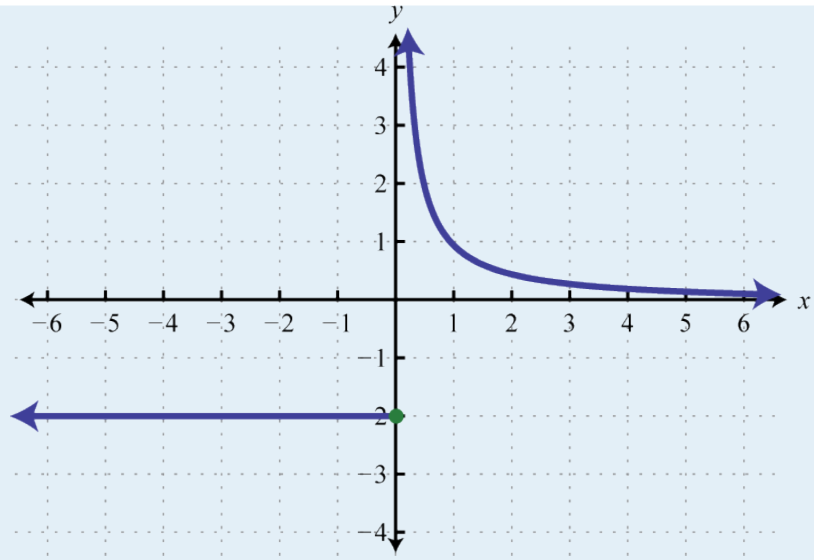
Find $f(-1.2)$, $f(0.4)$, and $f(2.6)$.

Evaluate given the graph of *f*.

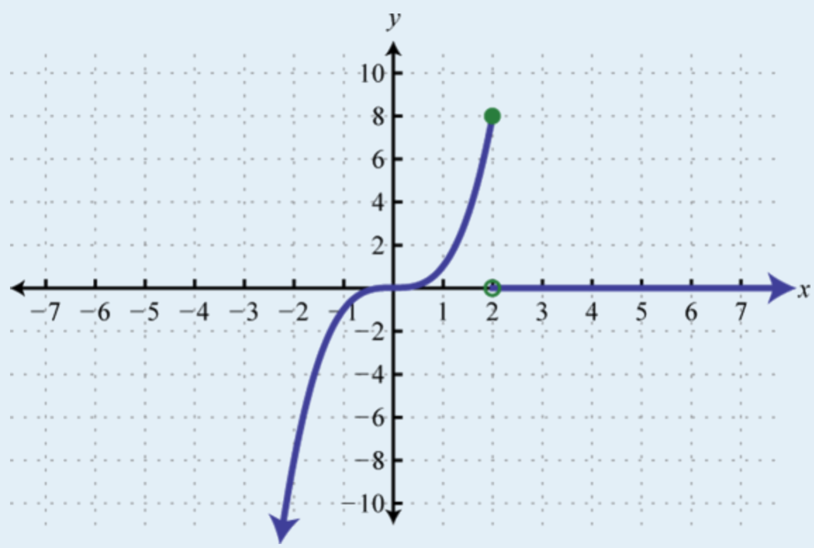


53. Find $f(-4)$, $f(-2)$, and $f(0)$.

54.

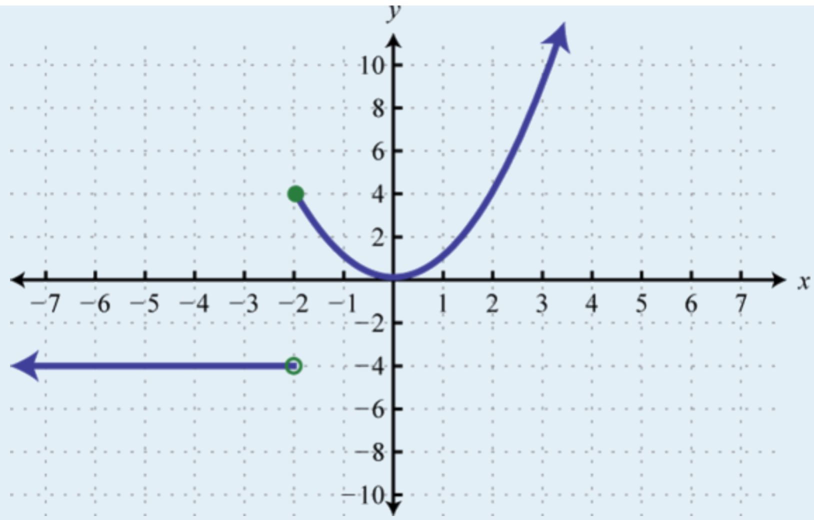


Find $f(-3)$, $f(0)$, and $f(1)$.

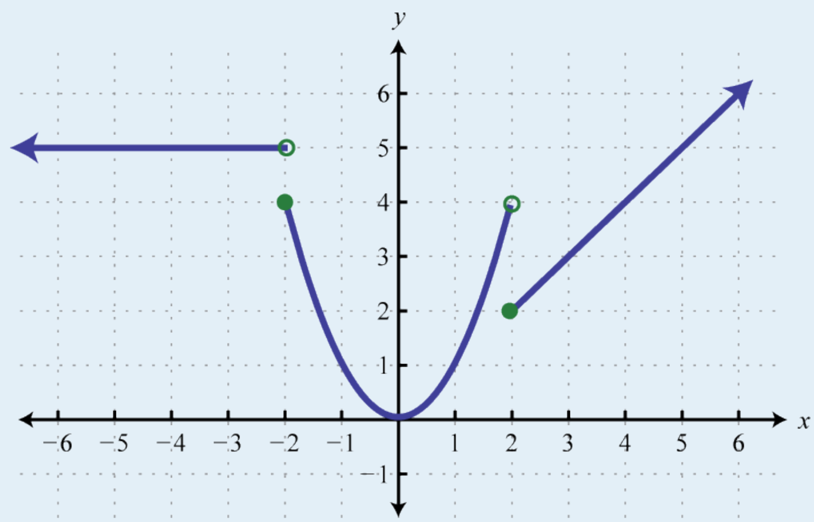


55. Find $f(0)$, $f(2)$, and $f(4)$.

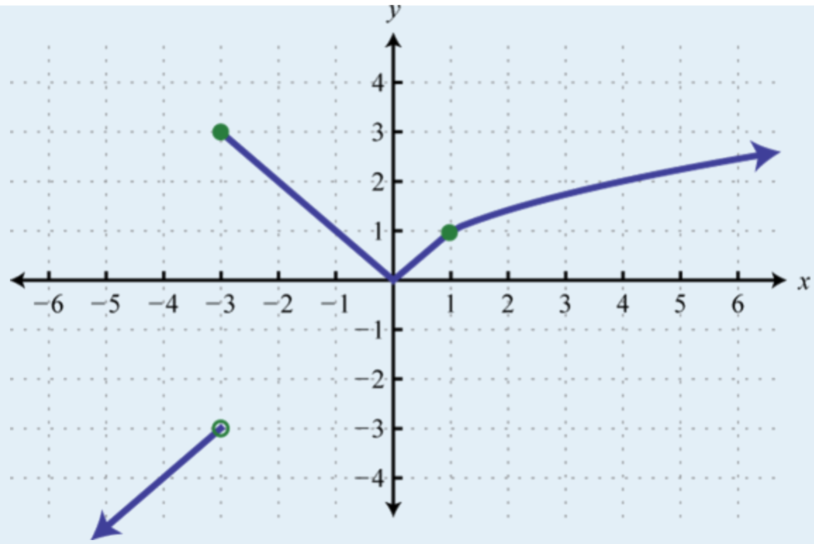
56.



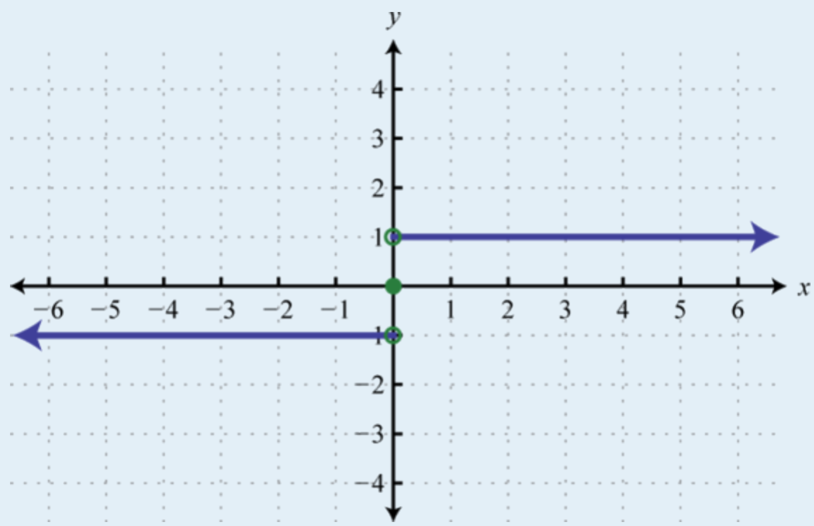
Find $f(-5)$, $f(-2)$, and $f(2)$.



57. Find $f(-3)$, $f(-2)$, and $f(2)$.
58.

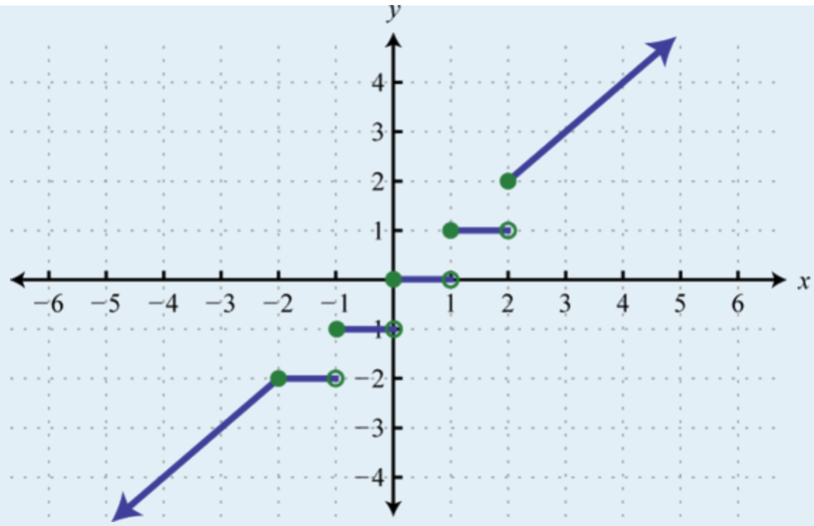


Find $f(-3)$, $f(0)$, and $f(4)$.



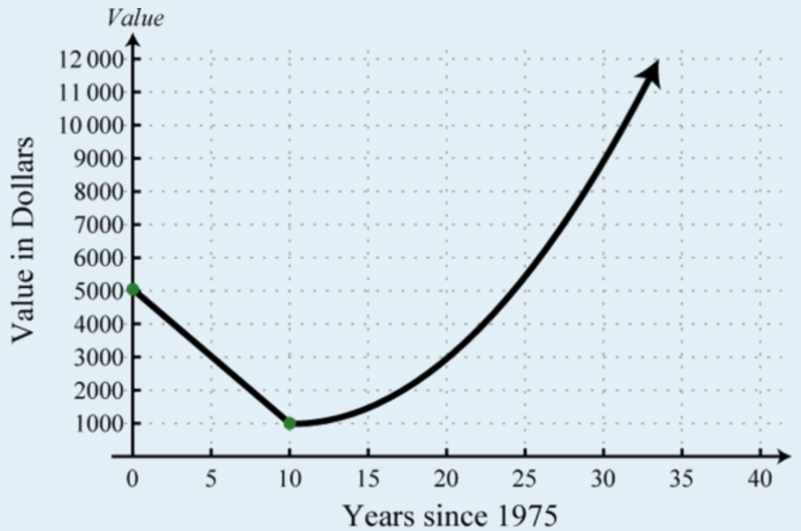
59. Find $f(-2)$, $f(0)$, and $f(2)$.

60.



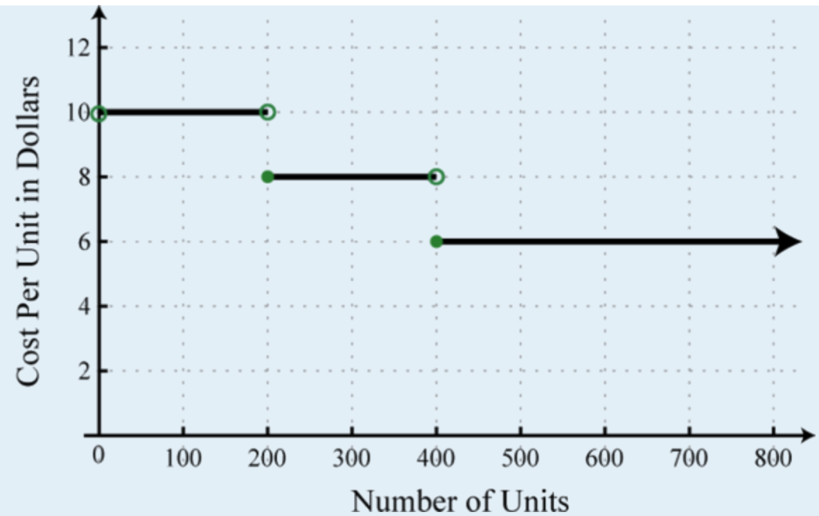
Find $f(-3)$, $f(1)$, and $f(2)$.

61. The value of an automobile in dollars is given in terms of the number of years since it was purchased new in 1975:



a. Determine the value of the automobile in the year 1980.
b. In what year is the automobile valued at $9,000?

62. The cost per unit in dollars of custom lamps depends on the number of units produced according to the following graph:



a. What is the cost per unit if 250 custom lamps are produced?
b. What level of production minimizes the cost per unit?

63. An automobile salesperson earns a commission based on total sales each month x according to the function:

$$g(x) = \begin{cases} 0.03x & if \quad 0 \leq x < \$20{,}000 \\ 0.05x & if \quad \$20{,}000 \leq x < \$50{,}000 \\ 0.07x & if \quad x \geq \$50{,}000 \end{cases}$$

a. If the salesperson's total sales for the month are \$35,500, what is her commission according to the function?
b. To reach the next level in the commission structure, how much more in sales will she need?

64. A rental boat costs \$32 for one hour, and each additional hour or partial hour costs \$8. Graph the cost of the rental boat and determine the cost to rent the boat for $4\,\frac{1}{2}$ hours.

PART C: DISCUSSION BOARD

65. Explain to a beginning algebra student what an asymptote is.

66. Research and discuss the difference between the floor and ceiling functions. What applications can you find that use these functions?

ANSWERS

1. b

3. c

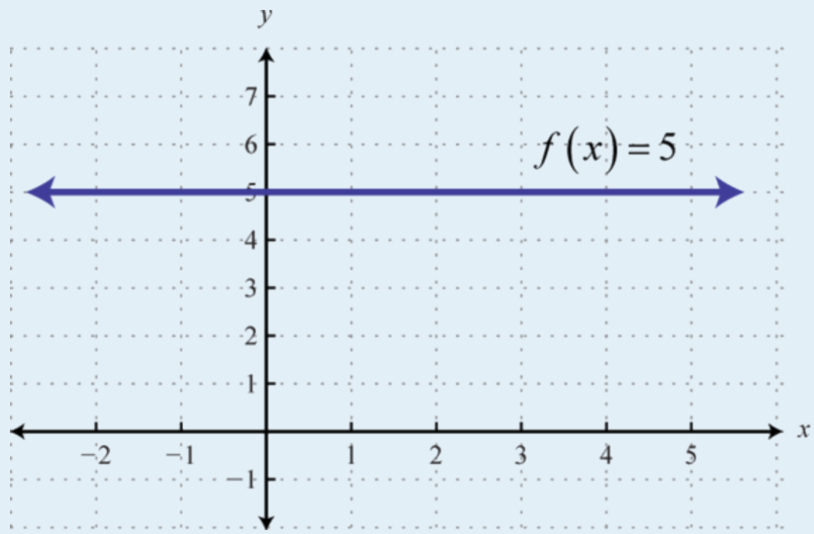
5. a

7. $f(-10) = -10, f(0) = 0, f(a) = a$

9. $f(-10) = -1{,}000, f(0) = 0, f(a) = a^3$

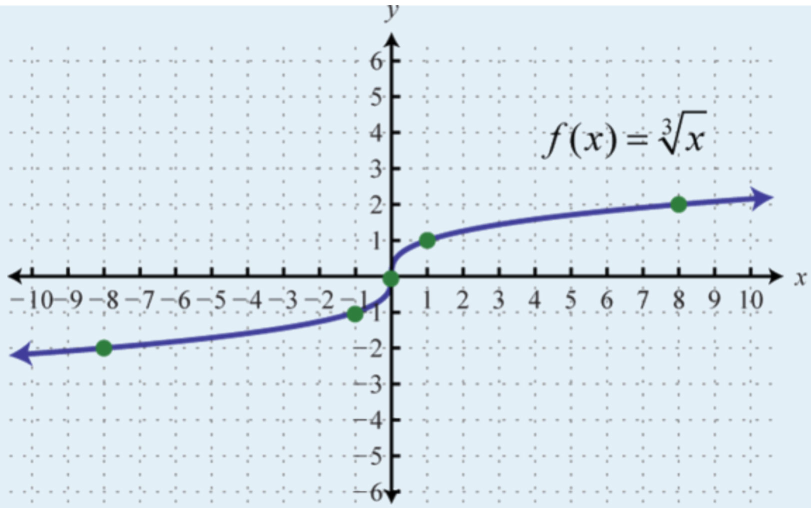
11. $f(25) = 5, f(0) = 0, f(a) = \sqrt{a}$

13. $f(-10) = 5, f(0) = 5, f(a) = 5$



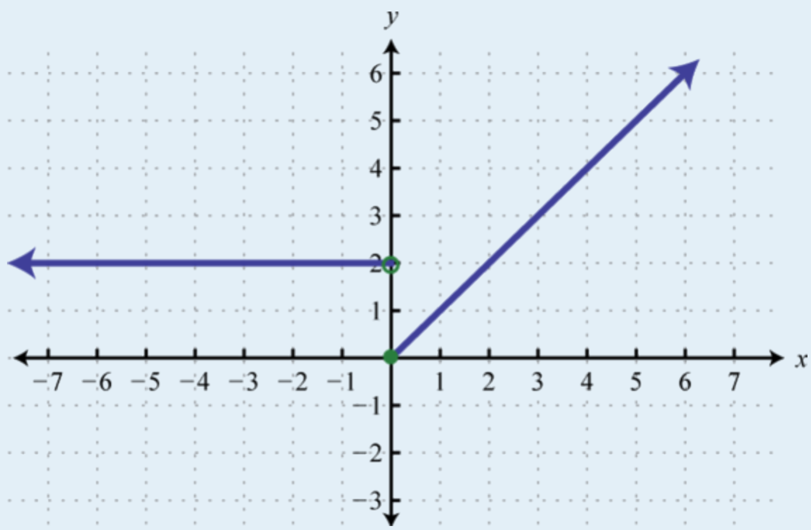
15. Domain: $\mathbb{R}$; range: $\{5\}$

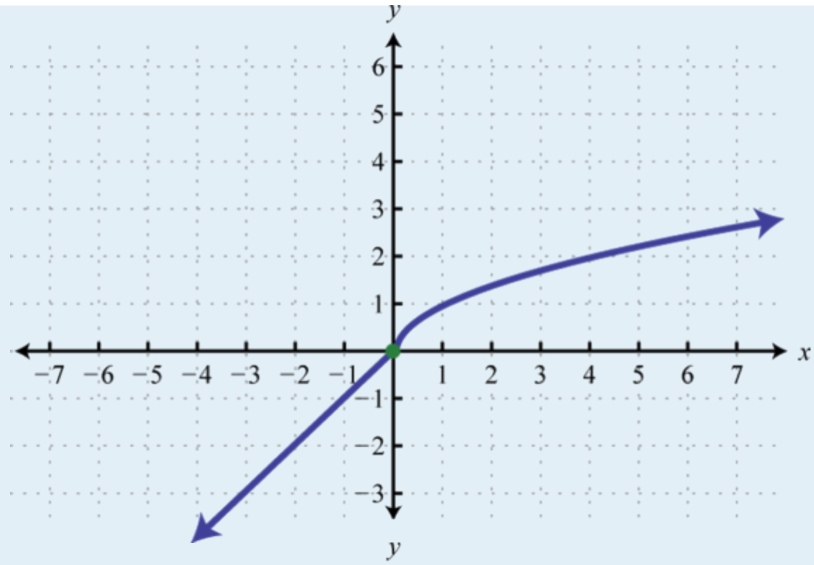
17. {(−8,−2), (−1,−1), (0,0), (1,1), (8,2)}

19. 

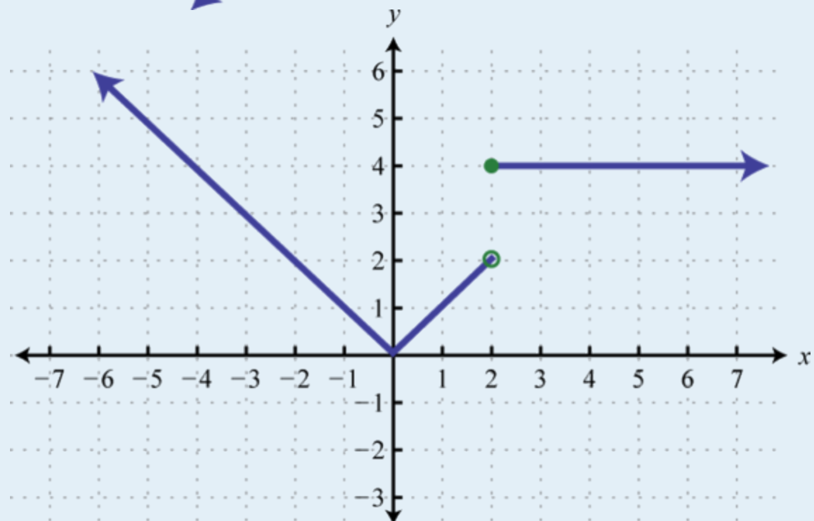
21. $\left(\frac{3}{2}, \frac{27}{8}\right)$

23. $\left(-\frac{5}{2}, -\frac{5}{2}\right)$

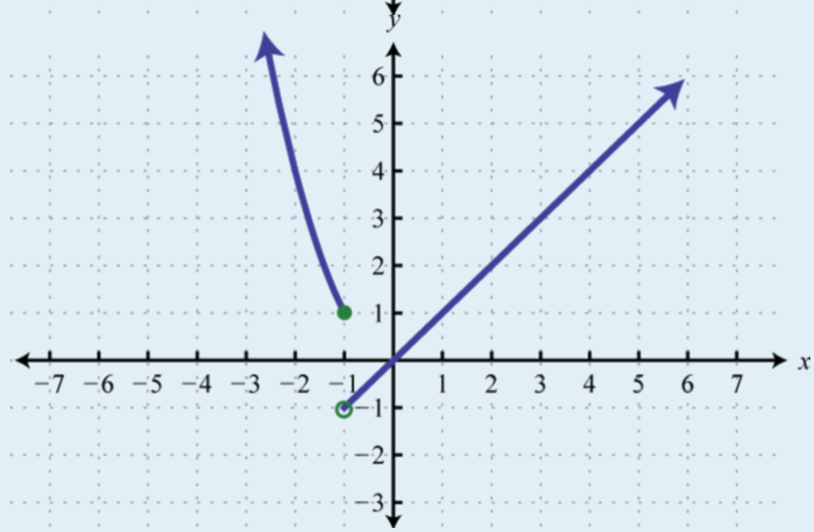
25. 


y
x
−7 −6 −5 −4 −3 −2 −1 1 2 3 4 5 6 7
6 5 4 3 2 1 −1 −2 −3

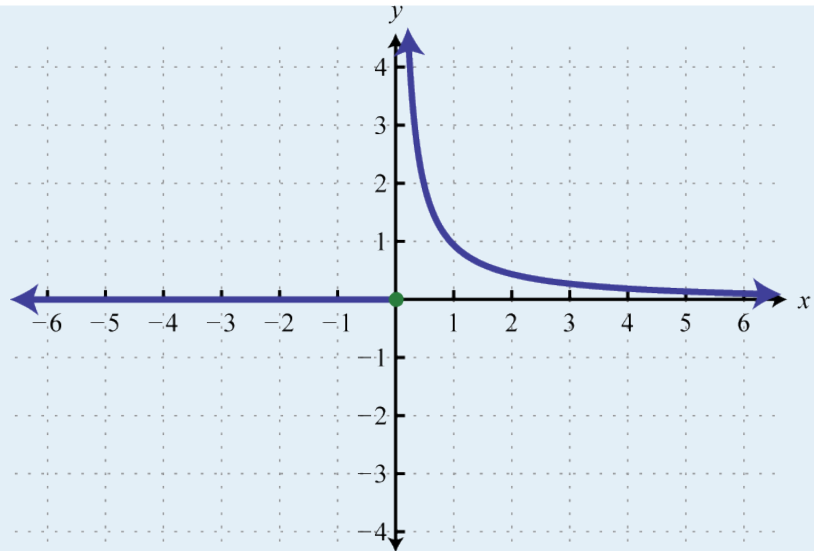
27.

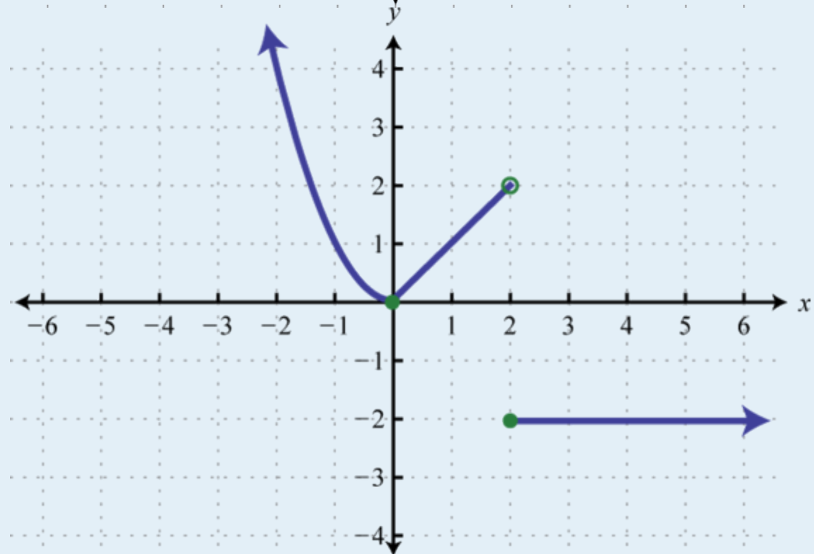

y
x
−7 −6 −5 −4 −3 −2 −1 1 2 3 4 5 6 7
6 5 4 3 2 1 −1 −2 −3

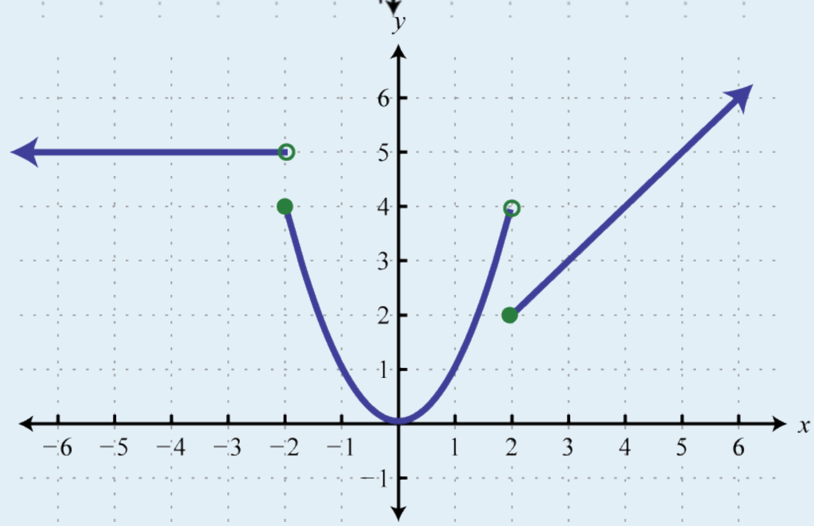
29.

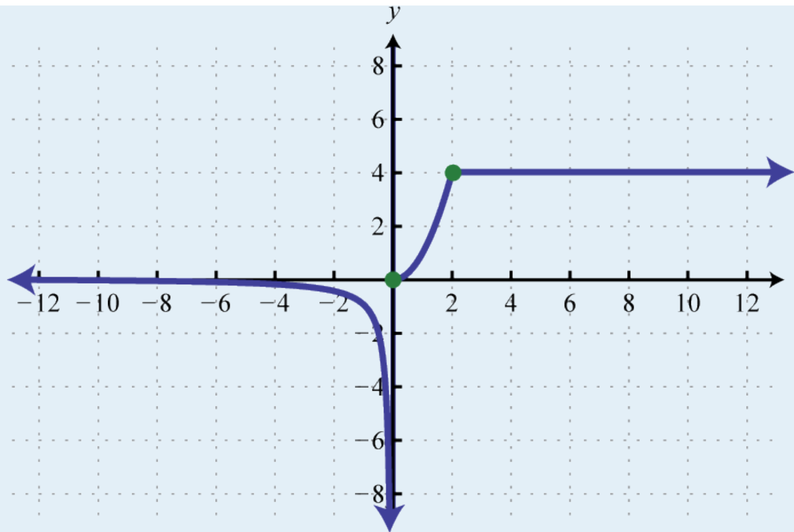

y
x
−7 −6 −5 −4 −3 −2 −1 1 2 3 4 5 6 7
6 5 4 3 2 1 −1 −2 −3

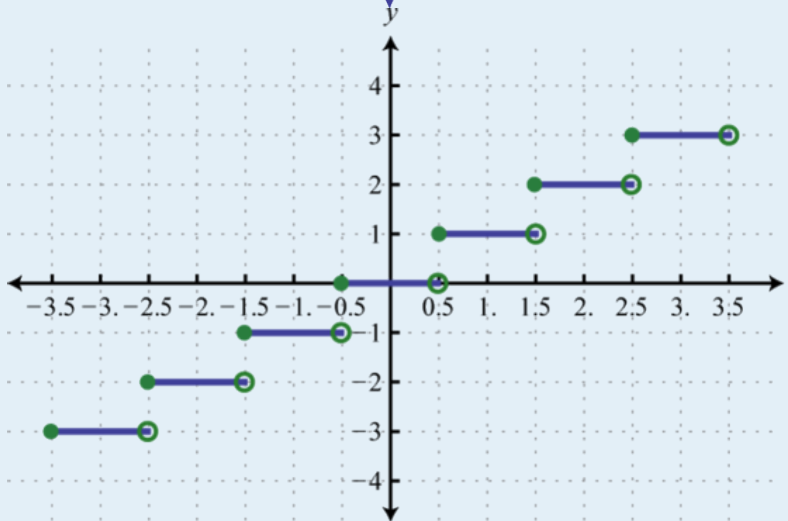
31.

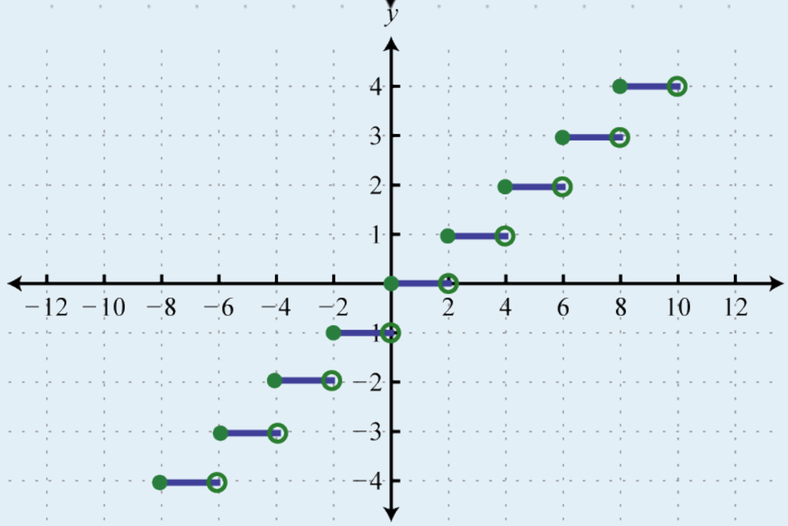
33. 
y
x
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6
4 3 2 1 −1 −2 −3 −4

35. 
y
x
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6
4 3 2 1 −1 −2 −3 −4

37. 
y
x
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6
6 5 4 3 2 1 −1

39. 

41. 

43. 

45. $f\left(-5\right) = 25$, $f(0) = 0$, and $f(3) = 5$

47. $g(-1) = -7, g(1) = 1$, and $g(4) = 2$

49. $h(-2) = -5, h(0) = -3$, and $h(4) = 16$

51. $f(-2) = -3, f(0) = -1$, and $f(3) = 2$

53. $f(-4) = 1, f(-2) = 1$, and $f(0) = 0$

55. $f(0) = 0, f(2) = 8$, and $f(4) = 0$

57. $f(-3) = 5, f(-2) = 4$, and $f(2) = 2$

59. $f(-2) = -1, f(0) = 0$, and $f(2) = 1$

61.
   a. $3,000;
   b. 2005

63.
   a. $1,775;
   b. $14,500

65. Answer may vary

2.5 Using Transformations to Graph Functions

LEARNING OBJECTIVES

1. Define the rigid transformations and use them to sketch graphs.
2. Define the non-rigid transformations and use them to sketch graphs.

Vertical and Horizontal Translations

When the graph of a function is changed in appearance and/or location we call it a transformation. There are two types of transformations. A **rigid transformation**[57] changes the location of the function in a coordinate plane, but leaves the size and shape of the graph unchanged. A **non-rigid transformation**[58] changes the size and/or shape of the graph.

A **vertical translation**[59] is a rigid transformation that shifts a graph up or down relative to the original graph. This occurs when a constant is added to any function. If we add a positive constant to each *y*-coordinate, the graph will shift up. If we add a negative constant, the graph will shift down. For example, consider the functions $g(x) = x^2 - 3$ and $h(x) = x^2 + 3$. Begin by evaluating for some values of the independent variable *x*.

$g(x) = x^2 - 3$

x	$g(x)$
-2	1
-1	-2
0	-3
1	-2
2	1

$h(x) = x^2 + 3$

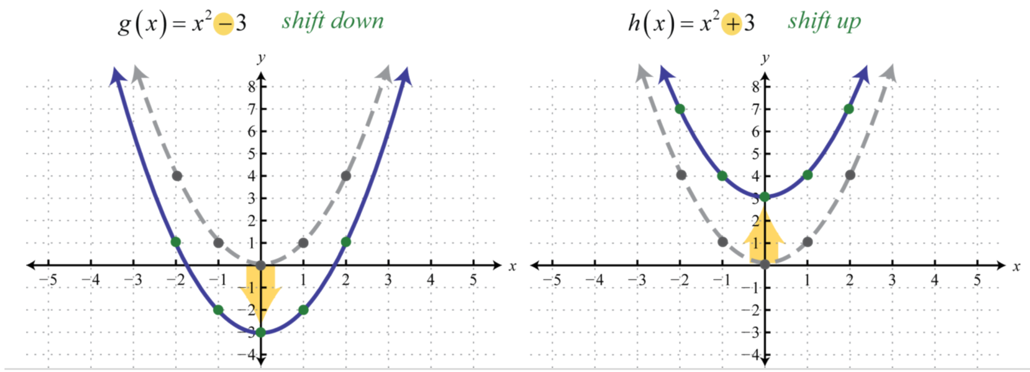
x	$h(x)$
-2	7
-1	4
0	3
1	4
2	7

Now plot the points and compare the graphs of the functions *g* and *h* to the basic graph of $f(x) = x^2$, which is shown using a dashed grey curve below.

57. A set of operations that change the location of a graph in a coordinate plane but leave the size and shape unchanged.

58. A set of operations that change the size and/or shape of a graph in a coordinate plane.

59. A rigid transformation that shifts a graph up or down.



The function g shifts the basic graph down 3 units and the function h shifts the basic graph up 3 units. In general, this describes the vertical translations; if k is any positive real number:

Vertical shift up k units:	$F(x) = f(x) + k$
Vertical shift down k units:	$F(x) = f(x) - k$

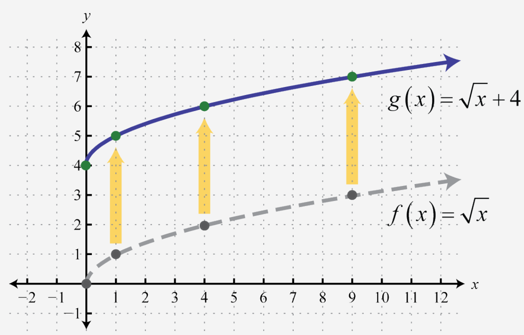
Example 1

Sketch the graph of $g(x) = \sqrt{x} + 4$.

Solution:

Begin with the basic function defined by $f(x) = \sqrt{x}$ and shift the graph up 4 units.

Answer:



A **horizontal translation**[60] is a rigid transformation that shifts a graph left or right relative to the original graph. This occurs when we add or subtract constants from the x-coordinate before the function is applied. For example, consider the functions defined by $g(x) = (x + 3)^2$ and $h(x) = (x - 3)^2$ and create the following tables:

$g(x) = (x+3)^2$

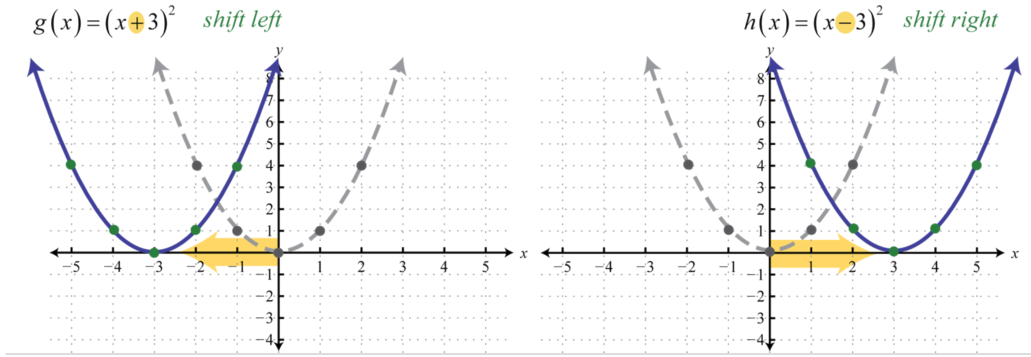
x	$g(x)$
-5	4
-4	1
-3	0
-2	1
-1	4

$h(x) = (x-3)^2$

x	$h(x)$
1	4
2	1
3	0
4	1
5	4

Here we add and subtract from the x-coordinates and then square the result. This produces a horizontal translation.

60. A rigid transformation that shifts a graph left or right.



Note that this is the opposite of what you might expect. In general, this describes the horizontal translations; if h is any positive real number:

Horizontal shift left *h* units:	$F(x) = f(x + h)$
Horizontal shift right *h* units:	$F(x) = f(x - h)$

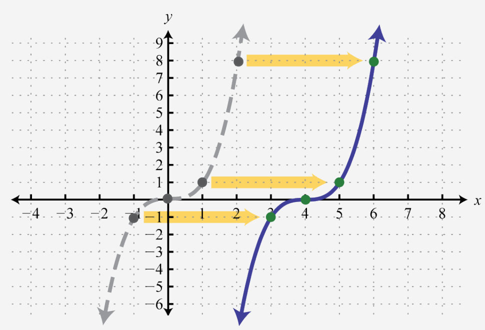
Example 2

Sketch the graph of $g(x) = (x - 4)^3$.

Solution:

Begin with a basic cubing function defined by $f(x) = x^3$ and shift the graph 4 units to the right.

Answer:



It is often the case that combinations of translations occur.

Example 3

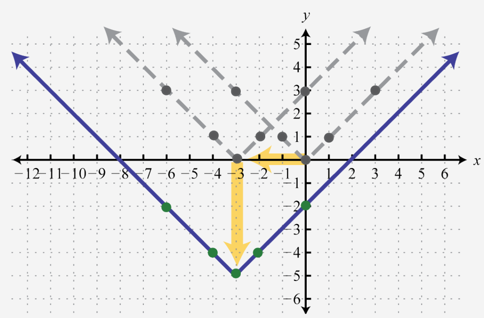
Sketch the graph of $g(x) = |x + 3| - 5$.

Solution:

Start with the absolute value function and apply the following transformations.

$$\begin{aligned} y &= |x| && \textit{Basic function} \\ y &= |x+3| && \textit{Horizontal shift left 3 units} \\ y &= |x+3| - 5 && \textit{Vertical shift down 5 units} \end{aligned}$$

Answer:



The order in which we apply horizontal and vertical translations does not affect the final graph.

Example 4

Sketch the graph of $g(x) = \frac{1}{x-5} + 3$.

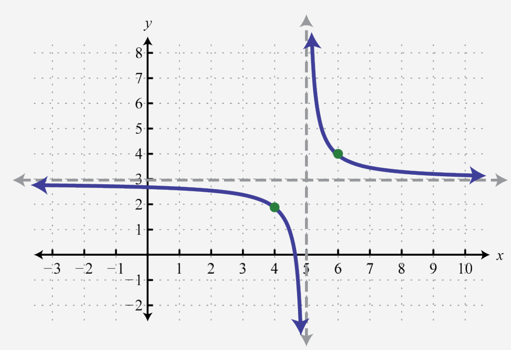
Solution:

Begin with the reciprocal function and identify the translations.

$$y = \frac{1}{x} \qquad \textit{Basic function}$$
$$y = \frac{1}{x-5} \qquad \textit{Horizontal shift right 5 units}$$
$$y = \frac{1}{x-5} + 3 \qquad \textit{Vertical shift up 3 units}$$

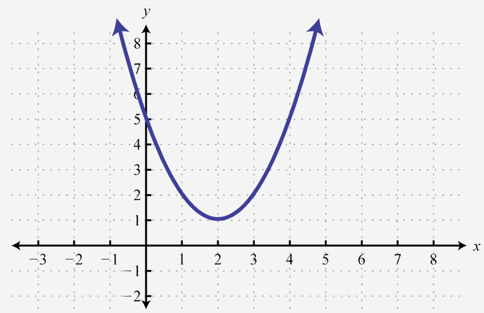
Take care to shift the vertical asymptote from the *y*-axis 5 units to the right and shift the horizontal asymptote from the *x*-axis up 3 units.

Answer:



Try this! Sketch the graph of $g(x) = (x - 2)^2 + 1$.

Answer:



(click to see video)

Reflections

A **reflection**[61] is a transformation in which a mirror image of the graph is produced about an axis. In this section, we will consider reflections about the x- and y-axis. The graph of a function is reflected about the x-axis if each y-coordinate is multiplied by −1. The graph of a function is reflected about the y-axis if each x-coordinate is multiplied by −1 before the function is applied. For example, consider $g(x) = \sqrt{-x}$ and $h(x) = -\sqrt{x}$.

$g(x) = \sqrt{-x}$

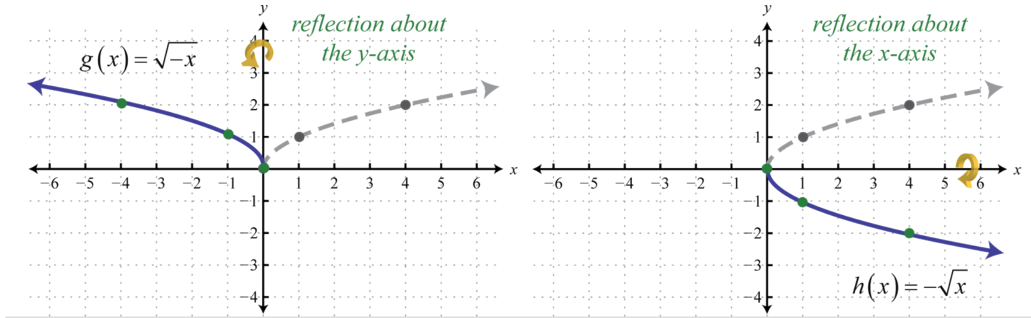
x	$g(x)$
-4	2
-1	1
0	0

$h(x) = -\sqrt{x}$

x	$h(x)$
4	-2
1	-1
0	0

Compare the graph of g and h to the basic square root function defined by $f(x) = \sqrt{x}$, shown dashed in grey below:

61. A transformation that produces a mirror image of the graph about an axis.



The first function g has a negative factor that appears "inside" the function; this produces a reflection about the y-axis. The second function h has a negative factor that appears "outside" the function; this produces a reflection about the x-axis. In general, it is true that:

Reflection about the *y*-axis:	$F(x) = f(-x)$
Reflection about the *x*-axis:	$F(x) = -f(x)$

When sketching graphs that involve a reflection, consider the reflection first and then apply the vertical and/or horizontal translations.

Example 5

Sketch the graph of $g(x) = -(x+5)^2 + 3$.

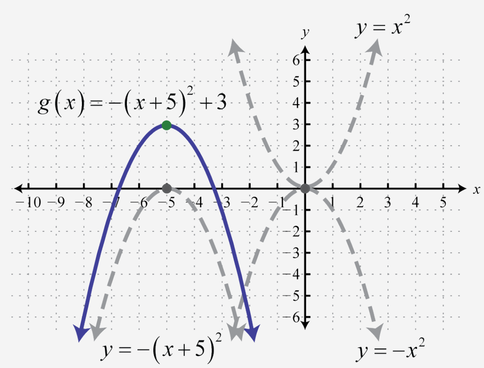
Solution:

Begin with the squaring function and then identify the transformations starting with any reflections.

$y = x^2$	*Basic function.*
$y = -x^2$	*Reflection about the x-axis.*
$y = -(x+5)^2$	*Horizontal shift left 5 units.*
$y = -(x+5)^2 + 3$	*Vertical shift up 3 units.*

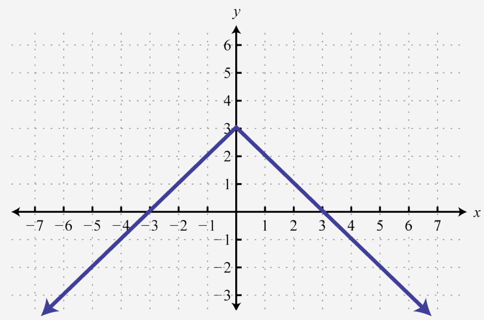
Use these translations to sketch the graph.

Answer:



Try this! Sketch the graph of $g(x) = -|x| + 3$.

Answer:



(click to see video)

Dilations

Horizontal and vertical translations, as well as reflections, are called rigid transformations because the shape of the basic graph is left unchanged, or rigid. Functions that are multiplied by a real number other than 1, depending on the real number, appear to be stretched vertically or stretched horizontally. This type of non-rigid transformation is called a **dilation**[62]. For example, we can multiply the squaring function $f(x) = x^2$ by 4 and $\frac{1}{4}$ to see what happens to the graph.

$$g(x) = 4x^2$$

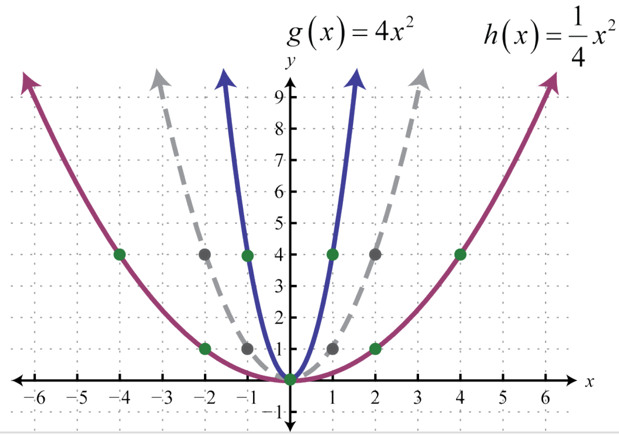
x	$g(x)$
-1	4
0	0
1	4

$$h(x) = \frac{1}{4}x^2$$

x	$h(x)$
-2	1
0	0
2	1

Compare the graph of g and h to the basic squaring function defined by $f(x) = x^2$, shown dashed in grey below:

62. A non-rigid transformation, produced by multiplying functions by a nonzero real number, which appears to stretch the graph either vertically or horizontally.



The function *g* is steeper than the basic squaring function and its graph appears to have been stretched vertically. The function *h* is not as steep as the basic squaring function and appears to have been stretched horizontally.

In general, we have:

Dilation:	$F(x) = a \cdot f(x)$

If the factor *a* is a nonzero fraction between -1 and 1, it will stretch the graph horizontally. Otherwise, the graph will be stretched vertically. If the factor *a* is negative, then it will produce a reflection as well.

Example 6

Sketch the graph of $g(x) = -2|x-5| - 3$.

Solution:

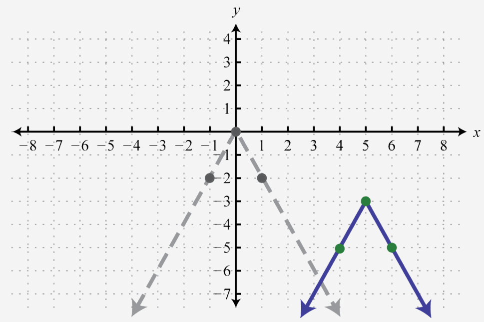
Here we begin with the product of -2 and the basic absolute value function: $y = -2|x|$. This results in a reflection and a dilation.

x	y	$y = -2\|x\|$ $\leftarrow$ *Dilation and reflection*
-1	-2	$y = -2\|-1\| = -2 \cdot 1 = -2$
0	0	$y = -2\|0\| = -2 \cdot 0 = 0$
1	-2	$y = -2\|1\| = -2 \cdot 1 = -2$

Use the points {(-1, -2), (0, 0), (1, -2)} to graph the reflected and dilated function $y = -2|x|$. Then translate this graph 5 units to the right and 3 units down.

$$\begin{aligned} y &= -2|x| && \textit{Basic graph with dilation and} \\ &&& \textit{reflection about the } x-\textit{axis.} \\ y &= -2|x-5| && \textit{Shift right 5 units.} \\ y &= -2|x-5| - 3 && \textit{Shift down 3 units.} \end{aligned}$$

Answer:



In summary, given positive real numbers *h* and *k*:

Vertical shift up *k* units:	$F(x) = f(x) + k$
Vertical shift down *k* units:	$F(x) = f(x) - k$

Horizontal shift left *h* units:	$F(x) = f(x + h)$
Horizontal shift right *h* units:	$F(x) = f(x - h)$

Reflection about the *y*-axis:	$F(x) = f(-x)$
Reflection about the *x*-axis:	$F(x) = -f(x)$

Dilation:	$F(x) = a \cdot f(x)$

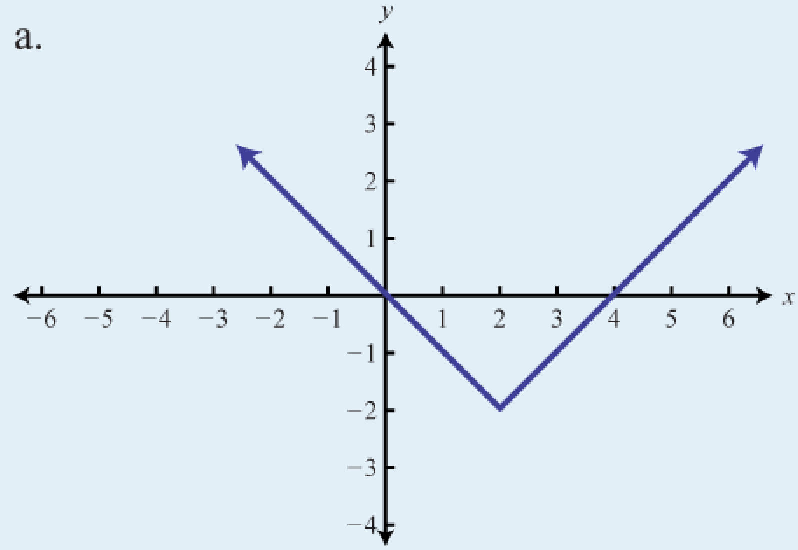
KEY TAKEAWAYS

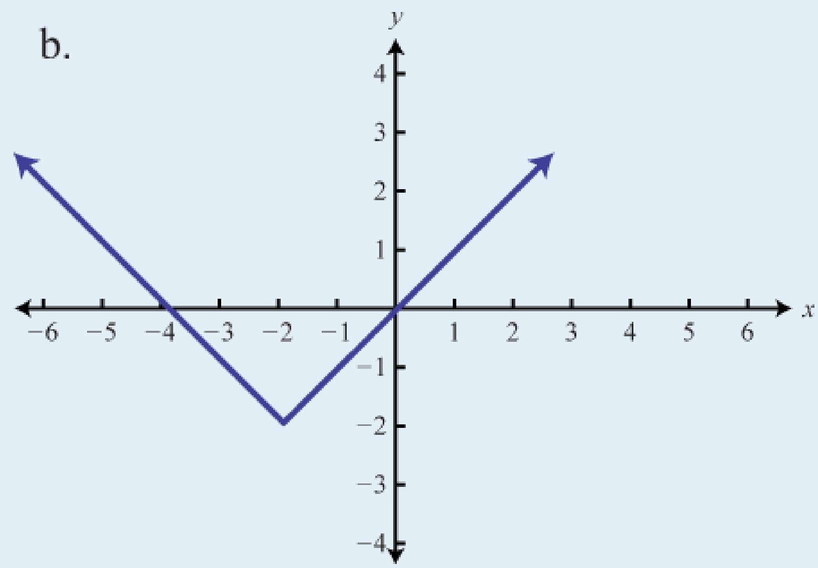
- Identifying transformations allows us to quickly sketch the graph of functions. This skill will be useful as we progress in our study of mathematics. Often a geometric understanding of a problem will lead to a more elegant solution.
- If a positive constant is added to a function, $f(x) + k$, the graph will shift up. If a positive constant is subtracted from a function, $f(x) - k$, the graph will shift down. The basic shape of the graph will remain the same.
- If a positive constant is added to the value in the domain before the function is applied, $f(x + h)$, the graph will shift to the left. If a positive constant is subtracted from the value in the domain before the function is applied, $f(x - h)$, the graph will shift right. The basic shape will remain the same.
- Multiplying a function by a negative constant, $-f(x)$, reflects its graph in the *x*-axis. Multiplying the values in the domain by −1 before applying the function, $f(-x)$, reflects the graph about the *y*-axis.
- When applying multiple transformations, apply reflections first.
- Multiplying a function by a constant other than 1, $a \cdot f(x)$, produces a dilation. If the constant is a positive number greater than 1, the graph will appear to stretch vertically. If the positive constant is a fraction less than 1, the graph will appear to stretch horizontally.

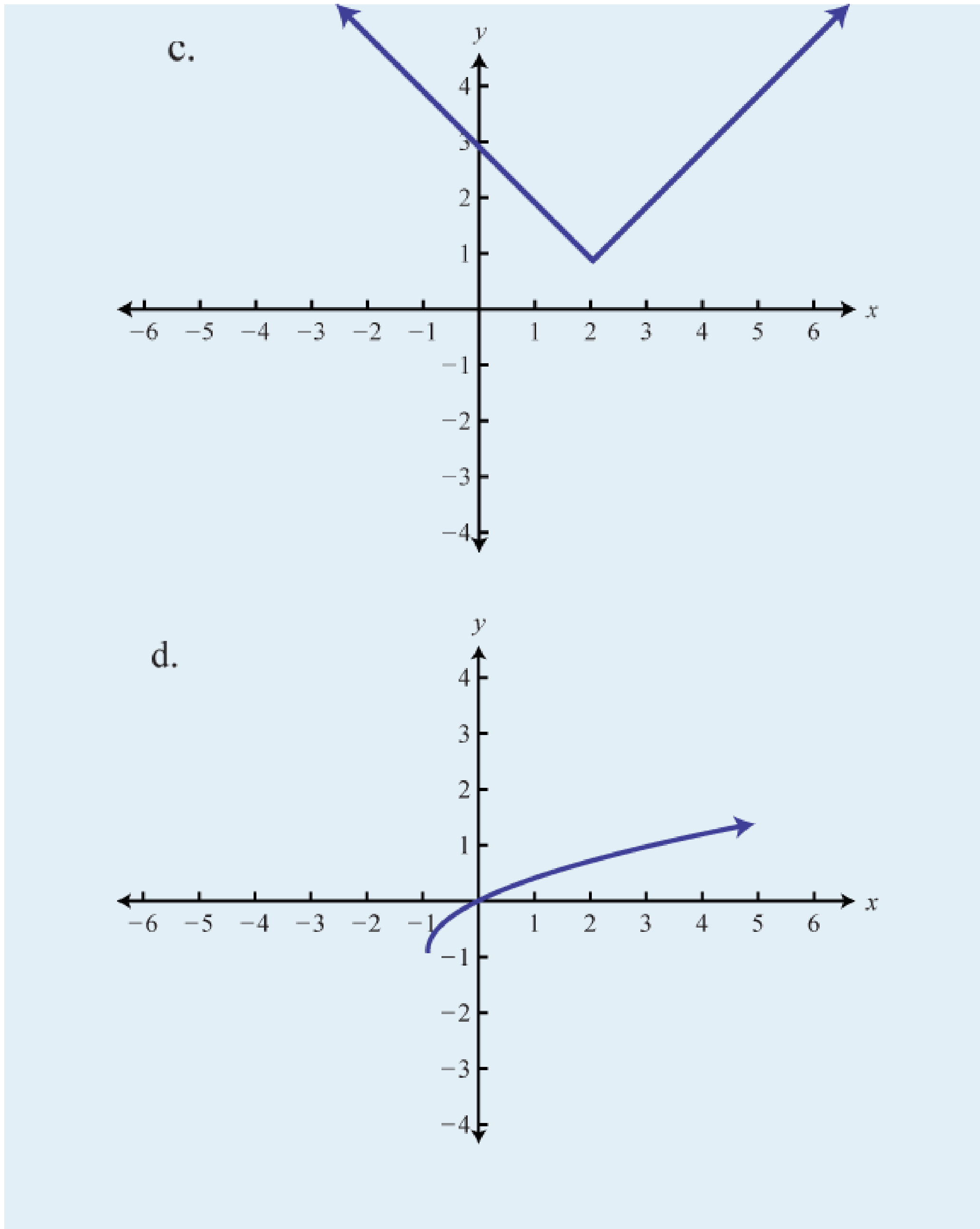
TOPIC EXERCISES

PART A: VERTICAL AND HORIZONTAL TRANSLATIONS

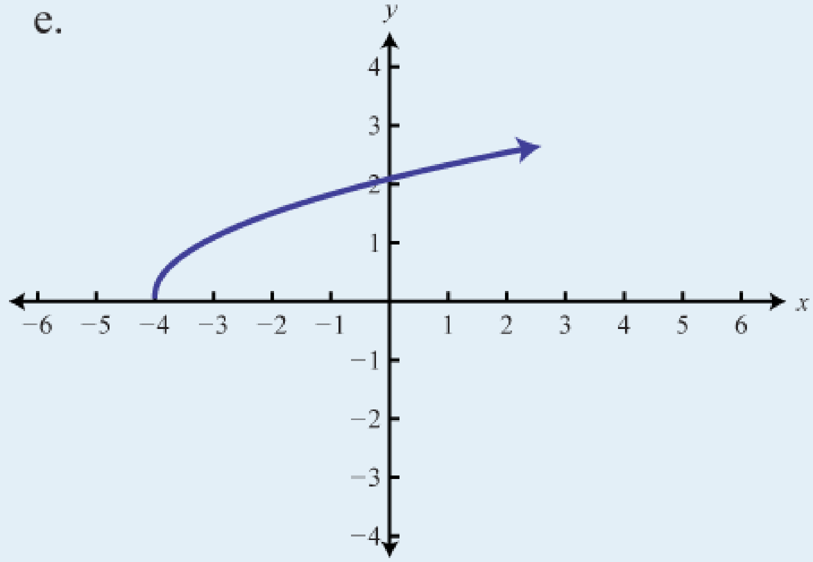
Match the graph to the function definition.



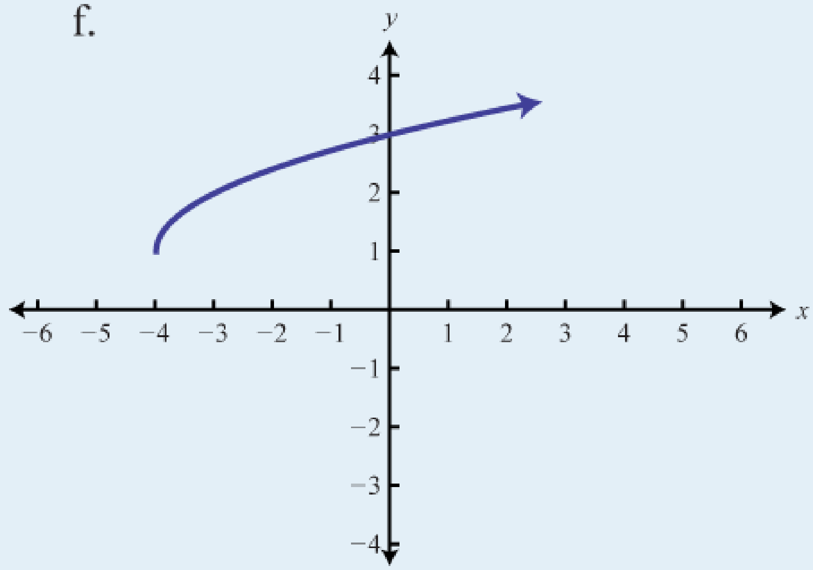



c.
d.
x
y
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6
4 3 2 1 −1 −2 −3 −4

e.



f.



1. $f(x) = \sqrt{x+4}$
2. $f(x) = |x-2| - 2$
3. $f(x) = \sqrt{x+1} - 1$
4. $f(x) = |x-2| + 1$
5. $f(x) = \sqrt{x+4} + 1$

6. $f(x) = |x + 2| - 2$

Graph the given function. Identify the basic function and translations used to sketch the graph. Then state the domain and range.

7. $f(x) = x + 3$
8. $f(x) = x - 2$
9. $g(x) = x^2 + 1$
10. $g(x) = x^2 - 4$
11. $g(x) = (x - 5)^2$
12. $g(x) = (x + 1)^2$
13. $g(x) = (x - 5)^2 + 2$
14. $g(x) = (x + 2)^2 - 5$
15. $h(x) = |x + 4|$
16. $h(x) = |x - 4|$
17. $h(x) = |x - 1| - 3$
18. $h(x) = |x + 2| - 5$
19. $g(x) = \sqrt{x} - 5$
20. $g(x) = \sqrt{x - 5}$
21. $g(x) = \sqrt{x - 2} + 1$
22. $g(x) = \sqrt{x + 2} + 3$
23. $h(x) = (x - 2)^3$
24. $h(x) = x^3 + 4$
25. $h(x) = (x - 1)^3 - 4$
26. $h(x) = (x + 1)^3 + 3$
27. $f(x) = \frac{1}{x-2}$

28. $f(x) = \frac{1}{x+3}$

29. $f(x) = \frac{1}{x} + 5$

30. $f(x) = \frac{1}{x} - 3$

31. $f(x) = \frac{1}{x+1} - 2$

32. $f(x) = \frac{1}{x-3} + 3$

33. $g(x) = -4$

34. $g(x) = 2$

35. $f(x) = \sqrt[3]{x-2} + 6$

36. $f(x) = \sqrt[3]{x+8} - 4$

Graph the piecewise functions.

37. $h(x) = \begin{cases} x^2 + 2 & \text{if } x < 0 \\ x + 2 & \text{if } x \geq 0 \end{cases}$

38. $h(x) = \begin{cases} x^2 - 3 & \text{if } x < 0 \\ \sqrt{x} - 3 & \text{if } x \geq 0 \end{cases}$

39. $h(x) = \begin{cases} x^3 - 1 & \text{if } x < 0 \\ |x-3| - 4 & \text{if } x \geq 0 \end{cases}$

40. $h(x) = \begin{cases} x^3 & \text{if } x < 0 \\ (x-1)^2 - 1 & \text{if } x \geq 0 \end{cases}$

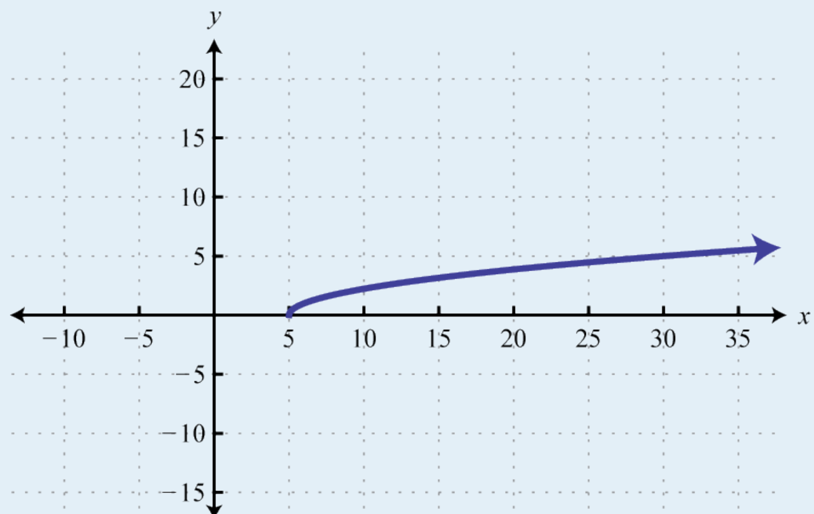
41. $h(x) = \begin{cases} x^2 - 1 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0 \end{cases}$

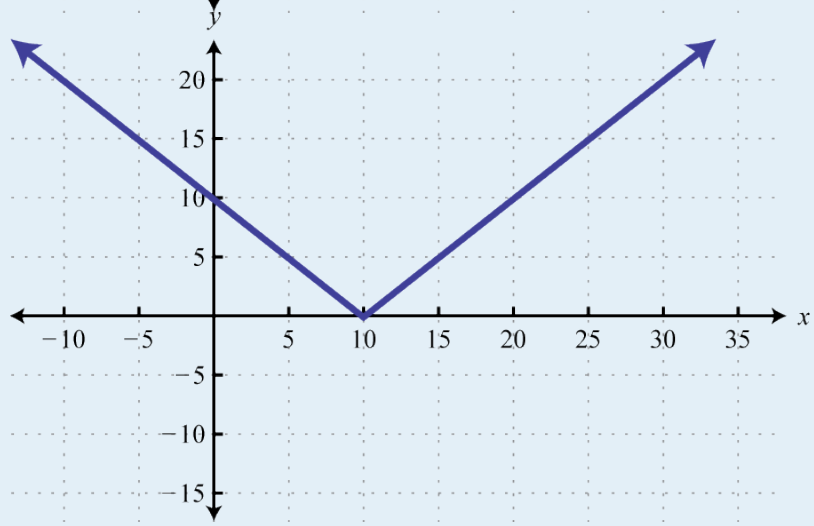
42. $h(x) = \begin{cases} x + 2 & \text{if} x < 0 \\ (x-2)^2 & \text{if} x \geq 0 \end{cases}$

43. $h(x) = \begin{cases} (x+10)^2 - 4 & \text{if } x < -8 \\ x + 4 & \text{if } -8 \leq x < -4 \\ \sqrt{x+4} & \text{if } x \geq -4 \end{cases}$

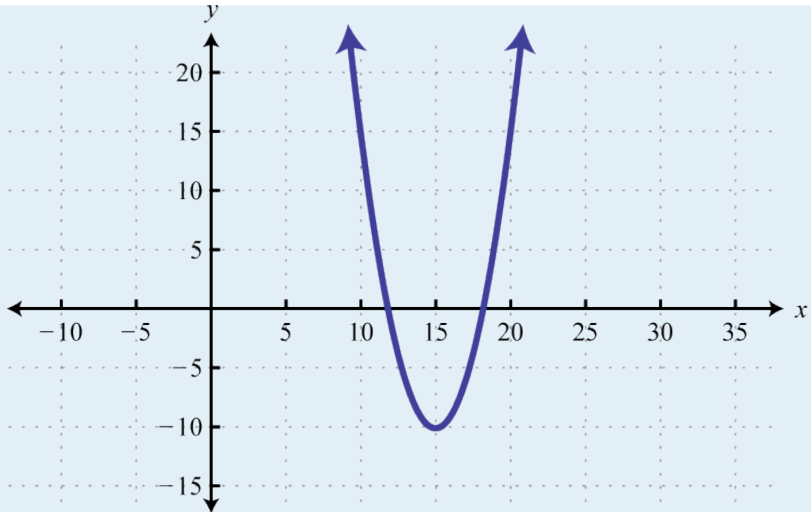
44. $$f(x) = \begin{cases} x + 10 & \text{if} \quad x \leq -10 \\ |x - 5| - 15 & \text{if} \quad -10 < x \leq 20 \\ 10 & \text{if} \quad x > 20 \end{cases}$$

Write an equation that represents the function whose graph is given.

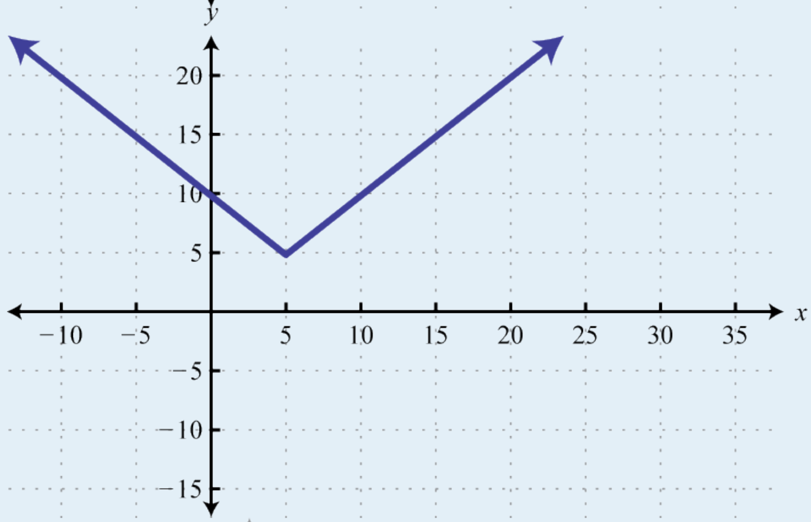
45. 

46. 

47.


y
20
15
10
5
x
−10
−5
5
10
15
20
25
30
35
−5
−10
−15

48.


y
20
15
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x
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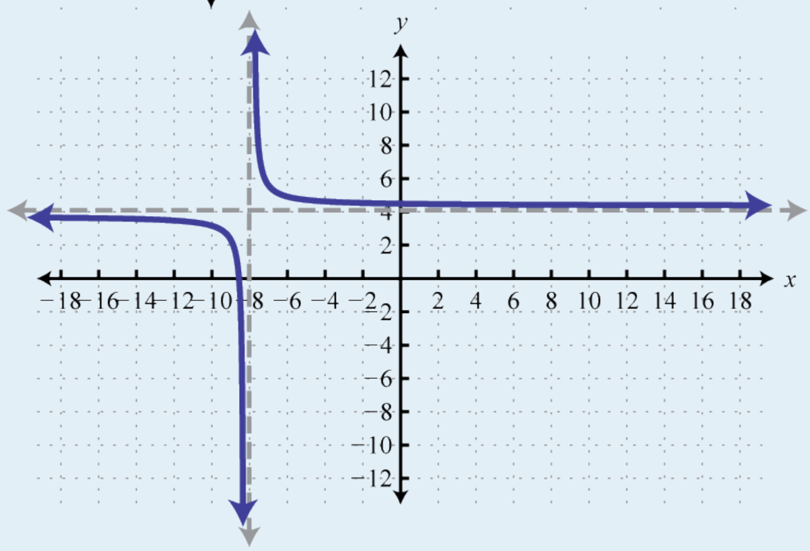
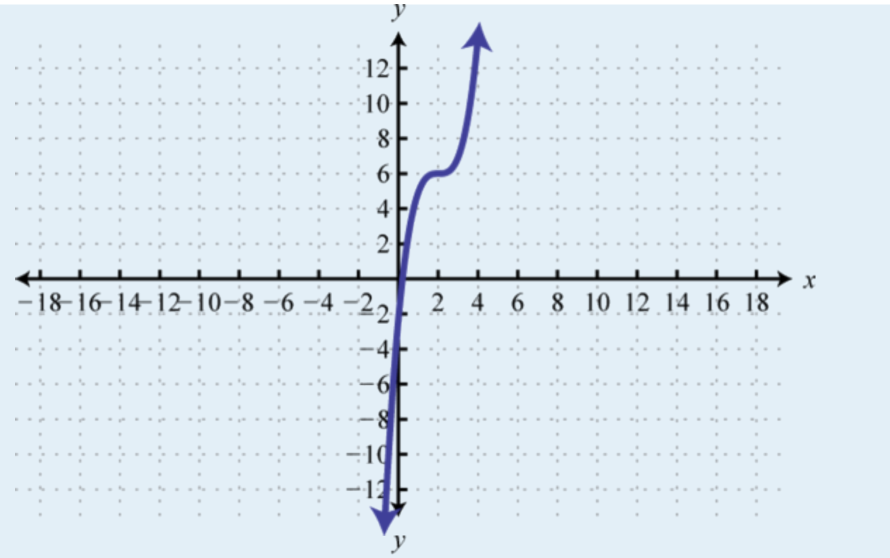
20
25
30
35
−5
−10
−15

49.


y
12
10
8
6
4
2
x
−18
−16
−14
−12
−10
−8
−6
−4
−2
2
4
6
8
10
12
14
16
18
−2
−4
−6
−8
−10
−12

50. 
y
12
10
8
6
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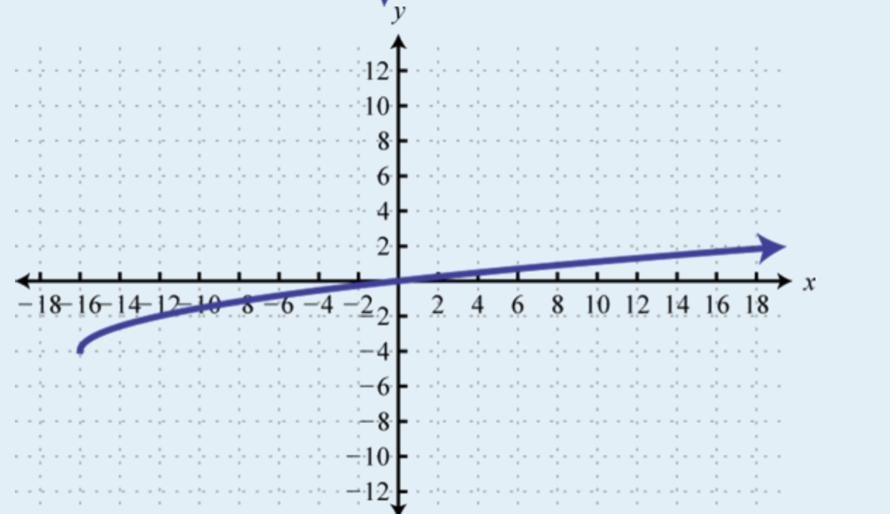
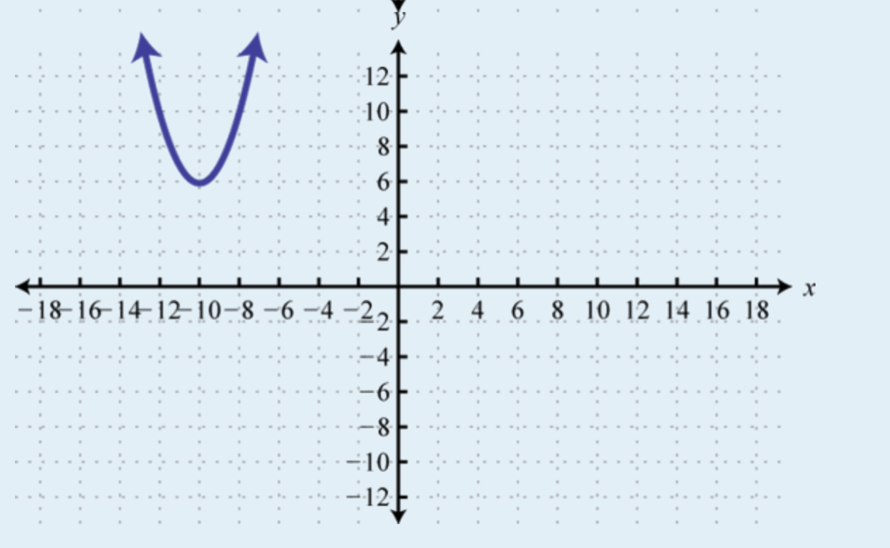
2
x
−18 −16 −14 −12 −10 −8 −6 −4 −2
2 4 6 8 10 12 14 16 18
−2
−4
−6
−8
−10
−12

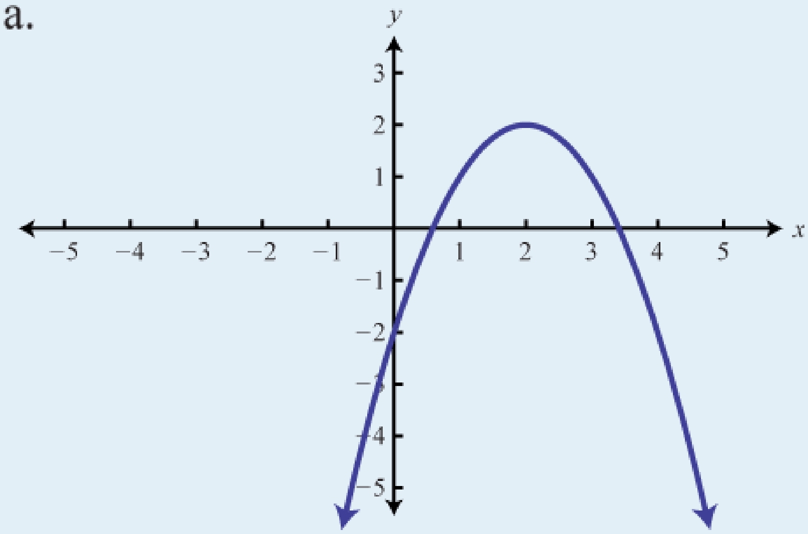
51. 
y
12
10
8
6
4
2
x
−18 −16 −14 −12 −10 −8 −6 −4 −2
2 4 6 8 10 12 14 16 18
−2
−4
−6
−8
−10
−12

52. 
y
12
10
8
6
4
2
x
−18 −16 −14 −12 −10 −8 −6 −4 −2
2 4 6 8 10 12 14 16 18
−2
−4
−6
−8
−10
−12

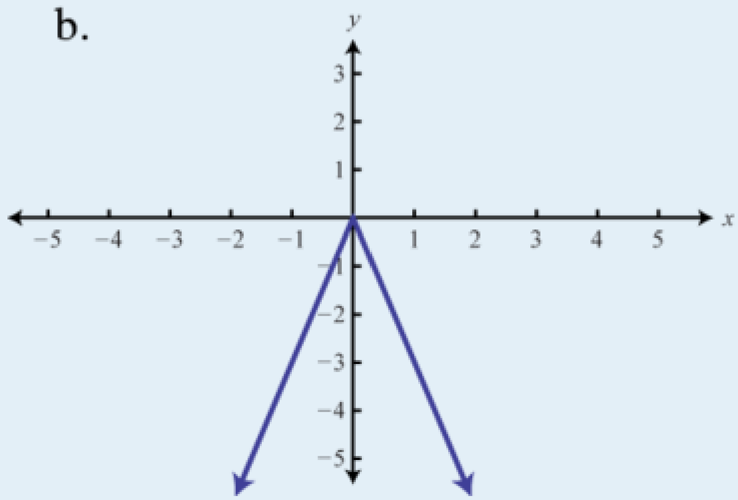
PART B: REFLECTIONS AND DILATIONS

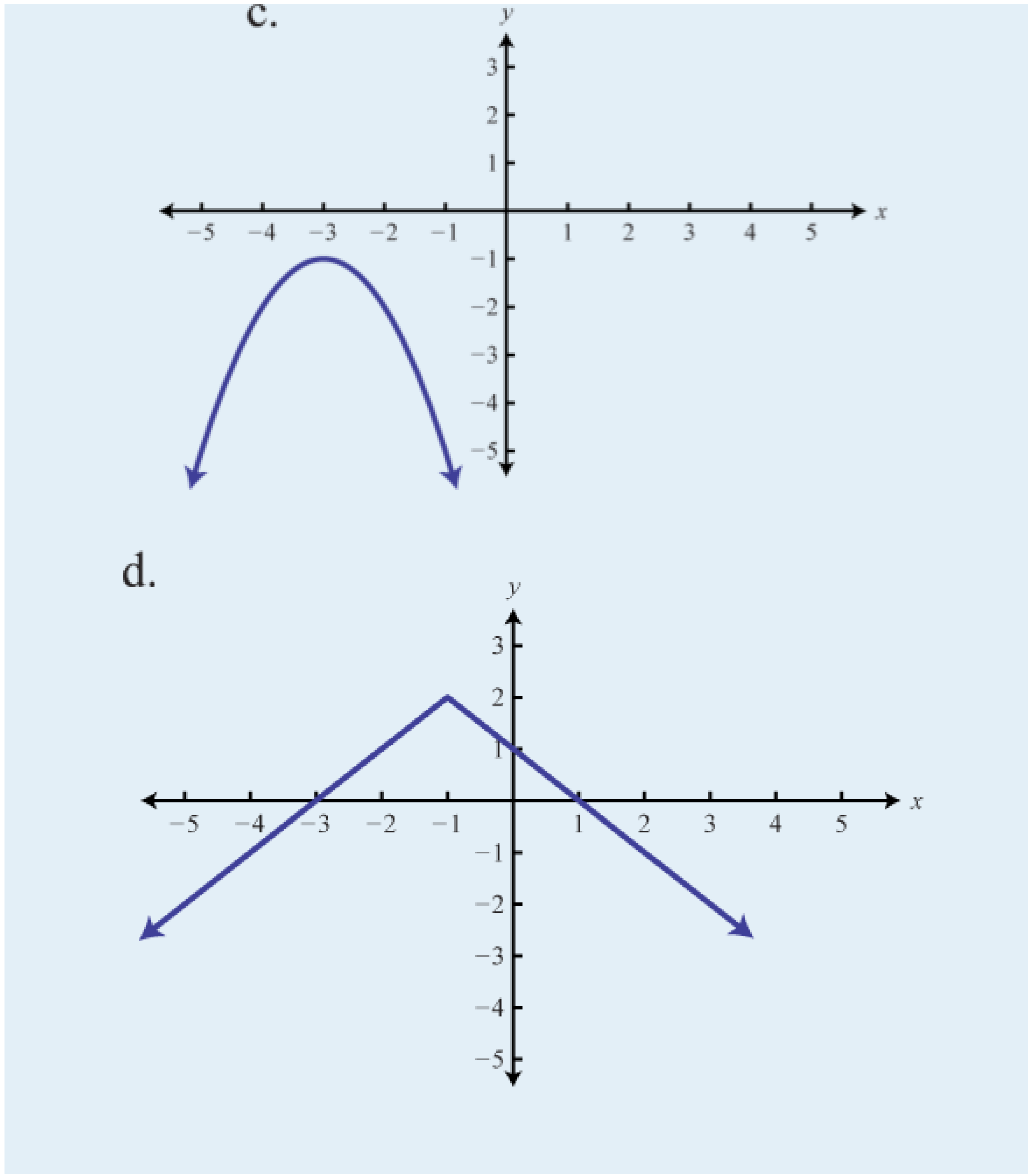
Match the graph the given function definition.

a.

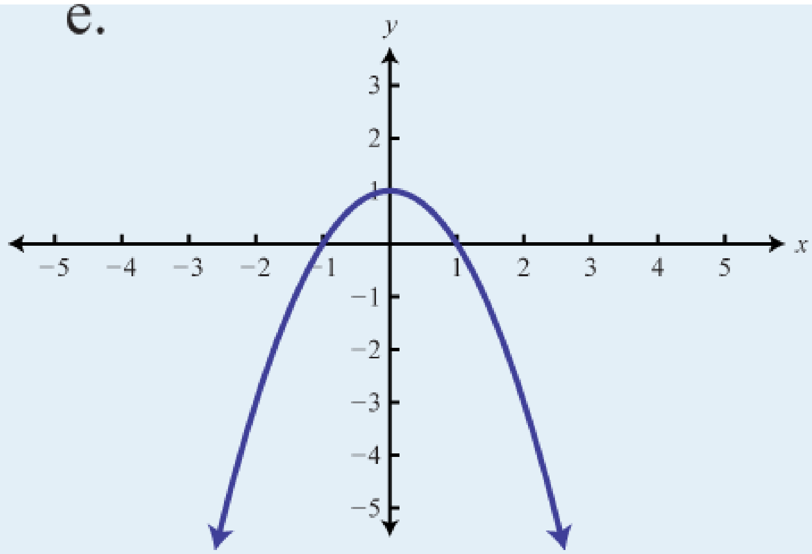


b.

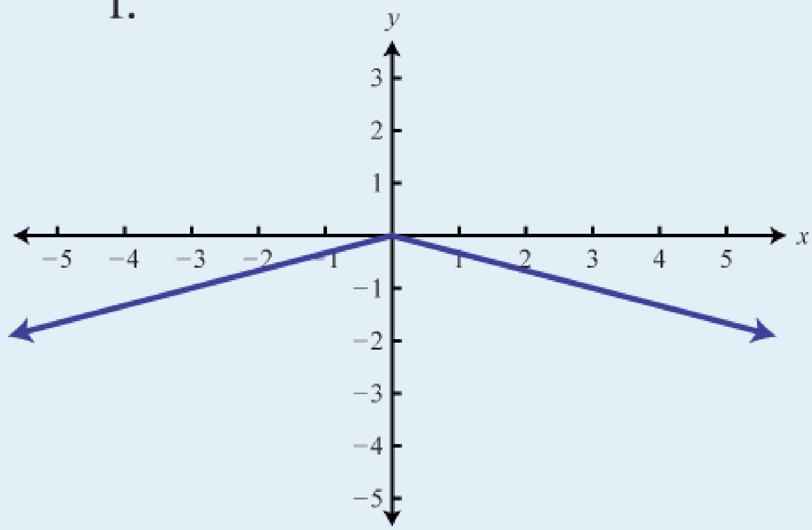



c.
y
x
d.
y
x

e.



f.



53. $f(x) = -3|x|$

54. $f(x) = -(x+3)^2 - 1$

55. $f(x) = -|x+1| + 2$

56. $f(x) = -x^2 + 1$

57. $f(x) = -\frac{1}{3}|x|$

58. $f(x) = -(x-2)^2 + 2$

Use the transformations to graph the following functions.

59. $f(x) = -x + 5$
60. $f(x) = -|x| - 3$
61. $g(x) = -|x - 1|$
62. $f(x) = -(x + 2)^2$
63. $h(x) = \sqrt{-x} + 2$
64. $g(x) = -\sqrt{x} + 2$
65. $g(x) = -(x + 2)^3$
66. $h(x) = -\sqrt{x - 2} + 1$
67. $g(x) = -x^3 + 4$
68. $f(x) = -x^2 + 6$
69. $f(x) = -3|x|$
70. $g(x) = -2x^2$
71. $h(x) = \frac{1}{2}(x - 1)^2$
72. $h(x) = \frac{1}{3}(x + 2)^2$
73. $g(x) = -\frac{1}{2}\sqrt{x - 3}$
74. $f(x) = -5\sqrt{x + 2}$
75. $f(x) = 4\sqrt{x - 1} + 2$
76. $h(x) = -2x + 1$
77. $g(x) = -\frac{1}{4}(x + 3)^3 - 1$
78. $f(x) = -5(x - 3)^2 + 3$
79. $h(x) = -3|x + 4| - 2$
80. $f(x) = -\frac{1}{x}$

81. $f(x) = -\frac{1}{x+2}$
82. $f(x) = -\frac{1}{x+1} + 2$

PART C: DISCUSSION BOARD

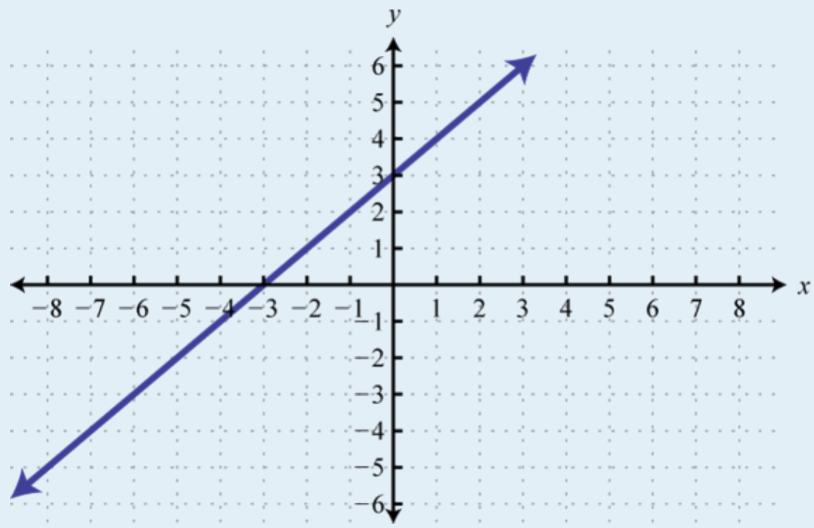
83. Use different colors to graph the family of graphs defined by $y = kx^2$, where $k \in \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right\}$. What happens to the graph when the denominator of k is very large? Share your findings on the discussion board.
84. Graph $f(x) = \sqrt{x}$ and $g(x) = -\sqrt{x}$ on the same set of coordinate axes. What does the general shape look like? Try to find a single equation that describes the shape. Share your findings.
85. Explore what happens to the graph of a function when the domain values are multiplied by a factor a before the function is applied, $f(ax)$. Develop some rules for this situation and share them on the discussion board.

ANSWERS

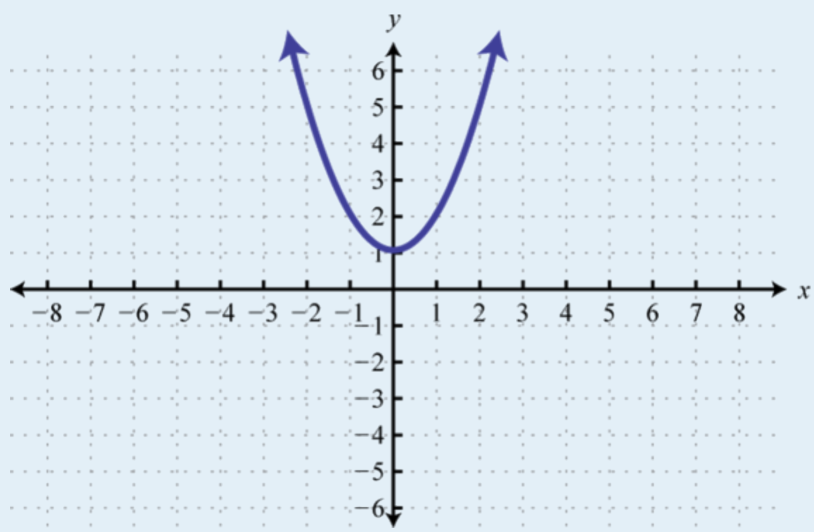
1. e

3. d

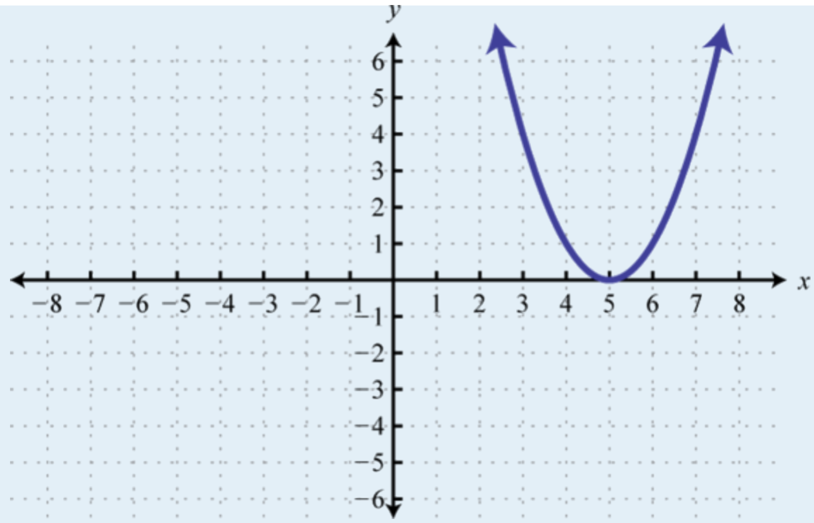
5. f



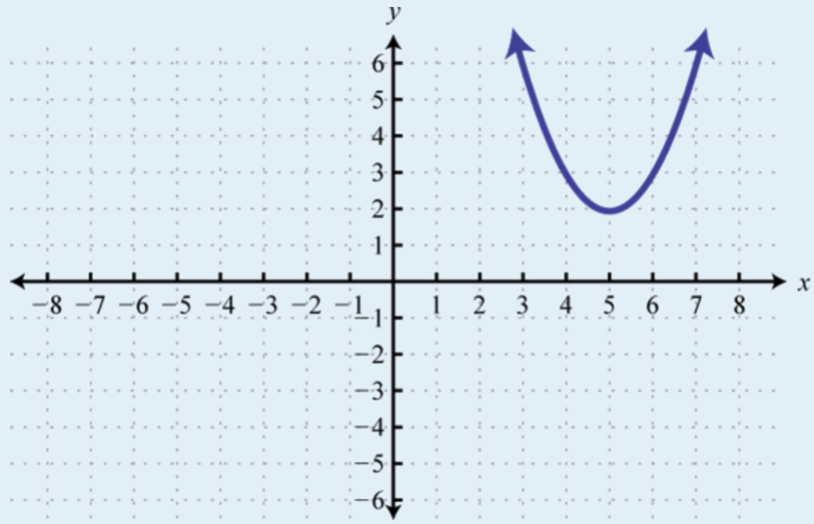
7. $y = x$; Shift up 3 units; domain: $\mathbb{R}$; range: $\mathbb{R}$



9. $y = x^2$; Shift up 1 unit; domain: $\mathbb{R}$; range: $[1, \infty)$

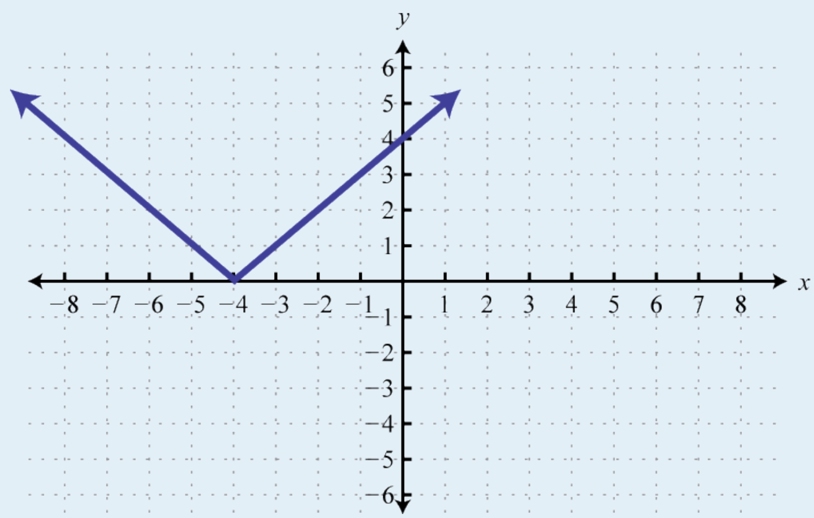


11. $y = x^2$; Shift right 5 units; domain: $\mathbb{R}$; range: $[0, \infty)$

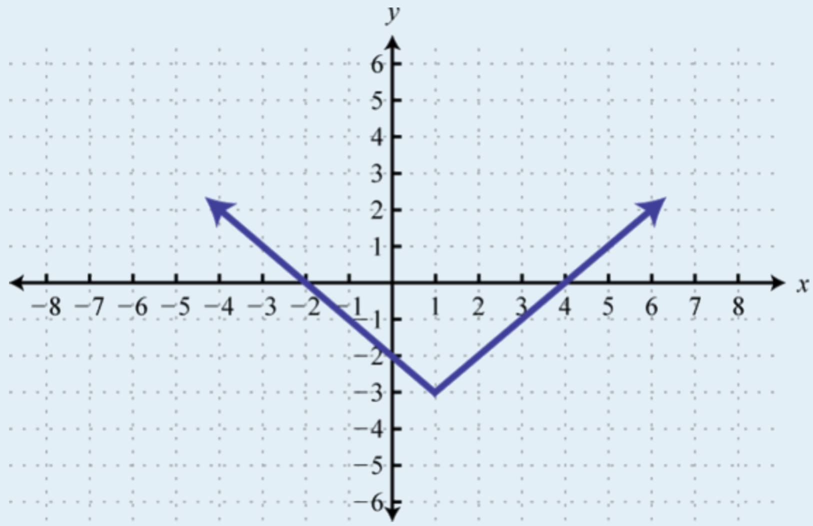


13. $y = x^2$; Shift right 5 units and up 2 units; domain: $\mathbb{R}$; range: $[2, \infty)$

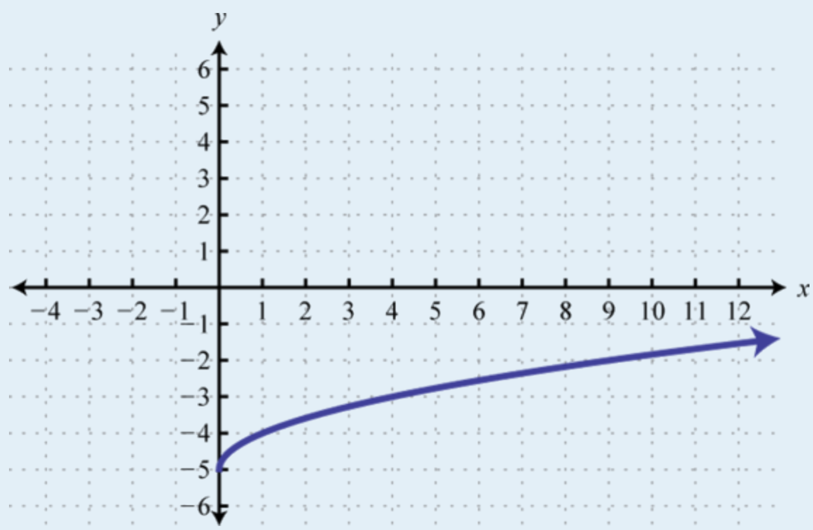
15.



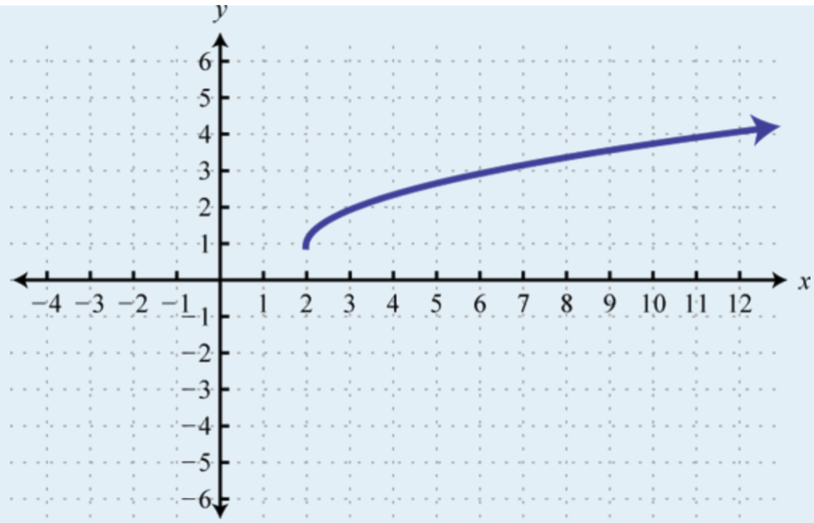
$y = |x|$; Shift left 4 units; domain: $\mathbb{R}$; range: $[0, \infty)$



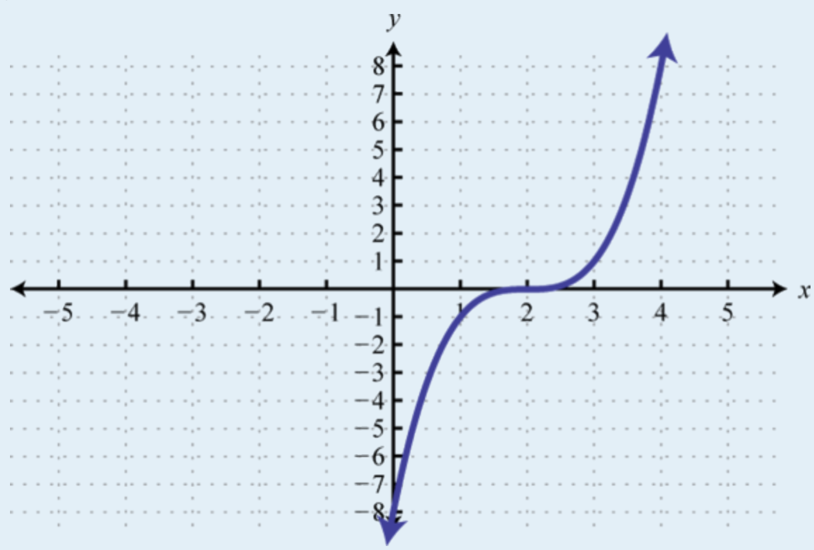
17. $y = |x|$; Shift right 1 unit and down 3 units; domain: $\mathbb{R}$; range: $[-3, \infty)$



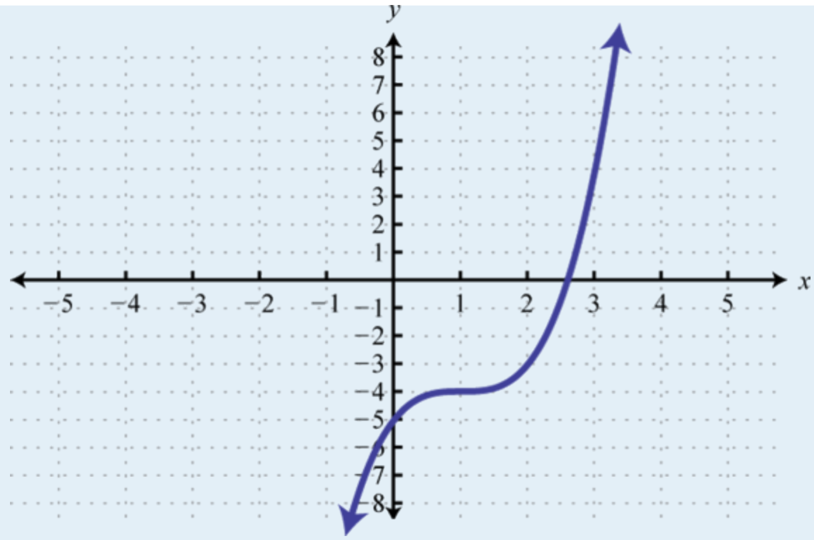
19. $y = \sqrt{x}$; Shift down 5 units; domain: $[0, \infty)$; range: $[-5, \infty)$



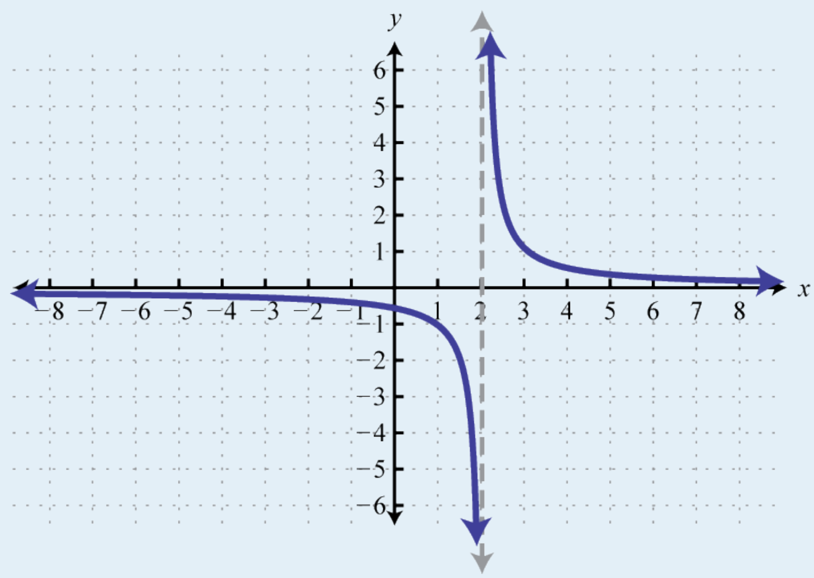
21. $y = \sqrt{x}$; Shift right 2 units and up 1 unit; domain: $[2, \infty)$; range: $[1, \infty)$



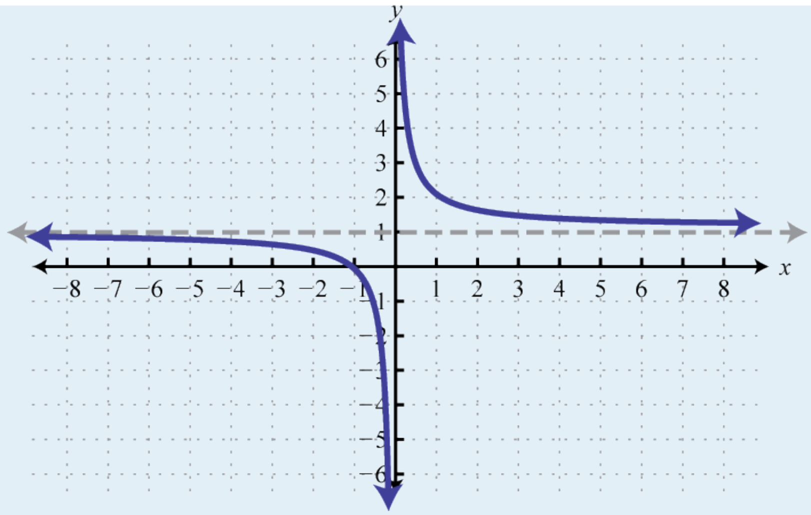
23. $y = x^3$; Shift right 2 units; domain: $\mathbb{R}$; range: $\mathbb{R}$



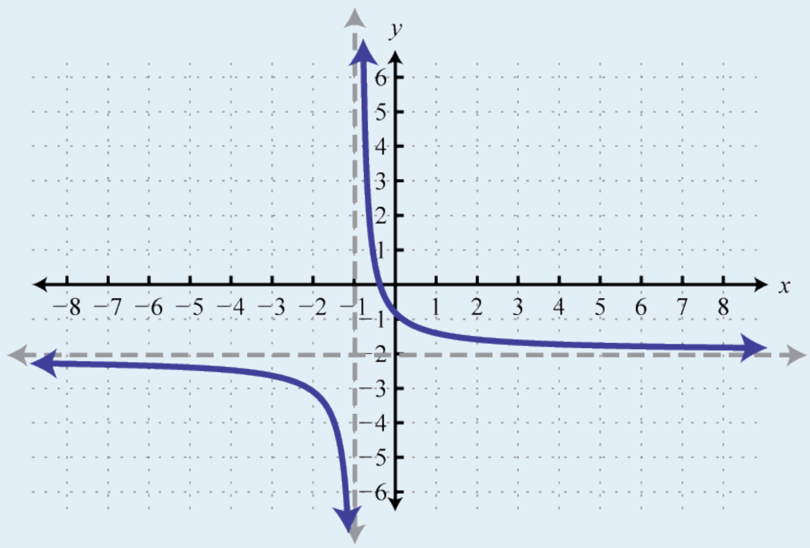
25. $y = x^3$; Shift right 1 unit and down 4 units; domain: $\mathbb{R}$; range: $\mathbb{R}$



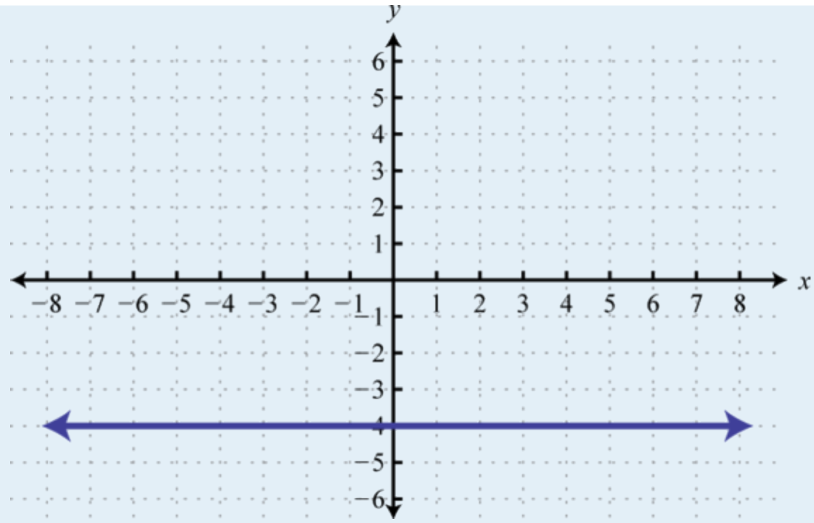
27. $y = \frac{1}{x}$; Shift right 2 units; domain: $(-\infty, 2) \cup (2, \infty)$; range: $(-\infty, 0) \cup (0, \infty)$



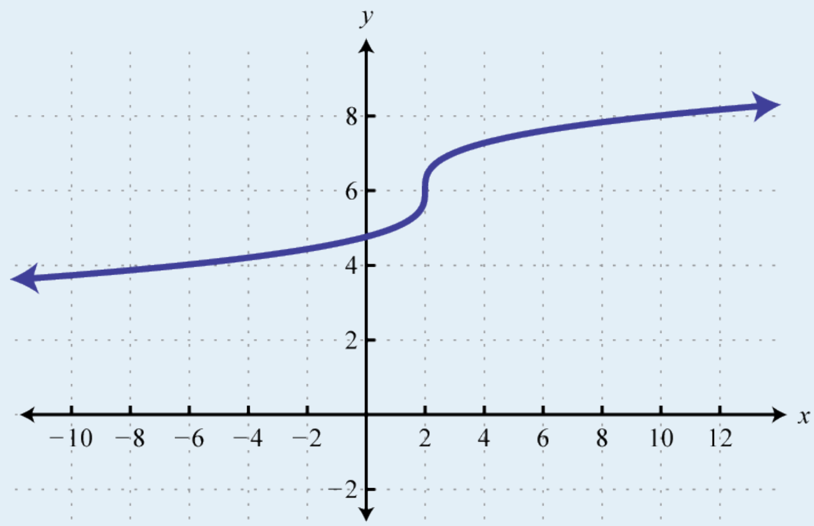
29. $y = \frac{1}{x}$; Shift up 5 units; domain: $(-\infty, 0) \cup (0, \infty)$; range: $(-\infty, 1) \cup (1, \infty)$



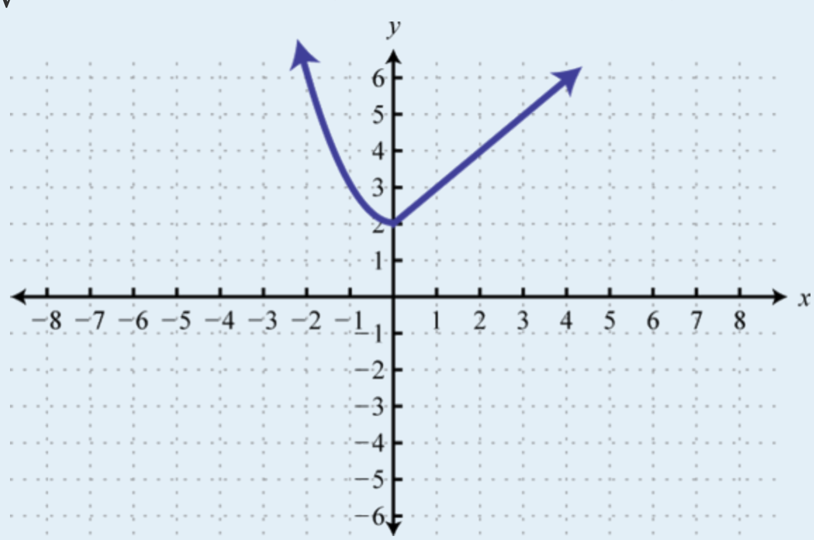
31. $y = \frac{1}{x}$; Shift left 1 unit and down 2 units; domain: $(-\infty, -1) \cup (-1, \infty)$; range: $(-\infty, -2) \cup (-2, \infty)$



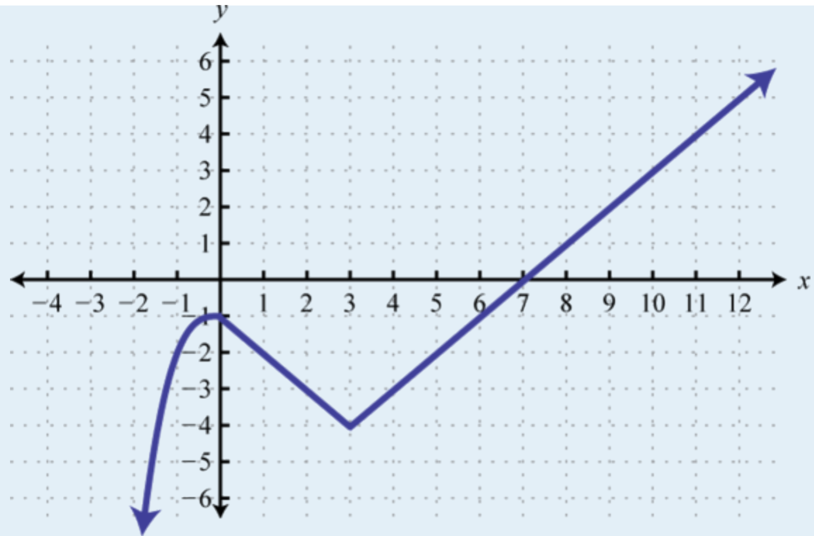
33. Basic graph $y = -4$; domain: $\mathbb{R}$; range: {-4}

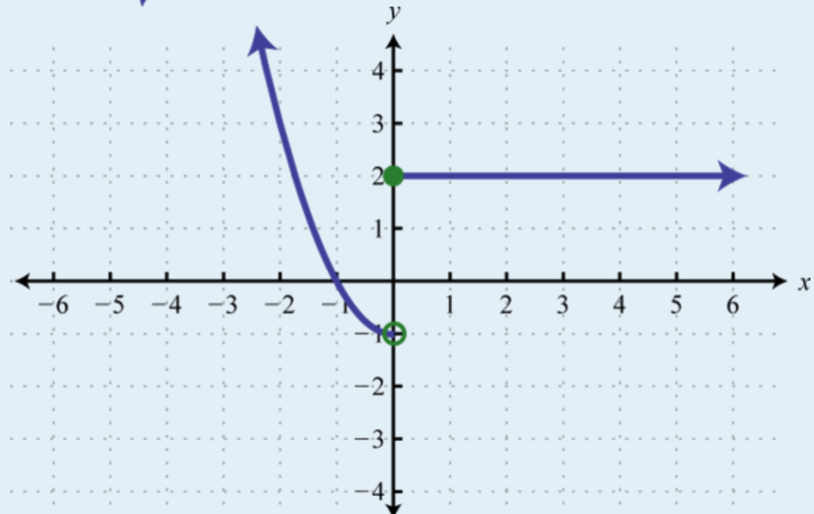


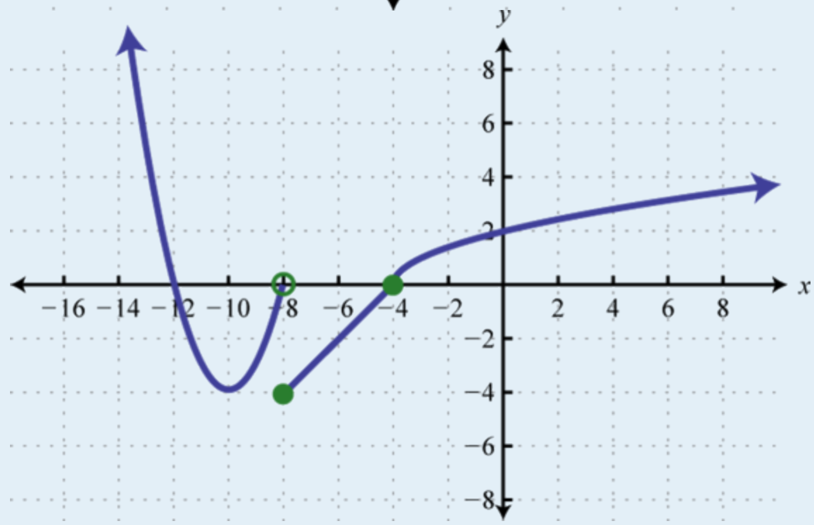
35. $y = \sqrt[3]{x}$; Shift up 6 units and right 2 units; domain: $\mathbb{R}$; range: $\mathbb{R}$



37.

39. 

41. 

43. 

45. $f(x) = \sqrt{x - 5}$

47. $f(x) = (x - 15)^2 - 10$

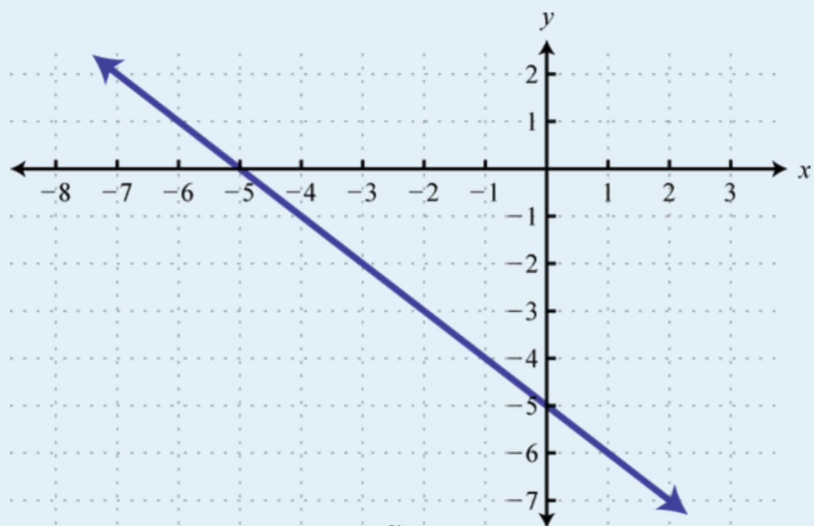
49. $f(x) = \frac{1}{x+8} + 4$

51. $f(x) = \sqrt{x + 16} - 4$

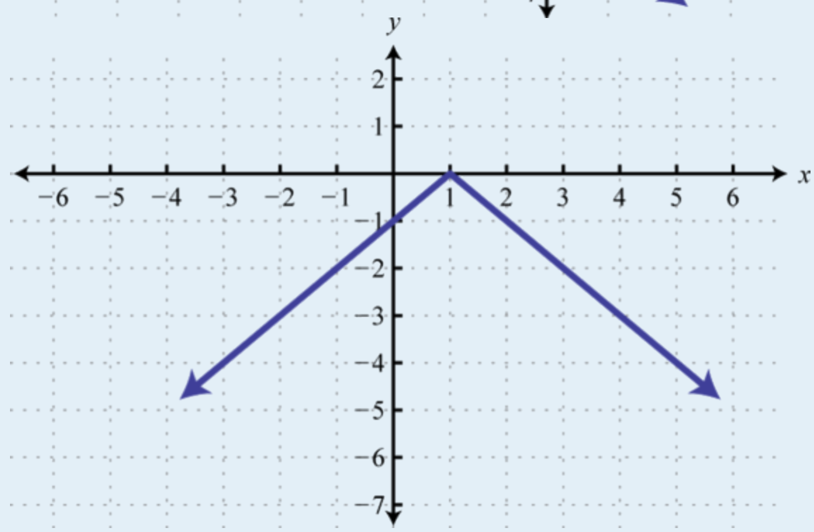
53. b

55. d

57. f

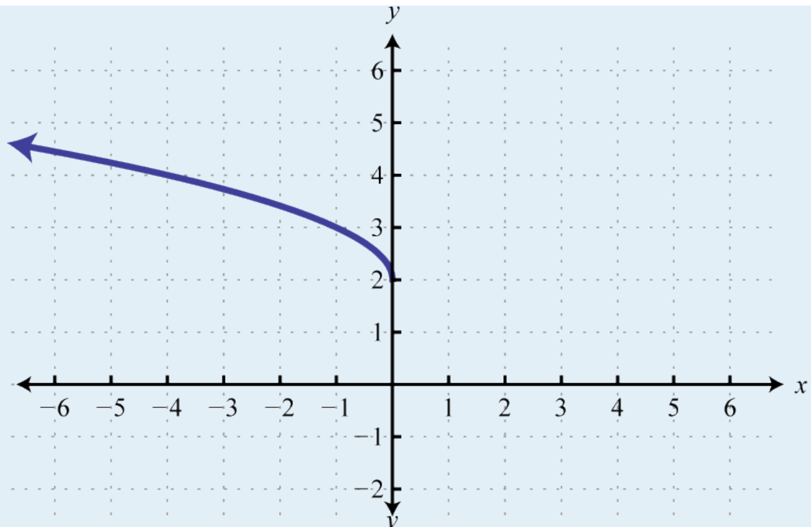


59.



61.

63.


y
x
6
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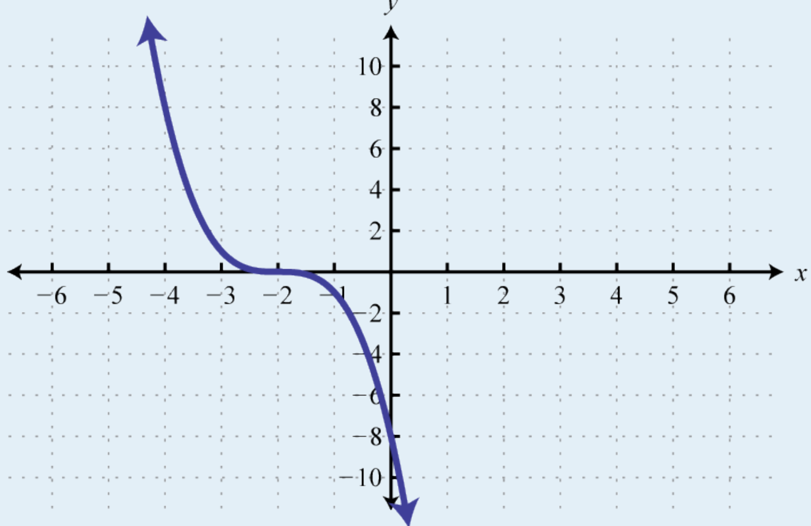
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6

65.


y
x
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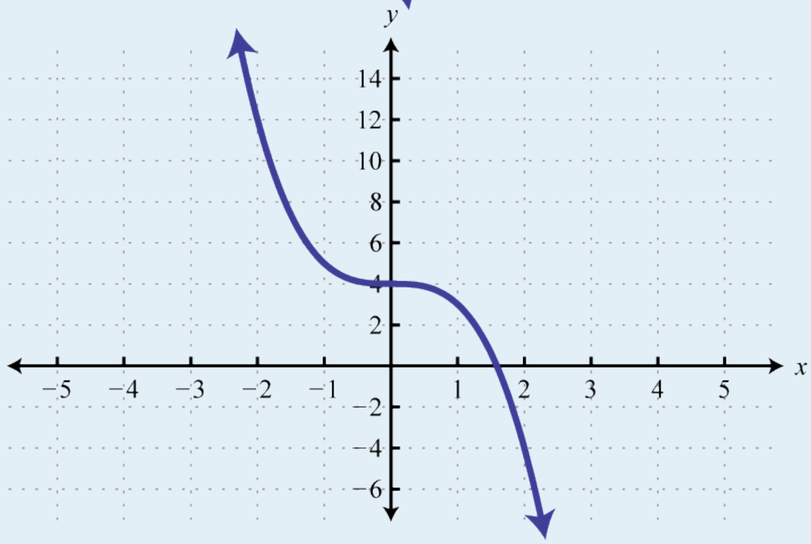
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6

67.


y
x
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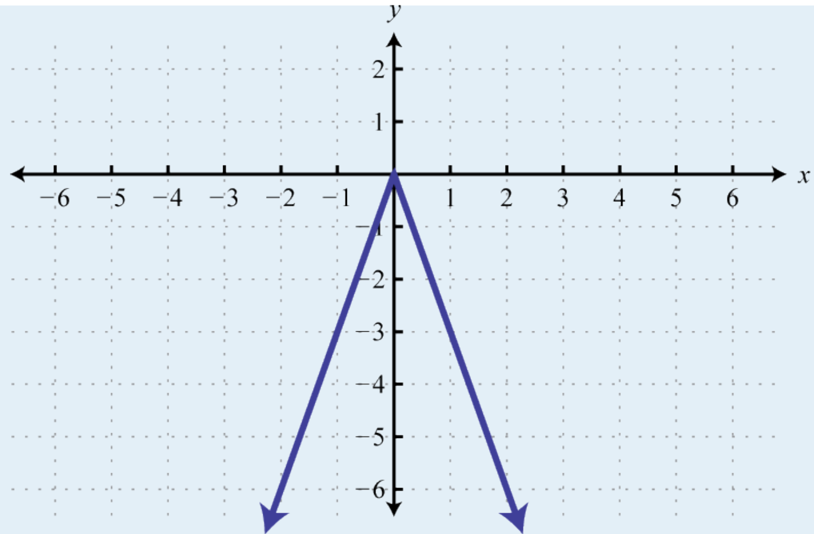
−5 −4 −3 −2 −1 1 2 3 4 5

69.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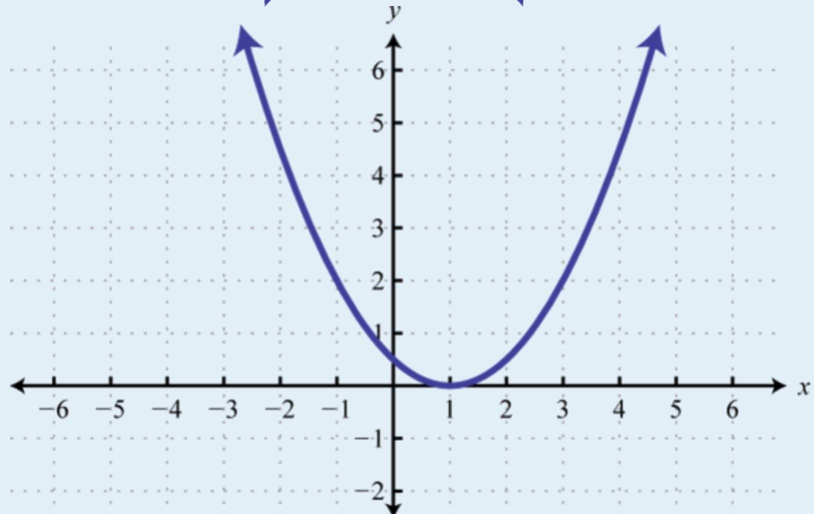
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6
2 1 −1 −2 −3 −4 −5 −6

71.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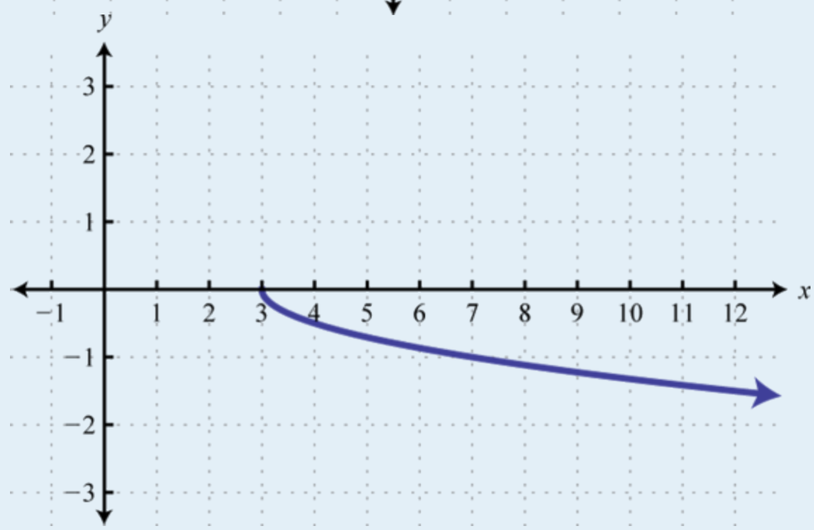
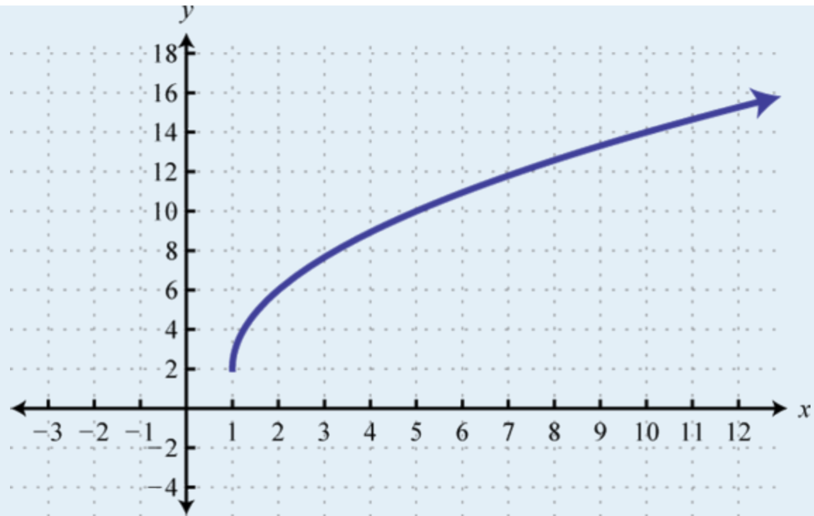

y
x
−1 1 2 3 4 5 6 7 8 9 10 11 12
3 2 1 −1 −2 −3

75. 
y
18
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x
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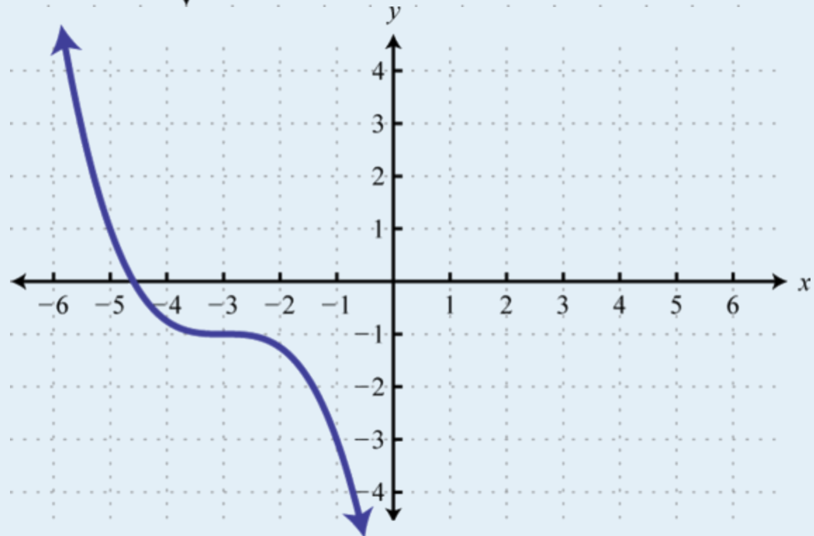
77. 
y
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3
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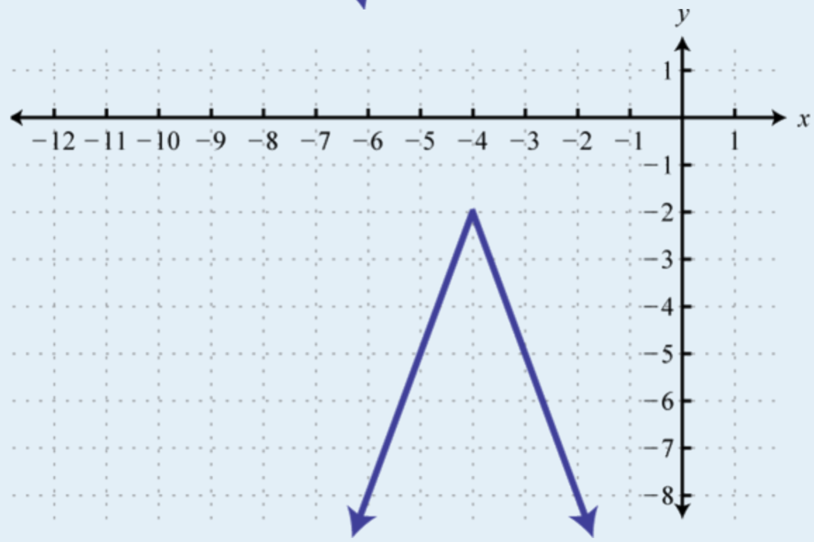
x
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6

79. 
y
1
−1
−2
−3
−4
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−6
−7
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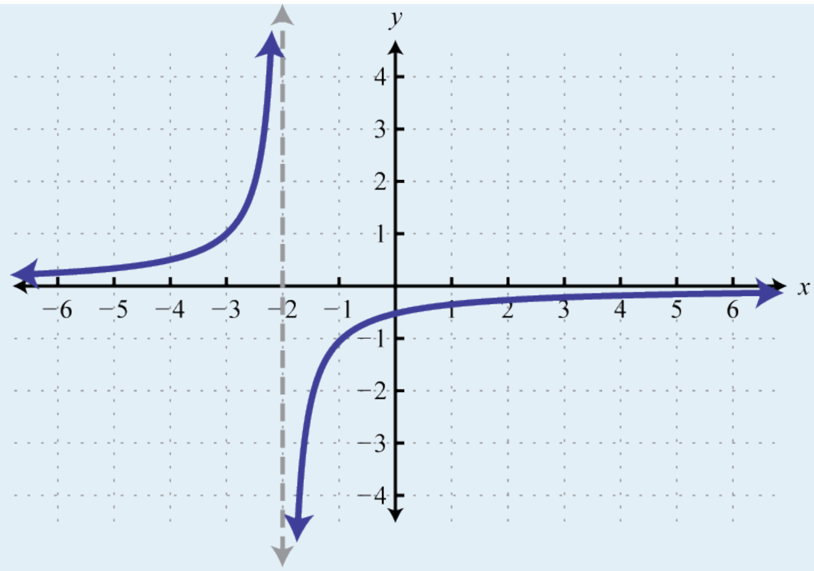
x
−12 −11 −10 −9 −8 −7 −6 −5 −4 −3 −2 −1 1

81. 

83. Answer may vary

85. Answer may vary

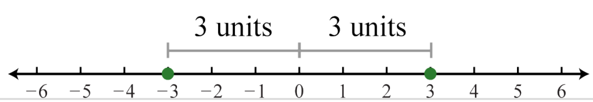
2.6 Solving Absolute Value Equations and Inequalities

LEARNING OBJECTIVES

1. Review the definition of absolute value.
2. Solve absolute value equations.
3. Solve absolute value inequalities.

Absolute Value Equations

Recall that the **absolute value**[63] of a real number a, denoted $|a|$, is defined as the distance between zero (the origin) and the graph of that real number on the number line. For example, $|-3| = 3$ and $|3| = 3$.



In addition, the absolute value of a real number can be defined algebraically as a piecewise function.

$$|a| = \begin{cases} a & if \ \ a \geq 0 \\ -a & if \ \ a < 0 \end{cases}$$

Given this definition, $|3| = 3$ and $|-3| = -(-3) = 3$. Therefore, the equation $|x| = 3$ has two solutions for x, namely $\{\pm 3\}$. In general, given any algebraic expression X and any positive number p:

$$If \ \ |X| = p \ \text{ then } \ X = -p \ \text{ or } \ X = p.$$

63. The distance from the graph of a number a to zero on a number line, denoted $|a|$.

In other words, the **argument of the absolute value**[64] X can be either positive or negative p. Use this theorem to solve absolute value equations algebraically.

Example 1

Solve: $|x + 2| = 3$.

Solution:

In this case, the argument of the absolute value is $x + 2$ and must be equal to 3 or −3.

$$\underbrace{|x+2|}_{\pm 3} = 3$$

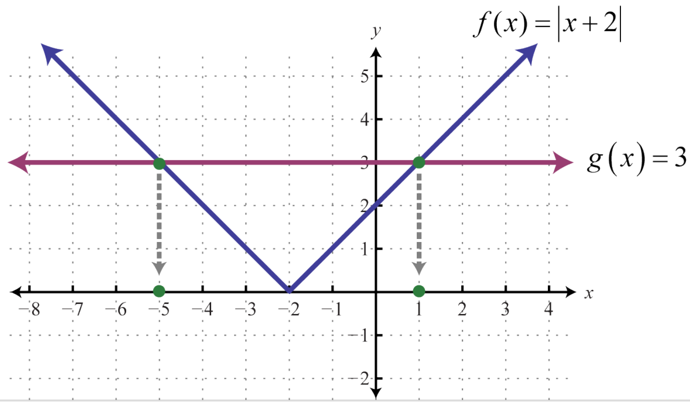
Therefore, to solve this absolute value equation, set $x + 2$ equal to ±3 and solve each linear equation as usual.

$$\begin{array}{ccc} & |x+2| = 3 & \\ x+2=-3 & \text{or} & x+2=3 \\ x=-5 & & x=1 \end{array}$$

Answer: The solutions are −5 and 1.

To visualize these solutions, graph the functions on either side of the equal sign on the same set of coordinate axes. In this case, $f(x) = |x + 2|$ is an absolute value function shifted two units horizontally to the left, and $g(x) = 3$ is a constant function whose graph is a horizontal line. Determine the x-values where $f(x) = g(x)$.

64. The number or expression inside the absolute value.



From the graph we can see that both functions coincide where $x = -5$ and $x = 1$. The solutions correspond to the points of intersection.

Example 2

Solve: $|2x + 3| = 4$.

Solution:

Here the argument of the absolute value is $2x + 3$ and can be equal to -4 or 4.

$$\begin{aligned} &|2x+3| = 4 \\ 2x + 3 &= -4 \quad \text{or} \quad 2x + 3 = 4 \\ 2x &= -7 \qquad\qquad 2x = 1 \\ x &= -\frac{7}{2} \qquad\qquad x = \frac{1}{2} \end{aligned}$$

Check to see if these solutions satisfy the original equation.

Check $x = -\frac{7}{2}$	Check $x = \frac{1}{2}$
$\begin{aligned} \|2x+3\| &= 4 \\ \left\|2\left(-\frac{7}{2}\right)+3\right\| &= 4 \\ \|-7+3\| &= 4 \\ \|-4\| &= 4 \\ 4 &= 4 \;\checkmark \end{aligned}$	$\begin{aligned} \|2x+3\| &= 4 \\ \left\|2\left(\frac{1}{2}\right)+3\right\| &= 4 \\ \|1+3\| &= 4 \\ \|4\| &= 4 \\ 4 &= 4 \;\checkmark \end{aligned}$

Answer: The solutions are $-\frac{7}{2}$ and $\frac{1}{2}$.

To apply the theorem, the absolute value must be isolated. The general steps for solving absolute value equations are outlined in the following example.

Example 3

Solve: $2\,|5x - 1| - 3 = 9$.

Solution:

Step 1: Isolate the absolute value to obtain the form $|X| = p$.

$$\begin{aligned} 2\,|5x - 1| - 3 &= 9 && \textit{Add 3 to both sides.} \\ 2\,|5x - 1| &= 12 && \textit{Divide both sides by 2.} \\ |5x - 1| &= 6 \end{aligned}$$

Step 2: Set the argument of the absolute value equal to $\pm p$. Here the argument is $5x - 1$ and $p = 6$.

$$5x - 1 = -6 \text{ or } 5x - 1 = 6$$

Step 3: Solve each of the resulting linear equations.

$$\begin{aligned} 5x - 1 &= -6 \quad \text{or} \quad & 5x - 1 &= 6 \\ 5x &= -5 & 5x &= 7 \\ x &= -1 & x &= \frac{7}{5} \end{aligned}$$

Step 4: Verify the solutions in the original equation.

Check $x = -1$	Check $x = \frac{7}{5}$
$2\|5x - 1\| - 3 = 9$ $2\|5(-1) - 1\| - 3 = 9$ $2\|-5 - 1\| - 3 = 9$ $2\|-6\| - 3 = 9$ $12 - 3 = 9$ $9 = 9$ ✓	$2\|5x - 1\| - 3 = 9$ $2\left\|5\left(\frac{7}{5}\right) - 1\right\| - 3 = 9$ $2\|7 - 1\| - 3 = 9$ $2\|6\| - 3 = 9$ $12 - 3 = 9$ $9 = 9$ ✓

Answer: The solutions are -1 and $\frac{7}{5}$.

Try this! Solve: $2 - 7\,|x + 4| = -12$.

Answer: -6, -2

(click to see video)

Not all absolute value equations will have two solutions.

Example 4

Solve: $|7x - 6| + 3 = 3$.

Solution:

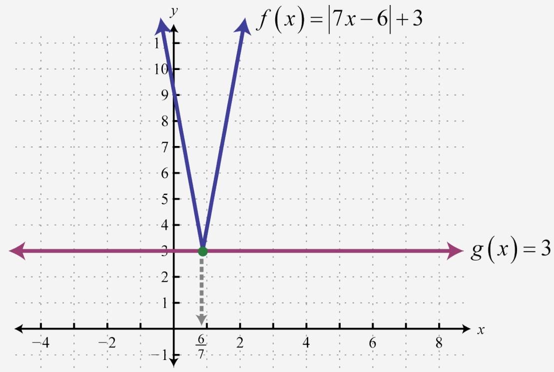
Begin by isolating the absolute value.

$$\begin{aligned} |7x - 6| + 3 &= 3 \quad \textit{Subtract } 3 \textit{ on both sides.} \\ |7x - 6| &= 0 \end{aligned}$$

Only zero has the absolute value of zero, $|0| = 0$. In other words, $|X| = 0$ has one solution, namely $X = 0$. Therefore, set the argument $7x - 6$ equal to zero and then solve for *x*.

$$\begin{aligned} 7x - 6 &= 0 \\ 7x &= 6 \\ x &= \frac{6}{7} \end{aligned}$$

Geometrically, one solution corresponds to one point of intersection.



Answer: The solution is $\frac{6}{7}$.

Example 5

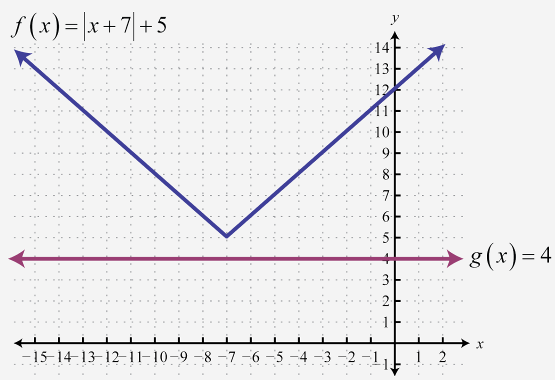
Solve: $|x + 7| + 5 = 4$.

Solution:

Begin by isolating the absolute value.

$$|x + 7| + 5 = 4 \quad \textit{Subtract 5 on both sides.}$$
$$|x + 7| = -1$$

In this case, we can see that the isolated absolute value is equal to a negative number. Recall that the absolute value will always be positive. Therefore, we conclude that there is no solution. Geometrically, there is no point of intersection.



Answer: There is no solution, Ø.

If given an equation with two absolute values of the form $|a| = |b|$, then b must be the same as a or opposite. For example, if $a = 5$, then $b = \pm 5$ and we have:

$$|5| = |-5| \text{ or } |5| = |+5|$$

In general, given algebraic expressions X and Y:

$$\text{If } |X| = |Y| \text{ then } X = -Y \text{ or } X = Y.$$

In other words, if two absolute value expressions are equal, then the arguments can be the same or opposite.

Example 6

Solve: $|2x - 5| = |x - 4|$.

Solution:

Set $2x - 5$ equal to $\pm(x - 4)$ and then solve each linear equation.

$$\begin{aligned} |2x - 5| &= |x - 4| \\ 2x - 5 = -(x - 4) \quad &\text{or} \quad 2x - 5 = +(x - 4) \\ 2x - 5 = -x + 4 \quad &\quad\ 2x - 5 = x - 4 \\ 3x = 9 \quad &\quad\ x = 1 \\ x = 3 \quad & \end{aligned}$$

To check, we substitute these values into the original equation.

Check $x = 1$	*Check* $x = 3$
$\begin{aligned} \lvert 2x - 5\rvert &= \lvert x - 4\rvert \\ \lvert 2(1) - 5\rvert &= \lvert(1) - 4\rvert \\ \lvert -3\rvert &= \lvert -3\rvert \\ 3 &= 3 \ \checkmark \end{aligned}$	$\begin{aligned} \lvert 2x - 5\rvert &= \lvert x - 4\rvert \\ \lvert 2(3) - 5\rvert &= \lvert(3) - 4\rvert \\ \lvert 1\rvert &= \lvert -1\rvert \\ 1 &= 1 \ \checkmark \end{aligned}$

As an exercise, use a graphing utility to graph both $f(x) = |2x - 5|$ and $g(x) = |x - 4|$ on the same set of axes. Verify that the graphs intersect where x is equal to 1 and 3.

Answer: The solutions are 1 and 3.

Try this! Solve: $|x+10| = |3x-2|$.

Answer: -2, 6

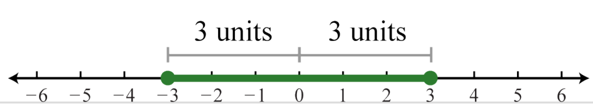
(click to see video)

Absolute Value Inequalities

We begin by examining the solutions to the following inequality:

$$|x| \leq 3$$

The absolute value of a number represents the distance from the origin. Therefore, this equation describes all numbers whose distance from zero is less than or equal to 3. We can graph this solution set by shading all such numbers.



Certainly we can see that there are infinitely many solutions to $|x| \leq 3$ bounded by -3 and 3. Express this solution set using set notation or interval notation as follows:

$$\begin{array}{ll} \{x \mid -3 \leq x \leq 3\} & \textit{Set Notation} \\ [-3, 3] & \textit{Interval Notation} \end{array}$$

In this text, we will choose to express solutions in interval notation. In general, given any algebraic expression X and any positive number p:

$$\text{If } |X| \leq p \text{ then } -p \leq X \leq p.$$

This theorem holds true for strict inequalities as well. In other words, we can convert any absolute value inequality involving "*less than*" into a compound inequality which can be solved as usual.

Example 7

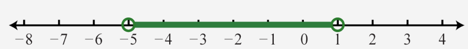
Solve and graph the solution set: $|x+2| < 3$.

Solution:

Bound the argument $x+2$ by -3 and 3 and solve.

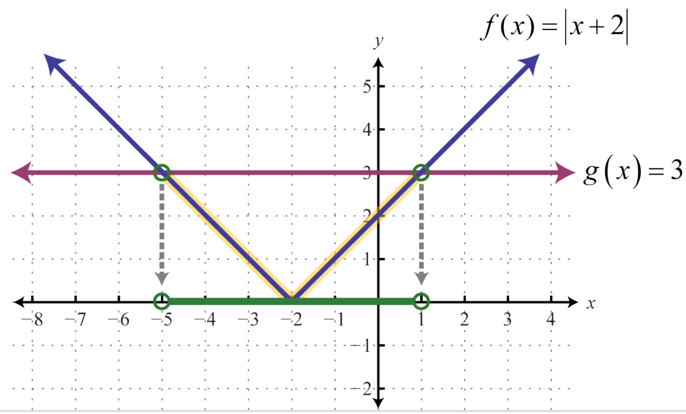
$$\begin{aligned} |x+2| &< 3 \\ -3 < x+2 &< 3 \\ -3 - 2 < x+2-2 &< 3-2 \\ -5 < x &< 1 \end{aligned}$$

Here we use open dots to indicate strict inequalities on the graph as follows.



Answer: Using interval notation, $(-5, 1)$.

The solution to $|x + 2| < 3$ can be interpreted graphically if we let $f(x) = |x + 2|$ and $g(x) = 3$ and then determine where $f(x) < g(x)$ by graphing both f and g on the same set of axes.



The solution consists of all x-values where the graph of f is below the graph of g. In this case, we can see that $|x + 2| < 3$ where the x-values are between −5 and 1. To apply the theorem, we must first isolate the absolute value.

Example 8

Solve: $4\,|x+3| - 7 \leq 5$.

Solution:

Begin by isolating the absolute value.

$$\begin{aligned} 4\,|x+3| - 7 &\leq 5 \\ 4\,|x+3| &\leq 12 \\ |x+3| &\leq 3 \end{aligned}$$

Next, apply the theorem and rewrite the absolute value inequality as a compound inequality.

$$\begin{gathered} |x+3| \leq 3 \\ -3 \leq x+3 \leq 3 \end{gathered}$$

Solve.

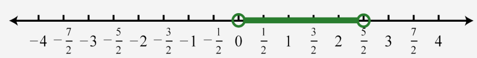
$$\begin{gathered} -3 \leq x+3 \leq 3 \\ -3 - 3 \leq x+3 - 3 \leq 3 - 3 \\ -6 \leq x \leq 0 \end{gathered}$$

Shade the solutions on a number line and present the answer in interval notation. Here we use closed dots to indicate inclusive inequalities on the graph as follows:

Answer: Using interval notation, $[-6, 0]$

Try this! Solve and graph the solution set: $3 + |4x - 5| < 8$.

Answer: Interval notation: $\left(0, \frac{5}{2}\right)$

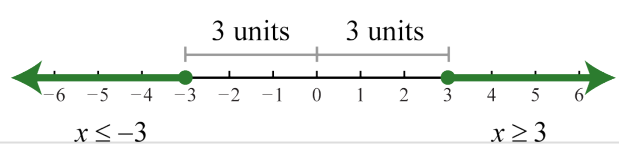


(click to see video)

Next, we examine the solutions to an inequality that involves "*greater than*," as in the following example:

$$|x| \geq 3$$

This inequality describes all numbers whose distance from the origin is greater than or equal to 3. On a graph, we can shade all such numbers.



There are infinitely many solutions that can be expressed using set notation and interval notation as follows:

$$\{x \mid x \leq -3 \text{ or } x \geq 3\}$$ *Set Notation*

$$(-\infty, -3] \cup [3, \infty)$$ *Interval Notation*

In general, given any algebraic expression *X* and any positive number *p*:

$$\text{If } |X| \geq p \text{ then } X \leq -p \text{ or } X \geq p.$$

The theorem holds true for strict inequalities as well. In other words, we can convert any absolute value inequality involving "*greater than*" into a compound inequality that describes two intervals.

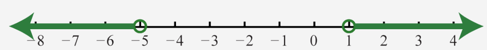
Example 9

Solve and graph the solution set: $|x + 2| > 3$.

Solve

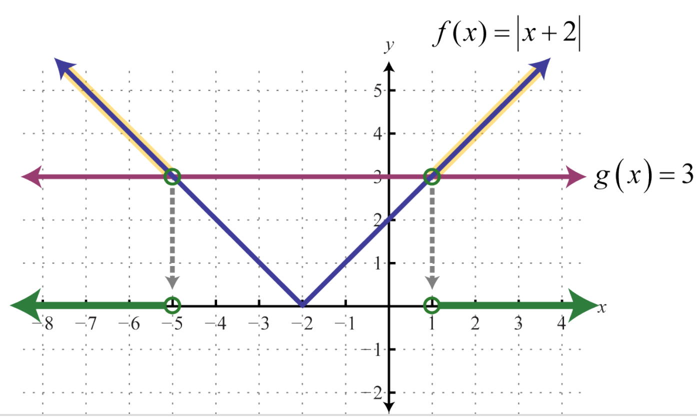
The argument $x + 2$ must be less than -3 or greater than 3.

$$|x + 2| > 3$$
$$\begin{array}{ccc} x + 2 < -3 & \text{or} & x + 2 > 3 \\ x < -5 & & x > 1 \end{array}$$



Answer: Using interval notation, $(-\infty, -5) \cup (1, \infty)$.

The solution to $|x + 2| > 3$ can be interpreted graphically if we let $f(x) = |x + 2|$ and $g(x) = 3$ and then determine where $f(x) > g(x)$ by graphing both f and g on the same set of axes.



The solution consists of all x-values where the graph of f is above the graph of g. In this case, we can see that $|x+2|>3$ where the x-values are less than –5 or are greater than 1. To apply the theorem we must first isolate the absolute value.

Example 10

Solve: $3 + 2\,|4x - 7| \geq 13$.

Solution:

Begin by isolating the absolute value.

$$\begin{aligned} 3 + 2\,|4x - 7| &\geq 13 \\ 2\,|4x - 7| &\geq 10 \\ |4x - 7| &\geq 5 \end{aligned}$$

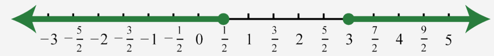
Next, apply the theorem and rewrite the absolute value inequality as a compound inequality.

$$|4x - 7| \geq 5$$
$$4x - 7 \leq -5 \qquad \text{or} \qquad 4x - 7 \geq 5$$

Solve.

$$\begin{aligned} 4x - 7 &\leq -5 \quad \text{or} \quad & 4x - 7 &\geq 5 \\ 4x &\leq 2 & 4x &\geq 12 \\ 4x &\leq \frac{2}{4} & x &\geq 3 \\ 4x &\leq \frac{1}{2} & & \end{aligned}$$

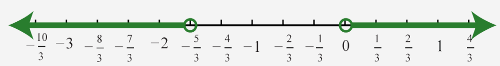
Shade the solutions on a number line and present the answer using interval notation.



Answer: Using interval notation, $\left(-\infty, \frac{1}{2}\right] \cup \left[3, \infty\right)$

Try this! Solve and graph: $3\,|6x + 5| - 2 > 13$.

Answer: Using interval notation, $\left(-\infty, -\frac{5}{3}\right) \cup \left(0, \infty\right)$



(click to see video)

Up to this point, the solution sets of linear absolute value inequalities have consisted of a single bounded interval or two unbounded intervals. This is not always the case.

Example 11

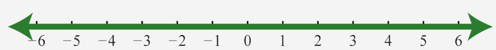
Solve and graph: $|2x - 1| + 5 > 2$.

Solution:

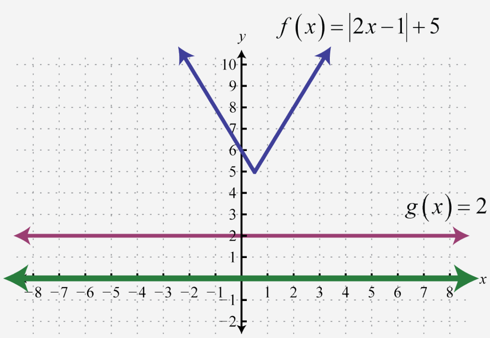
Begin by isolating the absolute value.

$$\begin{aligned} |2x - 1| + 5 &> 2 \\ |2x - 1| &> -3 \end{aligned}$$

Notice that we have an absolute value greater than a negative number. For any real number *x* the absolute value of the argument will always be positive. Hence, any real number will solve this inequality.



Geometrically, we can see that $f(x) = |2x - 1| + 5$ is always greater than $g(x) = 2$.



Answer: All real numbers, $\mathbb{R}$.

Example 12

Solve and graph: $|x + 1| + 4 \leq 3$.

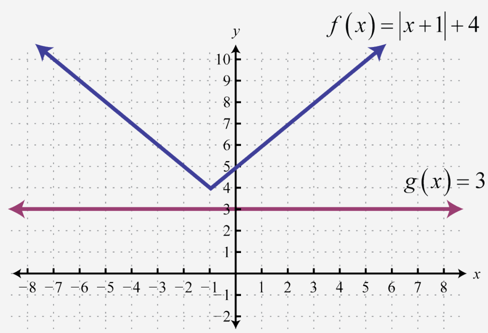
Solution:

Begin by isolating the absolute value.

$$\begin{aligned} |x + 1| + 4 &\leq 3 \\ |x + 1| &\leq -1 \end{aligned}$$

In this case, we can see that the isolated absolute value is to be less than or equal to a negative number. Again, the absolute value will always be positive; hence, we can conclude that there is no solution.

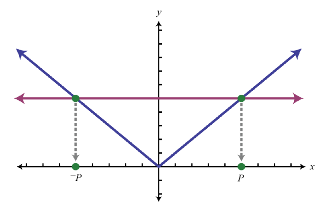
Geometrically, we can see that $f(x) = |x + 1| + 4$ is never less than $g(x) = 3$.



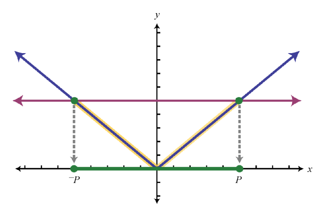
Answer: Ø

In summary, there are three cases for absolute value equations and inequalities. The relations =, $<$, $\leq$, $>$, and $\geq$ determine which theorem to apply.

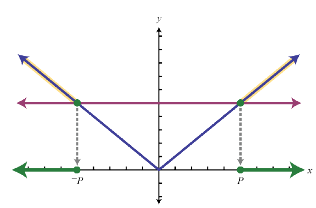
Case 1: An absolute value equation:

If $\|X\| = p$ then $X = -p$ or $X = p$	

Case 2: An absolute value inequality involving "*less than.*"

If $\|X\| \leq p$ then $-p \leq X \leq p$	

Case 3: An absolute value inequality involving "*greater than.*"

If $\|X\| \geq p$ then $X \leq -p$ or $X \geq p$	

KEY TAKEAWAYS

- To solve an absolute value equation, such as $|X| = p$, replace it with the two equations $X = -p$ and $X = p$ and then solve each as usual. Absolute value equations can have up to two solutions.
- To solve an absolute value inequality involving "less than," such as $|X| \leq p$, replace it with the compound inequality $-p \leq X \leq p$ and then solve as usual.
- To solve an absolute value inequality involving "greater than," such as $|X| \geq p$, replace it with the compound inequality $X \leq -p \quad or \quad X \geq p$ and then solve as usual.
- Remember to isolate the absolute value before applying these theorems.

TOPIC EXERCISES

PART A: ABSOLUTE VALUE EQUATIONS SOLVE.

1. $|x| = 9$
2. $|x| = 1$
3. $|x - 7| = 3$
4. $|x - 2| = 5$
5. $|x + 12| = 0$
6. $|x + 8| = 0$
7. $|x + 6| = -1$
8. $|x - 2| = -5$
9. $|2y - 1| = 13$
10. $|3y - 5| = 16$
11. $|-5t + 1| = 6$
12. $|-6t + 2| = 8$
13. $\left|\frac{1}{2}x - \frac{2}{3}\right| = \frac{1}{6}$
14. $\left|\frac{2}{3}x + \frac{1}{4}\right| = \frac{5}{12}$
15. $|0.2x + 1.6| = 3.6$
16. $|0.3x - 1.2| = 2.7$
17. $|5(y - 4) + 5| = 15$
18. $|2(y - 1) - 3y| = 4$
19. $|5x - 7| + 3 = 10$
20. $|3x - 8| - 2 = 6$
21. $9 + |7x + 1| = 9$
22. $4 - |2x - 3| = 4$

23. $3|x-8|+4=25$
24. $2|x+6|-3=17$
25. $9+5|x-1|=4$
26. $11+6|x-4|=5$
27. $8-2|x+1|=4$
28. $12-5|x-2|=2$
29. $\frac{1}{2}|x-5|-\frac{2}{3}=-\frac{1}{6}$
30. $\frac{1}{3}\left|x+\frac{1}{2}\right|+1=\frac{3}{2}$
31. $-2|7x+1|-4=2$
32. $-3|5x-3|+2=5$
33. $1.2|t-2.8|-4.8=1.2$
34. $3.6|t+1.8|-2.6=8.2$
35. $\frac{1}{2}|2(3x-1)-3|+1=4$
36. $\frac{2}{3}|4(3x+1)-1|-5=3$
37. $|5x-7|=|4x-2|$
38. $|8x-3|=|7x-12|$
39. $|5y+8|=|2y+3|$
40. $|7y+2|=|5y-2|$
41. $|5(x-2)|=|3x|$
42. $|3(x+1)|=|7x|$
43. $\left|\frac{2}{3}x+\frac{1}{2}\right|=\left|\frac{3}{2}x-\frac{1}{3}\right|$
44. $\left|\frac{3}{5}x-\frac{5}{2}\right|=\left|\frac{1}{2}x+\frac{2}{5}\right|$
45. $|1.5t-3.5|=|2.5t+0.5|$
46. $|3.2t-1.4|=|1.8t+2.8|$

47. $|5 - 3(2x + 1)| = |5x + 2|$
48. $|3 - 2(3x - 2)| = |4x - 1|$

Assume all variables in the denominator are nonzero.

49. Solve for x: $p|ax + b| - q = 0$
50. Solve for x: $|ax + b| = |p + q|$

PART B: ABSOLUTE VALUE INEQUALITIES

Solve and graph the solution set. In addition, give the solution set in interval notation.

51. $|x| < 5$
52. $|x| \leq 2$
53. $|x + 3| \leq 1$
54. $|x - 7| < 8$
55. $|x - 5| < 0$
56. $|x + 8| < -7$
57. $|2x - 3| \leq 5$
58. $|3x - 9| < 27$
59. $|5x - 3| \leq 0$
60. $|10x + 5| < 25$
61. $\left|\frac{1}{3}x - \frac{2}{3}\right| \leq 1$
62. $\left|\frac{1}{12}x - \frac{1}{2}\right| \leq \frac{3}{2}$
63. $|x| \geq 5$
64. $|x| > 1$
65. $|x + 2| > 8$
66. $|x - 7| \geq 11$
67. $|x + 5| \geq 0$

68. $|x - 12| > -4$
69. $|2x - 5| \geq 9$
70. $|2x + 3| \geq 15$
71. $|4x - 3| > 9$
72. $|3x - 7| \geq 2$
73. $\left|\frac{1}{7}x - \frac{3}{14}\right| > \frac{1}{2}$
74. $\left|\frac{1}{2}x + \frac{5}{4}\right| > \frac{3}{4}$

Solve and graph the solution set.

75. $|3(2x - 1)| > 15$
76. $|3(x - 3)| \leq 21$
77. $-5|x - 4| > -15$
78. $-3|x + 8| \leq -18$
79. $6 - 3|x - 4| < 3$
80. $5 - 2|x + 4| \leq -7$
81. $6 - |2x + 5| < -5$
82. $25 - |3x - 7| \geq 18$
83. $|2x + 25| - 4 \geq 9$
84. $|3(x - 3)| - 8 < -2$
85. $2|9x + 5| + 8 > 6$
86. $3|4x - 9| + 4 < -1$
87. $5|4 - 3x| - 10 \leq 0$
88. $6|1 - 4x| - 24 \geq 0$
89. $3 - 2|x + 7| > -7$
90. $9 - 7|x - 4| < -12$
91. $|5(x - 4) + 5| > 15$

92. $|3(x-9)+6| \leq 3$

93. $\left|\frac{1}{3}(x+2)-\frac{7}{6}\right|-\frac{2}{3} \leq -\frac{1}{6}$

94. $\left|\frac{1}{10}(x+3)-\frac{1}{2}\right|+\frac{3}{20} > \frac{1}{4}$

95. $12+4|2x-1| \leq 12$

96. $3-6|3x-2| \geq 3$

97. $\frac{1}{2}|2x-1|+3 < 4$

98. $2\left|\frac{1}{2}x+\frac{2}{3}\right|-3 \leq -1$

99. $7-|-4+2(3-4x)| > 5$

100. $9-|6+3(2x-1)| \geq 8$

101. $\frac{3}{2}-\left|2-\frac{1}{3}x\right| < \frac{1}{2}$

102. $\frac{5}{4}-\left|\frac{1}{2}-\frac{1}{4}x\right| < \frac{3}{8}$

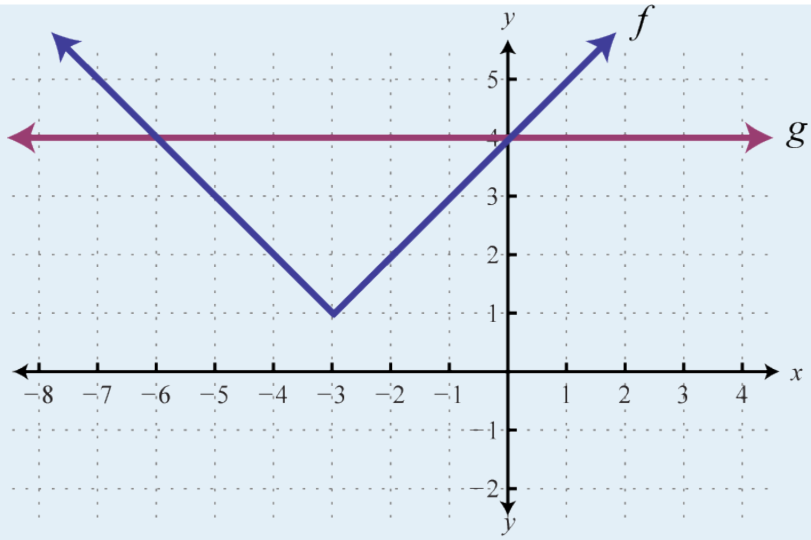
Assume all variables in the denominator are nonzero.

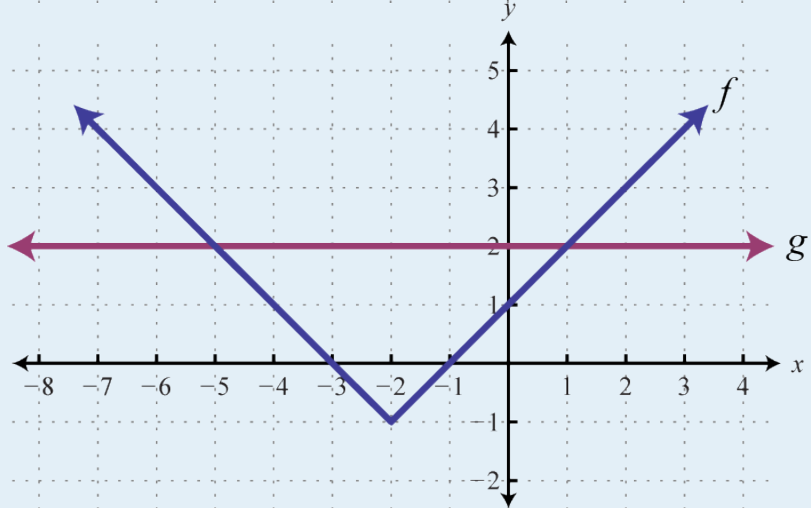
103. Solve for *x* where $a, p > 0$: $p|ax+b|-q \leq 0$

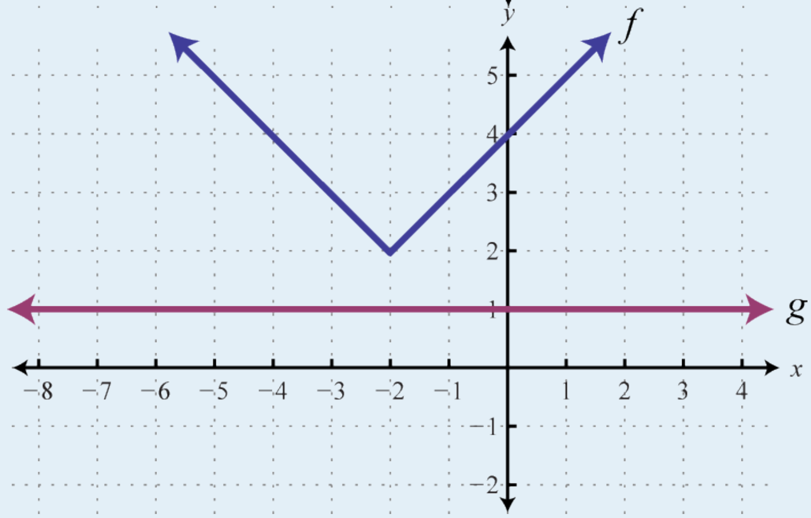
104. Solve for *x* where $a, p > 0$: $p|ax+b|-q \geq 0$

Given the graph of f and g, determine the *x*-values where:

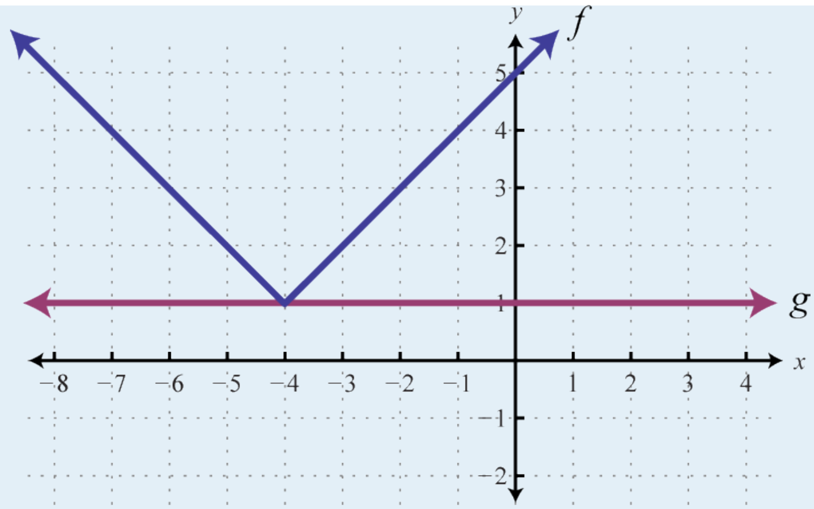
a. $f(x) = g(x)$
b. $f(x) > g(x)$
c. $f(x) < g(x)$

4. 

5. 

6. 

7.



PART C: DISCUSSION BOARD

109. Make three note cards, one for each of the three cases described in this section. On one side write the theorem, and on the other write a complete solution to a representative example. Share your strategy for identifying and solving absolute value equations and inequalities on the discussion board.

110. Make your own examples of absolute value equations and inequalities that have no solution, at least one for each case described in this section. Illustrate your examples with a graph.

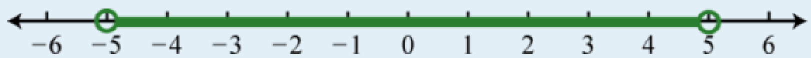
ANSWERS

1. $-9, 9$
3. $4, 10$
5. -12
7. $\varnothing$
9. $-6, 7$
11. $-1, \frac{7}{5}$
13. $1, \frac{5}{3}$
15. $-26, 10$
17. $0, 6$
19. $0, \frac{14}{5}$
21. $-\frac{1}{7}$
23. $1, 15$
25. $\varnothing$
27. $-3, 1$
29. $4, 6$
31. $\varnothing$
33. $-2.2, 7.8$
35. $-\frac{1}{6}, \frac{11}{6}$
37. $1, 5$
39. $-\frac{5}{3}, -\frac{11}{7}$
41. $\frac{5}{4}, 5$
43. $-\frac{1}{13}, 1$
45. $-4, 0.75$

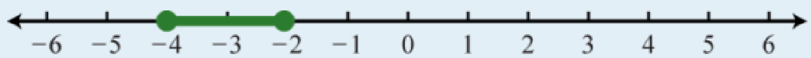
47. 0, 4

49. $x = \frac{-bq \pm q}{ap}$

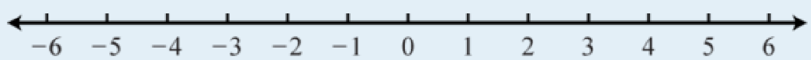
51. $(-5, 5)$;



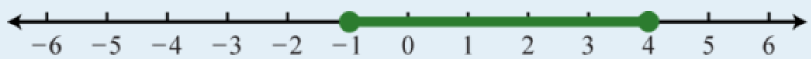
53. $[-4, -2]$;



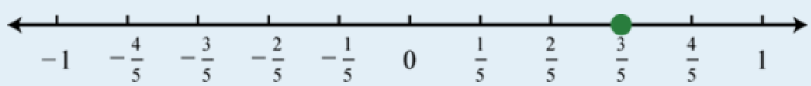
55. Ø;



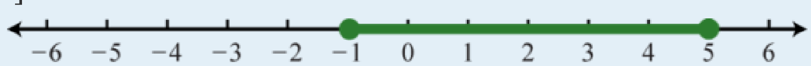
57. $[-1, 4]$;



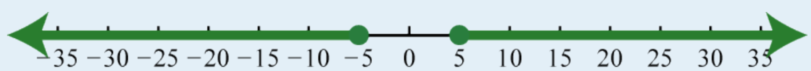
59. $\left\{\frac{3}{5}\right\}$;



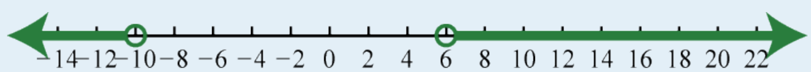
61. $[-1, 5]$;



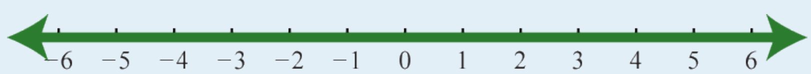
63. $(-\infty, -5] \cup [5, \infty)$;



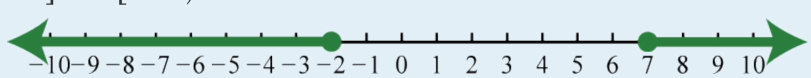
65. $(-\infty, -10) \cup (6, \infty)$;



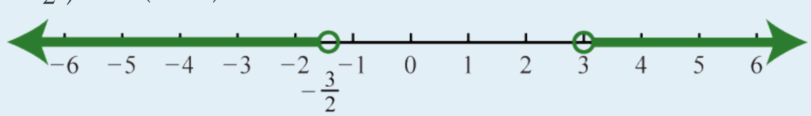
67. $\mathbb{R}$;



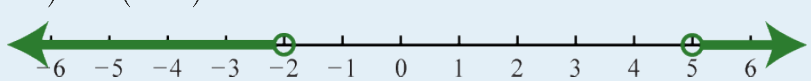
69. $(-\infty, -2] \cup [7, \infty)$;



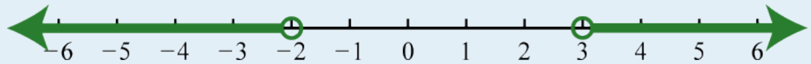
71. $\left(-\infty, -\frac{3}{2}\right) \cup (3, \infty)$;



73. $(-\infty, -2) \cup (5, \infty)$;

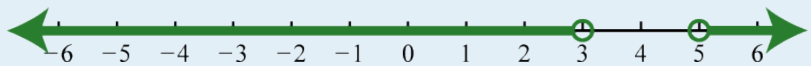


75. $(-\infty, -2) \cup (3, \infty)$;

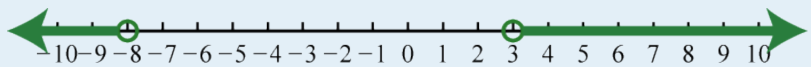


77. $(1, 7)$;

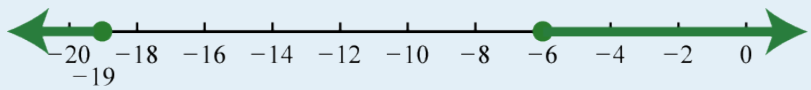
79. $(-\infty, 3) \cup (5, \infty)$;



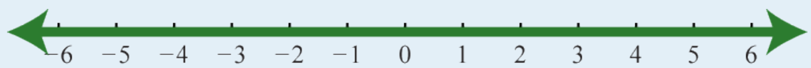
81. $(-\infty, -8) \cup (3, \infty)$;



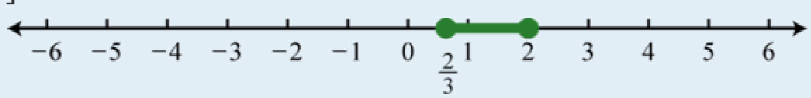
83. $(-\infty, -19] \cup [-6, \infty)$;



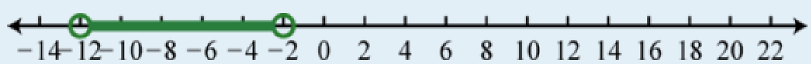
85. $\mathbb{R}$;



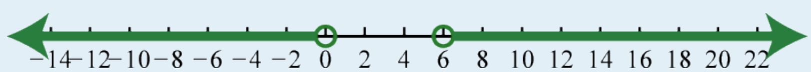
87. $\left[\frac{2}{3}, 2\right]$;



89. $(-12, -2)$;

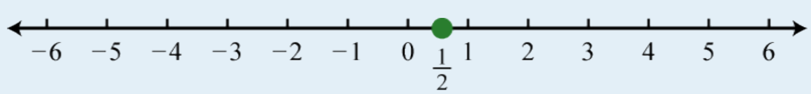


91. $(-\infty, 0) \cup (6, \infty)$;

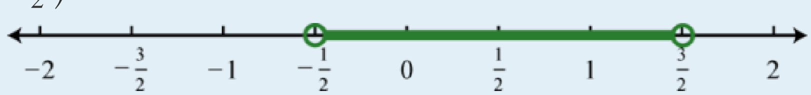


93. $[0, 3]$;

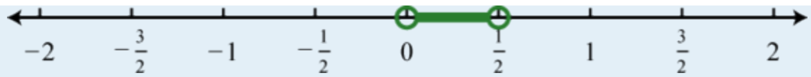
95. $\frac{1}{2}$;



97. $\left(-\frac{1}{2}, \frac{3}{2}\right)$;



99. $\left(0, \frac{1}{2}\right)$;



101. $\left(-\infty, 3\right) \cup \left(9, \infty\right)$;

103. $\frac{-q-bp}{ap} \leq x \leq \frac{q-bp}{ap}$

105.
a. −6, 0;
b. $\left(-\infty, -6\right) \cup \left(0, \infty\right)$;
c. $\left(-6, 0\right)$

107.
a. Ø;
b. $\mathbb{R}$;
c. Ø

109. Answer may vary

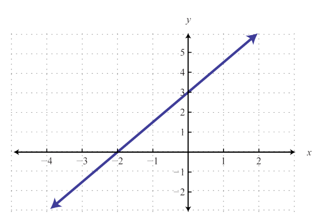
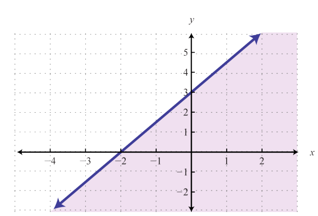
2.7 Solving Inequalities with Two Variables

LEARNING OBJECTIVES

1. Identify and check solutions to inequalities with two variables.
2. Graph solution sets of linear inequalities with two variables.

Solutions to Inequalities with Two Variables

We know that a linear equation with two variables has infinitely many ordered pair solutions that form a line when graphed. A **linear inequality with two variables**[65], on the other hand, has a solution set consisting of a region that defines half of the plane.

Linear Equation	Linear Inequality
$y = \frac{3}{2}x + 3$	$y \leq \frac{3}{2}x + 3$
	

For the inequality, the line defines the boundary of the region that is shaded. This indicates that any ordered pair in the shaded region, including the boundary line, will satisfy the inequality. To see that this is the case, choose a few **test points**[66] and substitute them into the inequality.

65. An inequality relating linear expressions with two variables. The solution set is a region defining half of the plane.

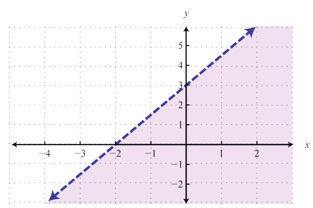
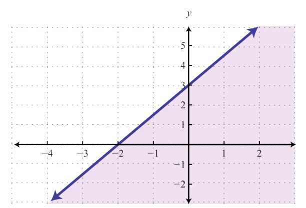
66. A point not on the boundary of the linear inequality used as a means to determine in which half-plane the solutions lie.

Test point	$y \leq \frac{3}{2}x + 3$
(0, 0)	$0 \leq \frac{3}{2}(0) + 3$ $0 \leq 3$ ✓
(2, 1)	$1 \leq \frac{3}{2}(2) + 3$ $1 \leq 3 + 3$ $1 \leq 6$ ✓
(-2, -1)	$-1 \leq \frac{3}{2}(-2) + 3$ $-1 \leq -3 + 3$ $-1 \leq 0$ ✓

Also, we can see that ordered pairs outside the shaded region do not solve the linear inequality.

Test point	$y \leq \frac{3}{2}x + 3$
(-2, 3)	$3 \leq \frac{3}{2}(-2) + 3$ $3 \leq -3 + 3$ $3 \leq 0$ ✗

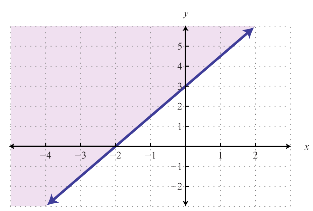
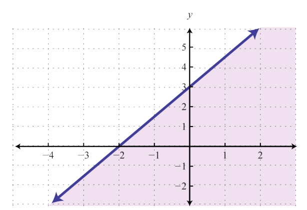
The graph of the solution set to a linear inequality is always a region. However, the boundary may not always be included in that set. In the previous example, the line was part of the solution set because of the “or equal to” part of the inclusive inequality $\leq$. If given a strict inequality $<$, we would then use a dashed line to indicate that those points are not included in the solution set.

Non-Inclusive Boundary	Inclusive Boundary
$y < \frac{3}{2}x + 3$	$y \leq \frac{3}{2}x + 3$
	

Consider the point (0, 3) on the boundary; this ordered pair satisfies the linear equation. It is the “or equal to” part of the inclusive inequality that makes the ordered pair part of the solution set.

$y < \frac{3}{2}x + 3$	$y \leq \frac{3}{2}x + 3$
$3 < \frac{3}{2}(0) + 3$ $3 < 0 + 3$ $3 < 3$ ✗	$3 \leq \frac{3}{2}(0) + 3$ $3 \leq 0 + 3$ $3 \leq 3$ ✓

So far we have seen examples of inequalities that were "less than." Now consider the following graphs with the same boundary:

Greater Than (Above)	Less Than (Below)
$y \geq \frac{3}{2}x + 3$	$y \leq \frac{3}{2}x + 3$
	

Given the graphs above, what might we expect if we use the origin (0, 0) as a test point?

$y \geq \frac{3}{2}x + 3$	$y \leq \frac{3}{2}x + 3$
$0 \geq \frac{3}{2}(0) + 3$ $0 \geq 0 + 3$ $0 \geq 3$ ✗	$0 \leq \frac{3}{2}(0) + 3$ $0 \leq 0 + 3$ $0 \leq 3$ ✓

Example 1

Determine whether or not $\left(2, \frac{1}{2}\right)$ is a solution to $5x - 2y < 10$.

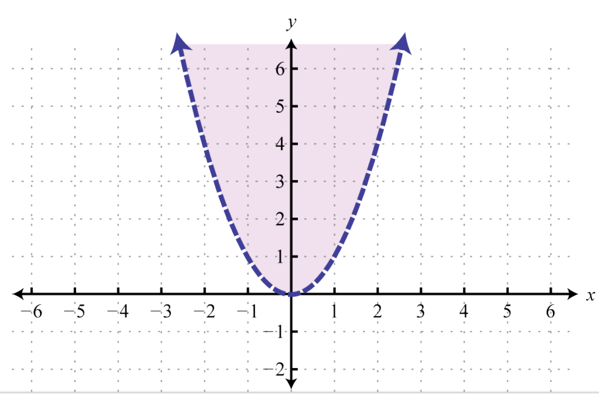
Solution:

Substitute the *x*- and *y*-values into the equation and see if a true statement is obtained.

$$\begin{aligned} 5x - 2y &< 10 \\ 5(2) - 2\left(\frac{1}{2}\right) &< 10 \\ 10 - 1 &< 10 \\ 9 &< 10 \quad \checkmark \end{aligned}$$

Answer: $\left(2, \frac{1}{2}\right)$ is a solution.

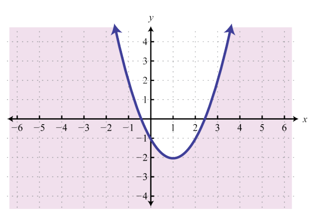
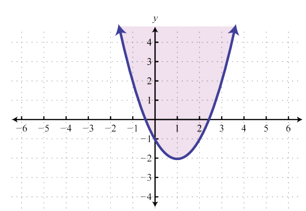
These ideas and techniques extend to nonlinear inequalities with two variables. For example, all of the solutions to $y > x^2$ are shaded in the graph below.



The boundary of the region is a parabola, shown as a dashed curve on the graph, and is not part of the solution set. However, from the graph we expect the ordered pair (−1,4) to be a solution. Furthermore, we expect that ordered pairs that are not in the shaded region, such as (−3, 2), will not satisfy the inequality.

Check (−1,4)	*Check* (−3, 2)
$y > x^2$ $4 > (-1)^2$ $4 > 1$ ✓	$y > x^2$ $2 > (-3)^2$ $2 > 9$ ✗

Following are graphs of solutions sets of inequalities with inclusive parabolic boundaries.

$y \leq (x-1)^2 - 2$	$y \geq (x-1)^2 - 2$
	

You are encouraged to test points in and out of each solution set that is graphed above.

Try this! Is $(-3, -2)$ a solution to $2x - 3y < 0$?

Answer: No

(click to see video)

Graphing Solutions to Inequalities with Two Variables

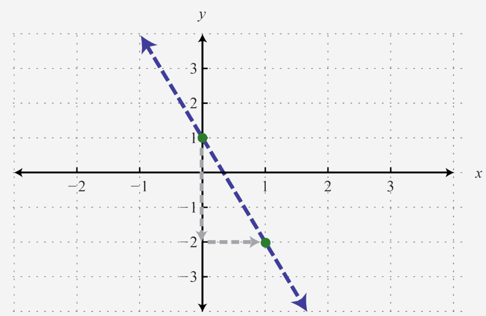
Solutions to linear inequalities are a shaded half-plane, bounded by a solid line or a dashed line. This boundary is either included in the solution or not, depending on the given inequality. If we are given a strict inequality, we use a dashed line to indicate that the boundary is not included. If we are given an inclusive inequality, we use a solid line to indicate that it is included. The steps for graphing the solution set for an inequality with two variables are shown in the following example.

Example 2

Graph the solution set $y > -3x + 1$.

Solution:

- **Step 1:** Graph the boundary. Because of the strict inequality, we will graph the boundary $y = -3x + 1$ using a dashed line. We can see that the slope is $m = -3 = \frac{-3}{1} = \frac{rise}{run}$and the y-intercept is (0, 1).

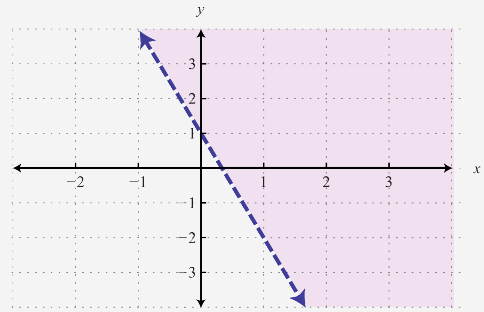


- **Step 2:** Test a point that is *not* on the boundary. A common test point is the origin, (0, 0). The test point helps us determine which half of the plane to shade.

Test point	$y > -3x + 1$
(0, 0)	$0 > -3(0) + 1$ $0 > 1$ ✗

- **Step 3:** Shade the region containing the solutions. Since the test point (0, 0) was not a solution, it does not lie in the region containing all the ordered pair solutions. Therefore, shade the half of the plane that does not contain this test point. In this case, shade above the boundary line.

Answer:



Consider the problem of shading above or below the boundary line when the inequality is in slope-intercept form. If $y > mx + b$, then shade above the line. If $y < mx + b$, then shade below the line. Shade with caution; sometimes the boundary is given in standard form, in which case these rules do not apply.

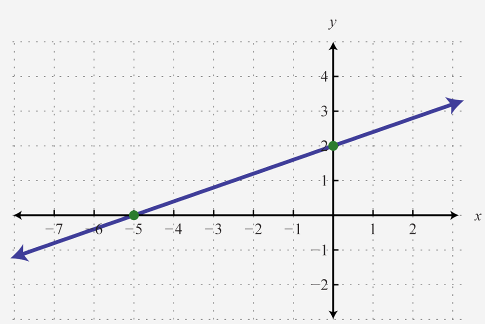
Example 3

Graph the solution set $2x - 5y \geq -10$.

Solution:

Here the boundary is defined by the line $2x - 5y = -10$. Since the inequality is inclusive, we graph the boundary using a solid line. In this case, graph the boundary line using intercepts.

To find the *x*-intercept, set *y* = 0.	To find the *y*-intercept, set *x* = 0.
$2x - 5y = -10$ $\begin{aligned} 2x - 5(0) &= -10 \\ 2x &= -10 \\ x &= -5 \end{aligned}$	$2x - 5y = -10$ $\begin{aligned} 2(0) - 5y &= -10 \\ -5y &= -10 \\ y &= 2 \end{aligned}$
x-intercept: (−5, 0)	*y*-intercept: (0, 2)

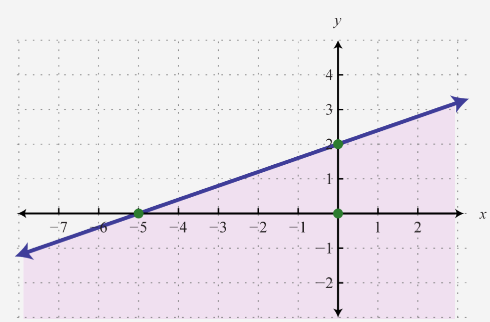


Next, test a point; this helps decide which region to shade.

Test point	$2x - 5y \geq -10$
(0, 0)	$2\,(0) - 5\,(0) \geq -10$ $0 \geq -10$ ✓

Since the test point is in the solution set, shade the half of the plane that contains it.

Answer:



In this example, notice that the solution set consists of all the ordered pairs below the boundary line. This may seem counterintuitive because the original inequality involved "greater than" ≥. This illustrates that it is a best practice to actually test a point. Solve for *y* and you see that the shading is correct.

$$\begin{aligned} 2x - 5y &\geq -10 \\ 2x - 5y - 2x &\geq -10 - 2x \\ -5y &\geq -2x - 10 \\ \frac{-5y}{-5} &\leq \frac{-2x - 10}{-5} \qquad \textit{Reverse the inequality}. \\ y &\leq \frac{2}{5}x + 2 \end{aligned}$$

In slope-intercept form, you can see that the region below the boundary line should be shaded. An alternate approach is to first express the boundary in slope-intercept form, graph it, and then shade the appropriate region.

Example 4

Graph the solution set $y < 2$.

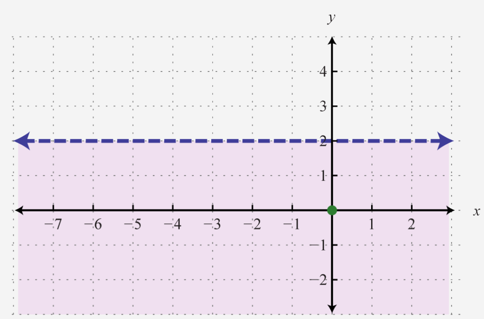
Solution:

First, graph the boundary line $y = 2$ with a dashed line because of the strict inequality. Next, test a point.

Test point	$y < 2$
(0, 0)	$0 < 2$ ✓

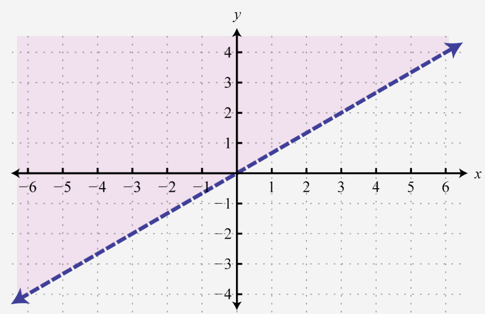
In this case, shade the region that contains the test point.

Answer:



Try this! Graph the solution set $2x - 3y < 0$.

Answer:



(click to see video)

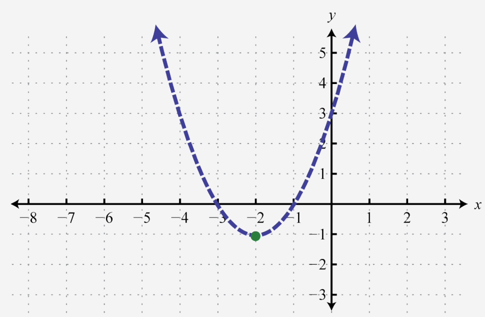
The steps are the same for nonlinear inequalities with two variables. Graph the boundary first and then test a point to determine which region contains the solutions.

Example 5

Graph the solution set $y < (x+2)^2 - 1$.

Solution:

The boundary is a basic parabola shifted 2 units to the left and 1 unit down. Begin by drawing a dashed parabolic boundary because of the strict inequality.

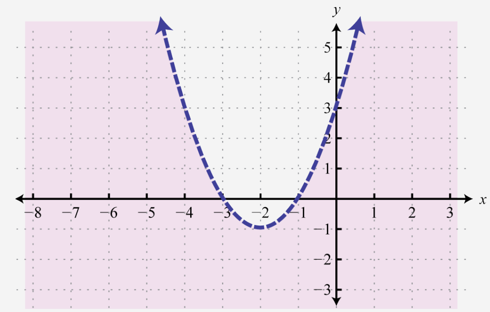


Next, test a point.

Test point	$y < (x+2)^2 - 1$
(0, 0)	$0 < (0+2)^2 - 1$ $0 < 4 - 1$ $0 < 3$ ✓

In this case, shade the region that contains the test point $(0, 0)$.

Answer:


y
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−2
−1
1
2
3
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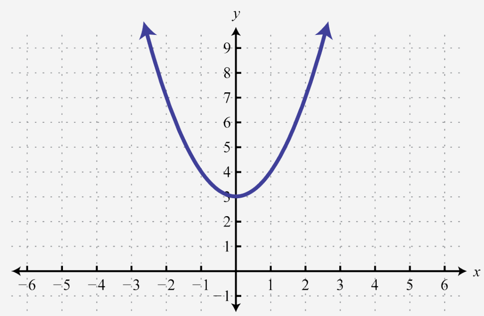
4
3
2
1
−1
−2
−3

Example 6

Graph the solution set $y \geq x^2 + 3$.

Solution:

The boundary is a basic parabola shifted 3 units up. It is graphed using a solid curve because of the inclusive inequality.

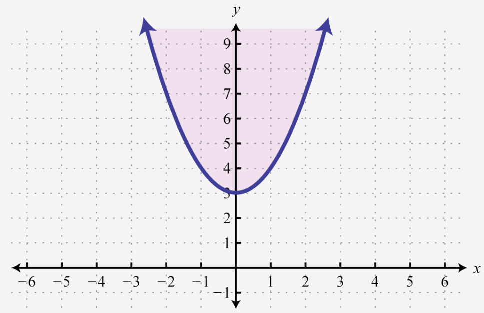


Next, test a point.

Test point	$y \geq x^2 + 3$
(0, 0)	$0 \geq 0^2 + 3$ $0 \geq 3$ ✗

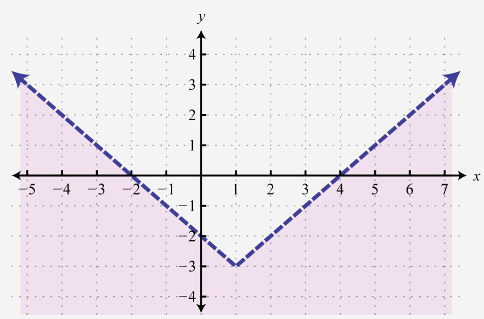
In this case, shade the region that does not contain the test point $(0, 0)$.

Answer:



Try this! Graph the solution set $y < |x - 1| - 3$.

Answer:



(click to see video)

KEY TAKEAWAYS

- Linear inequalities with two variables have infinitely many ordered pair solutions, which can be graphed by shading in the appropriate half of a rectangular coordinate plane.
- To graph the solution set of an inequality with two variables, first graph the boundary with a dashed or solid line depending on the inequality. If given a strict inequality, use a dashed line for the boundary. If given an inclusive inequality, use a solid line. Next, choose a test point not on the boundary. If the test point solves the inequality, then shade the region that contains it; otherwise, shade the opposite side.
- Check your answer by testing points in and out of the shading region to verify that they solve the inequality or not.

TOPIC EXERCISES

PART A: SOLUTIONS TO INEQUALITIES WITH TWO VARIABLES

Is the ordered pair a solution to the given inequality?

1. $5x - y > -2;\ (-3, -4)$
2. $4x - y < -8;\ (-3, -10)$
3. $6x - 15y \geq -1;\ \left(\frac{1}{2}, -\frac{1}{3}\right)$
4. $x - 2y \geq 2;\ \left(\frac{2}{3}, -\frac{5}{6}\right)$
5. $\frac{3}{4}x - \frac{2}{3}y < \frac{3}{2};\ (1, -1)$
6. $\frac{2}{5}x + \frac{4}{3}y > \frac{1}{2};\ (-2, 1)$
7. $y \leq x^2 - 1;\ (-1, 1)$
8. $y \geq x^2 + 3;\ (-2, 0)$
9. $y \geq (x - 5)^2 + 1;\ (3, 4)$
10. $y \leq 2(x + 1)^2 - 3;\ (-1, -2)$
11. $y > 3 - |x|;\ (-4, -3)$
12. $y < |x| - 8;\ (5, -7)$
13. $y > |2x - 1| - 3;\ (-1, 3)$
14. $y < |3x - 2| + 2;\ (-2, 10)$

PART B: GRAPHING SOLUTIONS TO INEQUALITIES WITH TWO VARIABLES.

Graph the solution set.

15. $y < 2x - 1$

16. $y > -4x + 1$
17. $y \geq -\frac{2}{3}x + 3$
18. $y \leq \frac{4}{3}x - 3$
19. $2x + 3y \leq 18$
20. $5x + 2y \leq 8$
21. $6x - 5y > 30$
22. $8x - 6y < 24$
23. $3x - 4y < 0$
24. $x - 3y > 0$
25. $x + y \geq 0$
26. $x - y \geq 0$
27. $y \leq -2$
28. $y > -3$
29. $x < -2$
30. $x \geq -3$
31. $\frac{1}{6}x + \frac{1}{10}y \leq \frac{1}{2}$
32. $\frac{3}{8}x + \frac{1}{2}y \geq \frac{3}{4}$
33. $\frac{1}{12}x - \frac{1}{6}y < \frac{2}{3}$
34. $\frac{1}{3}x - \frac{1}{9}y > \frac{4}{3}$
35. $5x \leq -4y - 12$
36. $-4x \leq 12 - 3y$
37. $4y + 2 < 3x$
38. $8x < 9 - 6y$
39. $5 \geq 3x - 15y$
40. $2x \geq 6 - 9y$

41. Write an inequality that describes all points in the upper half-plane above the x-axis.

42. Write an inequality that describes all points in the lower half-plane below the x-axis.

43. Write an inequality that describes all points in the half-plane left of the y-axis.

44. Write an inequality that describes all points in the half-plane right of the y-axis.

45. Write an inequality that describes all ordered pairs whose y-coordinate is at least k units.

46. Write an inequality that describes all ordered pairs whose x-coordinate is at most k units.

Graph the solution set.

47. $y \leq x^2 + 3$

48. $y > x^2 - 2$

49. $y \leq -x^2$

50. $y \geq -x^2$

51. $y > (x + 1)^2$

52. $y > (x - 2)^2$

53. $y \leq (x - 1)^2 + 2$

54. $y \leq (x + 3)^2 - 1$

55. $y < -x^2 + 1$

56. $y > -(x - 2)^2 + 1$

57. $y \geq |x| - 2$

58. $y < |x| + 1$

59. $y < |x - 3|$

60. $y \leq |x + 2|$

61. $y > -|x + 1|$

62. $y \leq -|x - 2|$

63. $y \geq |x + 3| - 2$

64. $y \geq |x - 2| - 1$

65. $y < -|x + 4| + 2$

66. $y > -|x - 4| - 1$

67. $y > x^3 - 1$

68. $y \leq x^3 + 2$

69. $y \leq \sqrt{x}$

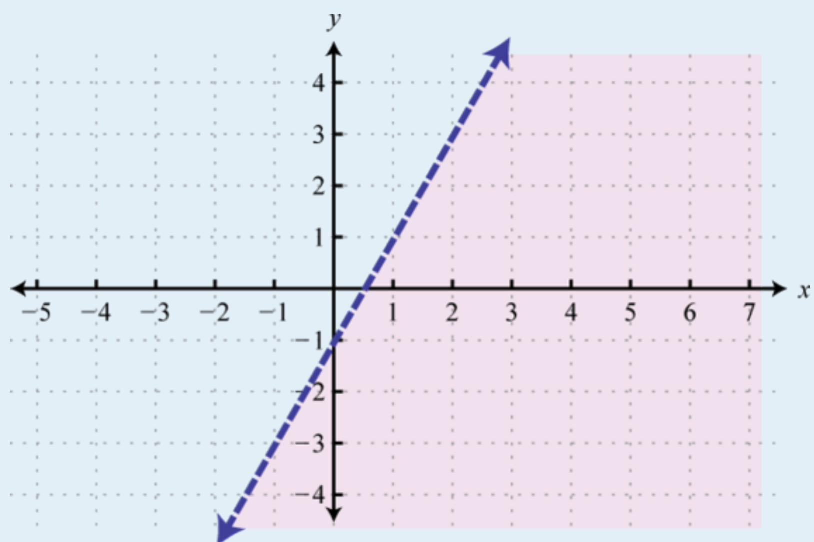
70. $y > \sqrt{x} - 1$

71. A rectangular pen is to be constructed with at most 200 feet of fencing. Write a linear inequality in terms of the length *l* and the width *w*. Sketch the graph of all possible solutions to this problem.

72. A company sells one product for \$8 and another for \$12. How many of each product must be sold so that revenues are at least \$2,400? Let *x* represent the number of products sold at \$8 and let *y* represent the number of products sold at \$12. Write a linear inequality in terms of *x* and *y* and sketch the graph of all possible solutions.

ANSWERS

1. No
3. Yes
5. Yes
7. No
9. No
11. No
13. Yes



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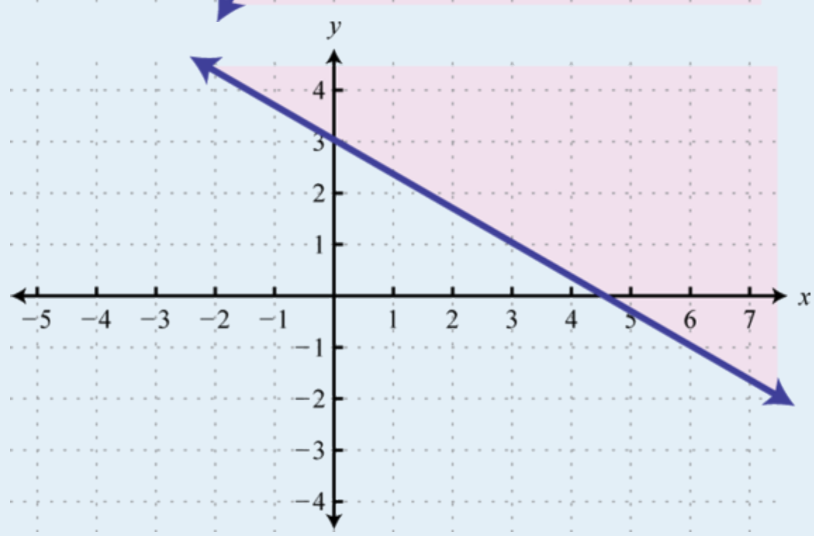

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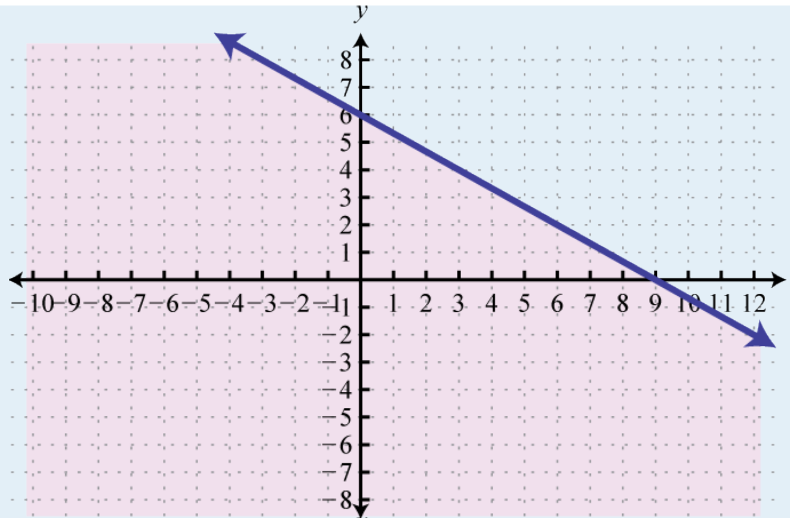
19. 
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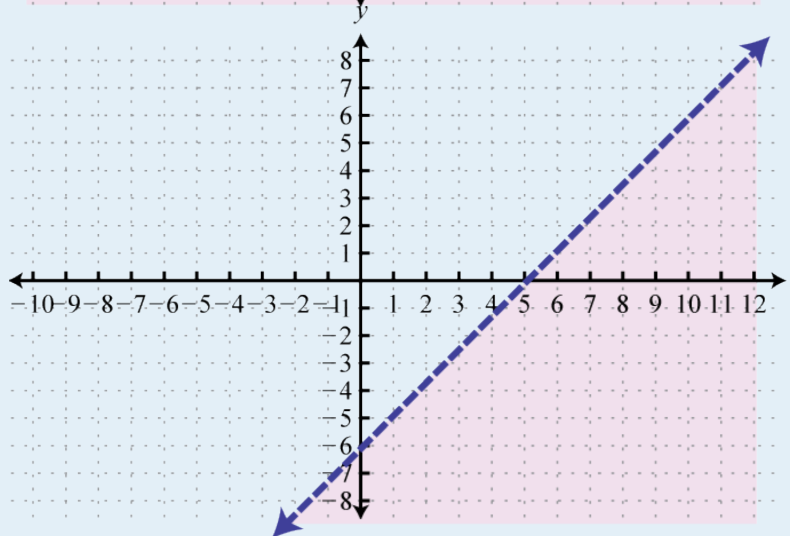
8 7 6 5 4 3 2 1 −2 −3 −4 −5 −6 −7 −8

21. 
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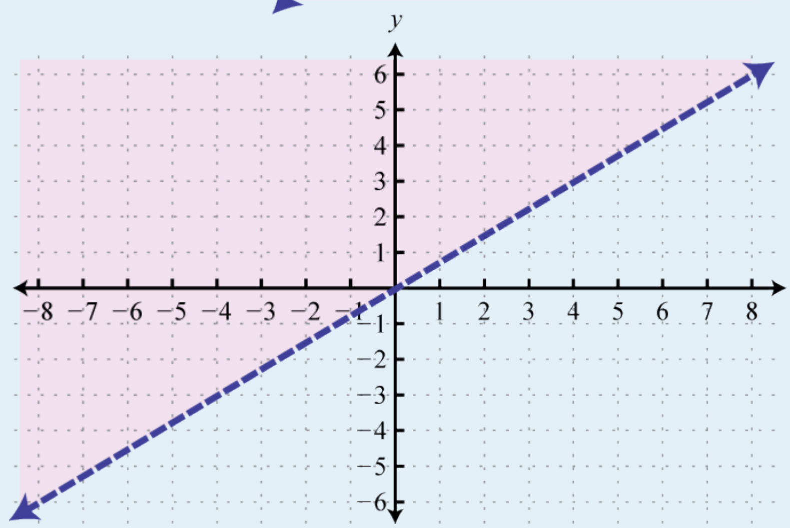
23. 
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6 5 4 3 2 1 −1 −2 −3 −4 −5 −6

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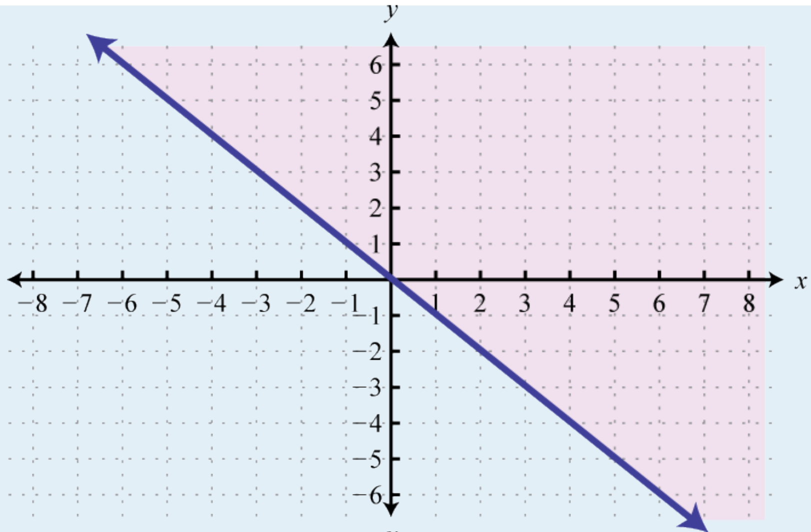
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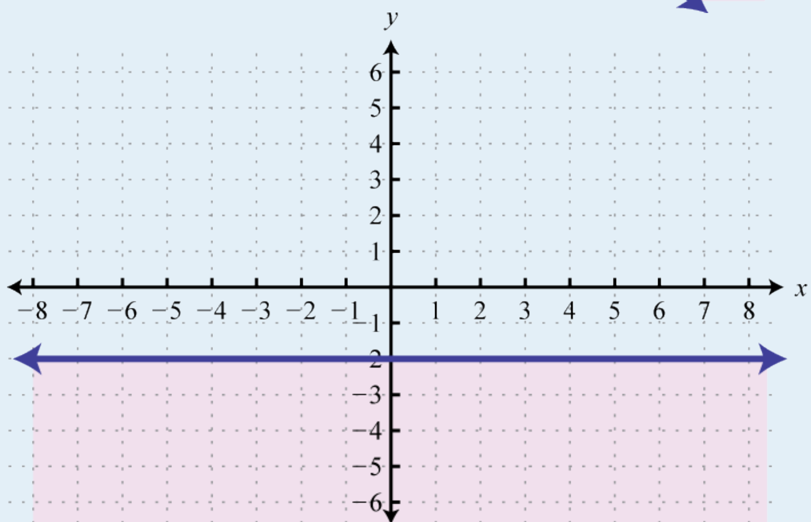
27.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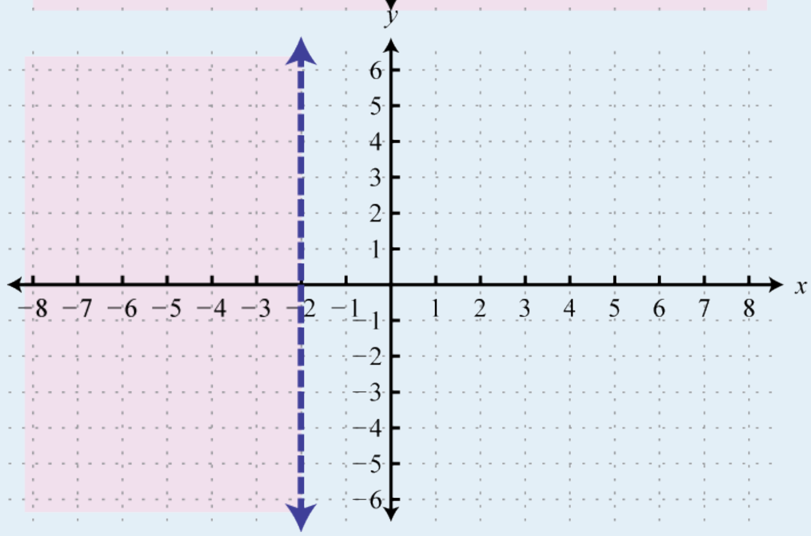
29.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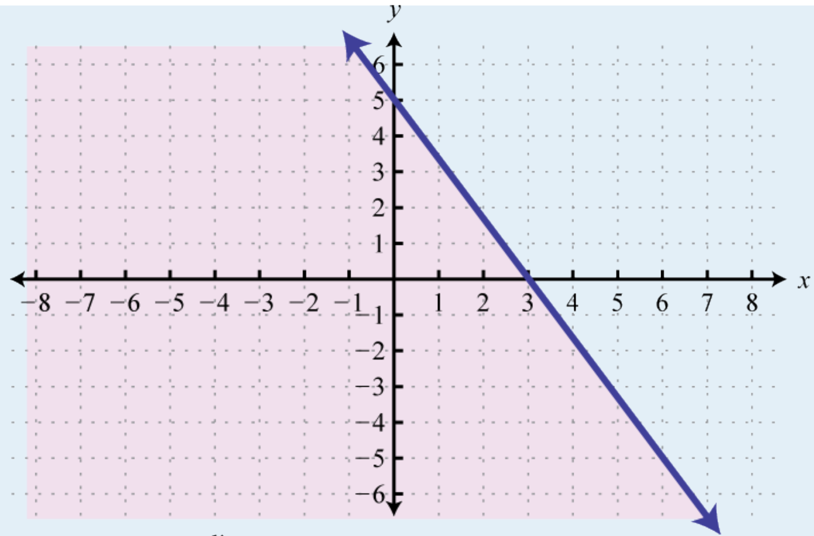
31.



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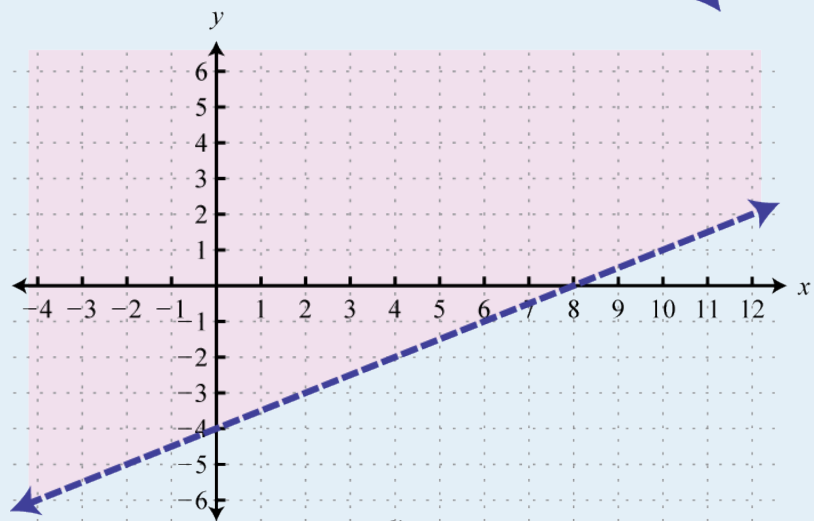

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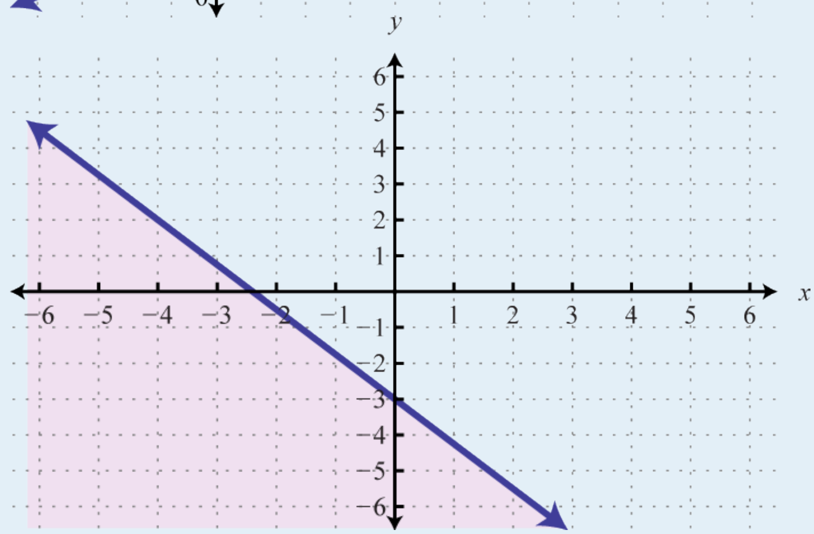
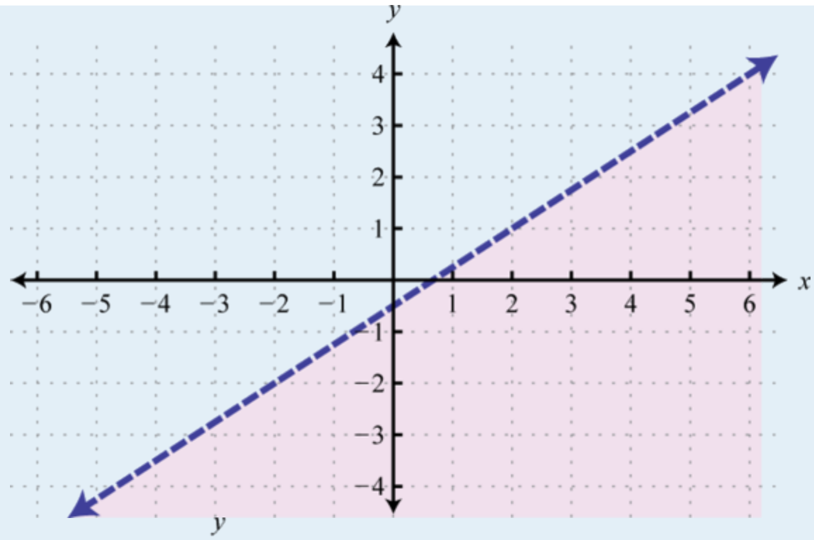
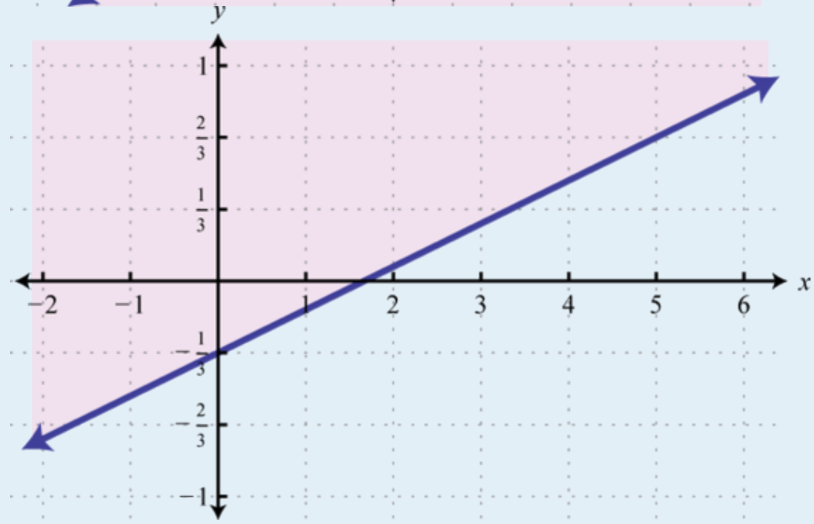


37. 

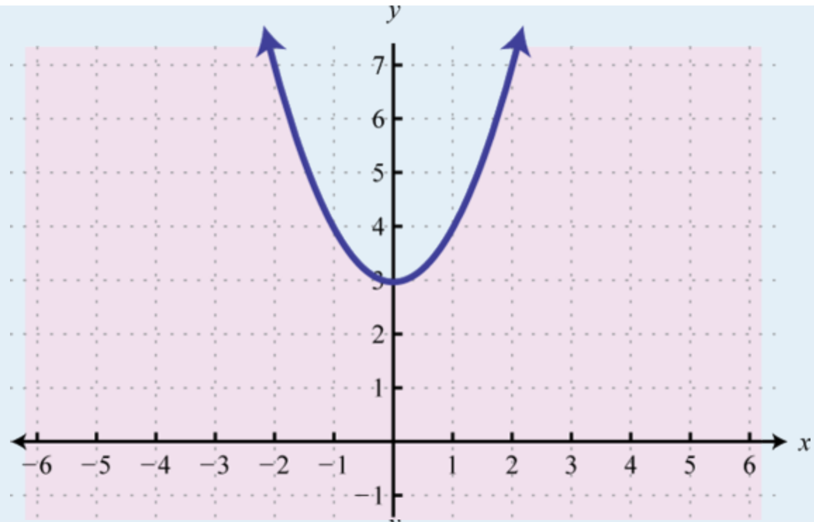
39. 

41. $y > 0$

43. $x < 0$

45. $y \geq k$

47.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x
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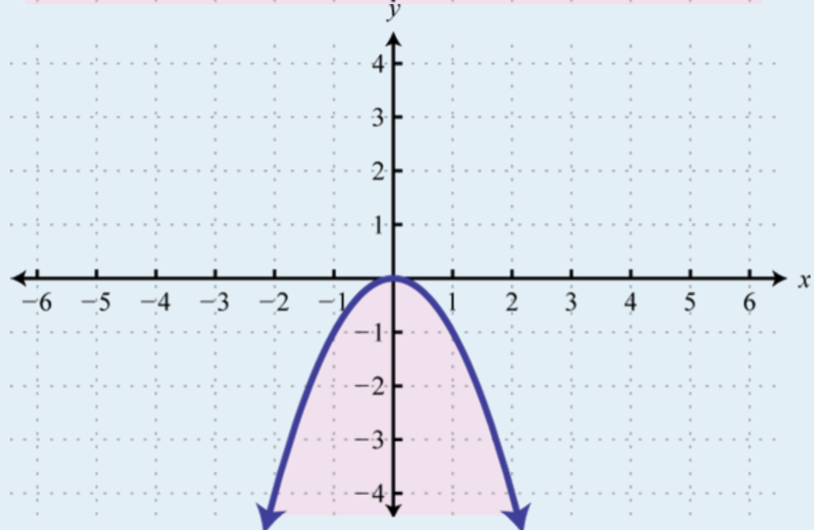
−1

49.


y
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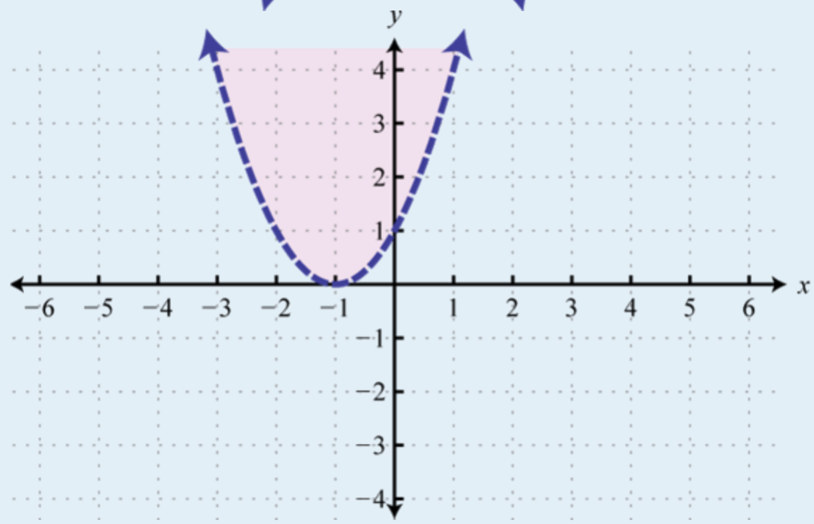
1
x
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6
−1
−2
−3
−4

51.


y
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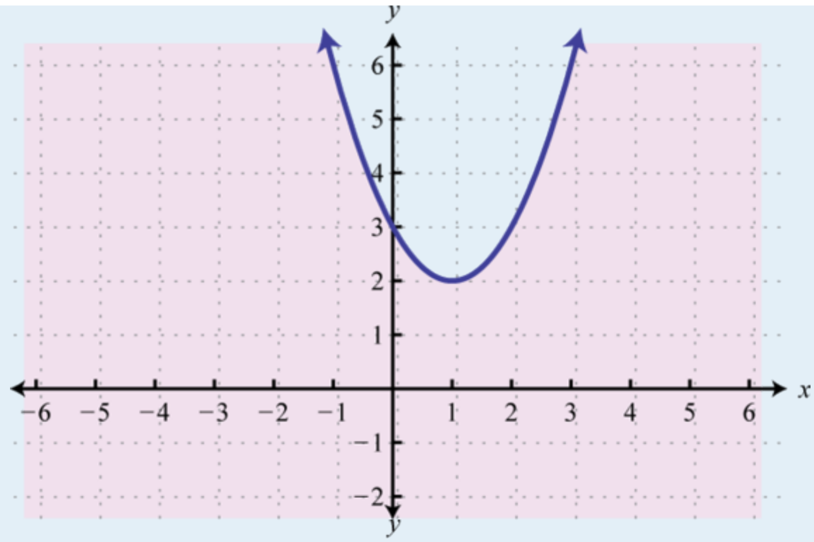
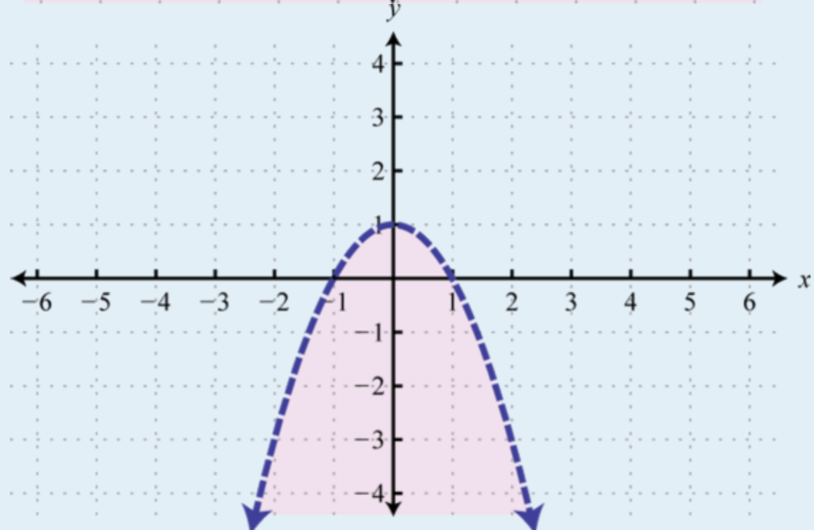
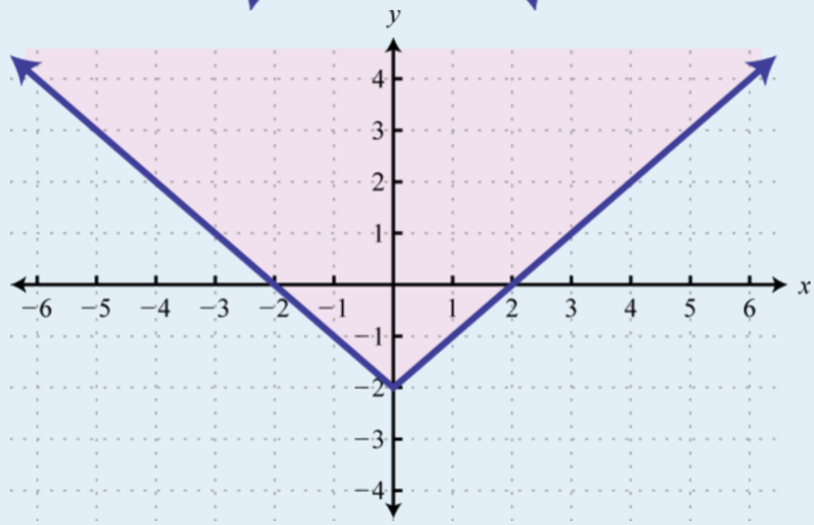
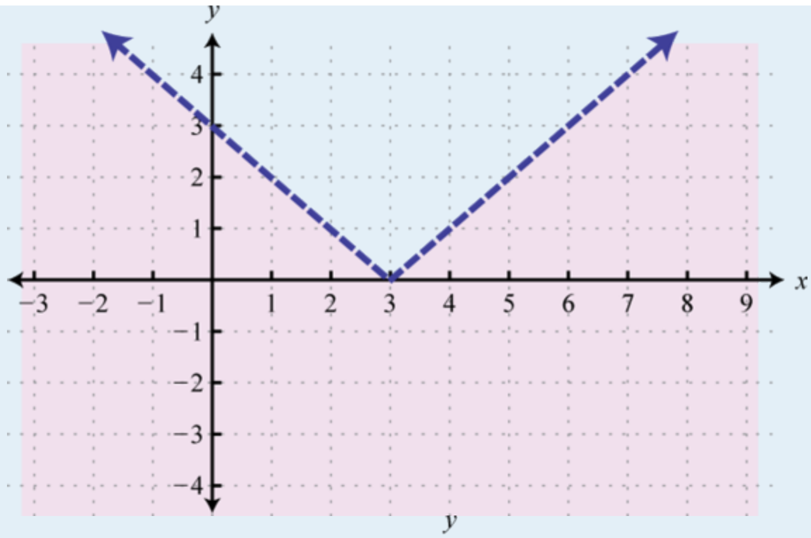
1
x
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6
−1
−2
−3
−4

53. 

55. 

57. 

59. 
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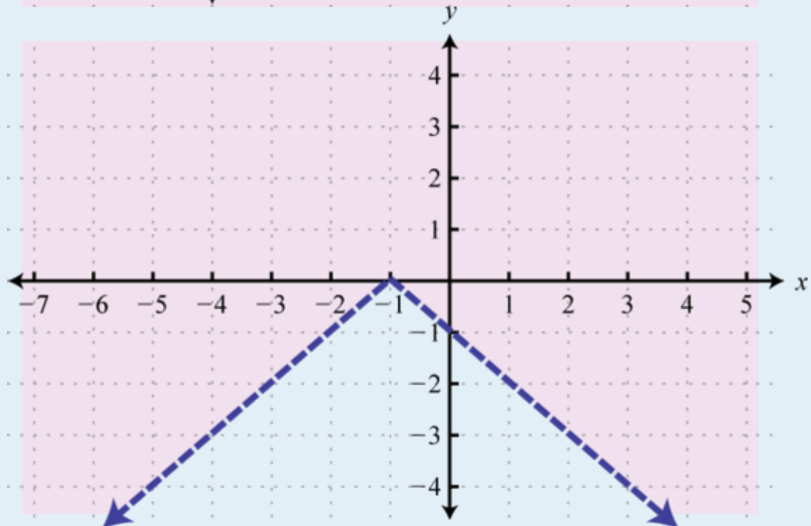
61. 
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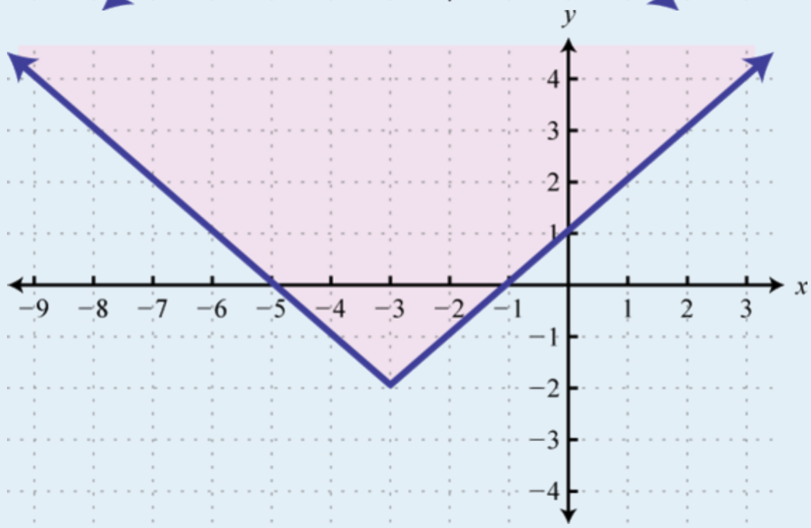
63. 
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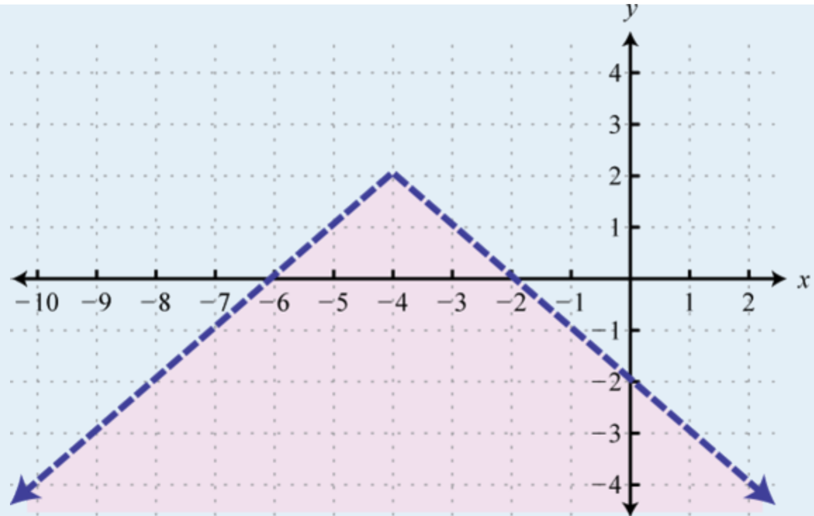
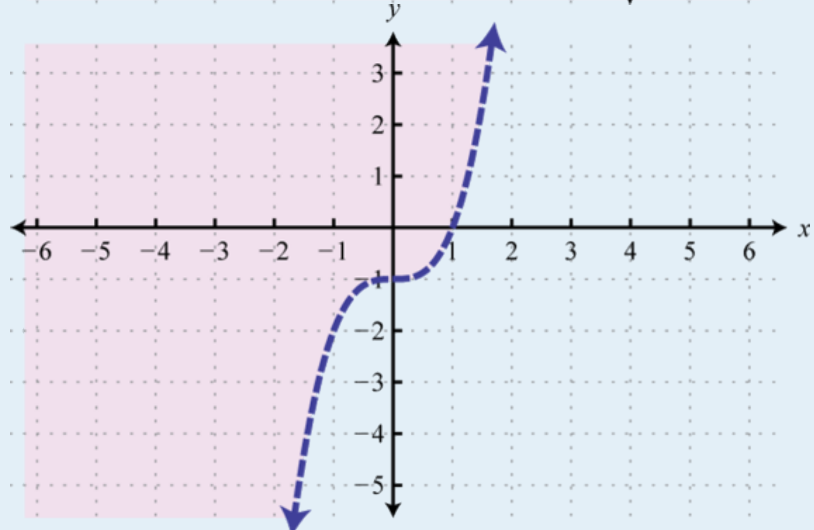
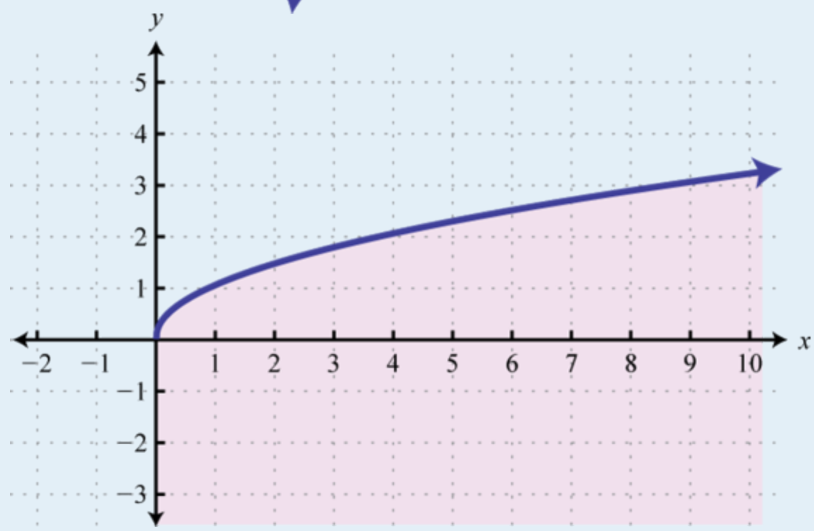
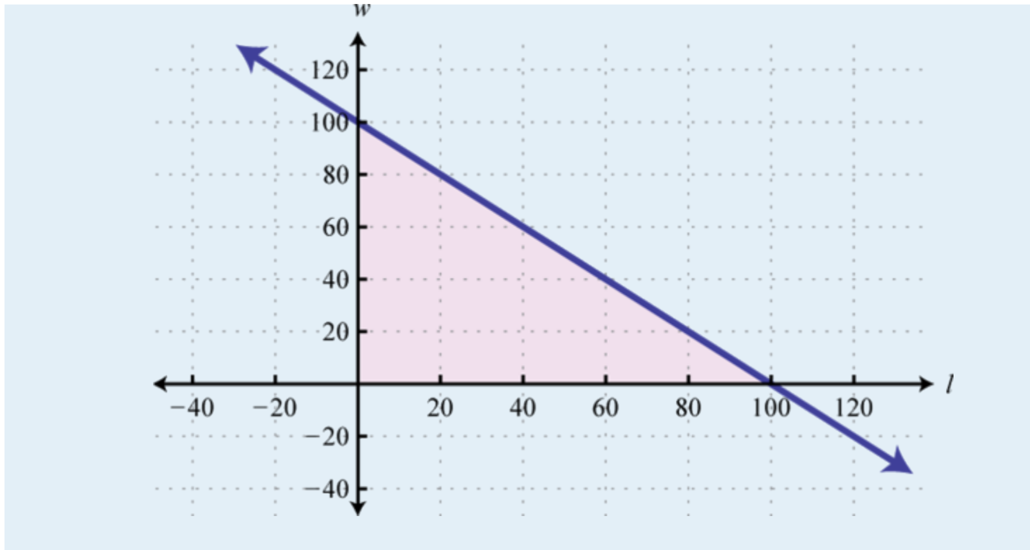
65. 

67. 

69. 

71. $l + w \leq 100;$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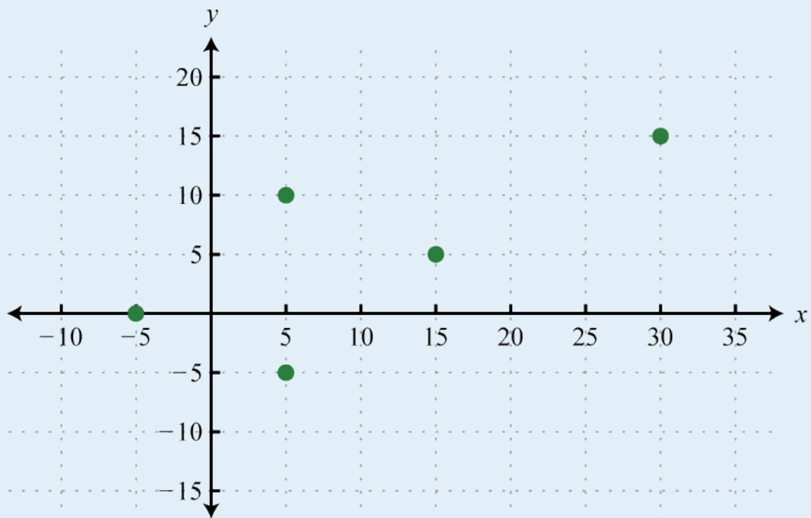
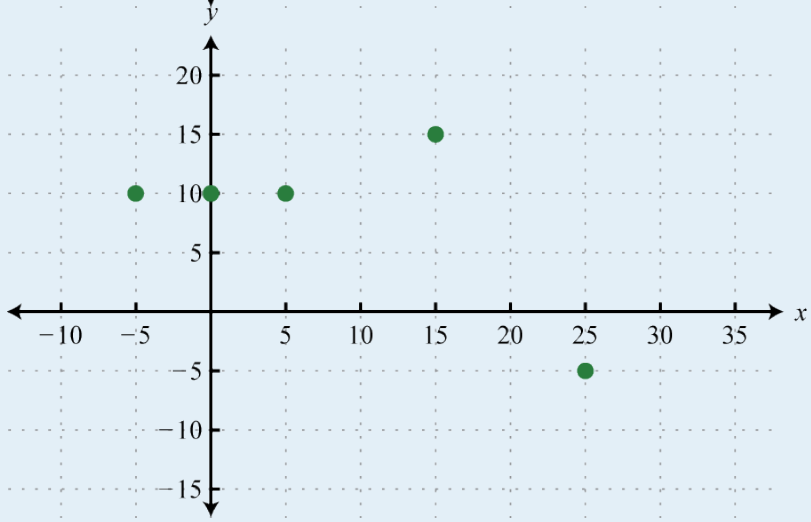
2.8 Review Exercises and Sample Exam

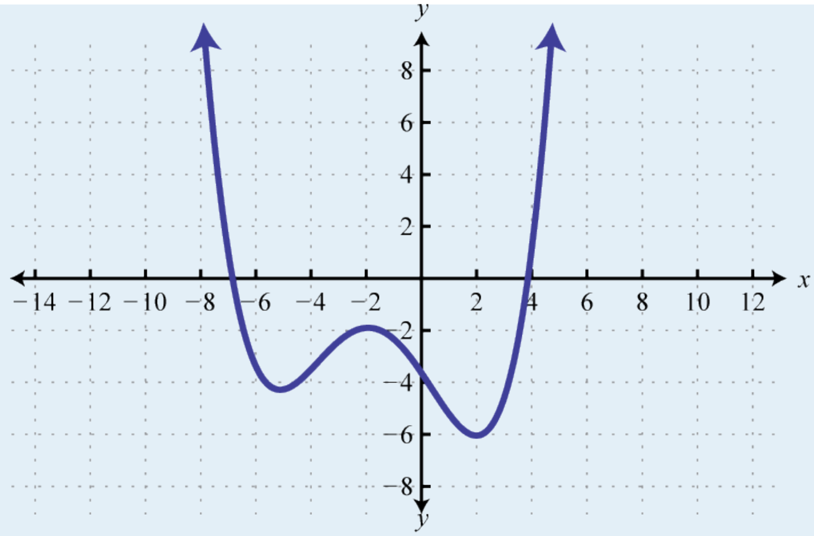
REVIEW EXERCISES

RELATIONS, GRAPHS, AND FUNCTIONS

Determine the domain and range and state whether the function is a relation or not.

1. {(−4, −1), (−5, 3), (10, 3), (11, 2), (15, 1)}
2. {(−3, 0), (−2, 1), (1, 3), (2, 7), (2, 5)}
3. 

4. 

5. 
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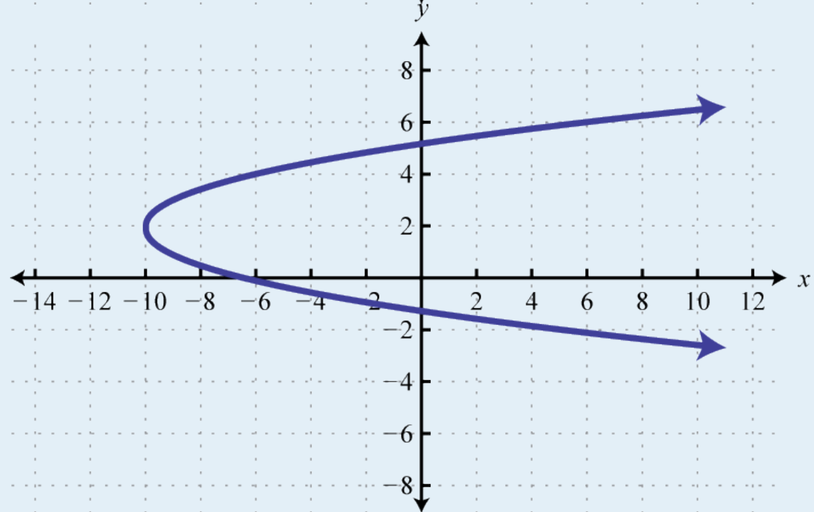
6. 
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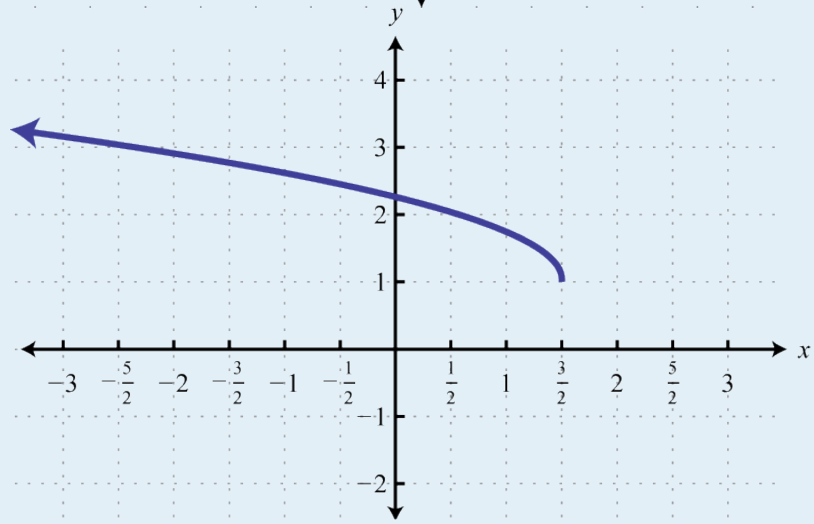
−14 −12 −10 −8 −6 −4 −2 2 4 6 8 10 12

7. 
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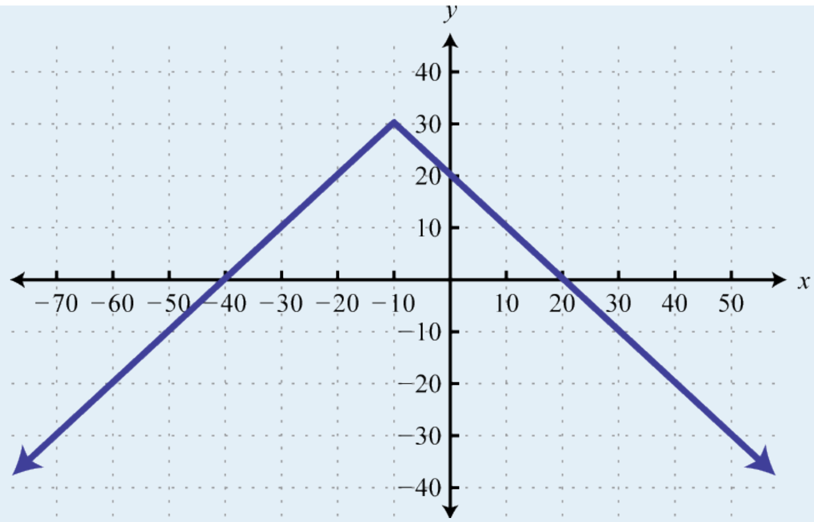
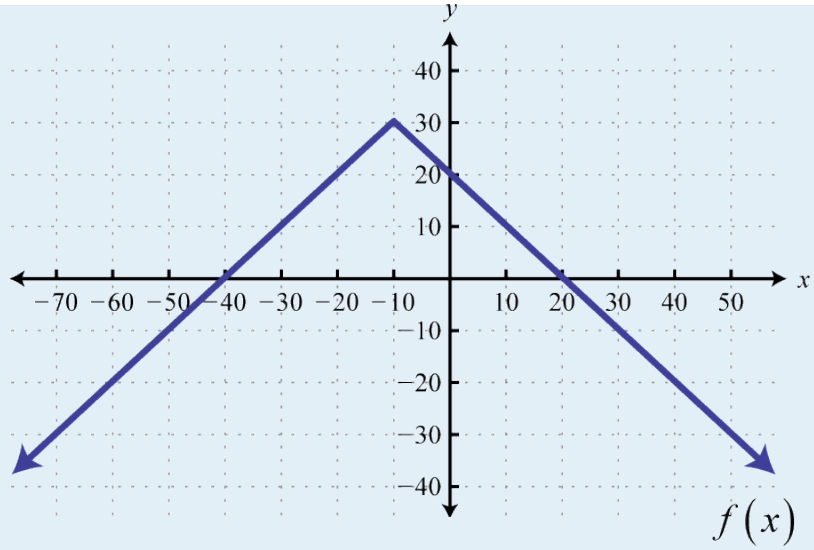
−3 −5/2 −2 −3/2 −1 −1/2 1/2 1 3/2 2 5/2 3

8.

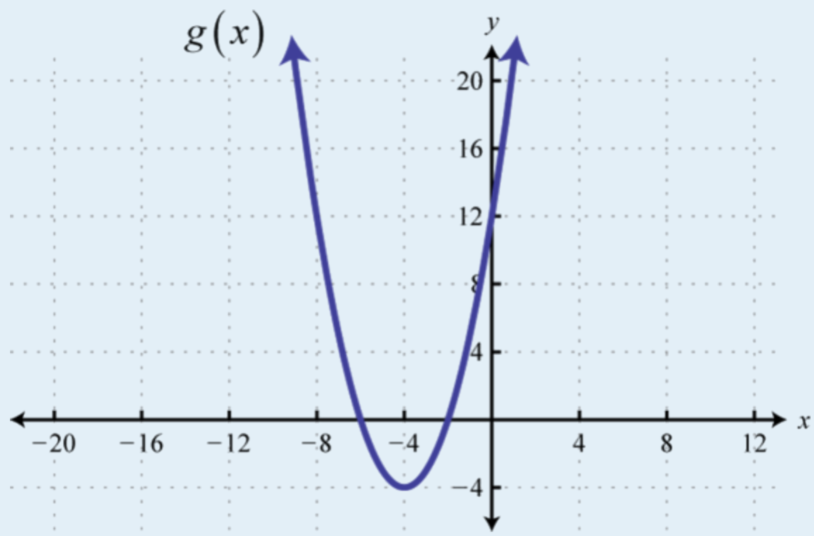


Evaluate.

9. $h(x) = \frac{1}{2}x - 3$; $h(-8)$, $h(3)$, and $h(4a + 1)$
10. $p(x) = 4 - x$; $p(-10)$, $p(0)$, and $p(5a - 1)$
11. $f(x) = 2x^2 - x + 3$; find $f(-5)$, $f(0)$, and $f(x + h)$
12. $g(x) = x^2 - 9$; find $f(-3)$, $f(2)$, and $f(x + h)$
13. $g(x) = \sqrt{2x - 1}$; find $g(5)$, $g(1)$, $g(13)$
14. $h(x) = \sqrt[3]{x + 6}$; find $h(-7)$, $h(-6)$, and $h(21)$
15. $f(x) = 8x + 3$; find x where $f(x) = 10$.
16. $g(x) = 5 - 3x$; find x where $g(x) = -4$.
17. Given the graph of $f(x)$ below, find $f(-60)$, $f(0)$, and $f(20)$.



18. Given the graph of $g(x)$ below, find x where $g(x) = -4$ and $g(x) = 12$.



LINEAR FUNCTIONS AND THEIR GRAPHS

Graph and label the intercepts.

19. $4x - 8y = 12$
20. $9x + 4y = 6$
21. $\frac{3}{8}x + \frac{1}{2}y = \frac{5}{4}$
22. $\frac{3}{4}x - \frac{1}{2}y = -1$

Graph the linear function and label the *x*-intercept.

23. $g(x) = \frac{5}{8}x + 10$
24. $g(x) = -\frac{1}{5}x - 3$
25. $f(x) = -4x + \frac{1}{2}$
26. $f(x) = 3x - 5$
27. $h(x) = -\frac{2}{3}x$
28. $h(x) = -6$

Find the slope of the line passing through the given points.

29. $(-5, 3)$ and $(-4, 1)$
30. $(7, -8)$ and $(-9, -2)$
31. $\left(-\frac{4}{5}, \frac{1}{3}\right)$ and $\left(-\frac{1}{10}, -\frac{3}{5}\right)$
32. $\left(\frac{3}{8}, -1\right)$ and $\left(-\frac{3}{4}, -\frac{1}{16}\right)$
33. $(-14, 7)$ and $(-10, 7)$
34. $(6, -5)$ and $(6, -2)$

Graph f and g on the same rectangular coordinate plane. Use the graph to find all values of x for which the given relation is true. Verify your answer algebraically.

35. $f(x) = \frac{1}{2}x - 2, g(x) = -\frac{5}{2}x + 4; f(x) = g(x)$
36. $f(x) = 5x - 2, g(x) = 3; f(x) \geq g(x)$
37. $f(x) = -4x + 3, g(x) = -x + 6; f(x) < g(x)$
38. $f(x) = \frac{3}{5}x - 1, g(x) = -\frac{3}{5}x + 5; f(x) \leq g(x)$

MODELING LINEAR FUNCTIONS

Find the linear function passing through the given points.

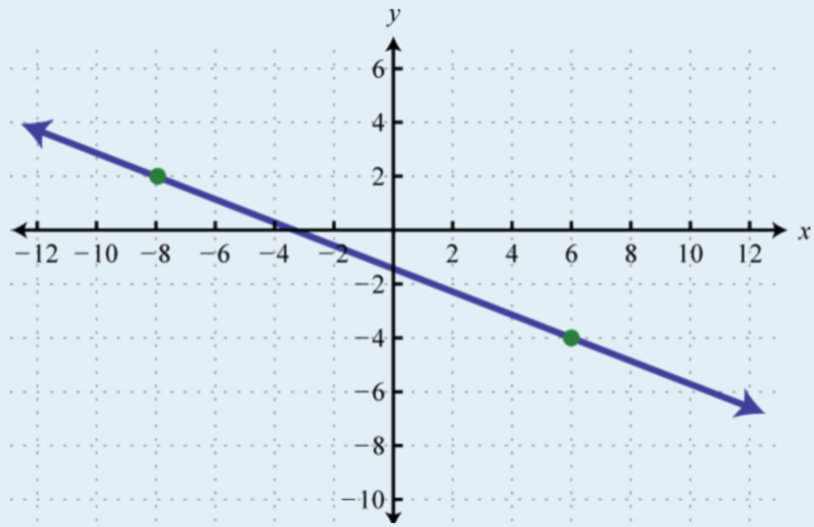
39. $(1, -5)$ and $\left(\frac{1}{2}, -4\right)$

40. $\left(\frac{5}{3}, -3\right)$ and $(-2, 8)$

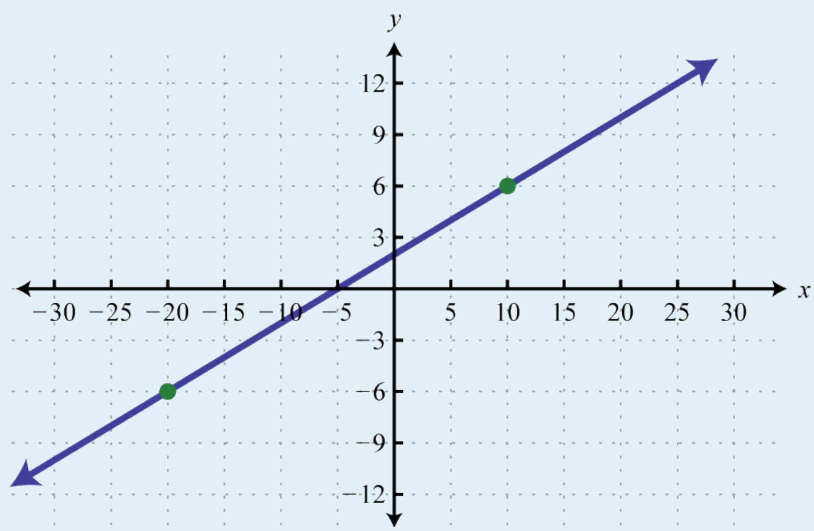
41. $(7, -6)$ and $(5, -7)$

42. $(-5, -6)$ and $(-3, -9)$

43. Find the equation of the given linear function:



44. Find the equation of the given linear function:



Find the equation of the line:

45. Parallel to $8x - 3y = 24$ and passing through $(-9, 4)$.

46. Parallel to $6x + 2y = 24$ and passing through $\left(\frac{1}{2}, -2\right)$.

47. Parallel to $\frac{1}{4}x - \frac{2}{3}y = 1$ and passing through $(-4, -1)$.

48. Perpendicular to $14x + 7y = 10$ and passing through $(8, -3)$.

49. Perpendicular to $15x - 3y = 6$ and passing through $(-3, 1)$.

50. Perpendicular to $\frac{2}{9}x + \frac{4}{3}y = \frac{1}{2}$ and passing through $(2, -7)$.

Use algebra to solve the following.

51. A taxi fare in a certain city includes an initial charge of $2.50 plus $2.00 per mile driven. Write a function that gives the cost of a taxi ride in terms of the number of miles driven. Use the function to determine the number of miles driven if the total fare is $9.70.

52. A salesperson earns a base salary of $1,800 per month and 4.2% commission on her total sales for that month. Write a function that gives her monthly salary based on her total sales. Use the function to determine the amount of sales for a month in which her salary was $4,824.

53. A certain automobile sold for $1,200 in 1980, after which it began to be considered a collector's item. In 1994, the same automobile sold at auction for $5,750. Write a linear function that models the value of the automobile in terms of the number of years since 1980. Use it to estimate the value of the automobile in the year 2000.

54. A specialized industrial robot was purchased new for $62,400. It has a lifespan of 12 years, after which it will be considered worthless. Write a linear function that models the value of the robot. Use the function to determine its value after 8 years of operation.

55. In 1950, the U.S. Census Bureau estimated the population of Detroit, MI to be 1.8 million people. In 1990, the population was estimated to have decreased to 1 million. Write a linear function that gives the population of Detroit in millions of people, in terms of years since 1950. Use the function to estimate the year in which the population decreased to 700,000 people.

56. Online sales of a particular product are related to the number of clicks on its advertisement. It was found that 100 clicks in a week result in $112 of online sales, and that 500 clicks result in $160 of online sales. Write a linear function that models the online sales of the product based on the number of clicks on its advertisement. How many clicks are needed to result in $250 of weekly online sales from this product?

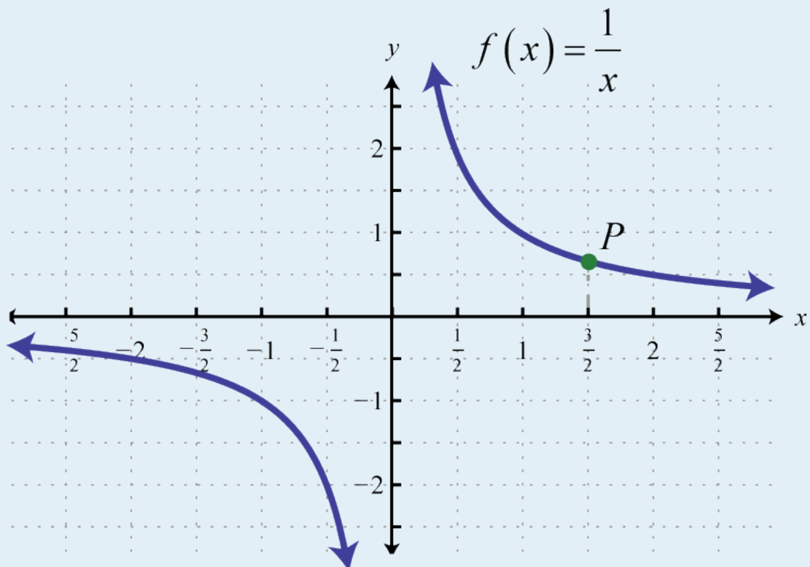
57. The cost in dollars of producing *n* bicycles is given by the formula $C(n) = 80n + 3{,}380$. If each bicycle can be sold for $132, write a function that gives the profit generated by producing and selling *n* bicycles.

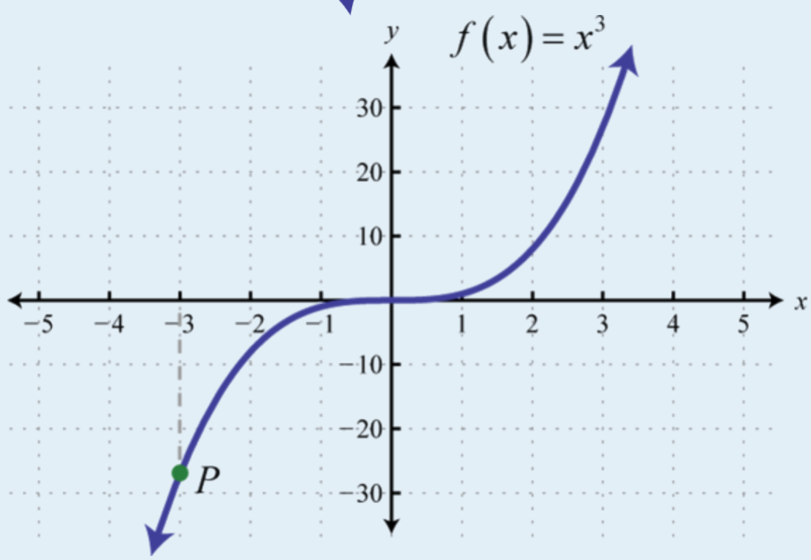
Use the formula to determine the number of bicycles that must be produced and sold to profit at least $10,000.

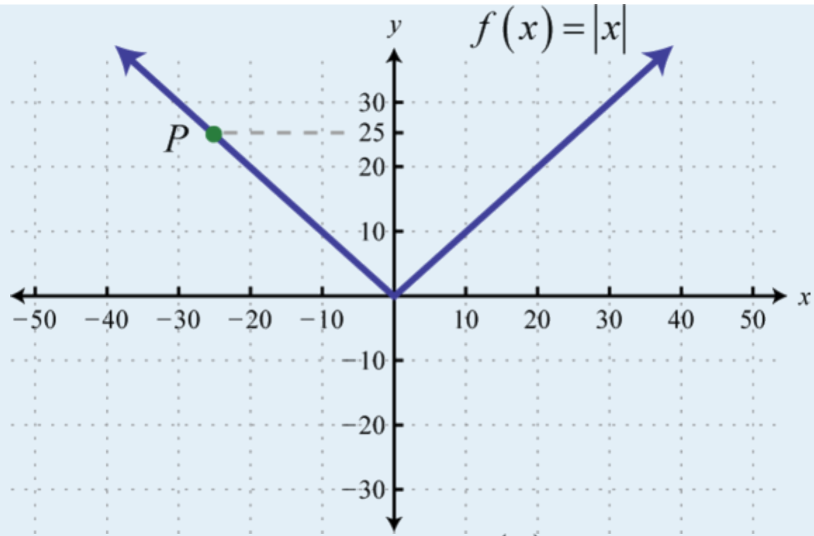
58. Determine the breakeven point from the previous exercise.

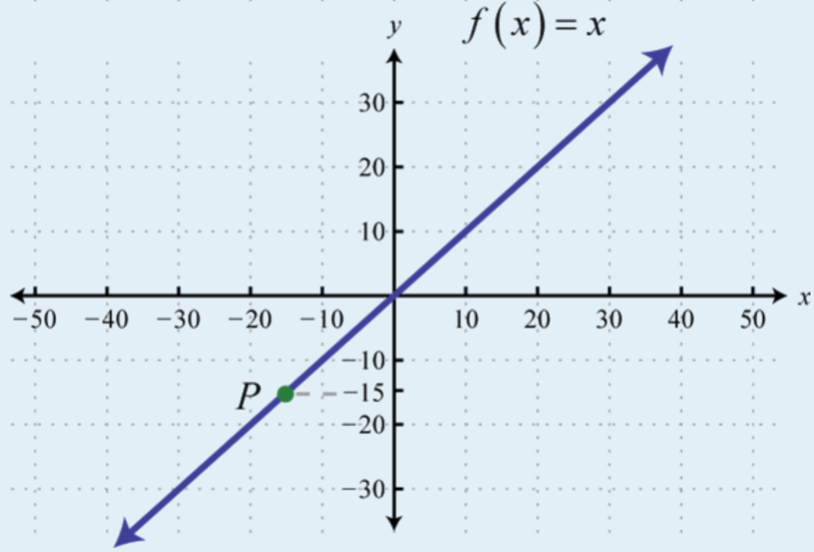
BASIC FUNCTIONS

Find the ordered pair that specifies the point *P*.

59. 

60. 

61. 

62. 

Graph the piecewise defined functions.

63. $g(x) = \begin{cases} x^2 & \text{if } x < 5 \\ 10 & \text{if } x \geq 5 \end{cases}$

64. $g(x) = \begin{cases} -5 & \text{if } x < -5 \\ |x| & \text{if } x \geq -5 \end{cases}$

65. $f(x) = \begin{cases} x & \text{if } x \leq -1 \\ x^3 & \text{if } x > -1 \end{cases}$

66. $f(x) = \begin{cases} x & \text{if } x \leq 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$

67. $h(x) = \begin{cases} x & \text{if } x < -3 \\ x^2 & \text{if } -3 \leq x < 3 \\ -6 & \text{if } x \geq 3 \end{cases}$

68. $f(x) = \begin{cases} 1 & \text{if } x < -1 \\ x^2 & \text{if } -1 \leq x \leq 0 \\ \frac{1}{x} & \text{if } x > 0 \end{cases}$

69. $g(x) = \begin{cases} 1 & \text{if } x \leq -1 \\ 0 & \text{if } -1 < x \leq 1 \\ -1 & \text{if } x > 1 \end{cases}$

70. $g(x) = [\![x]\!] + 2$

Evaluate.

71. $f(x) = \begin{cases} 5x - 2 & \text{if } x < -6 \\ x^2 & \text{if } x \geq -6 \end{cases}$

Find $f(-10), f(-6)$, and $f(0)$.

72. $h(x) = \begin{cases} 2 - 5x & \text{if } x \leq 0 \\ x^3 & \text{if } x > 0 \end{cases}$

Find $h(-1), h(0)$, and $h\left(\frac{1}{2}\right)$.

73. $g(x) = \begin{cases} -5 & \text{if } x < -4 \\ x - 9 & \text{if } -4 \leq x < 0 \\ \sqrt{x} & \text{if } x \geq 0 \end{cases}$

Find $g(-10), g(0)$ and $g(8)$.

74. $q(x) = \begin{cases} \frac{1}{x} & \text{if } x < -1 \\ 0 & \text{if } -1 \leq x \leq 1 \\ x & \text{if } x > 1 \end{cases}$

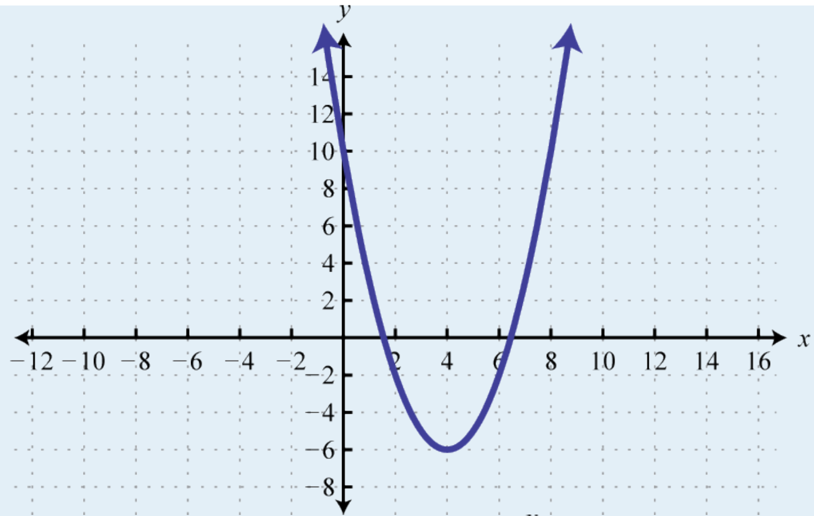
Find $q\left(-\frac{5}{3}\right), q(1)$ and $q(16)$.

TRANSFORMATIONS

Sketch the graph of the given function.

75. $f(x) = (x+5)^2 - 10$
76. $g(x) = \sqrt{x-6} + 9$
77. $p(x) = x - 9$
78. $h(x) = x^3 + 5$
79. $f(x) = |x - 20| - 40$
80. $f(x) = \frac{1}{x-3}$
81. $h(x) = \frac{1}{x+3} - 6$
82. $g(x) = \sqrt[3]{x-4} + 2$
83. $f(x) = \begin{cases} (x+4)^2 & \text{if } x < -2 \\ x+2 & \text{if } x \geq -2 \end{cases}$
84. $g(x) = \begin{cases} -2 & \text{if } x < 6 \\ |x-8| - 4 & \text{if } x \geq 6 \end{cases}$
85. $g(x) = -|x+4| - 8$
86. $h(x) = -x^2 + 16$
87. $f(x) = \sqrt{-x} - 2$
88. $r(x) = -\frac{1}{x} + 2$
89. $g(x) = -2|x+10| + 8$
90. $f(x) = -5\sqrt{x+1}$
91. $f(x) = -\frac{1}{4}x^2 + 1$
92. $h(x) = \frac{1}{3}(x-1)^3 + 2$

Write an equation that represents the function whose graph is given.

93. 
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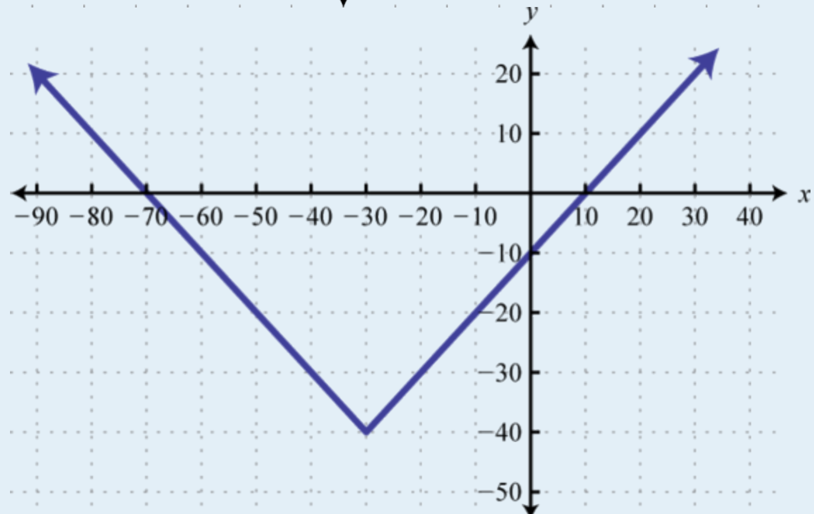
94. 
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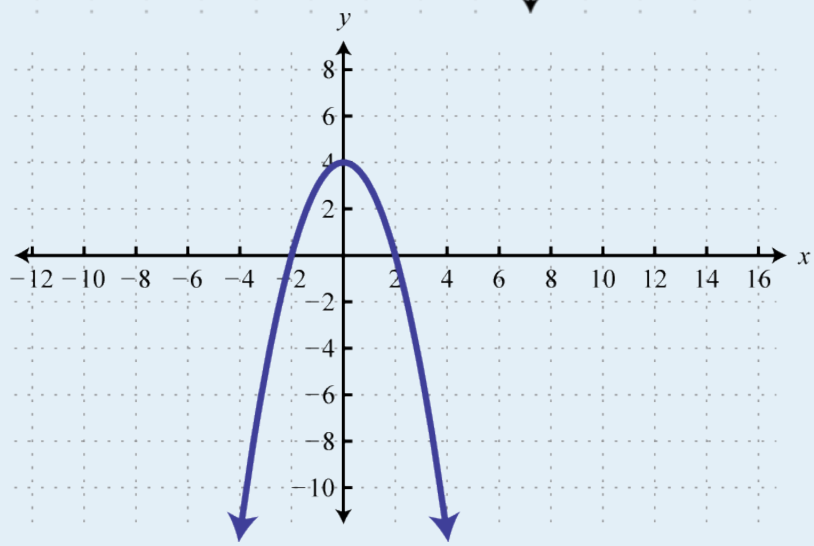
95. 
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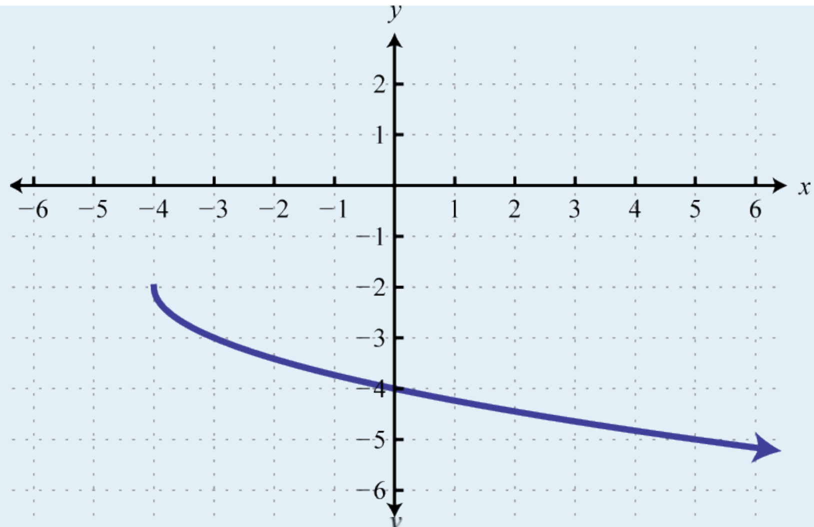
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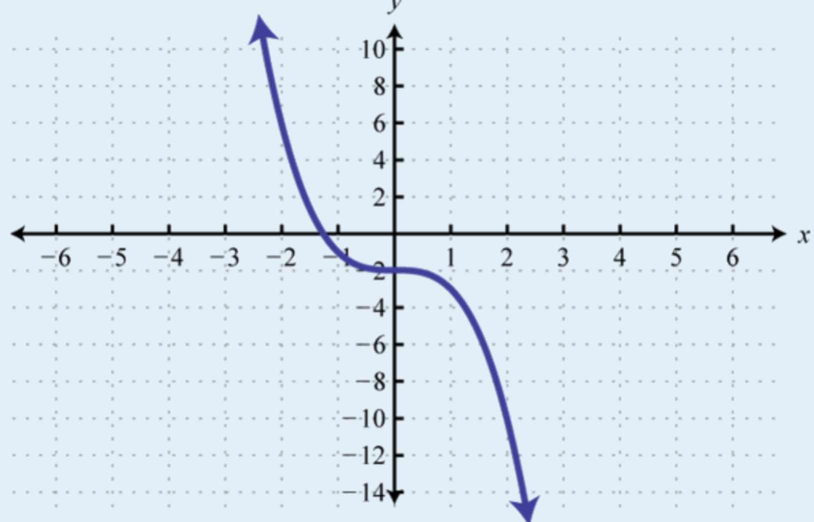
96.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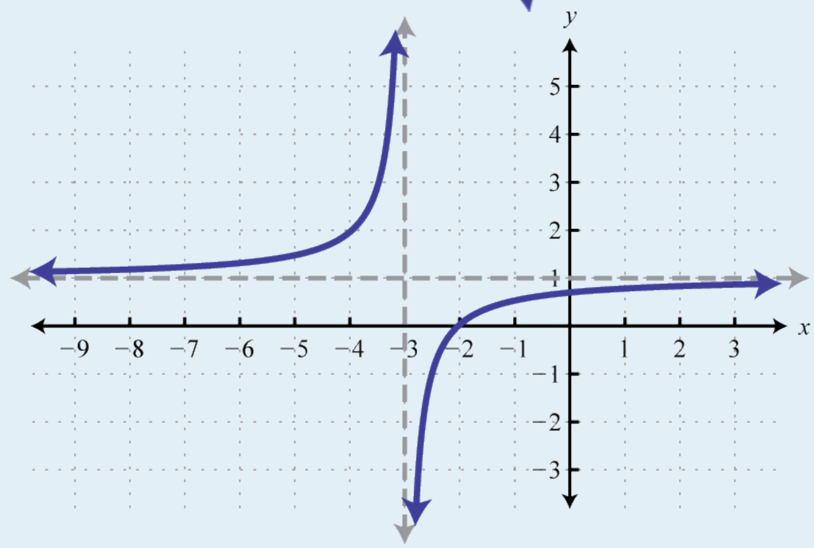
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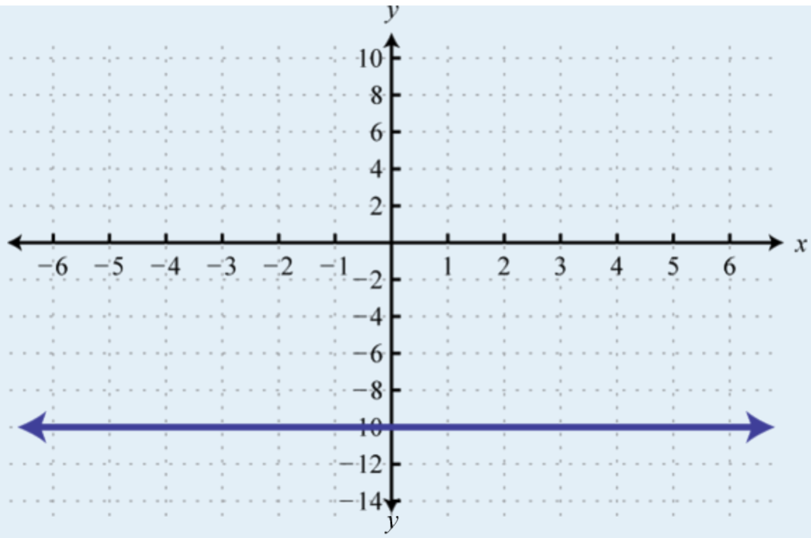
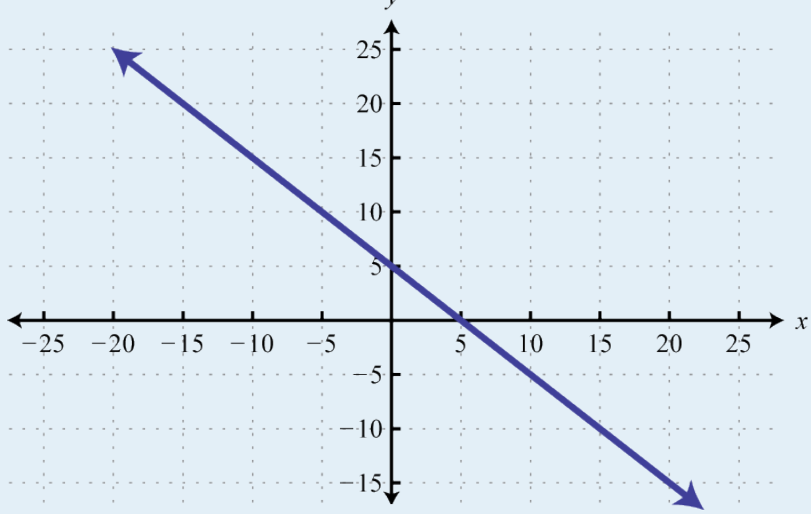
98.

99. 

100. 

SOLVING ABSOLUTE VALUE EQUATIONS AND INEQUALITIES

Solve.

101. $|5x - 4| = 14$

102. $|4 - 3x| = 4$

103. $9 - 5\,|x - 4| = 4$

104. $6 + 2\,|x + 10| = 12$

105. $|3x - 6| + 5 = 5$

106. $0.2\,|x - 1.8| = 4.6$

107. $\frac{2}{3}\left|2x-\frac{1}{2}\right|+\frac{1}{3}=2$
108. $\frac{1}{4}\left|x+\frac{5}{2}\right|-2=\frac{1}{8}$
109. $|3x-9|=|4x+3|$
110. $|9x-7|=|3+8x|$

Solve. Graph the solutions on a number line and give the corresponding interval notation.

111. $|2x+3|<1$
112. $|10x-15|\leq 25$
113. $|6x-1|\leq 11$
114. $|x-12|>7$
115. $6-4\left|x-\frac{1}{2}\right|\leq 2$
116. $5-|x+6|\geq 4$
117. $|3x+1|+7\leq 4$
118. $2|x-3|+6>4$
119. $5\left|\frac{1}{3}x-\frac{1}{2}\right|>\frac{5}{6}$
120. $6.4-3.2|x+1.6|>0$

INEQUALITIES WITH TWO VARIABLES

Is the ordered pair a solution to the given inequality?

121. $9x-2y<-1;(-1,-3)$
122. $4x+\frac{1}{3}y>0;(1,-12)$
123. $\frac{3}{4}x-y\geq\frac{1}{2};\left(\frac{1}{2},-\frac{1}{4}\right)$
124. $x-y\leq -6;(-1,7)$
125. $y\leq x^2-3;(-3,5)$
126. $y>|x-6|+10;(-4,12)$

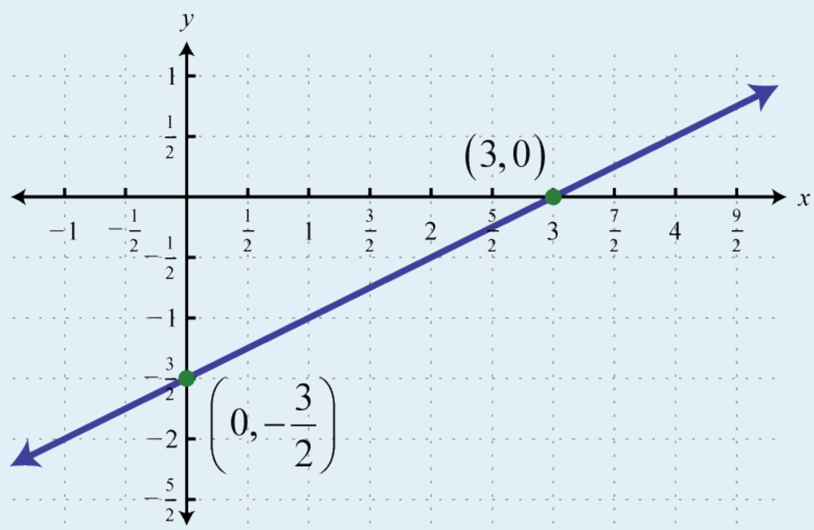
127. $y < (x - 1)^3 + 7; (-1, 0)$
128. $y \geq \sqrt{x + 4}; (-3, 4)$

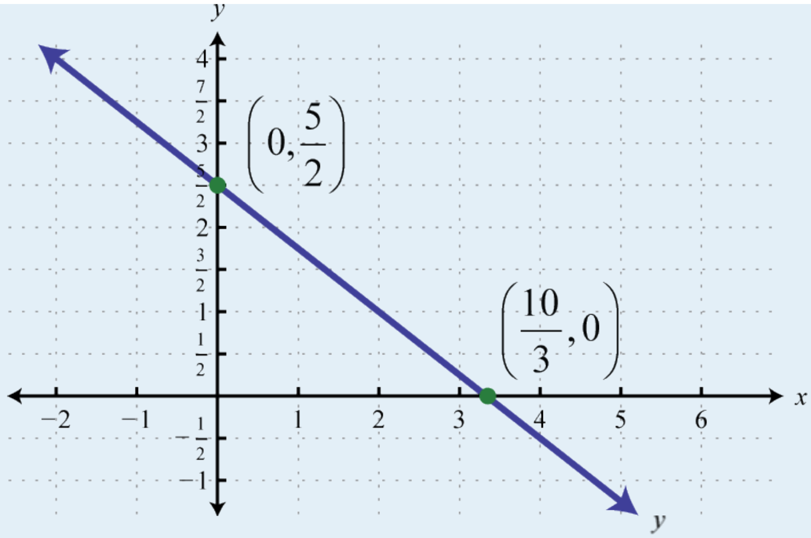
Graph the solution set.

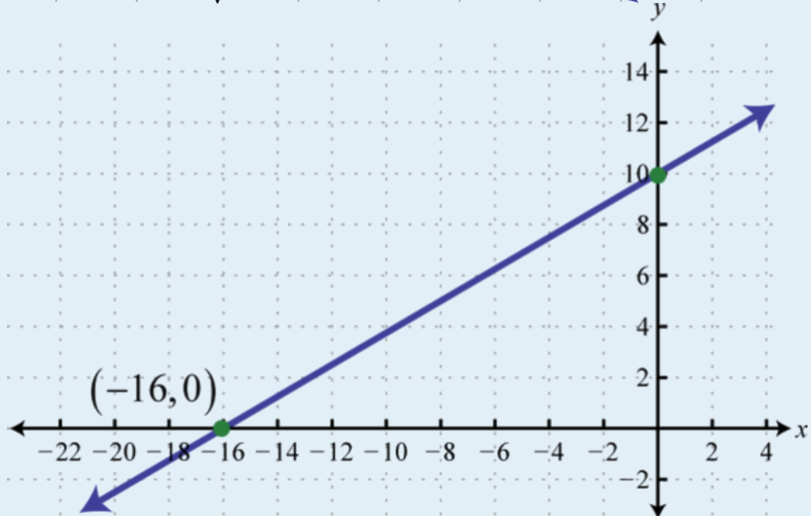
129. $x + y < 6$
130. $2x - 3y \geq 9$
131. $3x - y \leq 6$
132. $y + 4 > 0$
133. $x - 6 \geq 0$
134. $-\frac{1}{3}x + \frac{1}{6}y > \frac{1}{2}$
135. $y > (x - 2)^2 - 3$
136. $y \leq (x + 6)^2 + 3$
137. $y < -|x| + 9$
138. $y > |x - 12| + 3$
139. $y \geq x^3 + 8$
140. $y > -(x - 2)^3$

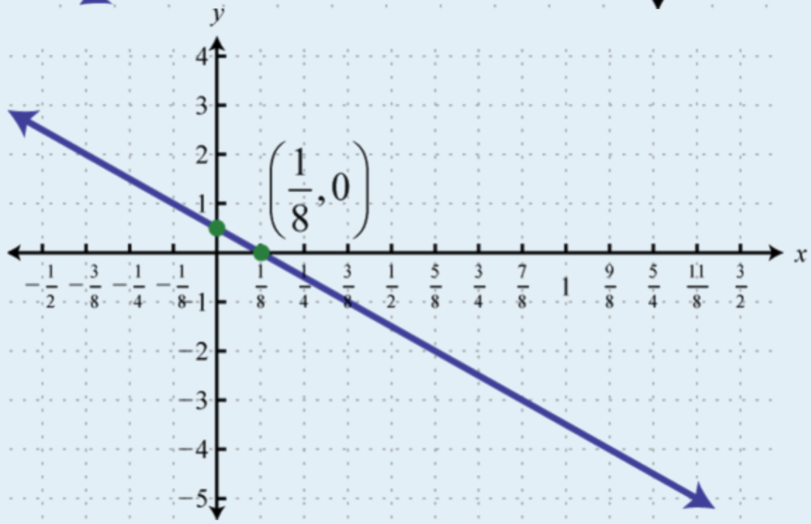
ANSWERS

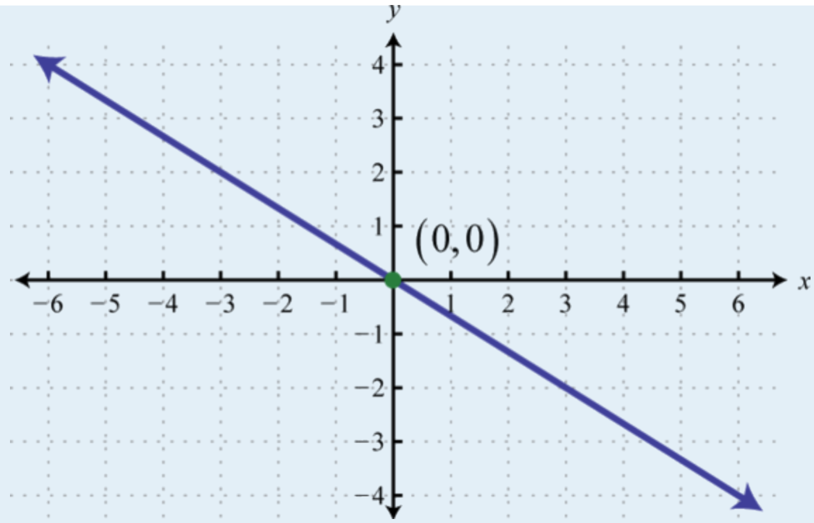
1. Domain: {-5, -4, 10, 11, 15}; range:{-1, 1, 2, 3}; function: yes
3. Domain: {-5, 5, 15, 30}; range: {-5, 0, 5, 10, 15}; function: no
5. Domain: $\left(-\infty, \infty\right)$; range: $\left[-6, \infty\right)$; function: yes
7. Domain: $\left(-\infty, \frac{3}{2}\right]$; range: $\left[1, \infty\right)$; function: yes
9. $h(-8) = -7, h(3) = -\frac{3}{2}$, and $h(4a + 1) = 2a - \frac{5}{2}$
11. $f\left(-5\right) = 58, f(0) = 3$, and $f\left(x + h\right) = 2x^2 + 4xh + 2h^2 - x - h + 3$
13. $g\left(5\right) = 3, g(1) = 1, g(13) = 5$
15. $f\left(\frac{7}{8}\right) = 10$
17. $f\left(-60\right) = -20, f(0) = 20, f(20) = 0$
19.



21. 
$\left(0, \frac{5}{2}\right)$
$\left(\frac{10}{3}, 0\right)$

23. 
$(-16, 0)$

25. 
$\left(\frac{1}{8}, 0\right)$

27. 

29. $m = -2$

31. $m = -\dfrac{4}{3}$

33. $m = 0$

35. $x = 2$

37. $(-1, \infty)$

39. $f(x) = -2x - 3$

41. $f(x) = \frac{1}{2}x - \frac{19}{2}$

43. $f(x) = -\frac{3}{7}x - \frac{10}{7}$

45. $y = \frac{8}{3}x + 28$

47. $y = \frac{3}{8}x - \frac{5}{2}$

49. $y = -\frac{1}{5}x + \frac{2}{5}$

51. $C(x) = 2x + 2.5$; 3.6 miles

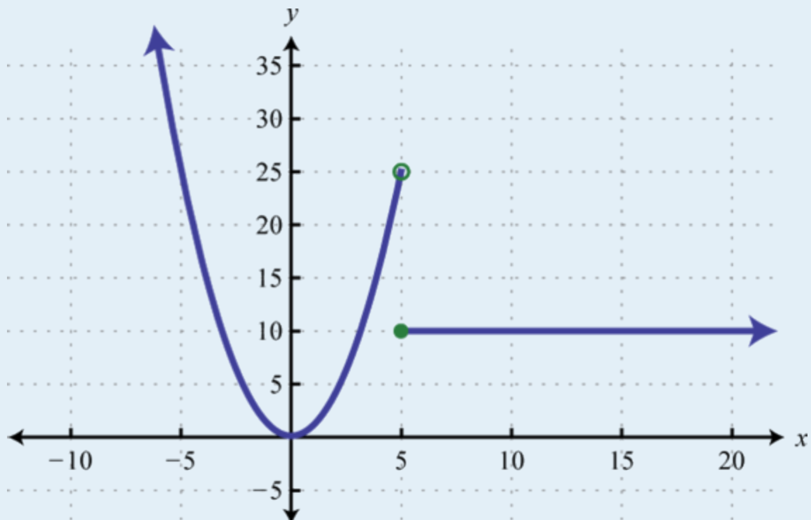
53. $V(t) = 325t + 1,200$; \$7,700

55. $p(x) = -0.02x + 1.8$; 2005

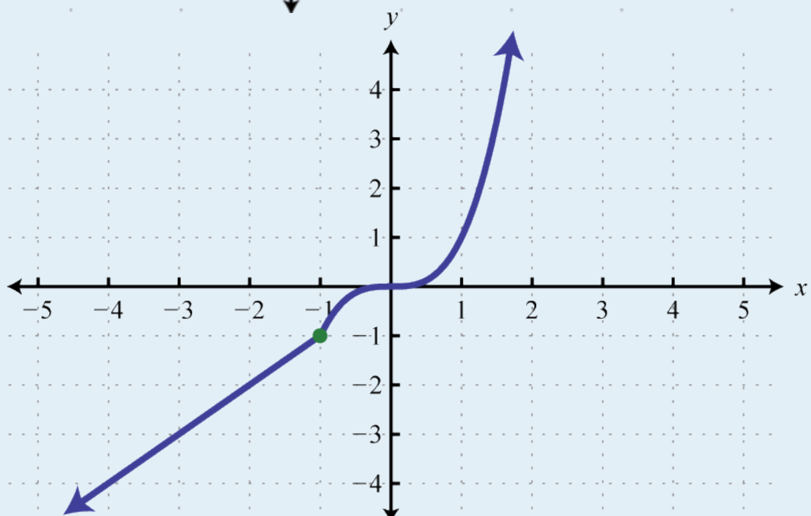
57. $P(n) = 52n - 3,380$; 258 bicycles

59. $\left(\frac{3}{2}, \frac{2}{3}\right)$

61. $(-25, 25)$



63.



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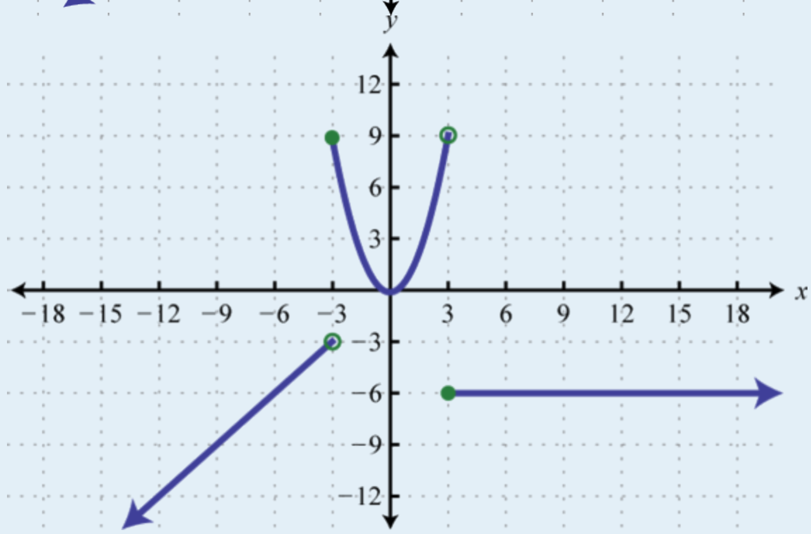
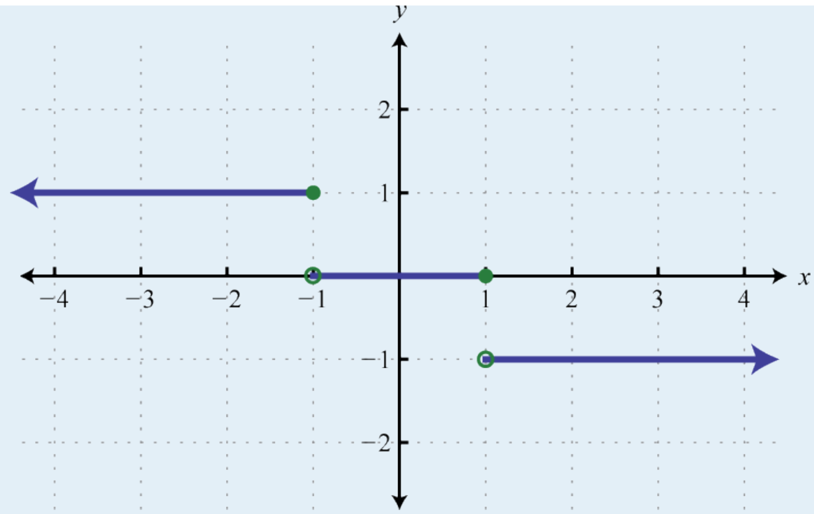
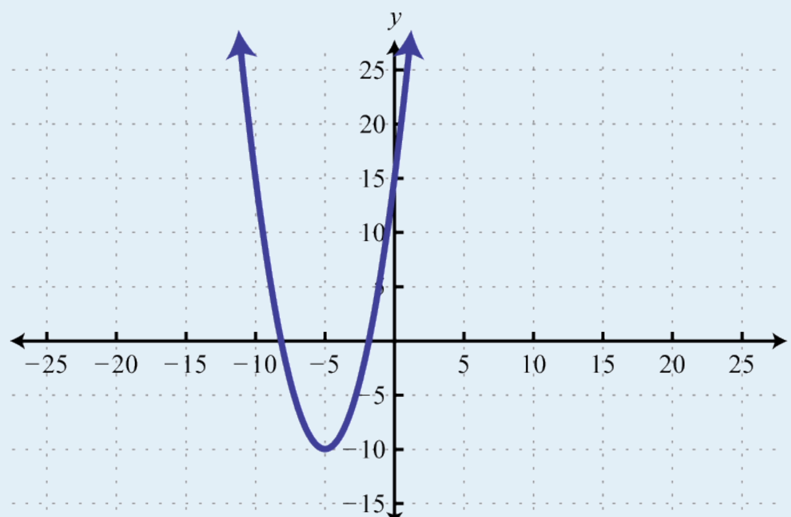


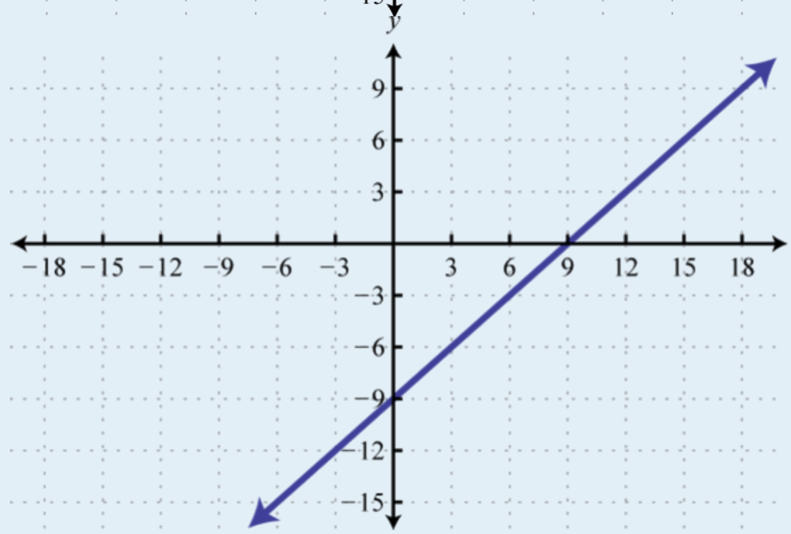
67.

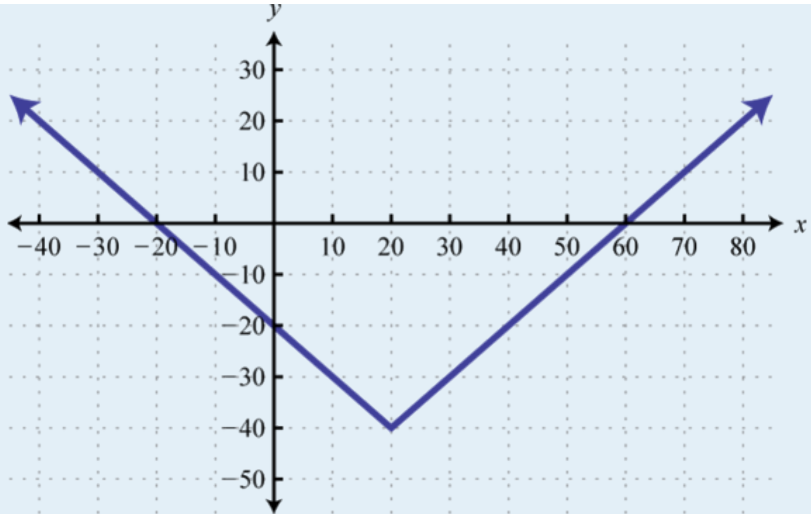
69. 

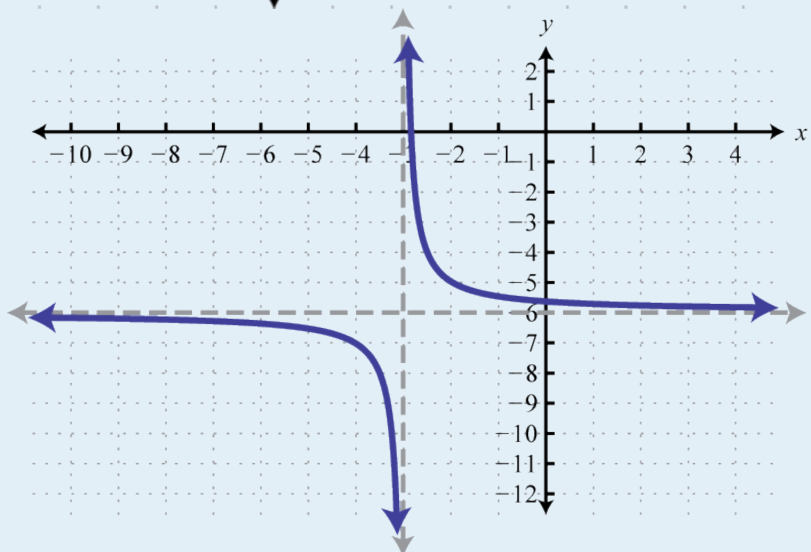
71. $f(-10) = -52, f\left(-6\right) = 36, f(0) = 0$

73. $g(-10) = -5, g(-4) = -13, g(8) = 2\sqrt{2}$

75. 

77. 

79. 
y
30
20
10
x
−40 −30 −20 −10 10 20 30 40 50 60 70 80
−10
−20
−30
−40
−50

81. 
y
2
1
x
−10 −9 −8 −7 −6 −5 −4 −2 −1 1 2 3 4
−1
−2
−3
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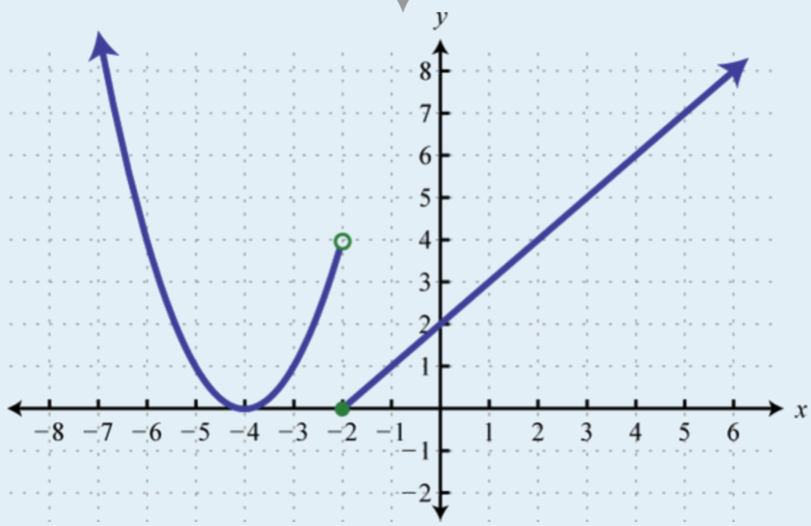
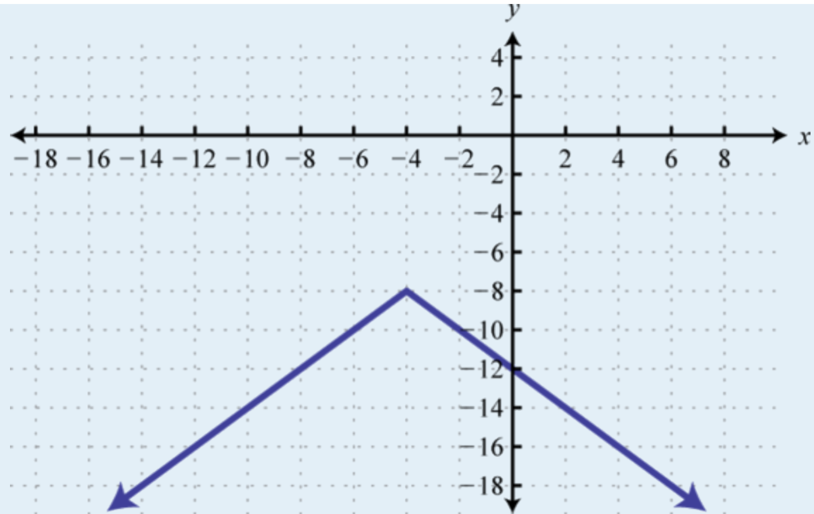
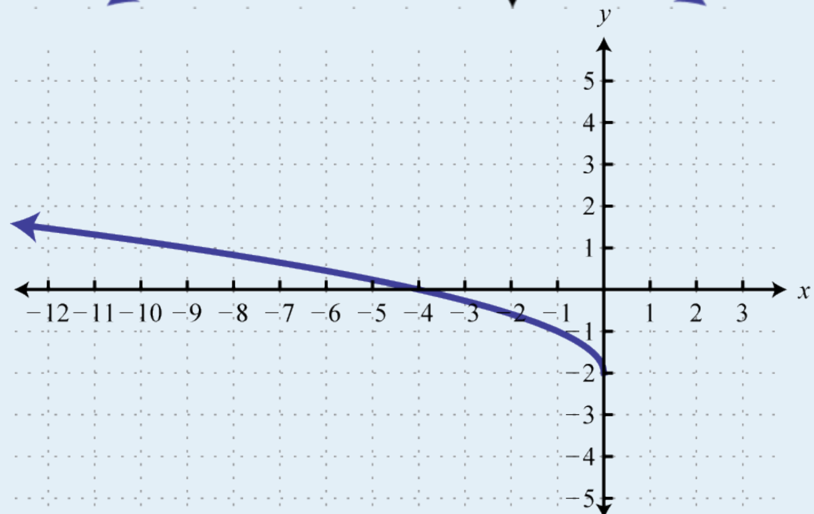
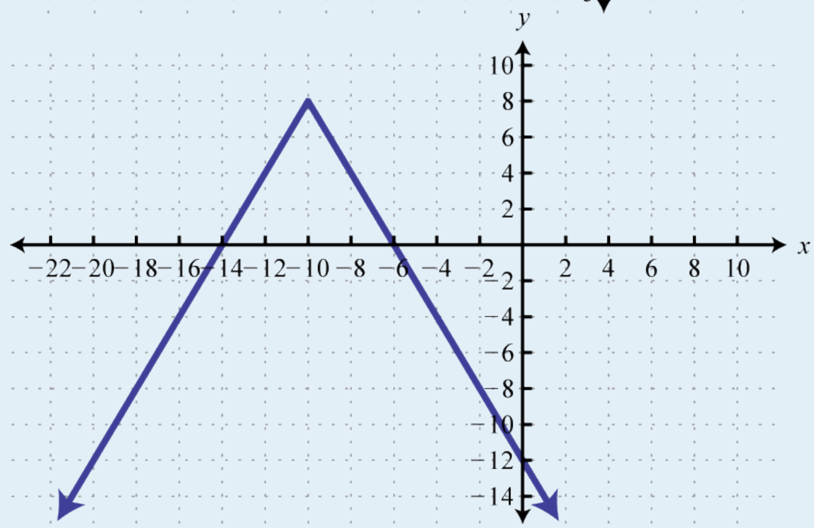
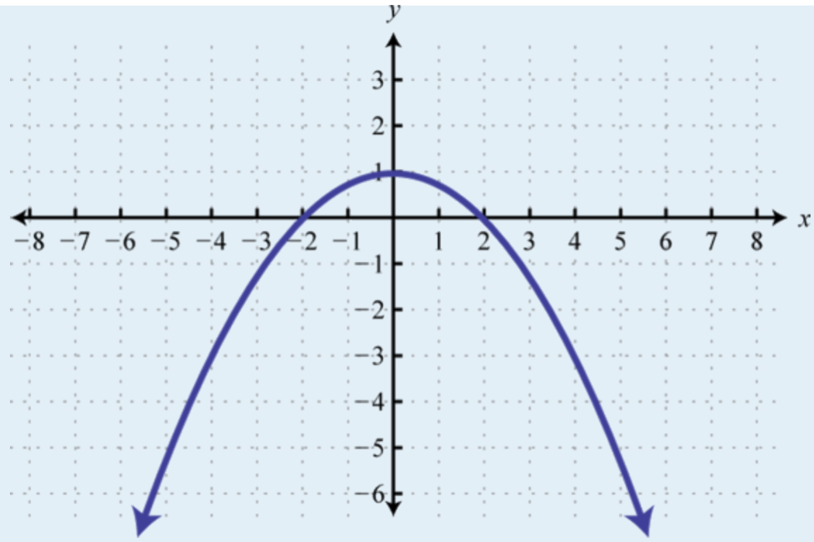
−5
−6
−7
−8
−9
−10
−11
−12

83. 
y
8
7
6
5
4
3
2
1
x
−8 −7 −6 −5 −4 −3 −2 −1 1 2 3 4 5 6
−1
−2

85. 
y
4
2
x
−18 −16 −14 −12 −10 −8 −6 −4 −2
2 4 6 8
−2
−4
−6
−8
−10
−12
−14
−16
−18

87. 
y
5
4
3
2
1
x
−12 −11 −10 −9 −8 −7 −6 −5 −4 −3 −2 −1
1 2 3
−1
−2
−3
−4
−5

89. 
y
10
8
6
4
2
x
−22 −20 −18 −16 −14 −12 −10 −8 −6 −4 −2
2 4 6 8 10
−2
−4
−6
−8
−10
−12
−14

91. 

93. $f(x) = (x - 4)^2 - 6$

95. $f(x) = -x^2 + 4$

97. $f(x) = -x^3 - 2$

99. $f(x) = -10$

101. -2, $\frac{18}{5}$

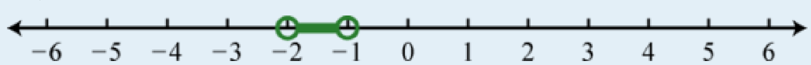
103. 3, 5

105. 2

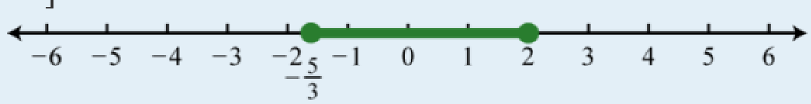
107. -1, $\frac{3}{2}$

109. -12, $\frac{6}{7}$

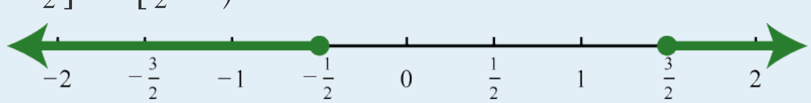
111. $(-2, -1)$;



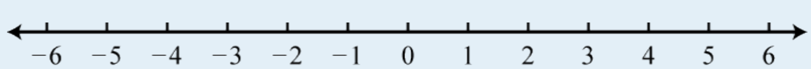
113. $\left[-\frac{5}{3}, 2\right]$;



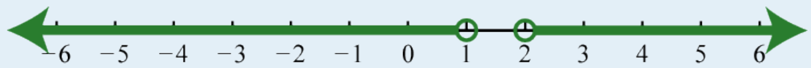
115. $\left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{3}{2}, \infty\right)$;



117. Ø;



119. $(-\infty, 1) \cup (2, \infty)$;



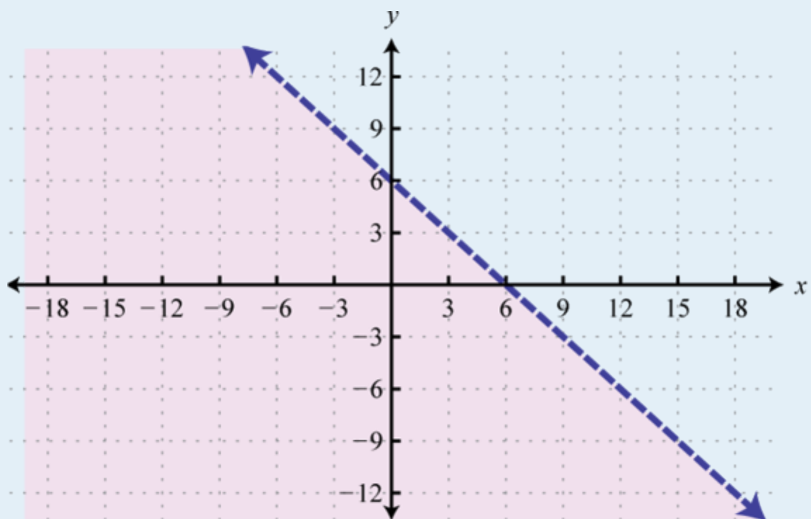
121. Yes

123. Yes

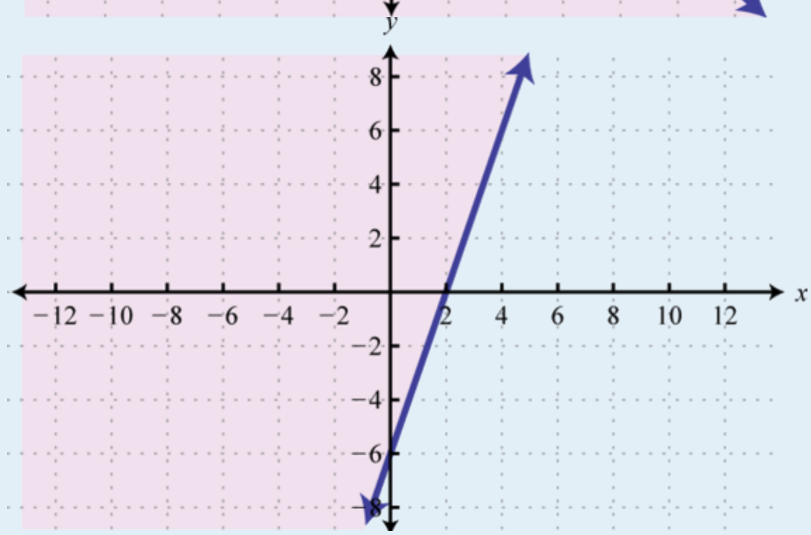
125. Yes

127. No

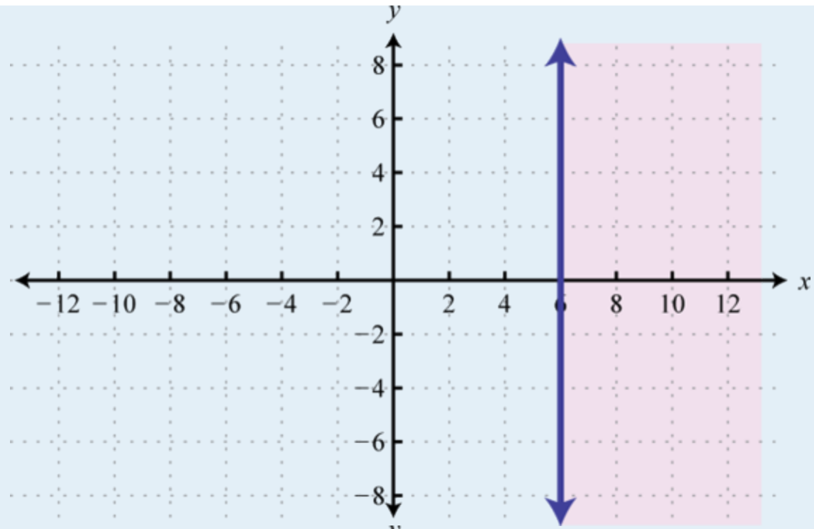
129.



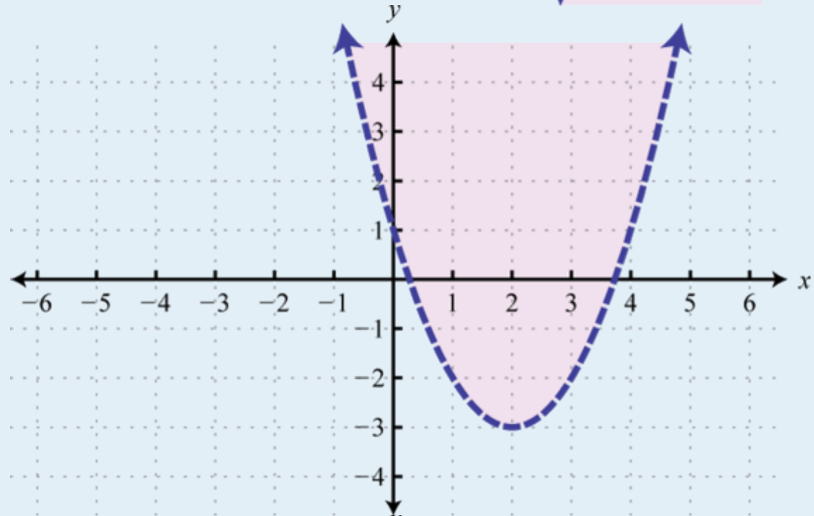
131.



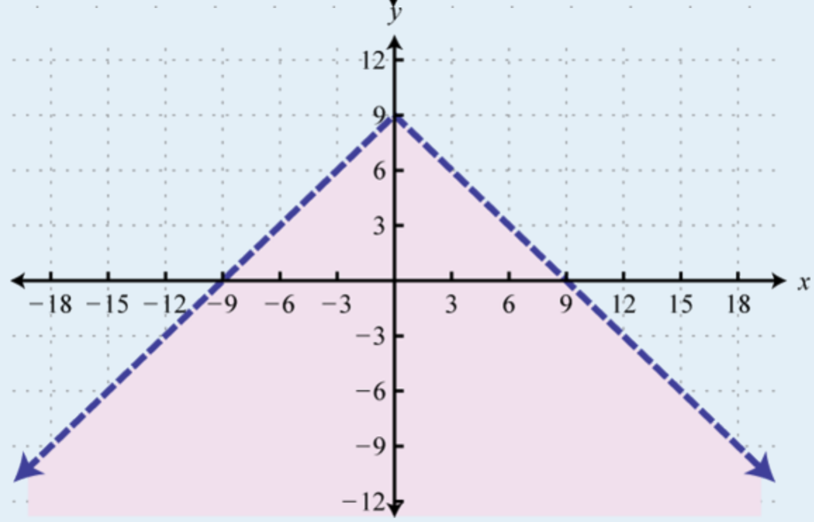
133.


y
x
8
6
4
2
−2
−4
−6
−8
−12 −10 −8 −6 −4 −2
2 4 6 8 10 12

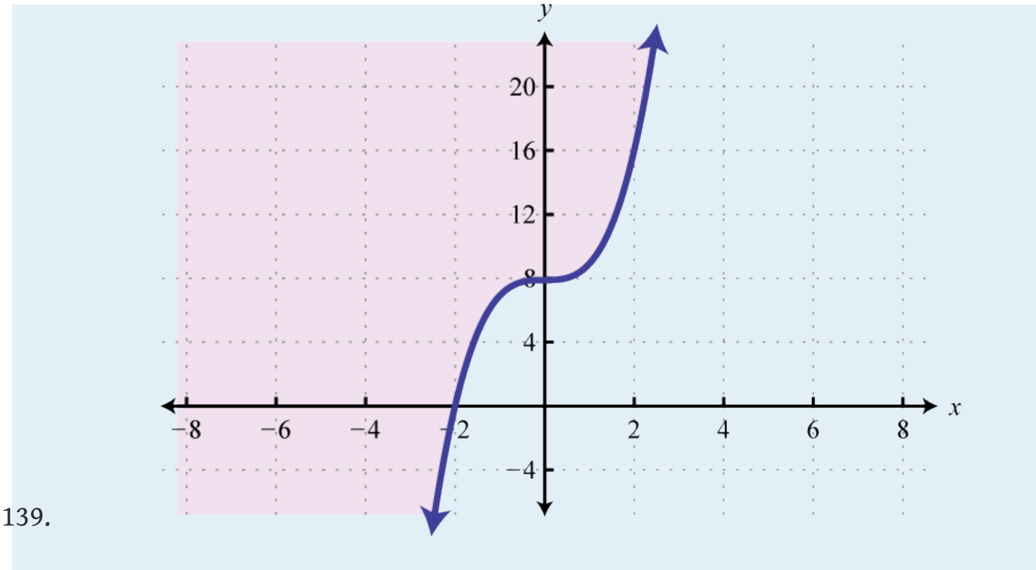
135.


y
x
4
3
2
1
−1
−2
−3
−4
−6 −5 −4 −3 −2 −1
1 2 3 4 5 6

137.

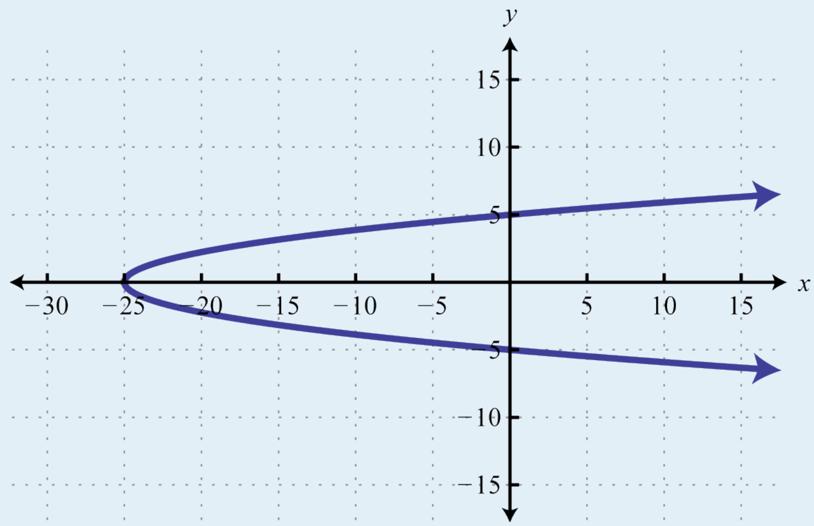

y
x
12
9
6
3
−3
−6
−9
−12
−18 −15 −12 −9 −6 −3
3 6 9 12 15 18

139.

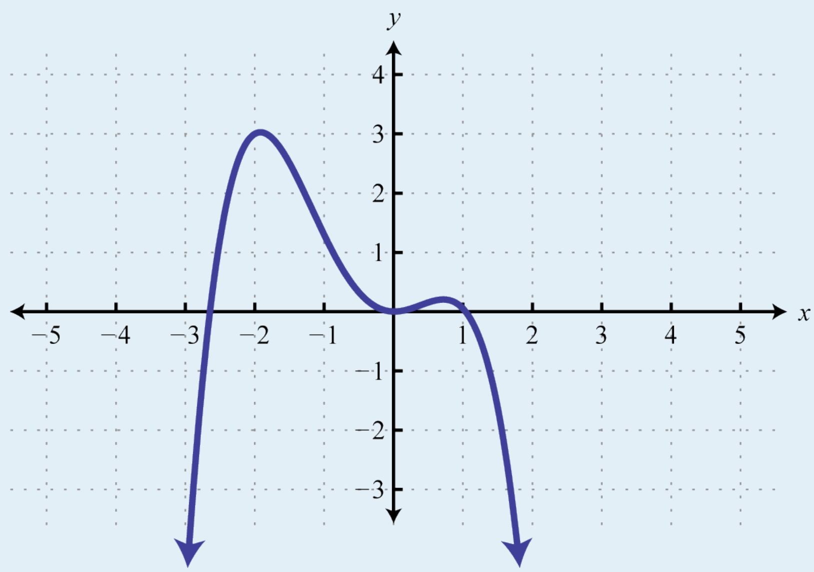


SAMPLE EXAM

1. Determine whether or not the following graph represents a function or not. Explain.

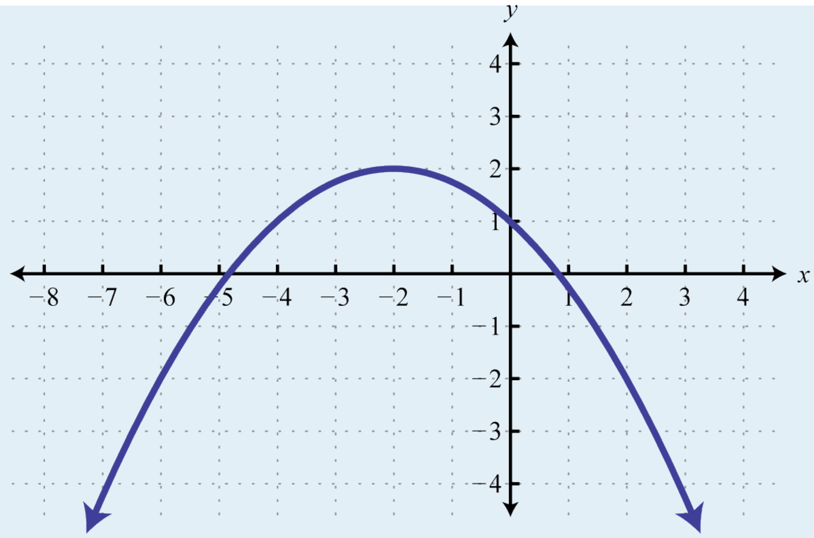


2. Determine the domain and range of the following function.



3. Given $g(x) = x^2 - 5x + 1$, find $g(-1)$, $g(0)$, and $g\left(x + h\right)$.

4. Given the graph of a function f:



a. Find $f(-6)$, $f(0)$, and $f(2)$.
b. Find x where $f(x) = 2$.

5. Graph $f(x) = -\frac{5}{2}x + 7$ and label the x-intercept.

6. Find a linear function passing through $\left(-\frac{1}{2}, -1\right)$ and $(2, -2)$.

7. Find the equation of the line parallel to $2x - 6y = 3$ and passing through $(-1, -2)$.

8. Find the equation of the line perpendicular to $3x - 4y = 12$ and passing through $(6, 1)$.

9. The annual revenue of a new web-services company in dollars is given by $R(n) = 125n$, where n represents the number of users the company has registered. The annual maintenance cost of the company's registered user base in dollars is given by the formula $C(n) = 85n + 22,480$ where n represents the users.

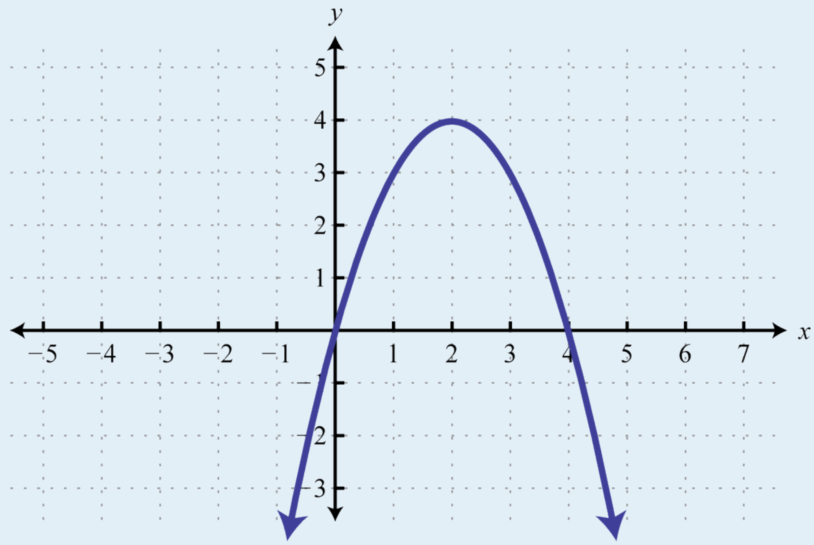
   a. Write a function that models the annual profit based on the number of registered users.
   b. Determine the number of registered users needed to break even.

10. A particular search engine assigns a ranking to a webpage based on the number of links that direct users to the webpage. If no links are found, the webpage is assigned a ranking of 1. If 40 links are found directing users to the webpage, the search engine assigns a page ranking of 5.

    a. Find a linear function that gives the webpage ranking based on the number of links that direct users to it.
    b. How many links will be needed to obtain a page ranking of 7?

Use the transformations to sketch the graph of the following functions and state the domain and range.

11. $g(x) = |x + 4| - 5$
12. $h(x) = \sqrt{x - 4} + 1$
13. $r(x) = -(x + 3)^3$
14. Given the graph, determine the function definition and its domain and range:



15. Sketch the graph: $h(x) = \begin{cases} -x & \text{if } x < 1 \\ \frac{1}{x} & \text{if } x \geq 1 \end{cases}$.
16. Sketch the graph: $g(x) = -\frac{1}{3}x^2 + 9$.

Solve.

17. $|2x - 1| + 2 = 7$
18. $10 - 5|2x - 3| = 0$
19. $|7x + 4| = |9x - 1|$

Solve and graph the solution set.

20. $|2x - 4| - 5 < 7$
21. $6 + |3x - 5| \geq 13$
22. $5 - 3|x - 4| \geq -10$

23. $3\left|7x-1\right|+5\leq 2$

Graph the solution set.

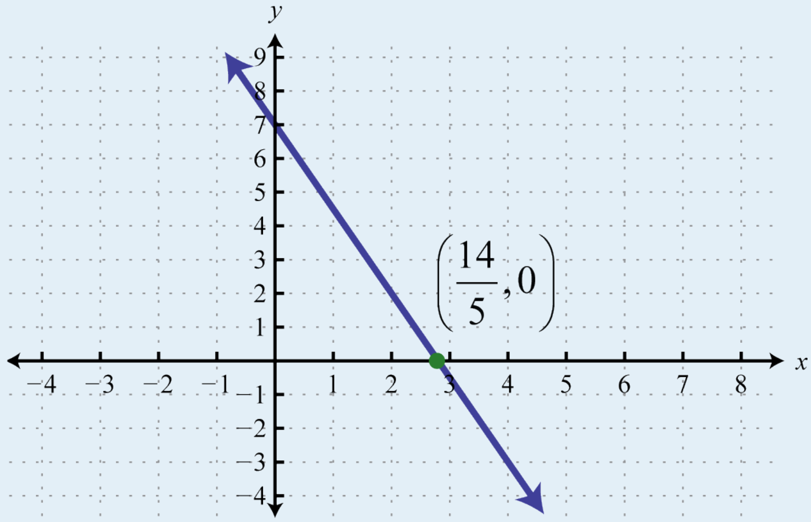
24. $\frac{1}{2}x-\frac{2}{3}y\geq 4$

25. $y>-(x-2)^2+4$

ANSWERS

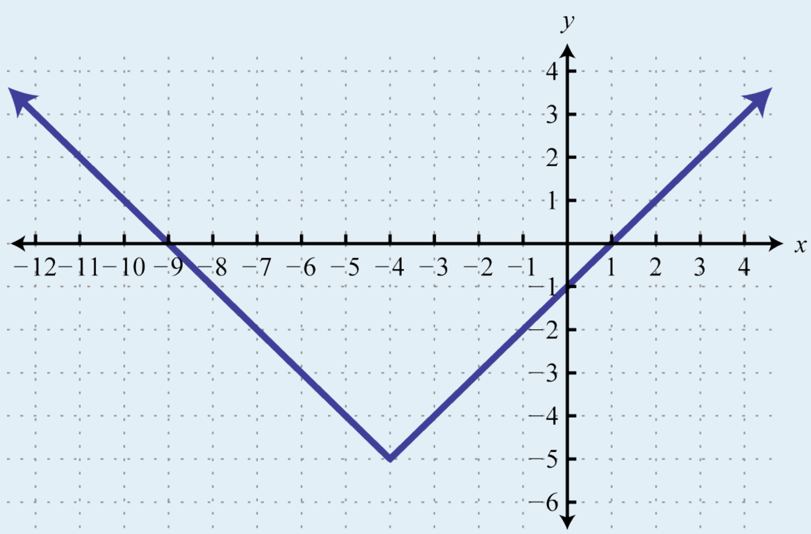
1. The graph is not a function; it fails the vertical line test.

3. $g(-1) = 7, g(0) = 1$, and
$g(x+h) = x^2 + 2xh + h^2 - 5x - 5h + 1$

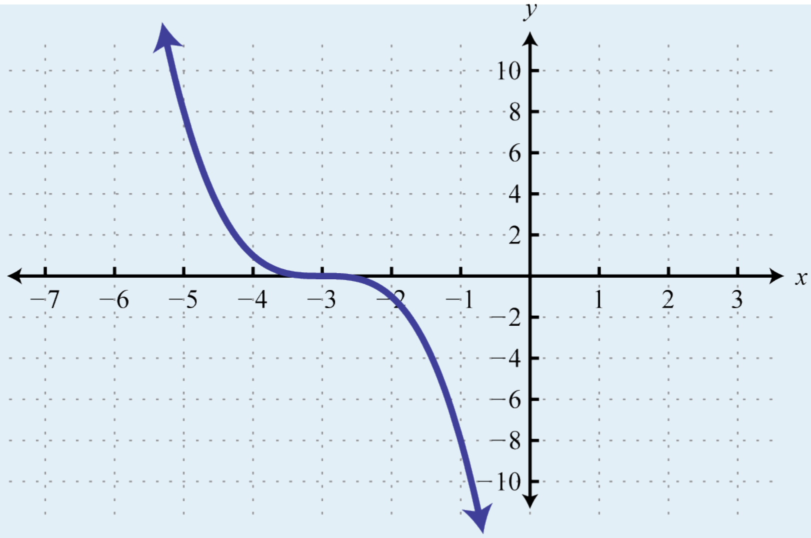
5. 

7. $y = \frac{1}{3}x - \frac{5}{3}$

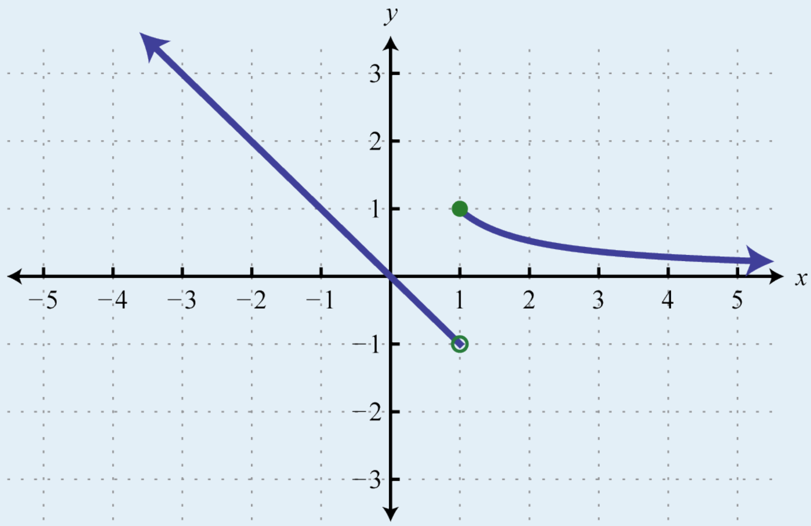
9.
a. $P(n) = 40n - 22{,}480$;
b. 562 users



11. Domain: $(-\infty, \infty)$; range: $[-5, \infty)$



13. Domain: $\left(-\infty, \infty\right)$; range: $\left(-\infty, \infty\right)$

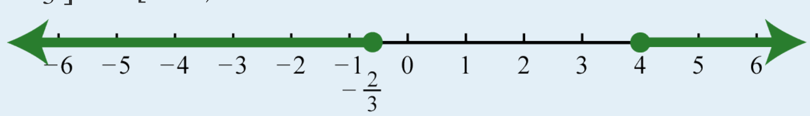


15.

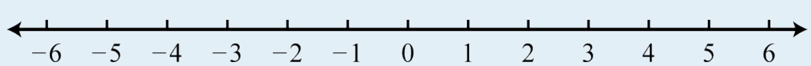
17. −2, 3

19. $-\frac{3}{16}$, $\frac{5}{2}$

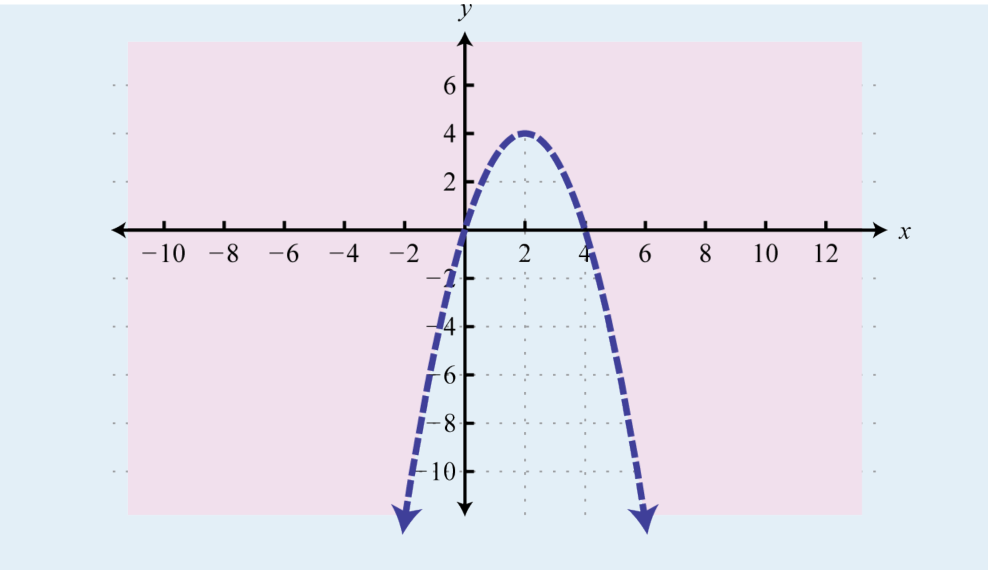
21. $\left(-\infty, -\frac{2}{3}\right] \cup \left[4, \infty\right)$;



23. Ø;



25.



Chapter 3

Solving Linear Systems

3.1 Linear Systems with Two Variables and Their Solutions

LEARNING OBJECTIVES

1. Check solutions to systems of linear equations.
2. Solve linear systems using the graphing method.
3. Identify dependent and inconsistent systems.

Definition of a Linear System with Two Variables

Real-world applications are often modeled using more than one variable and more than one equation. A **system of equations**[1] consists of a set of two or more equations with the same variables. In this section, we will study **linear systems**[2] consisting of two linear equations each with two variables. For example,

$$\begin{cases} 2x - 3y = 0 \\ -4x + 2y = -8 \end{cases}$$

A **solution to a linear system**[3], or **simultaneous solution**[4], is an ordered pair (x, y) that solves both of the equations. In this case, (3, 2) is the only solution. To check that an ordered pair is a solution, substitute the corresponding x- and y-values into each equation and then simplify to see if you obtain a true statement for both equations.

1. A set of two or more equations with the same variables.
2. A set of two or more linear equations with the same variables.
3. Given a linear system with two equations and two variables, a solution is an ordered pair that satisfies both equations and corresponds to a point of intersection.
4. Used when referring to a solution of a system of equations.

Check: (3, 2)	
Equation 1: $2x - 3y = 0$	*Equation 2:* $-4x + 2y = -8$
$\begin{aligned} 2(3) - 3(2) &= 0 \\ 6 - 6 &= 0 \\ 0 &= 0 \quad \checkmark \end{aligned}$	$\begin{aligned} -4(3) + 2(2) &= -8 \\ -12 + 4 &= -8 \\ -8 &= -8 \quad \checkmark \end{aligned}$

Example 1

Determine whether or not (1, 0) is a solution to the system $\begin{cases} x - y &= 1 \\ -2x + 3y &= 5 \end{cases}$.

Solution:

Substitute the appropriate values into both equations.

Check: (1, 0)	
Equation 1: $x - y = 1$	*Equation 2:* $-2x + 3y = 5$
$\begin{aligned} (1) - (0) &= 1 \\ 1 - 0 &= 1 \\ 1 &= 1 \ \checkmark \end{aligned}$	$\begin{aligned} -2(1) + 3(0) &= 5 \\ -2 + 0 &= 5 \\ -2 &= 5 \ ✗ \end{aligned}$

Answer: Since (1, 0) does not satisfy *both* equations, it is not a solution.

Try this! Is (−2, 4) a solution to the system $\begin{cases} x - y &= -6 \\ -2x + 3y &= 16 \end{cases}$?

Answer: Yes

(click to see video)

Solve by Graphing

Geometrically, a linear system consists of two lines, where a solution is a point of intersection. To illustrate this, we will graph the following linear system with a solution of (3, 2):

$$\begin{cases} 2x - 3y = 0 \\ -4x + 2y = -8 \end{cases}$$

First, rewrite the equations in slope-intercept form so that we may easily graph them.

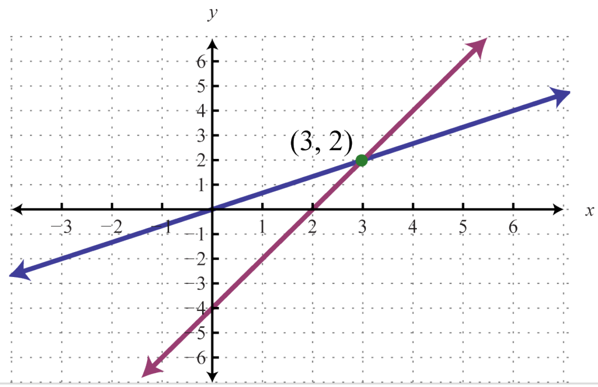
$2x - 3y = 0$ $2x - 3y - 2x = 0 - 2x$ $-3y = -2x$ $\frac{-3y}{-3} = \frac{-2x}{-3}$ $y = \frac{2}{3}x$	$-4x + 2y = -8$ $-4x + 2y + 4x = -8 + 4x$ $2y = 4x - 8$ $\frac{2y}{2} = \frac{4x-8}{2}$ $y = 2x - 4$

Next, replace these forms of the original equations in the system to obtain what is called an **equivalent system**[5]. Equivalent systems share the same solution set.

$$\begin{array}{ccc} \textit{Original system} & & \textit{Equivalent system} \\ \begin{cases} 2x - 3y = 0 \\ -4x + 2y = -8 \end{cases} & \Rightarrow & \begin{cases} y = \dfrac{2}{3}x \\ y = 2x - 4 \end{cases} \end{array}$$

5. A system consisting of equivalent equations that share the same solution set.

If we graph both of the lines on the same set of axes, then we can see that the point of intersection is indeed (3, 2), the solution to the system.



To summarize, linear systems described in this section consist of two linear equations each with two variables. A solution is an ordered pair that corresponds to a point where the two lines intersect in the rectangular coordinate plane. Therefore, one way to solve linear systems is by graphing both lines on the same set of axes and determining the point where they cross. This describes the **graphing method**[6] for solving linear systems.

When graphing the lines, take care to choose a good scale and use a straightedge to draw the line through the points; accuracy is very important here.

6. A means of solving a system by graphing the equations on the same set of axes and determining where they intersect.

Example 2

Solve by graphing: $\begin{cases} x - y = -4 \\ 2x + y = 1 \end{cases}$.

Solution:

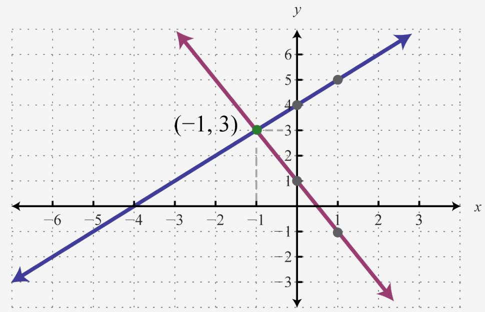
Rewrite the linear equations in slope-intercept form.

$x - y = -4$ $-y = -x - 4$ $\frac{-y}{-1} = \frac{-x-4}{-1}$ $y = x + 4$	$2x + y = 1$ $y = -2x + 1$

Write the equivalent system and graph the lines on the same set of axes.

$$\begin{cases} x - y = -4 \\ 2x + y = 1 \end{cases} \Rightarrow \begin{cases} y = x + 4 \\ y = -2x + 1 \end{cases}$$

$\text{Line 1}: y = x + 4$ $y\text{-intercept}: (0, 4)$ $\text{slope}: m = 1 = \frac{1}{1} = \frac{\text{rise}}{\text{run}}$	$\text{Line 2}: y = -2x + 1$ $y\text{-intercept}: (0, 1)$ $\text{slope}: m = -2 = \frac{-2}{1} = \frac{\text{rise}}{\text{run}}$



Use the graph to estimate the point where the lines intersect and check to see if it solves the original system. In the above graph, the point of intersection appears to be (-1, 3).

Check: (-1, 3)	
Line 1: $x - y = -4$	*Line 2:* $2x + y = 1$
$(-1) - (3) = -4$ $-1 - 3 = -4$ $-4 = -4$ ✓	$2(-1) + (3) = 1$ $-2 + 3 = 1$ $1 = 1$ ✓

Answer: (-1, 3)

Example 3

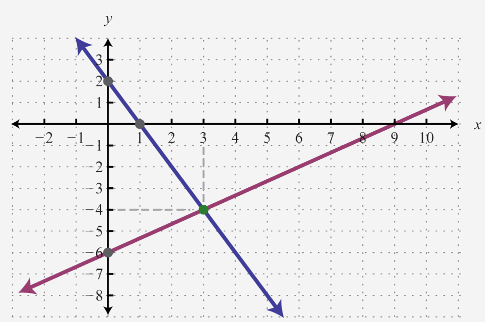
Solve by graphing: $\begin{cases} 2x + y = 2 \\ -2x + 3y = -18 \end{cases}$.

Solution:

We first solve each equation for y to obtain an equivalent system where the lines are in slope-intercept form.

$$\begin{cases} 2x + y = 2 \\ -2x + 3y = -18 \end{cases} \Rightarrow \begin{cases} y = -2x + 2 \\ y = \dfrac{2}{3}x - 6 \end{cases}$$

Graph the lines and determine the point of intersection.



Check: (3, -4)	
$2x+y=2$ $2(3)+(-4)=2$ $6-4=2$ $2=2$ ✓	$-2x+3y=-18$ $-2(3)+3(-4)=-18$ $-6-12=-18$ $-18=-18$ ✓

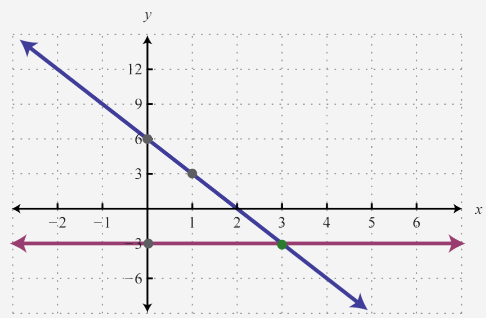
Answer: (3, -4)

Example 4

Solve by graphing: $\begin{cases} 3x + y = 6 \\ y = -3 \end{cases}$.

Solution:

$$\begin{cases} 3x + y = 6 \\ y = -3 \end{cases} \Rightarrow \begin{cases} y = -3x + 6 \\ y = -3 \end{cases}$$



Check: (3, -3)	
$3x + y = 6$ $3(3) + (-3) = 6$ $9 - 3 = 6$ $6 = 6$ ✓	$y = -3$ $(-3) = -3$ $-3 = -3$ ✓

Answer: (3, -3)

The graphing method for solving linear systems is not ideal when a solution consists of coordinates that are not integers. There will be more accurate algebraic methods in sections to come, but for now, the goal is to understand the geometry involved when solving systems. It is important to remember that the solutions to a system correspond to the point, or points, where the graphs of the equations intersect.

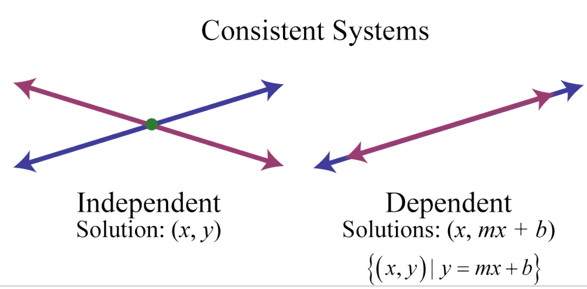
Try this! Solve by graphing: $\begin{cases} -x + y = 6 \\ 5x + 2y = -2 \end{cases}.$

Answer: (−2, 4)

(click to see video)

Dependent and Inconsistent Systems

A system with at least one solution is called a **consistent system**[7]. Up to this point, all of the examples have been of consistent systems with exactly one ordered pair solution. It turns out that this is not always the case. Sometimes systems consist of two linear equations that are equivalent. If this is the case, the two lines are the same and when graphed will coincide. Hence, the solution set consists of all the points on the line. This is a **dependent system**[8]. Given a consistent linear system with two variables, there are two possible results:



A solution to an **independent system**[9] is an ordered pair (x, y). The solution to a dependent system consists of infinitely many ordered pairs (x, y). Since any line can be written in slope-intercept form, $y = mx + b$, we can express these solutions, dependent on x, as follows:

7. A system with at least one solution.

8. A linear system with two variables that consists of equivalent equations. It has infinitely many ordered pair solutions, denoted by $(x, mx + b)$.

9. A linear system with two variables that has exactly one ordered pair solution.

$$\{(x, y) \mid y = mx + b\} \quad \textit{Set-Notation}$$
$$(x, mx + b) \quad \textit{Shortened Form}$$

In this text we will express all the ordered pair solutions $(x,\ y)$ in the shortened form $(x,\ \ mx + b)$, where x is any real number.

Example 5

Solve by graphing: $\begin{cases} -2x + 3y = -9 \\ 4x - 6y = 18 \end{cases}$.

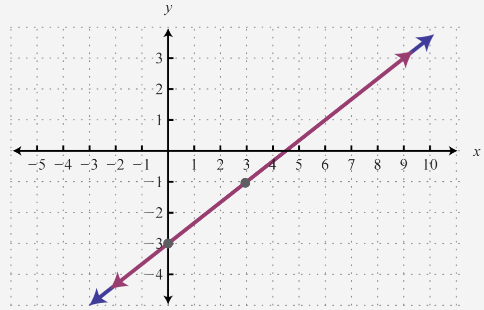
Solution:

Determine slope-intercept form for each linear equation in the system.

$-2x + 3y = -9$ $-2x + 3y = -9$ $3y = 2x - 9$ $y = \frac{2x-9}{3}$ $y = \frac{2}{3}x - 3$	$4x - 6y = 18$ $4x - 6y = 18$ $-6y = -4x + 18$ $y = \frac{-4x+18}{-6}$ $y = \frac{2}{3}x - 3$

$$\begin{cases} -2x + 3y = -9 \\ 4x - 6y = 18 \end{cases} \Rightarrow \begin{cases} y = \dfrac{2}{3}x - 3 \\ y = \dfrac{2}{3}x - 3 \end{cases}$$

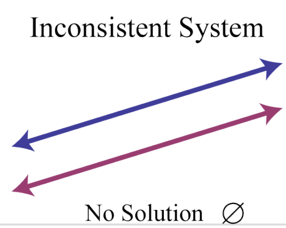
In slope-intercept form, we can easily see that the system consists of two lines with the same slope and same y-intercept. They are, in fact, the same line. And the system is dependent.



Answer: $\left(x,\ \ \frac{2}{3}x - 3\right)$

In this example, it is important to notice that the two lines have the same slope and same y-intercept. This tells us that the two equations are equivalent and that the simultaneous solutions are all the points on the line $y = \frac{2}{3}x - 3$. This is a dependent system, and the infinitely many solutions are expressed using the form $(x,\ \ mx + b)$. Other resources may express this set using set notation, $\{(x, y) \mid y = \frac{2}{3}x - 3\}$, which reads "*the set of all ordered pairs* (x, y) *such that* $y = \frac{2}{3}x - 3$."

Sometimes the lines do not cross and there is no point of intersection. Such a system has no solution, Ø, and is called an **inconsistent system**[10].



10. A system with no simultaneous solution.

Example 6

Solve by graphing: $\begin{cases} -2x + 5y = -15 \\ -4x + 10y = 10 \end{cases}$.

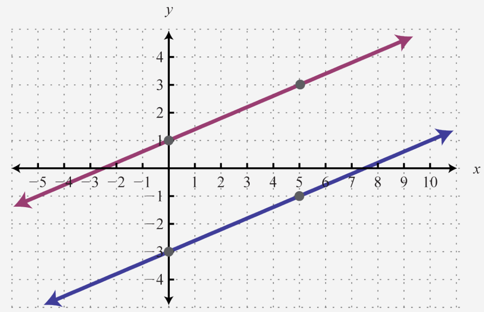
Solution:

Determine slope-intercept form for each linear equation.

$-2x + 5y = -15$	$-4x + 10y = 10$
$-2x + 5y = -15$	$-4x + 10y = 10$
$5y = 2x - 15$	$10y = 4x + 10$
$y = \frac{2x-15}{5}$	$y = \frac{4x+10}{10}$
$y = \frac{2}{5}x - 3$	$y = \frac{2}{5}x + 1$

$$\begin{cases} -2x + 5y = -15 \\ -4x + 10y = 10 \end{cases} \Rightarrow \begin{cases} y = \dfrac{2}{5}x - 3 \\ y = \dfrac{2}{5}x + 1 \end{cases}$$

In slope-intercept form, we can easily see that the system consists of two lines with the same slope and different y-intercepts. Therefore, the lines are parallel and will never intersect.



Answer: There is no simultaneous solution, Ø.

Try this! Solve by graphing: $\begin{cases} x + y = -1 \\ -2x - 2y = 2 \end{cases}$.

Answer: $(x, -x - 1)$

(click to see video)

KEY TAKEAWAYS

- In this section, we limit our study to systems of two linear equations with two variables. Solutions to such systems, if they exist, consist of ordered pairs that satisfy both equations. Geometrically, solutions are the points where the graphs intersect.
- The graphing method for solving linear systems requires us to graph both of the lines on the same set of axes as a means to determine where they intersect.
- The graphing method is not the most accurate method for determining solutions, particularly when a solution has coordinates that are not integers. It is a good practice to always check your solutions.
- Some linear systems have no simultaneous solution. These systems consist of equations that represent parallel lines with different *y*-intercepts and do not intersect in the plane. They are called inconsistent systems and the solution set is the empty set, $\emptyset$.
- Some linear systems have infinitely many simultaneous solutions. These systems consist of equations that are equivalent and represent the same line. They are called dependent systems and their solutions are expressed using the notation $(x, mx + b)$, where *x* is any real number.

TOPIC EXERCISES

PART A: DEFINITIONS

Determine whether or not the given ordered pair is a solution to the given system.

1. (3, -2);

$$\begin{cases} x + y = -1 \\ -2x - 2y = 2 \end{cases}$$

2. (-5, 0);

$$\begin{cases} x + y = -1 \\ -2x - 2y = 2 \end{cases}$$

3. (-2, -6);

$$\begin{cases} -x + y = -4 \\ 3x - y = -12 \end{cases}$$

4. (2, -7);

$$\begin{cases} 3x + 2y = -8 \\ -5x - 3y = 11 \end{cases}$$

5. (0, -3);

$$\begin{cases} 5x - 5y = 15 \\ -13x + 2y = -6 \end{cases}$$

6. $\left(-\frac{1}{2}, \frac{1}{4}\right)$;

$$\begin{cases} x + y = -\frac{1}{4} \\ -2x - 4y = 0 \end{cases}$$

7. $\left(\frac{3}{4}, \frac{1}{4}\right)$;

$$\begin{cases} -x - y = -1 \\ -4x - 8y = 5 \end{cases}$$

8. (-3, 4);

$$\begin{cases} \frac{1}{3}x + \frac{1}{2}y = 1 \\ \frac{2}{3}x - \frac{3}{2}y = -8 \end{cases}$$

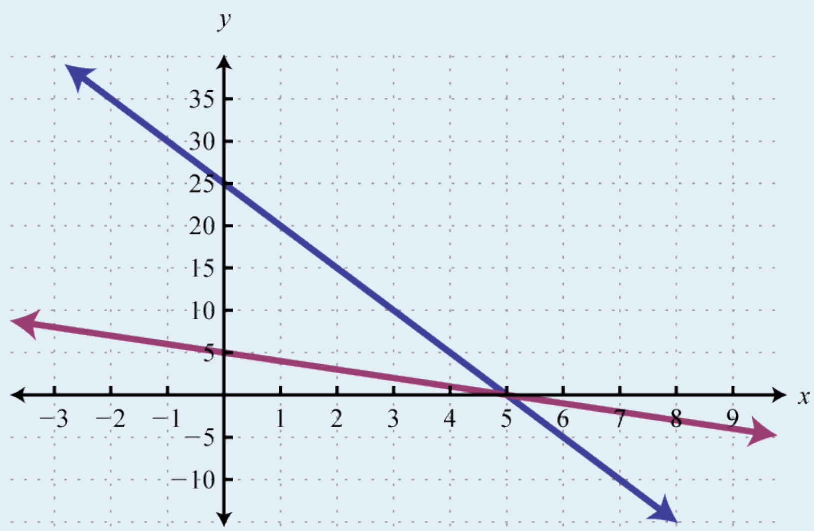
9. (-5, -3);

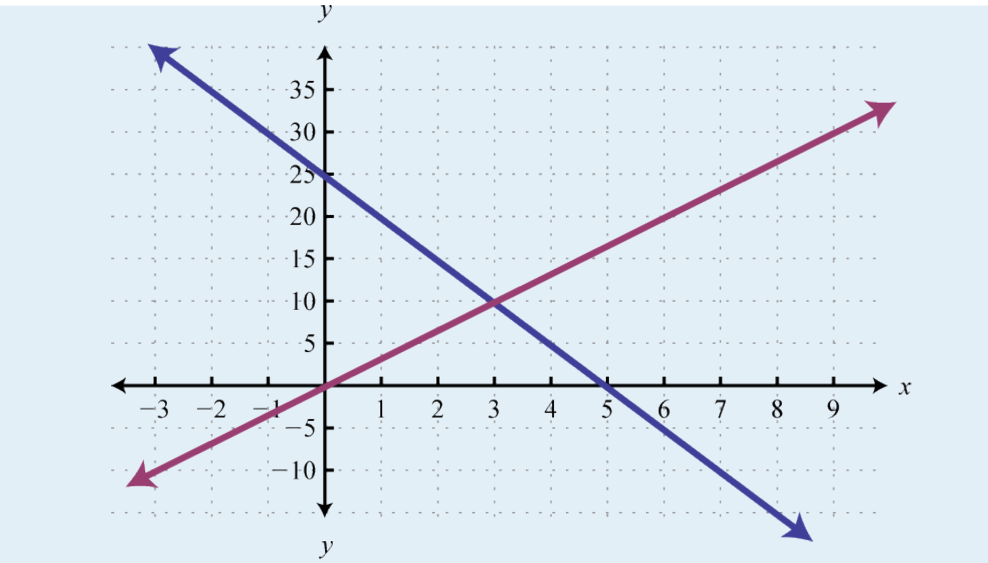
$$\begin{cases} y = -3 \\ 5x - 10y = 5 \end{cases}$$

10. (4, 2);

$$\begin{cases} x = 4 \\ -7x + 4y = 8 \end{cases}$$

Given the graphs, determine the simultaneous solution.

11. 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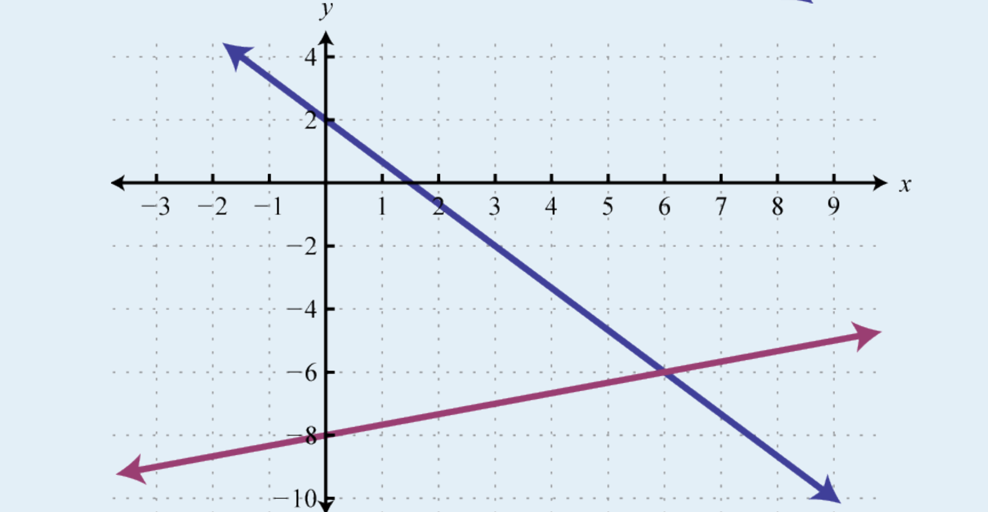
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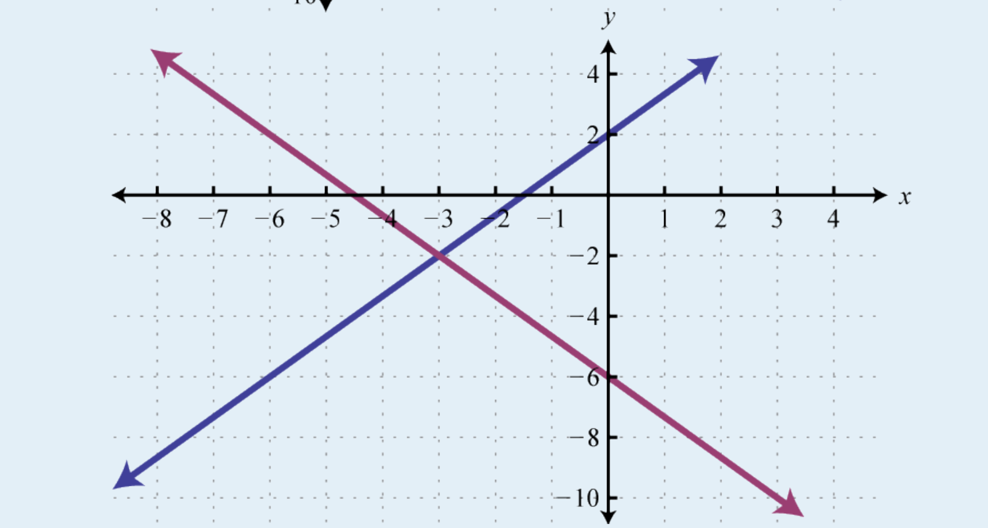
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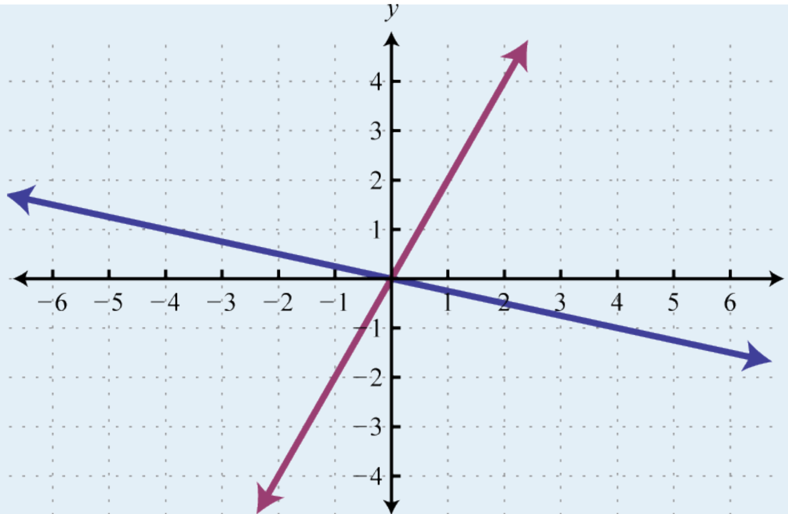
15. 
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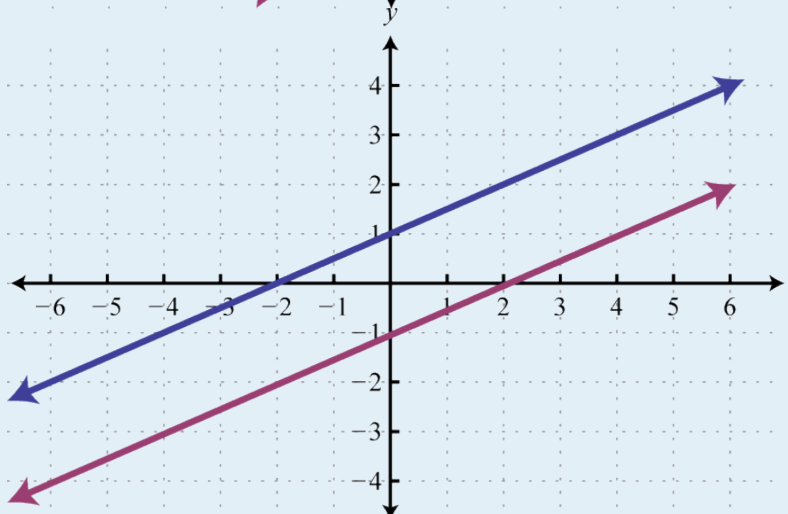
16. 
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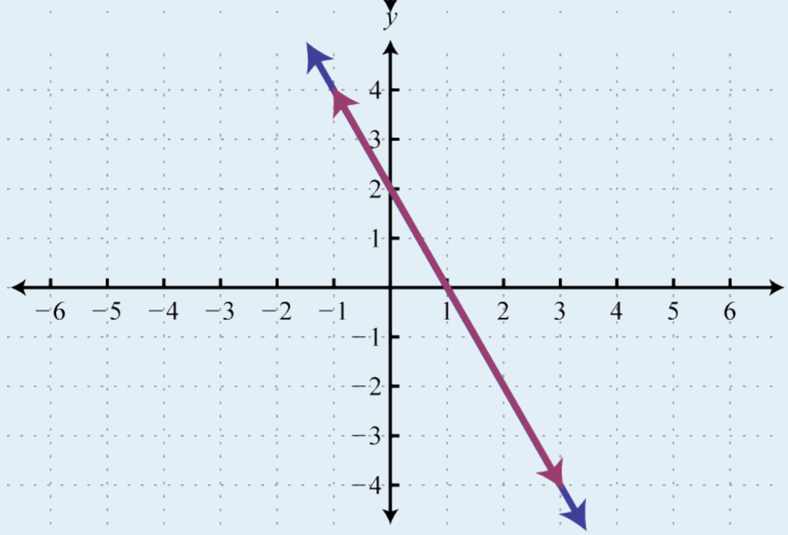
17. 
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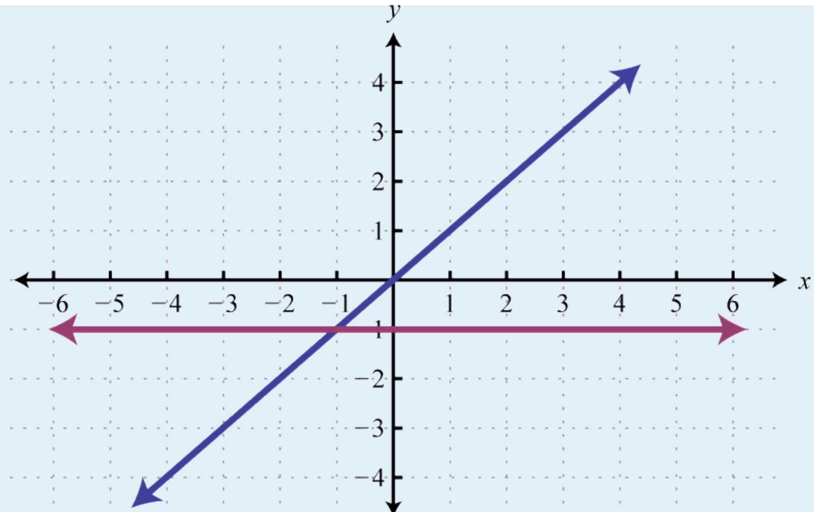
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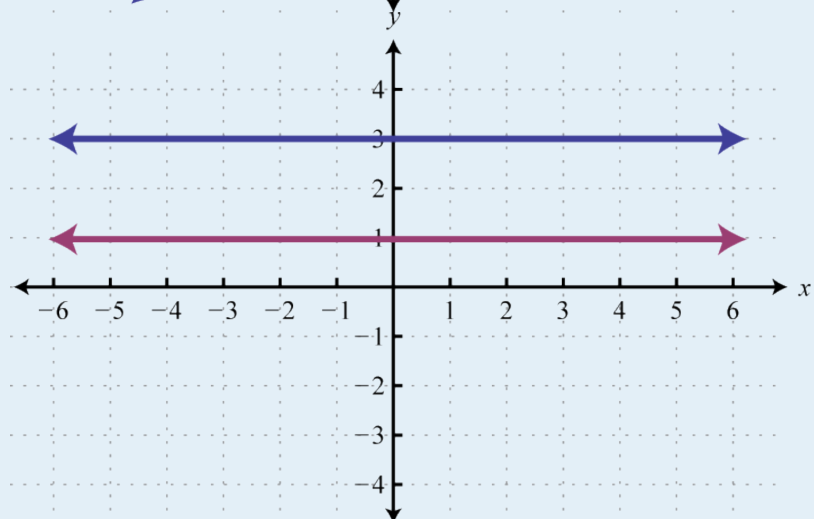
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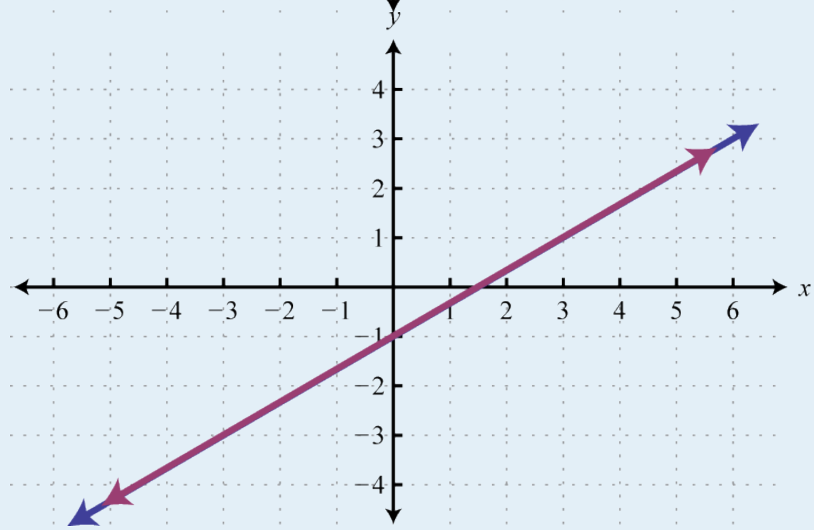
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PART B: SOLVE BY GRAPHING

Solve by graphing.

21. $\begin{cases} y = \dfrac{3}{2}x + 6 \\ y = -x + 1 \end{cases}$

22. $\begin{cases} y = \dfrac{3}{4}x + 2 \\ y = -\dfrac{1}{4}x - 2 \end{cases}$

23. $\begin{cases} y = x - 4 \\ y = -x + 2 \end{cases}$

24. $\begin{cases} y = -5x + 4 \\ y = 4x - 5 \end{cases}$

25. $\begin{cases} y = \dfrac{2}{5}x + 1 \\ y = \dfrac{3}{5}x \end{cases}$

26. $\begin{cases} y = -\dfrac{2}{5}x + 6 \\ y = \dfrac{2}{5}x + 10 \end{cases}$

27. $\begin{cases} y = -2 \\ y = x + 1 \end{cases}$

28. $\begin{cases} y = 3 \\ x = -3 \end{cases}$

29. $\begin{cases} y = 0 \\ y = \dfrac{2}{5}x - 4 \end{cases}$

30. $\begin{cases} x = 2 \\ y = 3x \end{cases}$

31. $\begin{cases} y = \frac{3}{5}x - 6 \\ y = \frac{3}{5}x - 3 \end{cases}$

32. $\begin{cases} y = -\frac{1}{2}x + 1 \\ y = -\frac{1}{2}x + 1 \end{cases}$

33. $\begin{cases} 2x + 3y = 18 \\ -6x + 3y = -6 \end{cases}$

34. $\begin{cases} -3x + 4y = 20 \\ 2x + 8y = 8 \end{cases}$

35. $\begin{cases} -2x + y = 1 \\ 2x - 3y = 9 \end{cases}$

36. $\begin{cases} x + 2y = -8 \\ 5x + 4y = -4 \end{cases}$

37. $\begin{cases} 4x + 6y = 36 \\ 2x - 3y = 6 \end{cases}$

38. $\begin{cases} 2x - 3y = 18 \\ 6x - 3y = -6 \end{cases}$

39. $\begin{cases} 3x + 5y = 30 \\ -6x - 10y = -10 \end{cases}$

40. $\begin{cases} -x + 3y = 3 \\ 5x - 15y = -15 \end{cases}$

41. $\begin{cases} x - y = 0 \\ -x + y = 0 \end{cases}$

42. $\begin{cases} y = x \\ y - x = 1 \end{cases}$

43. $\begin{cases} 3x + 2y = 0 \\ x = 2 \end{cases}$

44. $\begin{cases} 2x + \frac{1}{3}y = \frac{2}{3} \\ -3x + \frac{1}{2}y = -2 \end{cases}$

45. $\begin{cases} \frac{1}{10}x + \frac{1}{5}y = 2 \\ -\frac{1}{5}x + \frac{1}{5}y = -1 \end{cases}$

46. $\begin{cases} \frac{1}{3}x - \frac{1}{2}y = 1 \\ \frac{1}{3}x + \frac{1}{5}y = 1 \end{cases}$

47. $\begin{cases} \frac{1}{9}x + \frac{1}{6}y = 0 \\ \frac{1}{9}x + \frac{1}{4}y = \frac{1}{2} \end{cases}$

48. $\begin{cases} \frac{5}{16}x - \frac{1}{2}y = 5 \\ -\frac{5}{16}x + \frac{1}{2}y = \frac{5}{2} \end{cases}$

49. $\begin{cases} \frac{1}{6}x - \frac{1}{2}y = \frac{9}{2} \\ -\frac{1}{18}x + \frac{1}{6}y = -\frac{3}{2} \end{cases}$

50. $\begin{cases} \frac{1}{2}x - \frac{1}{4}y = -\frac{1}{2} \\ \frac{1}{3}x - \frac{1}{2}y = 3 \end{cases}$

51. $\begin{cases} y = 4 \\ x = -5 \end{cases}$

52. $\begin{cases} y = -3 \\ x = 2 \end{cases}$

53. $\begin{cases} y = 0 \\ x = 0 \end{cases}$

54. $\begin{cases} y = -2 \\ y = 3 \end{cases}$

55. $\begin{cases} y = 5 \\ y = -5 \end{cases}$

56. $\begin{cases} y = 2 \\ y - 2 = 0 \end{cases}$

57. $\begin{cases} x = -5 \\ x = 1 \end{cases}$

58. $\begin{cases} y = x \\ x = 0 \end{cases}$

59. $\begin{cases} 4x + 6y = 3 \\ -x + y = -2 \end{cases}$

60. $\begin{cases} -2x + 20y = 20 \\ 3x + 10y = -10 \end{cases}$

61. Assuming *m* is nonzero solve the system: $\begin{cases} y = mx + b \\ y = -mx + b \end{cases}$

62. Assuming *b* is nonzero solve the system: $\begin{cases} y = mx + b \\ y = mx - b \end{cases}$

63. Find the equation of the line perpendicular to $y = -2x + 4$ and passing through $(3, 3)$. Graph this line and the given line on the same set of axes and determine where they intersect.

64. Find the equation of the line perpendicular to $y - x = 2$ and passing through $(-5, 1)$. Graph this line and the given line on the same set of axes and determine where they intersect.

65. Find the equation of the line perpendicular to $y = -5$ and passing through $(2, -5)$. Graph both lines on the same set of axes.

66. Find the equation of the line perpendicular to the y-axis and passing through the origin.

67. Use the graph of $y = -\frac{2}{3}x + 3$ to determine the x-value where $y = -3$. Verify your answer using algebra.

68. Use the graph of $y = \frac{4}{5}x - 3$ to determine the x-value where $y = 5$. Verify your answer using algebra.

PART C: DISCUSSION BOARD TOPICS

69. Discuss the weaknesses of the graphing method for solving systems.

70. Explain why the solution set to a dependent linear system is denoted by $(x, mx + b)$.

71. Draw a picture of a dependent linear system as well as a picture of an inconsistent linear system. What would you need to determine the equations of the lines that you have drawn?

ANSWERS

1. No
3. No
5. Yes
7. No
9. Yes
11. (5, 0)
13. (6, −6)
15. (0, 0)
17. $(x, -2x + 2)$
19. $\emptyset$
21. (−2, 3)
23. (3, −1)
25. (5, 3)
27. (−3, −2)
29. (10, 0)
31. $\emptyset$
33. (3, 4)
35. (−3, −5)
37. (6, 2)
39. $\emptyset$
41. $(x,\ x)$
43. (2, −3)
45. (10, 5)
47. (−9, 6)
49. $(x,\ \frac{1}{3}x - 9)$

51. (–5, 4)

53. (0, 0)

55. $\emptyset$

57. $\emptyset$

59. $\left(\frac{3}{2}, -\frac{1}{2}\right)$

61. $(0, b)$

63. $y = \frac{1}{2}x + \frac{3}{2};\ (1, 2)$

65. $x = 2$

67. $x = 9$

69. Answer may vary

71. Answer may vary

3.2 Solving Linear Systems with Two Variables

LEARNING OBJECTIVES

1. Solve linear systems using the substitution method.
2. Solve linear systems using the elimination method.
3. Identify the strengths and weaknesses of each method.

The Substitution Method

In this section, we review a completely algebraic technique for solving systems, the **substitution method**[11]. The idea is to solve one equation for one of the variables and substitute the result into the other equation. After performing this substitution step, we are left with a single equation with one variable, which can be solved using algebra.

11. A means of solving a linear system by solving for one of the variables and substituting the result into the other equation.

Example 1

Solve by substitution: $\begin{cases} 2x + y = -3 \\ 3x - 2y = -8 \end{cases}$.

Solution:

Solve for either variable in either equation. If you choose the first equation, you can isolate y in one step.

$$\begin{aligned} 2x + y &= -3 \\ y &= -2x - 3 \end{aligned}$$

Substitute the expression $-2x - 3$ for the variable y in the *other* equation.

$$\begin{cases} 2x + y = -3 \\ 3x - 2y = -8 \end{cases} \quad \Rightarrow \quad y = -2x - 3$$

$$3x - 2\,(-2x - 3) = -8$$

This leaves us with an equivalent equation with one variable, which can be solved using the techniques learned up to this point. Solve for the remaining variable.

$$\begin{aligned} 3x - 2(-2x - 3) &= -8 \\ 3x + 4x + 6 &= -8 \\ 7x + 6 &= -8 \\ 7x &= -14 \\ x &= -2 \end{aligned}$$

Back substitute[12] to find the other coordinate. Substitute $x = -2$ into either of the original equations or their equivalents. Typically, we use the equivalent equation that we found when isolating a variable in the first step.

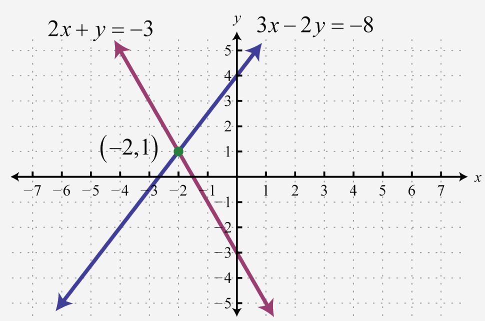
$$\begin{aligned} y &= -2x - 3 \\ &= -2(-2) - 3 \\ &= 4 - 3 \\ &= 1 \end{aligned}$$

Remember to present the solution as an ordered pair: $(-2, 1)$. Verify that these coordinates solve both equations of the original system:

12. Once a value is found for a variable, substitute it back into one of the original equations, or its equivalent, to determine the corresponding value of the other variable.

Check: (-2, 1)	
Equation 1	*Equation 2*
$\begin{aligned} 2x + y &= -3 \\ 2(-2) + (1) &= -3 \\ -4 + 1 &= -3 \\ -3 &= -3 \; \checkmark \end{aligned}$	$\begin{aligned} 3x - 2y &= -8 \\ 3(-2) - 2(1) &= -8 \\ -6 - 2 &= -8 \\ -8 &= -8 \; \checkmark \end{aligned}$

The graph of this linear system follows:



The substitution method for solving systems is a completely algebraic method. Thus graphing the lines is not required.

Answer: (-2, 1)

Example 2

Solve by substitution: $\begin{cases} 3x - 5y & = & 9 \\ 4x + 2y & = & -1 \end{cases}$.

Solution:

It does not matter which variable we choose to isolate first. In this case, begin by solving for x in the first equation.

$$\begin{aligned} 3x - 5y &= 9 \\ 3x &= 5y + 9 \\ x &= \frac{5y + 9}{3} \\ x &= \frac{5}{3}y + 3 \end{aligned}$$

$$\begin{cases} 3x - 5y = \cancel{9} \Longrightarrow x = \frac{5}{3}y + 3 \\ 4x + 2y = -1 \end{cases}$$

Next, substitute into the second equation and solve for y.

$$\begin{aligned} 4\left(\frac{5}{3}y+3\right)+2y &= -1 \\ \frac{20}{3}y+12+2y &= -1 \\ \frac{26}{3}y &= -13 \\ y &= -13\left(\frac{3}{26}\right) \\ y &= -\frac{3}{2} \end{aligned}$$

Back substitute into the equation used in the substitution step:

$$\begin{aligned} x &= \frac{5}{3}y+3 \\ &= \frac{5}{3}\left(-\frac{3}{2}\right)+3 \\ &= -\frac{5}{2}+3 \\ &= \frac{1}{2} \end{aligned}$$

Answer: $\left(\frac{1}{2}, -\frac{3}{2}\right)$

Try this! Solve by substitution: $\begin{cases} 5x - 4y = 3 \\ x + 2y = 2 \end{cases}$.

Answer: $\left(1,\ \frac{1}{2}\right)$

(click to see video)

As we know, not all linear systems have only one ordered pair solution. Next, we explore what happens when using the substitution method to solve a dependent system.

Example 3

Solve by substitution: $\begin{cases} -5x+y=-1 \\ 10x-2y=2 \end{cases}$.

Solution:

Since the first equation has a term with coefficient 1, we choose to solve for that first.

$$\begin{cases} -5x+y=-1 \\ 10x-2y=2 \end{cases} \quad \Rightarrow \quad y=5x-1$$

Next, substitute this expression in for y in the second equation.

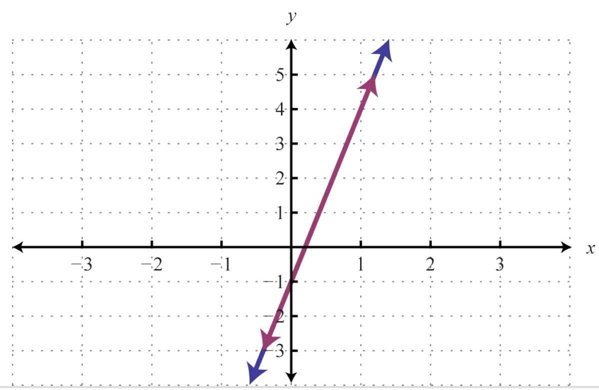
$$\begin{aligned} 10x-2y&=2 \\ 10x-2(5x-1)&=2 \\ 10x-10x+2&=2 \\ 2&=2 \quad \textit{True} \end{aligned}$$

This process led to a true statement; hence the equation is an identity and any real number is a solution. This indicates that the system is dependent. The simultaneous solutions take the form $(x, mx+b)$, or in this case, $(x,\ 5x-1)$, where x is any real number.

Answer: $(x,\ 5x-1)$

To have a better understanding of the previous example, rewrite both equations in slope-intercept form and graph them on the same set of axes.

$$\left\{\begin{array}{l}-5x+y=-1\\10x-2y=2\end{array}\right. \Rightarrow \left\{\begin{array}{l}y=5x-1\\y=5x-1\end{array}\right.$$



We can see that both equations represent the same line, and thus the system is dependent. Now explore what happens when solving an inconsistent system using the substitution method.

Example 4

Solve by substitution: $\begin{cases} -7x + 3y = 3 \\ 14x - 6y = -16 \end{cases}$.

Solution:

Solve for y in the first equation.

$$\begin{aligned} -7x + 3y &= 3 \\ -7x + 3y &= 3 \\ 3y &= 7x + 3 \\ y &= \frac{7x+3}{3} \\ y &= \frac{7}{3}x + 1 \end{aligned}$$

$$\begin{cases} -7x + 3y = 3 \\ 14x - 6y = -16 \end{cases} \quad \Rightarrow \quad y = \frac{7}{3}x + 1$$

Substitute into the second equation and solve.

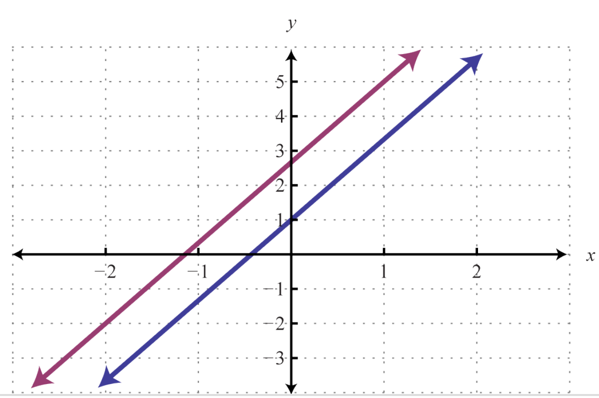
$$14x - 6y = -16$$
$$14x - 6\left(\frac{7}{3}x + 1\right) = -16$$
$$14x - \overset{2}{\cancel{6}} \cdot \frac{7}{\underset{1}{\cancel{3}}}x - 6 = -16$$
$$14x - 14x - 6 = -16$$
$$-6 = -16 \quad \textit{False}$$

Solving leads to a false statement. This indicates that the equation is a contradiction. There is no solution for x and hence no solution to the system.

Answer: Ø

A false statement indicates that the system is inconsistent, or in geometric terms, that the lines are parallel and do not intersect. To illustrate this, determine the slope-intercept form of each line and graph them on the same set of axes.

$$\begin{cases} -7x + 3y = 3 \\ 14x - 6y = -16 \end{cases} \Rightarrow \begin{cases} y = \frac{7}{3}x + 1 \\ y = \frac{7}{3}x + \frac{8}{3} \end{cases}$$



In slope-intercept form, it is easy to see that the two lines have the same slope but different y-intercepts.

Try this! Solve by substitution: $\begin{cases} 2x - 5y = 3 \\ 4x - 10y = 6 \end{cases}$.

Answer: $\left(x, \ \frac{2}{5}x - \frac{3}{5}\right)$

(click to see video)

The Elimination Method

In this section, the goal is to review another completely algebraic method for solving a system of linear equations called the **elimination method**[13] or **addition method**[14]. This method depends on the **addition property of equations**[15]: given algebraic expressions A, B, C, and D we have

$$\text{If } A = B \text{ and } C = D, \text{ then } A + C = B + D$$

Consider the following system:

$$\begin{cases} x + y = 5 \\ x - y = 1 \end{cases}$$

We can add the equations together to eliminate the variable y.

13. A means of solving a system by adding equivalent equations in such a way as to eliminate a variable.

14. Often used when referring to the elimination method for solving systems.

15. If A, B, C, and D are algebraic expressions, where $A = B$ and $C = D$, then $A + C = B + D$.

$$\begin{array}{r} x + y = 5 \\ \underline{+\ x - y = 1} \\ 2x \qquad = 6 \end{array}$$

This leaves us with a linear equation with one variable that can be easily solved:

$$\begin{aligned} 2x &= 6 \\ x &= 3 \end{aligned}$$

At this point, we have the *x*-coordinate of the simultaneous solution, so all that is left to do is back substitute to find the corresponding *y*-value.

$$\begin{aligned} x + y &= 5 \\ 3 + y &= 5 \\ y &= 2 \end{aligned}$$

The solution to the system is (3, 2). Of course, the variable is not always so easily eliminated. Typically, we have to find an equivalent system by applying the multiplication property of equality to one or both of the equations as a means to line up one of the variables to eliminate. The goal is to arrange that either the *x* terms or the *y* terms are opposites, so that when the equations are added, the terms eliminate.

Example 5

Solve by elimination: $\begin{cases} 5x - 3y = -1 \\ 3x + 2y = 7 \end{cases}$.

Solution:

We choose to eliminate the terms with variable y because the coefficients have different signs. To do this, we first determine the least common multiple of the coefficients; in this case, the LCM(3, 2) is 6. Therefore, multiply both sides of both equations by the appropriate values to obtain coefficients of –6 and 6. This results in the following equivalent system:

$$\begin{cases} 5x - 3y = -1 \\ 3x + 2y = 7 \end{cases} \begin{array}{c} \overset{\times 2}{\Rightarrow} \\ \underset{\times 3}{\Rightarrow} \end{array} \begin{cases} 10x - 6y = -2 \\ 9x + 6y = 21 \end{cases}$$

The terms involving y are now lined up to eliminate. Add the equations together and solve for x.

$$\begin{array}{rrl} & 10x - 6y & = -2 \\ + & 9x + 6y & = 21 \\ \hline & 19x & = 19 \\ & x & = 1 \end{array}$$

Back substitute.

$$\begin{aligned} 3x + 2y &= 7 \\ 3(1) + 2y &= 7 \\ 3 + 2y &= 7 \\ 2y &= 4 \\ y &= 2 \end{aligned}$$

Therefore the simultaneous solution is (1, 2). The check follows.

Check: (1, 2)	
Equation 1:	*Equation 2:*
$\begin{aligned} 5x - 3y &= -1 \\ 5(1) - 3(2) &= -1 \\ 5 - 6 &= -1 \\ -1 &= -1 \ \checkmark \end{aligned}$	$\begin{aligned} 3x + 2y &= 7 \\ 3(1) + 2(2) &= 7 \\ 3 + 4 &= 7 \\ 7 &= 7 \ \checkmark \end{aligned}$

Answer: (1, 2)

Sometimes linear systems are not given in standard form $ax + by = c$. When this is the case, it is best to rearrange the equations before beginning the steps to solve by elimination. Also, we can eliminate either variable. The goal is to obtain a solution for one of the variables and then back substitute to find a solution for the other.

Example 6

Solve by elimination: $\begin{cases} 12x + 5y = 11 \\ 3x = 4y + 1 \end{cases}$.

Solution:

First, rewrite the second equation in standard form.

$$\begin{aligned} 3x &= 4y + 1 \\ 3x - 4y &= 1 \end{aligned}$$

This results in an equivalent system in standard form, where like terms are aligned in columns.

$$\begin{cases} 12x + 5y = 11 \\ 3x = 4y + 1 \end{cases} \Rightarrow \begin{cases} 12x + 5y = 11 \\ 3x - 4y = 1 \end{cases}$$

We can eliminate the term with variable *x* if we multiply the second equation by −4.

$$\begin{cases} 12x + 5y = 11 \\ 3x - 4y = 1 \end{cases} \underset{\times(-4)}{\Rightarrow} \begin{cases} 12x + 5y = 11 \\ -12x + 16y = -4 \end{cases}$$

Next, we add the equations together,

$$\begin{array}{rrl} & 12x + 5y = & 11 \\ + & -12x + 16y = & -4 \\ \hline & 21y = & 7 \\ & y = & \frac{7}{21} = \frac{1}{3} \end{array}$$

Back substitute.

$$\begin{aligned} 3x &= 4y + 1 \\ 3x &= 4\left(\frac{1}{3}\right) + 1 \\ 3x &= \frac{4}{3} + 1 \\ 3x &= \frac{7}{3} \\ x &= \frac{7}{3} \cdot \frac{1}{3} \\ x &= \frac{7}{9} \end{aligned}$$

Answer: $\left(\frac{7}{9}, \frac{1}{3}\right)$

Try this! Solve by elimination: $\begin{cases} 2x + 5y = 5 \\ 3x + 2y = -9 \end{cases}$.

Answer: (−5, 3)

(click to see video)

At this point, we explore what happens when solving dependent and inconsistent systems using the elimination method.

Example 7

Solve by elimination: $\begin{cases} 3x - y = 7 \\ 6x - 2y = 14 \end{cases}$.

Solution:

To eliminate the variable x, we could multiply the first equation by -2.

$$\begin{cases} 3x - y = 7 \\ 6x - 2y = 14 \end{cases} \overset{\times(-2)}{\Rightarrow} \begin{cases} -6x + 2y = -14 \\ 6x - 2y = 14 \end{cases}$$

Now adding the equations we have

$$\begin{array}{rr} & -6x + 2y = -14 \\ + & 6x - 2y = 14 \\ \hline & 0 = 0 \quad \textit{True} \end{array}$$

A true statement indicates that this is a dependent system. The lines coincide, and we need y in terms of x to present the solution set in the form $(x,\ mx + b)$. Choose one of the original equations and solve for y. Since the equations are equivalent, it does not matter which one we choose.

$$\begin{aligned} 3x - y &= 7 \\ -y &= -3x + 7 \\ -1\left(-y\right) &= -1\left(-3x + 7\right) \\ y &= 3x - 7 \end{aligned}$$

Answer: $(x, \;\; 3x - 7)$

Try this! Solve by elimination: $\begin{cases} 3x + 15y = -15 \\ 2x + 10y = 30 \end{cases}.$

Answer: No solution, $\varnothing$

(click to see video)

Given a linear system where the equations have fractional coefficients, it is usually best to clear the fractions before beginning the elimination method.

Example 8

Solve: $\begin{cases} -\frac{1}{10}x+\frac{1}{2}y=\frac{4}{5} \\ \frac{1}{7}x+\frac{1}{3}y=-\frac{2}{21} \end{cases}$.

Solution:

Recall that we can clear fractions by multiplying both sides of an equation by the least common multiple of the denominators (LCD). Take care to distribute and then simplify.

Equation 1	*Equation 2*
$10\left(-\frac{1}{10}x+\frac{1}{2}y\right)=10\left(\frac{4}{5}\right)$ $10\cdot\left(-\frac{1}{10}x\right)+10\cdot\frac{1}{2}y=10\cdot\frac{4}{5}$ $-x+5y=8$	$21\left(\frac{1}{7}x+\frac{1}{3}y\right)=21\left(-\frac{2}{21}\right)$ $21\cdot\frac{1}{7}x+21\cdot\frac{1}{3}y=21\left(-\frac{2}{21}\right)$ $3x+7y=-2$

This results in an equivalent system where the equations have integer coefficients,

$$\begin{cases} -\dfrac{1}{10}x+\dfrac{1}{2}y=\dfrac{4}{5} \\ \dfrac{1}{7}x+\dfrac{1}{3}y=-\dfrac{2}{21} \end{cases} \quad \begin{matrix}\xRightarrow{\times 10} \\ \xRightarrow[\times 21]{}\end{matrix} \quad \begin{cases} -x+5y=8 \\ 3x+7y=-2 \end{cases}$$

Solve using the elimination method.

$$\begin{cases} -x+5y=8 \\ 3x+7y=-2 \end{cases} \overset{\times 3}{\Rightarrow} \begin{cases} -3x+15y=24 \\ 3x+7y=-2 \end{cases}$$

$$\begin{aligned} -3x + 15y &= 24 \\ + \quad 3x + 7y &= -2 \\ \hline 22y &= 22 \\ y &= 1 \end{aligned}$$

Back substitute.

$$\begin{aligned} 3x + 7y &= -2 \\ 3x + 7(1) &= -2 \\ 3x + 7 &= -2 \\ 3x &= -9 \\ x &= -3 \end{aligned}$$

Answer: (-3, 1)

We can use a similar technique to clear decimals before solving.

Try this! Solve using elimination: $\begin{cases} \frac{1}{3}x - \frac{2}{3}y &= 3 \\ \frac{1}{3}x - \frac{1}{2}y &= \frac{8}{3} \end{cases}$.

Answer: (5, -2)

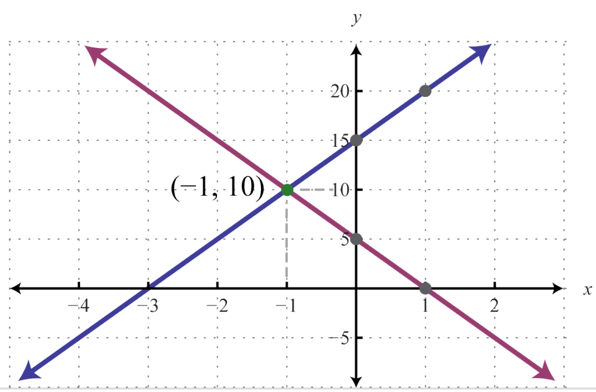
(click to see video)

Summary of the Methods for Solving Linear Systems

We have reviewed three methods for solving linear systems of two equations with two variables. Each method is valid and can produce the same correct result. In this section, we summarize the strengths and weaknesses of each method.

The graphing method is useful for understanding what a system of equations is and what the solutions must look like. When the equations of a system are graphed on the same set of axes, we can see that the solution is the point where the graphs intersect. The graphing is made easy when the equations are in slope-intercept form. For example,

$$\begin{cases} y = 5x + 15 \\ y = -5x + 5 \end{cases}$$



The simultaneous solution (−1, 10) corresponds to the point of intersection. One drawback of this method is that it is very inaccurate. When the coordinates of the solution are not integers, the method is practically unusable. If we have a choice, we typically avoid this method in favor of the more accurate algebraic techniques.

The substitution method, on the other hand, is a completely algebraic method. It requires you to solve for one of the variables and substitute the result into the other equation. The resulting equation has one variable for which you can solve. This method is particularly useful when there is a variable within the system with coefficient of 1. For example,

$$\begin{cases} 10x + y = 20 \\ 7x + 5y = 14 \end{cases} \quad \textit{Choose the substitution method.}$$

In this case, it is easy to solve for y in the first equation and then substitute the result into the other equation. One drawback of this method is that it often leads to equivalent equations with fractional coefficients, which are tedious to work with. If there is not a coefficient of 1, then it usually is best to choose the elimination method.

The elimination method is a completely algebraic method which makes use of the addition property of equations. We multiply one or both of the equations to obtain equivalent equations where one of the variables is eliminated if we add them together. For example,

$$\begin{cases} 2x - 3y = 9 \\ 5x - 8y = -16 \end{cases} \quad \textit{Choose the elimination method.}$$

To eliminate the terms involving x, we would multiply both sides of the first equation by 5 and both sides of the second equation by −2. This results in an equivalent system where the variable x is eliminated when we add the equations together. Of course, there are other combinations of numbers that achieve the same result. We could even choose to eliminate the variable y. No matter which variable is eliminated first, the solution will be the same. Note that the substitution method, in this case, would require tedious calculations with fractional coefficients. One weakness of the elimination method, as we will see later in our study of algebra, is that it does not always work for nonlinear systems.

KEY TAKEAWAYS

- The substitution method requires that we solve for one of the variables and then substitute the result into the other equation. After performing the substitution step, the resulting equation has one variable and can be solved using the techniques learned up to this point.
- The elimination method is another completely algebraic method for solving a system of equations. Multiply one or both of the equations in a system by certain numbers to obtain an equivalent system where at least one variable in both equations have opposite coefficients. Adding these equivalent equations together eliminates that variable, and the resulting equation has one variable for which you can solve.
- It is a good practice to first rewrite the equations in standard form before beginning the elimination method.
- Solutions to systems of two linear equations with two variables, if they exist, are ordered pairs (x, y).
- If the process of solving a system of equations leads to a false statement, then the system is inconsistent and there is no solution, $\emptyset$.
- If the process of solving a system of equations leads to an identity, then the system is dependent and there are infinitely many solutions that can be expressed using the form $(x, mx + b)$.

TOPIC EXERCISES

PART A: SUBSTITUTION METHOD

Solve by substitution.

1. $\begin{cases} y = -5x + 1 \\ 4x - 3y = -41 \end{cases}$
2. $\begin{cases} x = 2y - 3 \\ x + 3y = -8 \end{cases}$
3. $\begin{cases} y = x \\ 2x + 3y = 10 \end{cases}$
4. $\begin{cases} y = \frac{1}{2}x + \frac{1}{3} \\ x - 6y = 4 \end{cases}$
5. $\begin{cases} y = 4x + 1 \\ -4x + y = 2 \end{cases}$
6. $\begin{cases} y = -3x + 5 \\ 3x + y = 5 \end{cases}$
7. $\begin{cases} y = 2x + 3 \\ 2x - y = -3 \end{cases}$
8. $\begin{cases} y = \frac{2}{3}x - 1 \\ 6x - 9y = 0 \end{cases}$
9. $\begin{cases} y = -2 \\ -2x - y = -6 \end{cases}$
10. $\begin{cases} y = -\frac{1}{5}x + 3 \\ 7x - 5y = 9 \end{cases}$
11. $\begin{cases} x + y = 1 \\ 3x - 5y = 19 \end{cases}$

12. $$\begin{cases} x - y = 3 \\ -2x + 3y = -2 \end{cases}$$

13. $$\begin{cases} 2x + y = 2 \\ 3x - 2y = 17 \end{cases}$$

14. $$\begin{cases} x - 3y = -11 \\ 3x + 5y = -5 \end{cases}$$

15. $$\begin{cases} x + 2y = -3 \\ 3x - 4y = -2 \end{cases}$$

16. $$\begin{cases} 5x - y = 12 \\ 9x - y = 10 \end{cases}$$

17. $$\begin{cases} x + 2y = -6 \\ -4x - 8y = 24 \end{cases}$$

18. $$\begin{cases} x + 3y = -6 \\ -2x - 6y = -12 \end{cases}$$

19. $$\begin{cases} -3x + y = -4 \\ 6x - 2y = -2 \end{cases}$$

20. $$\begin{cases} x - 5y = -10 \\ 2x - 10y = -20 \end{cases}$$

21. $$\begin{cases} 3x - y = 9 \\ 4x + 3y = -1 \end{cases}$$

22. $$\begin{cases} 2x - y = 5 \\ 4x + 2y = -2 \end{cases}$$

23. $$\begin{cases} 2x - 5y = 1 \\ 4x + 10y = 2 \end{cases}$$

24. $$\begin{cases} 3x - 7y = -3 \\ 6x + 14y = 0 \end{cases}$$

25. $\begin{cases} 10x - y = 3 \\ -5x + \frac{1}{2}y = 1 \end{cases}$

26. $\begin{cases} -\frac{1}{3}x + \frac{1}{6}y = \frac{2}{3} \\ \frac{1}{2}x - \frac{1}{3}y = -\frac{3}{2} \end{cases}$

27. $\begin{cases} \frac{1}{3}x + \frac{2}{3}y = 1 \\ \frac{1}{4}x - \frac{1}{3}y = -\frac{1}{12} \end{cases}$

28. $\begin{cases} \frac{1}{7}x - y = \frac{1}{2} \\ \frac{1}{4}x + \frac{1}{2}y = 2 \end{cases}$

29. $\begin{cases} -\frac{3}{5}x + \frac{2}{5}y = \frac{1}{2} \\ \frac{1}{3}x - \frac{1}{12}y = -\frac{1}{3} \end{cases}$

30. $\begin{cases} \frac{1}{2}x = \frac{2}{3}y \\ x - \frac{2}{3}y = 2 \end{cases}$

31. $\begin{cases} -\frac{1}{2}x + \frac{1}{2}y = \frac{5}{8} \\ \frac{1}{4}x + \frac{1}{2}y = \frac{1}{4} \end{cases}$

32. $\begin{cases} x - y = 0 \\ -x + 2y = 3 \end{cases}$

33. $\begin{cases} y = 3x \\ 2x - 3y = 0 \end{cases}$

34. $\begin{cases} -3x + 4y = 20 \\ 2x + 8y = 8 \end{cases}$

35. $\begin{cases} 5x - 3y = -1 \\ 3x + 2y = 7 \end{cases}$

36. $\begin{cases} -3x + 7y = 2 \\ 2x + 7y = 1 \end{cases}$

37. $\begin{cases} x = 5 \\ x = -2 \end{cases}$

38. $\begin{cases} y = 4 \\ 5y = 20 \end{cases}$

PART B: ELIMINATION METHOD

Solve by elimination.

39. $\begin{cases} 6x + y = 3 \\ 3x - y = 0 \end{cases}$

40. $\begin{cases} x + y = 3 \\ 2x - y = 9 \end{cases}$

41. $\begin{cases} x - y = -6 \\ 5x + y = -18 \end{cases}$

42. $\begin{cases} x + 3y = 5 \\ -x - 2y = 0 \end{cases}$

43. $\begin{cases} -x + 4y = 4 \\ x - y = -7 \end{cases}$

44. $\begin{cases} -x + y = 2 \\ x - y = -3 \end{cases}$

45. $\begin{cases} 3x - y = -2 \\ 6x + 4y = 2 \end{cases}$

46. $\begin{cases} 5x + 2y = -3 \\ 10x - y = 4 \end{cases}$

47. $\begin{cases} -2x + 14y = 28 \\ x - 7y = 21 \end{cases}$

48. $\begin{cases} -2x + y = 4 \\ 12x - 6y = -24 \end{cases}$

49. $\begin{cases} x + 8y = 3 \\ 3x + 12y = 6 \end{cases}$

50. $\begin{cases} 2x - 3y = 15 \\ 4x + 10y = 14 \end{cases}$

51. $\begin{cases} 4x + 3y = -10 \\ 3x - 9y = 15 \end{cases}$

52. $\begin{cases} -4x - 5y = -3 \\ 8x + 3y = -15 \end{cases}$

53. $\begin{cases} -2x + 7y = 56 \\ 4x - 2y = -112 \end{cases}$

54. $\begin{cases} -9x - 15y = -15 \\ 3x + 5y = -10 \end{cases}$

55. $\begin{cases} 6x - 7y = 4 \\ 2x + 6y = -7 \end{cases}$

56. $\begin{cases} 4x + 2y = 4 \\ -5x - 3y = -7 \end{cases}$

57. $\begin{cases} 5x - 3y = -1 \\ 3x + 2y = 7 \end{cases}$

58. $\begin{cases} 7x + 3y = 9 \\ 2x + 5y = -14 \end{cases}$

59. $\begin{cases} 9x - 3y = 3 \\ 7x + 2y = -15 \end{cases}$

60. $\begin{cases} 5x - 3y = -7 \\ -7x + 6y = 11 \end{cases}$

61. $\begin{cases} 2x + 9y = 8 \\ 3x + 7y = -1 \end{cases}$

62. $\begin{cases} 2x + 2y = 5 \\ 3x + 3y = -5 \end{cases}$

63. $\begin{cases} -3x + 6y = -12 \\ 2x - 4y = 8 \end{cases}$

64. $\begin{cases} 25x + 15y = -1 \\ 15x + 10y = -1 \end{cases}$

65. $\begin{cases} 2x - 3y = 2 \\ 18x - 12y = 5 \end{cases}$

66. $\begin{cases} y = -2x - 3 \\ -3x - 2y = 4 \end{cases}$

67. $\begin{cases} 28x + 6y = 9 \\ 6y = 4x - 15 \end{cases}$

68. $\begin{cases} y = 5x + 15 \\ y = -5x + 5 \end{cases}$

69. $\begin{cases} 2x - 3y = 9 \\ 5x - 8y = -16 \end{cases}$

70. $\begin{cases} \frac{1}{2}x - \frac{1}{3}y = \frac{1}{6} \\ \frac{5}{2}x + y = \frac{7}{2} \end{cases}$

71. $\begin{cases} \frac{1}{4}x - \frac{1}{9}y = 1 \\ x + y = \frac{3}{4} \end{cases}$

72. $\begin{cases} \frac{1}{2}x - \frac{1}{4}y = \frac{1}{3} \\ \frac{1}{4}x + \frac{1}{2}y = -\frac{19}{6} \end{cases}$

73. $\begin{cases} -\frac{14}{3}x + 2y = 4 \\ -\frac{1}{3}x + \frac{1}{7}y = \frac{4}{21} \end{cases}$

74. $\begin{cases} 0.025x + 0.1y = 0.5 \\ 0.11x + 0.04y = -0.2 \end{cases}$

75. $\begin{cases} 1.3x + 0.1y = 0.35 \\ 0.5x + y = -2.75 \end{cases}$

76. $\begin{cases} x + y = 5 \\ 0.02x + 0.03y = 0.125 \end{cases}$

PART C: MIXED PRACTICE

Solve using any method.

77. $\begin{cases} 6x = 12y + 7 \\ 6x + 24y + 5 = 0 \end{cases}$

78. $\begin{cases} y = 2x - 3 \\ 3x + y = 12 \end{cases}$

79. $\begin{cases} x + 3y = -5 \\ y = \frac{1}{3}x + 5 \end{cases}$

80. $\begin{cases} y = 1 \\ x = -4 \end{cases}$

81. $\begin{cases} y = \frac{1}{2} \\ x + 9 = 0 \end{cases}$

82. $\begin{cases} y = x \\ -x + y = 1 \end{cases}$

83. $\begin{cases} y = 5x \\ y = -10 \end{cases}$

84. $\begin{cases} y = -\dfrac{3}{2}x + 1 \\ -2y + 2 = 3x \end{cases}$

85. $\begin{cases} 7y = -2x - 1 \\ 7x = 2y + 23 \end{cases}$

86. $\begin{cases} 5x + 9y - 14 = 0 \\ 3x + 2y - 5 = 0 \end{cases}$

87. $\begin{cases} y = -\dfrac{5}{16}x + 10 \\ y = \dfrac{5}{16}x - 10 \end{cases}$

88. $\begin{cases} y = -\dfrac{6}{5}x + 12 \\ x = 6 \end{cases}$

89. $\begin{cases} 2(x - 3) + y = 0 \\ 3(2x + y - 1) = 15 \end{cases}$

90. $\begin{cases} 3 - 2(x - y) = -3 \\ 4x - 3(y + 1) = 8 \end{cases}$

91. $\begin{cases} 2(x + 1) = 3(2y - 1) - 21 \\ 3(x + 2) = 1 - (3y - 2) \end{cases}$

92. $\begin{cases} \dfrac{x}{2} - \dfrac{y}{3} = -7 \\ \dfrac{x}{3} - \dfrac{y}{2} = -8 \end{cases}$

93. $\begin{cases} -\dfrac{1}{7}x + y = -\dfrac{2}{3} \\ -\dfrac{1}{14}x + \dfrac{1}{2}y = \dfrac{1}{3} \end{cases}$

94. $\begin{cases} \frac{x}{4} - \frac{y}{2} = \frac{3}{4} \\ \frac{x}{3} + \frac{y}{6} = \frac{1}{6} \end{cases}$

95. $\begin{cases} y = -\frac{5}{3}x + \frac{1}{2} \\ \frac{1}{3}x + \frac{1}{5}y = \frac{1}{10} \end{cases}$

96. $\begin{cases} \frac{1}{15}x - \frac{1}{12}y = \frac{1}{3} \\ -\frac{3}{10}x + \frac{3}{8}y = -\frac{3}{2} \end{cases}$

97. $\begin{cases} 0.2x - 0.05y = 0.43 \\ 0.3x + 0.1y = -0.3 \end{cases}$

98. $\begin{cases} 0.1x + 0.3y = 0.3 \\ 0.05x - 0.5y = -0.63 \end{cases}$

99. $\begin{cases} 0.15x - 0.25y = -0.3 \\ -0.75x + 1.25y = -4 \end{cases}$

100. $\begin{cases} -0.15x + 1.25y = 0.4 \\ -0.03x + 0.25y = 0.08 \end{cases}$

PART D: DISCUSSION BOARD

101. Explain to a beginning algebra student how to choose a method for solving a system of two linear equations. Also, explain what solutions look like and why.

102. Make up your own linear system with two variables and solve it using all three methods. Explain which method was preferable in your exercise.

ANSWERS

1. (-2, 11)
3. (2, 2)
5. $\emptyset$
7. $(x,\ 2x+3)$
9. (4, -2)
11. (3, -2)
13. (3, -4)
15. $\left(-\frac{8}{5}, -\frac{7}{10}\right)$
17. $(x, -\frac{1}{2}x - 3)$
19. $\emptyset$
21. (2, -3)
23. $\left(\frac{1}{2}, 0\right)$
25. $\emptyset$
27. (1, 1)
29. $\left(-\frac{11}{10}, -\frac{2}{5}\right)$
31. $\left(-\frac{1}{2}, \frac{3}{4}\right)$
33. (0, 0)
35. (1, 2)
37. $\emptyset$
39. $\left(\frac{1}{3}, 1\right)$
41. (-4, 2)
43. (-8, -1)

45. $\left(-\frac{1}{3}, 1\right)$

47. $\varnothing$

49. $\left(1, \frac{1}{4}\right)$

51. (−1, −2)

53. (−28, 0)

55. $\left(-\frac{1}{2}, -1\right)$

57. (1, 2)

59. (−1, −4)

61. (−5, 2)

63. $(x, \frac{1}{2}x - 2)$

65. $\left(-\frac{3}{10}, -\frac{13}{15}\right)$

67. $\left(\frac{3}{4}, -2\right)$

69. (120, 77)

71. $\left(3, -\frac{9}{4}\right)$

73. $\varnothing$

75. (0.5, −3)

77. $\left(\frac{1}{2}, -\frac{1}{3}\right)$

79. $\left(-10, \frac{5}{3}\right)$

81. $\left(-9, \frac{1}{2}\right)$

83. (−2, −10)

85. (3, −1)

87. (32, 0)
89. $(x, -2x + 6)$
91. (−4, 3)
93. $\varnothing$
95. $\left(x, -\frac{5}{3}x + \frac{1}{2}\right)$
97. (0.8, −5.4)
99. Ø
101. Answer may vary

3.3 Applications of Linear Systems with Two Variables

LEARNING OBJECTIVES

1. Set up and solve applications involving relationships between two variables.
2. Set up and solve mixture problems.
3. Set up and solve uniform motion problems (distance problems).

Problems Involving Relationships between Two Variables

If we translate an application to a mathematical setup using two variables, then we need to form a linear system with two equations. Setting up word problems with two variables often simplifies the entire process, particularly when the relationships between the variables are not so clear.

Example 1

The sum of 4 times a larger integer and 5 times a smaller integer is 7. When twice the smaller integer is subtracted from 3 times the larger, the result is 11. Find the integers.

Solution:

Begin by assigning variables to the larger and smaller integer.

Let x represent the larger integer.

Let y represent the smaller integer.

When using two variables, we need to set up two equations. The first sentence describes a sum and the second sentence describes a difference.

$$\overbrace{4x}^{4\text{ times a larger}} + \overbrace{5y}^{5\text{ times a smaller}} = 7$$

$$\overbrace{3x}^{3\text{ times the larger}} - \overbrace{2y}^{\text{twice the smaller}} = 11$$

This leads to the following system:

$$\begin{cases} 4x + 5y = 7 \\ 3x - 2y = 11 \end{cases}$$

Solve using the elimination method. To eliminate the variable y multiply the first equation by 2 and the second by 5.

$$\begin{cases} 4x + 5y = 7 \\ 3x - 2y = 11 \end{cases} \begin{array}{c} \overset{\times 2}{\Rightarrow} \\ \underset{\times 5}{\Rightarrow} \end{array} \begin{cases} 8x + 10y = 14 \\ 15x - 10y = 55 \end{cases}$$

Add the equations in the equivalent system and solve for x.

$$\begin{aligned} 8x \; + 10y &= 14 \\ \underline{+\,15x \; - 10y} &\underline{= 55} \\ 23x \qquad\quad &= 69 \\ x &= \frac{69}{23} \\ x &= 3 \end{aligned}$$

Back substitute to find y.

$$\begin{aligned} 4x + 5y &= 7 \\ 4\,(3) + 5y &= 7 \\ 12 + 5y &= 7 \\ 5y &= -5 \\ y &= -1 \end{aligned}$$

Answer: The larger integer is 3 and the smaller integer is -1.

Try this! An integer is 1 less than twice that of another. If their sum is 20, find the integers.

Answer: The two integers are 7 and 13.

(click to see video)

Next consider applications involving simple interest and money.

Example 2

A total of $12,800 was invested in two accounts. Part was invested in a CD at a $3\frac{1}{8}$%annual interest rate and part was invested in a money market fund at a $4\frac{3}{4}$%annual interest rate. If the total simple interest for one year was $465, then how much was invested in each account?

Solution:

Begin by identifying two variables.

Let x represent the amount invested at $3\frac{1}{8}\% = 3.125\% = 0.03125$

Let y represent the amount invested at $4\frac{3}{4}\% = 4.75\% = 0.0475$

The total amount in both accounts can be expressed as

$$x + y = 12{,}800$$

To set up a second equation, use the fact that the total interest was $465. Recall that the interest for one year is the interest rate times the principal $(I = prt = pr \cdot 1 = pr)$. Use this to add the interest in both accounts. Be sure to use the decimal equivalents for the interest rates given as percentages.

$$\begin{array}{ccccc} \textit{interest from the CD} & + & \textit{interest from the fund} & = & \textit{total interest} \\ 0.03125x & + & 0.0475y & = & 465 \end{array}$$

These two equations together form the following linear system:

$$\begin{cases} x + y = 12{,}800 \\ 0.03125x + 0.0475y = 465 \end{cases}$$

Eliminate *x* by multiplying the first equation by −0.03125.

$$\begin{cases} x + y = 12{,}800 \\ 0.03125x + 0.0475y = 465 \end{cases} \overset{\times(-0.03125)}{\Rightarrow} \begin{cases} -0.03125x - 0.03125y = -400 \\ 0.03125x + 0.0475y = 465 \end{cases}$$

Next, add the resulting equations.

$$\begin{aligned} -0.03125x - 0.03125y &= -400 \\ + \quad 0.03125x + 0.0475y &= 465 \\ \hline 0.01625y &= 65 \\ y &= \frac{65}{0.01625} \\ y &= 4{,}000 \end{aligned}$$

Back substitute to find *x*.

$$\begin{aligned} x + y &= 12{,}800 \\ x + 4000 &= 12{,}800 \\ x &= 8{,}800 \end{aligned}$$

Answer: \$4,000 was invested at $4\frac{3}{4}$ %and \$8,800 was invested at $3\frac{1}{8}$ %

Example 3

A jar consisting of only nickels and dimes contains 58 coins. If the total value is \$4.20, how many of each coin is in the jar?

Solution:

Let n represent the number of nickels in the jar.

Let d represent the number of dimes in the jar.

The total number of coins in the jar can be expressed using the following equation:

$$n + d = 58$$

Next, use the value of each coin to determine the total value \$4.20.

$$\begin{array}{ccccc} \textit{value of nickels} & + & \textit{value of dimes} & = & \textit{total value} \\ 0.05n & + & 0.10d & = & 4.20 \end{array}$$

This leads us to following linear system:

$$\begin{cases} n + d = 58 \\ 0.05n + 0.10d = 4.20 \end{cases}$$

Here we will solve using the substitution method. In the first equation, we can solve for n.

$$\begin{cases} n + d = 58 \\ 0.05n + 0.10d = 4.20 \end{cases} \Rightarrow \quad n = 58 - d$$

Substitute $n = 58 - d$ into the second equation and solve for d.

$$\begin{aligned} 0.05\left(58 - d\right) + 0.10d &= 4.20 \\ 2.9 - 0.05d + 0.10d &= 4.20 \\ 2.9 + 0.05d &= 4.20 \\ 0.05d &= 1.3 \\ d &= 26 \end{aligned}$$

Now back substitute to find the number of nickels.

$$\begin{aligned} n &= 58 - d \\ &= 58 - 26 \\ &= 32 \end{aligned}$$

Answer: There are 32 nickels and 26 dimes in the jar.

Try this! Joey has a jar full of 40 coins consisting of only quarters and nickels. If the total value is $5.00, how many of each coin does Joey have?

Answer: Joey has 15 quarters and 25 nickels.

(click to see video)

Mixture Problems

Mixture problems often include a percentage and some total amount. It is important to make a distinction between these two types of quantities. For example, if a problem states that a 20-ounce container is filled with a 2% saline (salt) solution, then this means that the container is filled with a mixture of salt and water as follows:

	Percentage	**Amount**
Salt	2% = 0.02	0.02(20 ounces) = 0.4 ounces
Water	98% = 0.98	0.98(20 ounces) = 19.6 ounces

In other words, we multiply the percentage times the total to get the amount of each part of the mixture.

Example 4

A 1.8% saline solution is to be combined and mixed with a 3.2% saline solution to produce 35 ounces of a 2.2% saline solution. How much of each is needed?

Solution:

Let x represent the amount of 1.8% saline solution needed.

Let y represent the amount of 3.2% saline solution needed.

The total amount of saline solution needed is 35 ounces. This leads to one equation,

$$x + y = 35$$

The second equation adds up the amount of salt in the correct percentages. The amount of salt is obtained by multiplying the percentage times the amount, where the variables x and y represent the amounts of the solutions. The amount of salt in the end solution is 2.2% of the 35 ounces, or .022(35).

$$\begin{array}{ccccc} \textit{salt in } 1.8\% \textit{ solution} & + & \textit{salt in } 3.2\% \textit{ solution} & = & \textit{salt in the end solution} \\ 0.018x & + & 0.032y & = & 0.022(35) \end{array}$$

The algebraic setup consists of both equations presented as a system:

$$\begin{cases} x + y = 35 \\ 0.018x + 0.032y = 0.022(35) \end{cases}$$

Solve.

$$\begin{cases} x + y = 35 \\ 0.018x + 0.032y = 0.022(35) \end{cases} \xRightarrow{\times(-0.018)} \begin{cases} -0.018x - 0.018y = -0.63 \\ 0.018x + 0.032y = 0.77 \end{cases}$$

Add the resulting equations together

$$\begin{array}{rrl} & -0.018x - 0.018y = & -0.63 \\ + & \underline{0.018x + 0.032y =} & \underline{0.77} \\ \hline & 0.014y = & 0.14 \\ & y = & \dfrac{0.14}{0.014} \\ & y = & 10 \end{array}$$

Back substitute to find *x*.

$$\begin{aligned} x + y &= 35 \\ x + 10 &= 35 \\ x &= 25 \end{aligned}$$

Answer: We need 25 ounces of the 1.8% saline solution and 10 ounces of the 3.2% saline solution.

Example 5

An 80% antifreeze concentrate is to be mixed with water to produce a 48-liter mixture containing 25% antifreeze. How much water and antifreeze concentrate is needed?

Solution:

Let x represent the amount of 80% antifreeze concentrate needed.

Let y represent the amount of water needed.

The total amount of the mixture must be 48 liters.

$$x + y = 48$$

The second equation adds up the amount of antifreeze from each solution in the correct percentages. The amount of antifreeze in the end result is 25% of 48 liters, or 0.25(48).

$$\begin{array}{ccccc} \textit{antifreeze in } 80\% \textit{ concentrate} & + & \textit{antifreeze in water} & = & \textit{antifreeze in the end mixture} \\ 0.80x & + & 0 & = & 0.25(48) \end{array}$$

Now we can form a system of two linear equations and two variables as follows:

$$\left\{\begin{aligned} x + y &= 48 \\ 0.80x &= 0.25(48) \end{aligned}\right. \Rightarrow \left\{\begin{aligned} x + y &= 48 \\ 0.80x &= 12 \end{aligned}\right.$$

Use the second equation to find *x*:

$$\begin{aligned} 0.80x &= 12 \\ x &= \frac{12}{0.80} \\ x &= 15 \end{aligned}$$

Back substitute to find *y*.

$$\begin{aligned} x + y &= 48 \\ {\color{green}15} + y &= 48 \\ y &= 33 \end{aligned}$$

Answer: We need to mix 33 liters of water with 15 liters of antifreeze concentrate.

Try this! A chemist wishes to create 100 ml of a solution with 12% acid content. He uses two types of stock solutions, one with 30% acid content and another with 10% acid content. How much of each does he need?

Answer: The chemist will need to mix 10 ml of the 30% acid solution with 90 ml of the 10% acid solution.

(click to see video)

Uniform Motion Problems (Distance Problems)

Recall that the distance traveled is equal to the average rate times the time traveled at that rate, $D = r \cdot t$. These uniform motion problems usually have a lot of data, so it helps to first organize that data in a chart and then set up a linear system. In this section, you are encouraged to use two variables.

Example 6

An executive traveled a total of 4 hours and 875 miles by car and by plane. Driving to the airport by car, she averaged 50 miles per hour. In the air, the plane averaged 320 miles per hour. How long did it take her to drive to the airport?

Solution:

We are asked to find the time it takes her to drive to the airport; this indicates that time is the unknown quantity.

Let x represent the time it took to drive to the airport.

Let y represent the time spent in the air.

Fill in the chart with the given information.

	Distance =	***Rate*** ×	***Time***
Travel by car		50 mph	x
Travel by air		320 mph	y
Total	875 mi		4 hours

Use the formula $D = r \cdot t$ to fill in the unknown distances.

$$\begin{aligned}&\text{Distance traveled in the car:} \quad D = r \cdot t = 50 \cdot x\\&\text{Distance traveled in the air:} \quad D = r \cdot t = 320 \cdot y\end{aligned}$$

	Distance =	***Rate*** ×	***Time***
Travel by car	$50x$	50 mph	x
Travel by air	$320y$	320 mph	y
Total	875 mi		4 hours

The distance column and the time column of the chart help us to set up the following linear system.

	Distance =	*Rate* ×	*Time*
Travel by car	$50x$	50 mph	x
Travel by air	$320y$	320 mph	y
Total	875 mi		4 hours

$$50x + 320y = 875 \qquad x + y = 4$$

$$\begin{cases} x + y = 4 & \leftarrow \textit{total time traveled} \\ 50x + 320y = 875 & \leftarrow \textit{total distance traveled} \end{cases}$$

Solve.

$$\begin{cases} x + y = 4 \\ 50x + 320y = 875 \end{cases} \stackrel{\times(-50)}{\Rightarrow} \begin{cases} -50x - 50y = -200 \\ 50x + 320y = 875 \end{cases}$$

$$\begin{array}{rr} & -50x - 50y = -200 \\ + & 50x + 320y = 875 \\ \hline & 270y = 675 \\ & y = \dfrac{675}{270} \\ & y = \dfrac{5}{2} \end{array}$$

Now back substitute to find the time x it took to drive to the airport:

$$
\begin{aligned}
x + y &= 4 \\
x + \frac{5}{2} &= 4 \\
x &= \frac{8}{2} - \frac{5}{2} \\
x &= \frac{3}{2}
\end{aligned}
$$

Answer: It took her $1\frac{1}{2}$ hours to drive to the airport.

It is not always the case that time is the unknown quantity. Read the problem carefully and identify what you are asked to find; this defines your variables.

16

17

18

16. Applications involving simple interest and money.
17. Applications involving a mixture of amounts usually given as a percentage of some total.
18. Applications relating distance, average rate, and time.

Example 7

Flying with the wind, a light aircraft traveled 240 miles in 2 hours. The aircraft then turned against the wind and traveled another 135 miles in $1\frac{1}{2}$ hours. Find the speed of the airplane and the speed of the wind.

Solution:

Begin by identifying variables.

Let x represent the speed of the airplane.

Let w represent the speed of the wind.

Use the following chart to organize the data:

	Distance =	*Rate* ×	*Time*
Flight with wind	240 mi		2 hrs
Flight against wind	135 mi		1.5 hrs
Total			

With the wind, the airplane's total speed is $x + w$. Flying against the wind, the total speed is $x - w$.

	Distance =	*Rate* ×	*Time*
Flight with wind	240 mi	$x+w$	2 hrs
Flight against wind	135 mi	$x-w$	1.5 hrs
Total			

Use the rows of the chart along with the formula $D = r \cdot t$ to construct a linear system that models this problem. Take care to group the quantities that represent the rate in parentheses.

	Distance =	*Rate* ×	*Time*	
Flight with wind	240 mi	$x+w$	2 hrs	$240 = (x+w)\cdot 2$
Flight against wind	135 mi	$x-w$	1.5 hrs	$135 = (x-w)\cdot 1.5$
Total				

$$\begin{cases} 240 = (x+w)\cdot 2 & \leftarrow \textit{distance traveled with the wind} \\ 135 = (x-w)\cdot 1.5 & \leftarrow \textit{distance traveled against the wind} \end{cases}$$

If we divide both sides of the first equation by 2 and both sides of the second equation by 1.5, then we obtain the following equivalent system:

$$\begin{cases} 240 = (x+w)\cdot 2 \\ 135 = (x-w)\cdot 1.5 \end{cases} \begin{matrix} \stackrel{\div 2}{\Longrightarrow} \\ \underset{\div 1.5}{\Longrightarrow} \end{matrix} \begin{cases} 120 = x+w \\ 90 = x-w \end{cases}$$

Here w is lined up to eliminate.

$$\begin{aligned} x + w &= 120 \\ \underline{+\, x - w} &\underline{= 90} \\ 2x &= 210 \\ x &= \frac{210}{2} \\ x &= 105 \end{aligned}$$

Back substitute.

$$\begin{aligned} x + w &= 120 \\ 105 + w &= 120 \\ w &= 15 \end{aligned}$$

Answer: The speed of the airplane is 105 miles per hour and the speed of the wind is 15 miles per hour.

Try this! A boat traveled 27 miles downstream in 2 hours. On the return trip, which was against the current, the boat was only able to travel 21 miles in 2 hours. What were the speeds of the boat and of the current?

Answer: The speed of the boat was 12 miles per hour and the speed of the current was 1.5 miles per hour.

(click to see video)

KEY TAKEAWAYS

- Use two variables as a means to simplify the algebraic setup of applications where the relationship between unknowns is unclear.
- Carefully read the problem several times. If two variables are used, then remember that you need to set up two linear equations in order to solve the problem.
- Be sure to answer the question in sentence form and include the correct units for the answer.

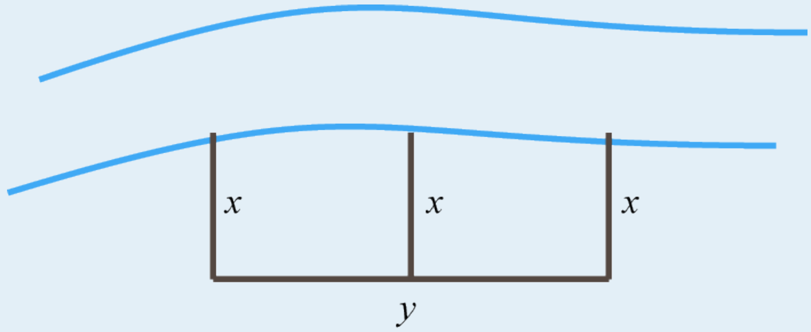
TOPIC EXERCISES

PART A: APPLICATIONS INVOLVING TWO VARIABLES

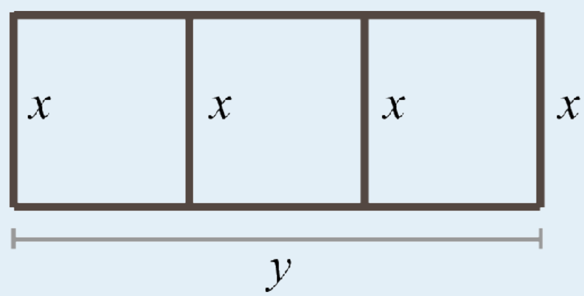
Set up a linear system and solve.

1. The sum of two integers is 45. The larger integer is 3 less than twice the smaller. Find the two integers.
2. The sum of two integers is 126. The larger is 18 less than 5 times the smaller. Find the two integers.
3. The sum of two integers is 41. When 3 times the smaller is subtracted from the larger the result is 17. Find the two integers.
4. The sum of two integers is 46. When the larger is subtracted from twice the smaller the result is 2. Find the two integers.
5. The difference of two integers is 11. When twice the larger is subtracted from 3 times the smaller, the result is 3. Find the integers.
6. The difference of two integers is 6. The sum of twice the smaller and the larger is 72. Find the integers.
7. The sum of 3 times a larger integer and 2 times a smaller is 15. When 3 times the smaller integer is subtracted from twice the larger, the result is 23. Find the integers.
8. The sum of twice a larger integer and 3 times a smaller is 10. When the 4 times the smaller integer is added to the larger, the result is 0. Find the integers.
9. The difference of twice a smaller integer and 7 times a larger is 4. When 5 times the larger integer is subtracted from 3 times the smaller, the result is −5. Find the integers.
10. The difference of a smaller integer and twice a larger is 0. When 3 times the larger integer is subtracted from 2 times the smaller, the result is −5. Find the integers.
11. The length of a rectangle is 5 more than twice its width. If the perimeter measures 46 meters, then find the dimensions of the rectangle.
12. The width of a rectangle is 2 centimeters less than one-half its length. If the perimeter measures 62 centimeters, then find the dimensions of the rectangle.

13. A partitioned rectangular pen next to a river is constructed with a total 136 feet of fencing (see illustration). If the outer fencing measures 114 feet, then find the dimensions of the pen.



14. A partitioned rectangular pen is constructed with a total 168 feet of fencing (see illustration). If the perimeter measures 138 feet, then find the dimensions of the pen.



15. Find a and b such that the system $\begin{cases} ax + by = 8 \\ bx + ay = 7 \end{cases}$ has solution $(2, 1)$.

(Hint: Substitute the given x- and y-values and solve the resulting linear system in terms of a and b.)

16. Find a and b such that the system $\begin{cases} ax - by = 11 \\ bx + ay = 13 \end{cases}$ has solution $(3, -1)$.

17. A line passes through two points $(5, -9)$ and $(-3, 7)$. Use these points and $y = mx + b$ to construct a system of two linear equations in terms of m and b and solve it.

18. A line passes through two points $(2, 7)$ and $\left(\frac{1}{2}, -2\right)$. Use these points and $y = mx + b$ to construct a system of two linear equations in terms of m and b and solve it.

19. A $5,200 principal is invested in two accounts, one earning 3% interest and another earning 6% interest. If the total interest for the year is $210, then how much is invested in each account?
20. Harry's $2,200 savings is in two accounts. One account earns 2% annual interest and the other earns 4%. His total interest for the year is $69. How much does he have in each account?
21. Janine has two savings accounts totaling $6,500. One account earns $2\frac{3}{4}\%$ annual interest and the other earns $3\frac{1}{2}\%$. If her total interest for the year is $211, then how much is in each account?
22. Margaret has her total savings of $24,200 in two different CD accounts. One CD earns 4.6% interest and another earns 3.4% interest. If her total interest for the year is $1,007.60, then how much does she have in each CD account?
23. Last year Mandy earned twice as much interest in her Money Market fund as she did in her regular savings account. The total interest from the two accounts was $246. How much interest did she earn in each account?
24. A small business invested $120,000 in two accounts. The account earning 4% annual interest yielded twice as much interest as the account earning 3% annual interest. How much was invested in each account?
25. Sally earns $1,000 per month plus a commission of 2% of sales. Jane earns $200 per month plus 6% of her sales. At what monthly sales figure will both Sally and Jane earn the same amount of pay?
26. The cost of producing specialty book shelves includes an initial set-up fee of $1,200 plus an additional $20 per unit produced. Each shelf can be sold for $60 per unit. Find the number of units that must be produced and sold where the costs equal the revenue generated.
27. Jim was able to purchase a pizza for $12.35 with quarters and dimes. If he uses 71 coins to buy the pizza, then how many of each did he have?
28. A cash register contains $5 bills and $10 bills with a total value of $350. If there are 46 bills total, then how many of each does the register contain?
29. Two families bought tickets for the home basketball game. One family ordered 2 adult tickets and 4 children's tickets for a total of $36.00. Another family ordered 3 adult tickets and 2 children's tickets for a total of $32.00. How much did each ticket cost?

30. Two friends found shirts and shorts on sale at a flea market. One bought 4 shirts and 2 shorts for a total of \$28.00. The other bought 3 shirts and 3 shorts for a total of \$30.75. How much was each shirt and each pair of shorts?

31. A community theater sold 140 tickets to the evening musical for a total of \$1,540. Each adult ticket was sold for \$12 and each child ticket was sold for \$8. How many adult tickets were sold?

32. The campus bookstore sells graphing calculators for \$110 and scientific calculators for \$16. On the first day of classes 50 calculators were sold for a total of \$1,646. How many of each were sold?

33. A jar consisting of only nickels and quarters contains 70 coins. If the total value is \$9.10, how many of each coin are in the jar?

34. Jill has \$9.20 worth of dimes and quarters. If there are 68 coins in total, how many of each does she have?

PART B: MIXTURE PROBLEMS

Set up a linear system and solve.

35. A 17% acid solution is to be mixed with a 9% acid solution to produce 8 gallons of a 10% acid solution. How much of each is needed?

36. A nurse wishes to obtain 28 ounces of a 1.5% saline solution. How much of a 1% saline solution must she mix with a 4.5% saline solution to achieve the desired mixture?

37. A customer ordered 4 pounds of a mixed peanut product containing 12% cashews. The inventory consists of only two mixes containing 10% and 26% cashews. How much of each type must be mixed to fill the order?

38. One alcohol solution contains 10% alcohol and another contains 25% alcohol. How much of each should be mixed together to obtain 2 gallons of a 13.75% alcohol solution?

39. How much cleaning fluid concentrate, with 60% alcohol content, must be mixed with water to obtain a 24-ounce mixture with 15% alcohol content?

40. How many pounds of pure peanuts must be combined with a 20% peanut mix to produce 2 pounds of a 50% peanut mix?

41. A 50% fruit juice concentrate can be purchased wholesale. Best taste is achieved when water is mixed with the concentrate in such a way as to obtain

a 15% fruit juice mixture. How much water and concentrate is needed to make a 60-ounce fruit juice drink?

42. Pure sugar is to be mixed with a fruit salad containing 10% sugar to produce 65 ounces of a salad containing 18% sugar. How much pure sugar is required?

43. A custom aluminum alloy is created by mixing 150 grams of a 15% aluminum alloy and 350 grams of a 55% aluminum alloy. What percentage of aluminum is in the resulting mixture?

44. A research assistant mixed 500 milliliters of a solution that contained a 12% acid with 300 milliliters of water. What percentage of acid is in the resulting solution?

PART C: UNIFORM MOTION PROBLEMS

Set up a linear system and solve.

45. The two legs of a 432-mile trip took 8 hours. The average speed for the first leg of the trip was 52 miles per hour and the average speed for the second leg of the trip was 60 miles per hour. How long did each leg of the trip take?

46. Jerry took two buses on the 265-mile trip from Los Angeles to Las Vegas. The first bus averaged 55 miles per hour and the second bus was able to average 50 miles per hour. If the total trip took 5 hours, then how long was spent in each bus?

47. An executive was able to average 48 miles per hour to the airport in her car and then board an airplane that averaged 210 miles per hour. The 549-mile business trip took 3 hours. How long did it take her to drive to the airport?

48. Joe spends 1 hour each morning exercising by jogging and then cycling for a total of 15 miles. He is able to average 6 miles per hour jogging and 18 miles per hour cycling. How long does he spend jogging each morning?

49. Swimming with the current Jack can swim 2.5 miles in $\frac{1}{2}$ hour. Swimming back, against the same current, he can only swim 2 miles in the same amount of time. How fast is the current?

50. A light aircraft flying with the wind can travel 180 miles in $1\frac{1}{2}$ hours. The aircraft can fly the same distance against the wind in 2 hours. Find the speed of the wind.

51. A light airplane flying with the wind can travel 600 miles in 4 hours. On the return trip, against the wind, it will take 5 hours. What are the speeds of the airplane and of the wind?

52. A boat can travel 15 miles with the current downstream in $1\frac{1}{4}$ hours. Returning upstream against the current, the boat can only travel $8\frac{3}{4}$ miles in the same amount of time. Find the speed of the current.

53. Mary jogged the trail from her car to the cabin at the rate of 6 miles per hour. She then walked back to her car at a rate of 4 miles per hour. If the entire trip took 1 hour, then how long did it take her to walk back to her car?

54. Two trains leave the station traveling in opposite directions. One train is 8 miles per hour faster than the other and in $2\frac{1}{2}$ hours they are 230 miles apart. Determine the average speed of each train.

55. Two trains leave the station traveling in opposite directions. One train is 12 miles per hour faster than the other and in 3 hours they are 300 miles apart. Determine the average speed of each train.

56. A jogger can sustain an average running rate of 8 miles per hour to his destination and 6 miles an hour on the return trip. Find the total distance the jogger ran if the total time running was $1\frac{3}{4}$ hour.

PART E: DISCUSSION BOARD

57. Compose a number or money problem of your own and share it on the discussion board.

58. Compose a mixture problem of your own and share it on the discussion board.

59. Compose a uniform motion problem of your own and share it on the discussion board.

ANSWERS

1. The integers are 16 and 29.
3. The integers are 6 and 35.
5. The integers are 25 and 36.
7. The integers are −3 and 7.
9. The integers are −5 and −2.
11. Length: 17 meters; width: 6 meters
13. Width: 22 feet; length: 70 feet
15. $a = 3, b = 2$
17. $m = -2, b = 1$
19. \$3,400 at 3% and \$1,800 at 6%
21. \$2,200 at $2\frac{3}{4}\%$ and \$4,300 at $3\frac{1}{2}\%$
23. Savings: \$82; Money Market: \$164.
25. \$20,000
27. 35 quarters and 36 dimes
29. Adults \$7.00 each and children \$5.50 each.
31. 105 adult tickets were sold.
33. The jar contains 42 nickels and 28 quarters.
35. 7 gallons of the 9% acid solution and 1 gallon of the 17% acid solution
37. 3.5 pounds of the 10% cashew mix and 0.5 pounds of the 26% cashew mix
39. 6 ounces of cleaning fluid concentrate
41. 18 ounces of fruit juice concentrate and 42 ounces of water
43. 43%
45. The first leg of the trip took 6 hours and the second leg took 2 hours.
47. It took her $\frac{1}{2}$ hour to drive to the airport.
49. 0.5 miles per hour.

51. Airplane: 135 miles per hour; wind: 15 miles per hour
53. $\frac{3}{5}$ hour
55. One train averaged 44 miles per hour and the other averaged 56 miles per hour.
57. Answer may vary
59. Answer may vary

3.4 Solving Linear Systems with Three Variables

LEARNING OBJECTIVES

1. Check solutions to linear systems with three variables.
2. Solve linear systems with three variables by elimination.
3. Identify dependent and inconsistent systems.
4. Solve applications involving three unknowns.

Solutions to Linear Systems with Three Variables

Real-world applications are often modeled using more than one variable and more than one equation. In this section, we will study linear systems consisting of three linear equations each with three variables. For example,

$$\begin{cases} 3x + 2y - z = -7 & (1) \\ 6x - y + 3z = -4 & (2) \\ x + 10y - 2z = 2 & (3) \end{cases}$$

A solution to such a linear system is an **ordered triple**[19] (x, y, z) that solves all of the equations. In this case, (−2, 1, 3) is the only solution. To check that an ordered triple is a solution, substitute in the corresponding x-, y-, and z-values and then simplify to see if you obtain a true statement from all three equations.

19. Triples (x, y, z) that identify position relative to the origin in three-dimensional space.

Check: $(-2, 1, 3)$		
Equation (1) : $3x + 2y + z = -7$ $3(-2) + 2(1) - (3) = -7$ $-6 + 2 - 3 = -7$ $-7 = -7$ ✓	*Equation* (2) : $6x - y + 3z = -4$ $6(-2) - (1) - 3(3) = -4$ $-12 - 1 - 9 = -4$ $-4 = -4$ ✓	*Equatio* $x + 10y$ $(-2) + 10(1)$ - $-2 + 1$

Because the ordered triple satisfies all three equations we conclude that it is indeed a solution.

Example 1

Determine whether or not $\left(1, 4, \frac{4}{3}\right)$ is a solution to the following linear system:

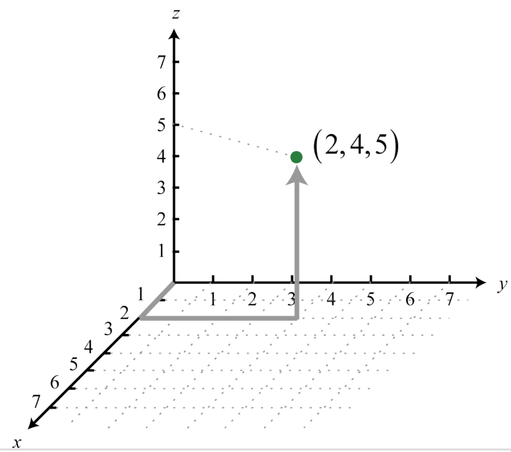
$$\begin{cases} 9x + y - 6z = 5 \\ -6x - 3y + 3z = -14 \\ 3x + 2y - 7z = 15 \end{cases}.$$

Solution:

Check: $\left(1, 4, \frac{4}{3}\right)$		
Equation (1) : $9x + y - 6z = 5$ $9(1) + (4) - 6\left(\frac{4}{3}\right) = 5$ $9 + 4 - 8 = 5$ $5 = 5$ ✓	*Equation* (2) : $-6x - 3y + 3z = -14$ $6(1) - 3(4) + 3\left(\frac{4}{3}\right) = 14$ $-6 - 12 + 4 = -14$ $-14 = -14$ ✓	*Equat* $3x +$ $3(1) + 2(4) -$ $3 +$

Answer: The point does not satisfy all of the equations and thus is not a solution.

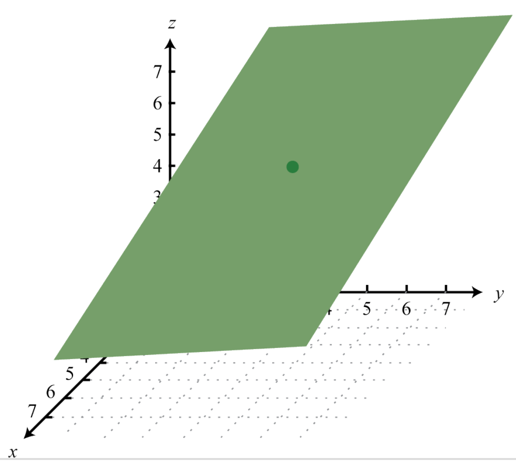
An ordered triple such as $(2, 4, 5)$ can be graphed in three-dimensional space as follows:



The ordered triple indicates position relative to the origin (0, 0, 0), in this case, 2 units along the x-axis, 4 units parallel to the y-axis, and 5 units parallel to the z-axis. A **linear equation with three variables**[20] is in standard form if

$$ax + by + cz = d$$

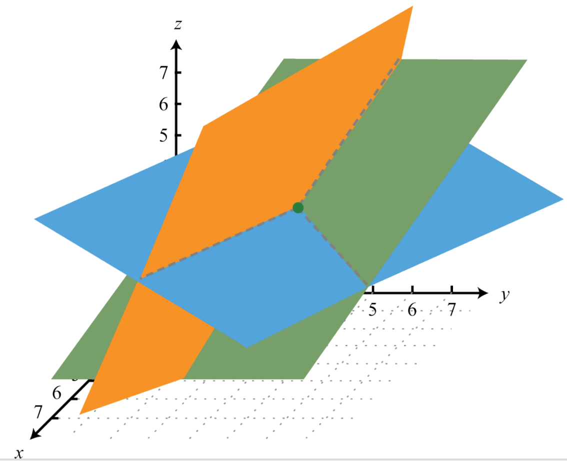
where a, b, c, and d are real numbers. For example, $6x + y + 2z = 26$ is in standard form. Solving for z, we obtain $z = -3x - \frac{1}{2}y + 13$ and can consider both x and y to be the independent variables. When graphed in three-dimensional space, its graph will form a straight flat surface called a **plane**[21].



Therefore, the graph of a system of three linear equations and three unknowns will consist of three planes in space. If there is a simultaneous solution, the system is consistent and the solution corresponds to a point where the three planes intersect.

20. An equation that can be written in the standard form $ax + by + cz = d$ where a, b, c, and d are real numbers.

21. Any flat two-dimensional surface.



Graphing planes in three-dimensional space is not within the scope of this textbook. However, it is always important to understand the geometric interpretation.

Try this! Determine whether or not (3, -1, 2) a solution to the system:

$$\begin{cases} 2x - 3y - z = 7 \\ 3x + 5y - 3z = -2. \\ 4x - y + 2z = 17 \end{cases}$$

Answer: Yes, it is a solution.

(click to see video)

Solve Linear Systems with Three Variables by Elimination

In this section, the elimination method is used to solve systems of three linear equations with three variables. The idea is to eliminate one of the variables and resolve the original system into a system of two linear equations, after which we can then solve as usual. The steps are outlined in the following example.

Example 2

Solve: $\begin{cases} 3x + 2y - z = -7 & \color{#1a7fc1}{(1)} \\ 6x - y + 3z = -4 & \color{#1a7fc1}{(2)} \\ x + 10y - 2z = 2 & \color{#1a7fc1}{(3)} \end{cases}$.

Solution:

All three equations are in standard form. If this were not the case, it would be a best practice to rewrite the equations in standard form before beginning this process.

Step 1: Choose any two of the equations and eliminate a variable. In this case, we can line up the variable z to eliminate if we group 3 times the first equation with the second equation.

$$\begin{array}{l} (1) \\ (2) \end{array} \begin{cases} 3x + 2y - z = -7 \\ 6x - y + 3z = -4 \end{cases} \overset{\times 3}{\Rightarrow} \begin{cases} 9x + 6y - 3z = -21 \\ 6x - y + 3z = -4 \end{cases}$$

Next, add the equations together.

$$\begin{array}{rrrl} & 9x + 6y & \color{red}{-\ 3z} & = 21 \\ + & 6x - y & \color{red}{+\ 3z} & = -4 \\ \hline & 15x + 5y & & = -25 \quad \checkmark \end{array}$$

Step 2: Choose any other two equations and eliminate the same variable. We can line up z to eliminate again if we group -2 times the first equation with the third equation.

$$\begin{array}{l} (1) \\ (3) \end{array} \begin{cases} 3x + 2y - z = -7 \\ x + 10y - 2z = 2 \end{cases} \overset{\times(-2)}{\Rightarrow} \begin{cases} -6x - 4y + 2z = 14 \\ x + 10y - 2z = 2 \end{cases}$$

And then add,

$$\begin{array}{rrrl} & -6x - 4y & + 2z & = 14 \\ + & x + 10y & - 2z & = 2 \\ \hline & -5x + 6y & & = 16 \quad \checkmark \end{array}$$

Step 3: Solve the resulting system of two equations with two unknowns. Here we solve by elimination. Multiply the second equation by 3 to line up the variable *x* to eliminate.

$$\begin{cases} 15x + 5y = -25 \\ -5x + 6y = 16 \end{cases} \underset{\times 3}{\Rightarrow} \begin{cases} 15x + 5y = -25 \\ -15x + 18y = 48 \end{cases}$$

Next, add the equations together.

$$\begin{array}{rl} 15x + 5y & = -25 \\ + \; -15x + 18y & = 48 \\ \hline 23y & = 23 \\ y & = 1 \end{array}$$

Step 4: Back substitute and determine all of the coordinates. To find *x* use the following,

$$\begin{aligned} 15x + 5y &= -25 \\ 15x + 5(1) &= -25 \\ 15x &= -30 \\ x &= -2 \end{aligned}$$

Now choose one of the original equations to find z,

$$\begin{aligned}3x+2y-z&=-7 \qquad \textit{(1)}\\3(-2)+2(1)-z&=-7\\-6+2-z&=-7\\-4-z&=-7\\-z&=-3\\z&=3\end{aligned}$$

Hence the solution, presented as an ordered triple (x, y, z), is (−2, 1, 3). This is the same system that we checked in the beginning of this section.

Answer: (−2, 1, 3)

It does not matter which variable we initially choose to eliminate, as long as we eliminate it twice with two different sets of equations.

Example 3

Solve: $\begin{cases} -6x - 3y + 3z = -14 \\ 9x + y - 6z = 5 \\ 3x + 2y - 7z = 15 \end{cases}$.

Solution:

Because y has coefficient 1 in the second equation, choose to eliminate this variable. Use equations 1 and 2 to eliminate y.

$$\begin{array}{l} (1) \\ (2) \end{array} \begin{cases} -6x - 3y + 3z = -14 \\ 9x + y - 6z = 5 \end{cases} \underset{\times 3}{\Rightarrow} \begin{array}{r} \begin{cases} -6x - 3y + 3z = -14 \\ 27x + 3y - 18z = 15 \end{cases} \\ \hline 21x \quad -15z = 1 \quad \checkmark \end{array}$$

Next use equations 2 and 3 to eliminate y again.

$$\begin{array}{l} (2) \\ (3) \end{array} \begin{cases} 9x + y - 6z = 5 \\ 3x + 2y - 7z = 15 \end{cases} \overset{\times(-2)}{\Rightarrow} \begin{array}{r} \begin{cases} -18x - 2y + 12z = -10 \\ 3x + 2y - 7z = 15 \end{cases} \\ \hline -15x \quad + 5z = 5 \quad \checkmark \end{array}$$

This leaves a system of two equations with two variables x and z,

$$\begin{cases} 21x - 15z = 1 \\ -15x + 5z = 5 \end{cases}$$

Multiply the second equation by 3 and eliminate the variable z.

$$\begin{cases} 21x - 15z = 1 \\ -15x + 5z = 5 \end{cases} \underset{\times 3}{\Rightarrow} \begin{array}{r} \begin{cases} 21x - 15z = 1 \\ -45x + 15z = 15 \end{cases} \\ \hline -24x \qquad = 16 \\ x = -\frac{16}{24} \\ x = -\frac{2}{3} \end{array}$$

Now back substitute to find z.

$$\begin{aligned} 21x - 15z &= 1 \\ 21\left(-\frac{2}{3}\right) - 15z &= 1 \\ -14 - 15z &= 1 \\ -15z &= 15 \\ z &= -1 \end{aligned}$$

Finally, choose one of the original equations to find y.

$$\begin{aligned} -6x - 3y + 3z &= -14 \\ -6\left(-\frac{2}{3}\right) - 3y + 3(-1) &= -14 \\ 4 - 3y - 3 &= -14 \\ 1 - 3y &= -14 \\ -3y &= -15 \\ y &= 5 \end{aligned}$$

Answer: $\left(-\frac{2}{3}, 5, -1\right)$

Example 4

Solve: $\begin{cases} 2x + 6y + 7z = 4 \\ -3x - 4y + 5z = 12 \\ 5x + 10y - 3z = -13 \end{cases}$.

Solution:

In this example, there is no obvious choice of variable to eliminate. We choose to eliminate *x*.

$$\begin{array}{l} (1) \\ (2) \end{array} \begin{cases} 2x + 6y + 7z = 4 & \stackrel{\times 3}{\Longrightarrow} \\ -3x - 4y + 5z = 12 & \underset{\times 2}{\Longrightarrow} \end{cases} \begin{cases} 6x + 18y + 21z = 12 \\ -6x - 8y + 10z = 24 \end{cases}$$

$$\overline{\quad 10y + 31z = 36 \checkmark}$$

Next use equations 2 and 3 to eliminate *x* again.

$$\begin{array}{l} (2) \\ (3) \end{array} \begin{cases} -3x - 4y + 5z = 12 & \stackrel{\times 5}{\Longrightarrow} \\ 5x + 10y - 3z = -13 & \underset{\times 3}{\Longrightarrow} \end{cases} \begin{cases} -15x - 20y + 25z = 60 \\ 15x + 30y - 9z = -39 \end{cases}$$

$$\overline{\quad 10y + 16z = 21 \checkmark}$$

This leaves a system of two equations with two variables *y* and *z*,

$$\begin{cases} 10y + 31z = 36 \\ 10y + 16z = 21 \end{cases}$$

Multiply the first equation by −1 as a means to eliminate the variable *y*.

$$\begin{cases}10y+31z=36\\10y+16z=21\end{cases}\overset{\times(-1)}{\Rightarrow}\begin{cases}-10y-31z=-36\\10y+16z=21\end{cases}$$

$$\begin{aligned}-15z&=-15\\z&=1\end{aligned}$$

Now back substitute to find *y*.

$$\begin{aligned}10y+31z&=36\\10y+31\,(1)&=36\\10y+31&=36\\10y&=5\\y&=\frac{5}{10}\\y&=\frac{1}{2}\end{aligned}$$

Choose any one of the original equations to find *x*.

$$\begin{aligned}2x+6y+7z&=4\\2x+6\left(\frac{1}{2}\right)+7\,(1)&=4\\2x+3+7&=4\\2x+10&=4\\2x&=-6\\x&=-3\end{aligned}$$

Answer: $\left(-3, \frac{1}{2}, 1\right)$

Try this! Solve: $\begin{cases} 2x - 3y - z = 7 \\ 3x + 5y - 3z = -2. \\ 4x - y + 2z = 17 \end{cases}$

Answer: (3, −1, 2)

(click to see video)

Dependent and Inconsistent Systems

Just as with linear systems with two variables, not all linear systems with three variables have a single solution. Sometimes there are no simultaneous solutions.

Example 5

Solve the system: $\begin{cases} 4x - y + 3z = 5 \\ 21x - 4y + 18z = 7 \\ -9x + y - 9z = -8 \end{cases}$.

Solution:

In this case we choose to eliminate the variable y.

$$\begin{array}{r} (1) \\ (3) \end{array} \begin{cases} 4x - y + 3z = 5 \\ -9x + y - 9z = -8 \end{cases}$$
$$\overline{\quad -5x - 6z = -3 \quad} \checkmark$$

Next use equations 2 and 3 to eliminate y again.

$$\begin{array}{r} (2) \\ (3) \end{array} \begin{cases} 21x - 4y + 18z = 7 \\ -9x + y - 9z = -8 \end{cases} \overset{}{\underset{\times 4}{\Rightarrow}} \begin{cases} 21x - 4y + 18z = 7 \\ -36x + 4y - 36z = -32 \end{cases}$$
$$\overline{\quad -15x \quad - 18z = -25 \quad} \checkmark$$

This leaves a system of two equations with two variables x and z,

$$\begin{cases} -5x - 6z = -3 \\ -15x - 18z = -25 \end{cases}$$

Multiply the first equation by -3 and eliminate the variable z.

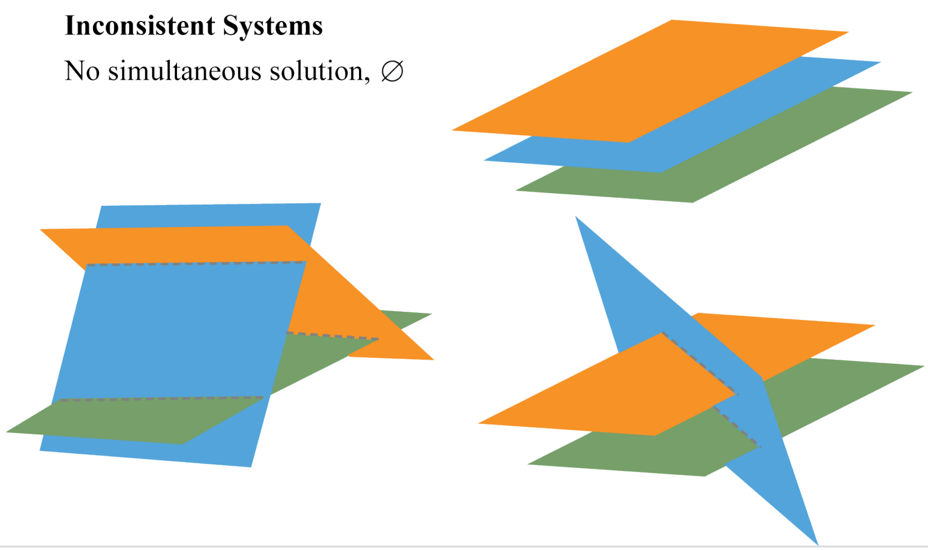
$$\begin{cases} -5x-6z=-3 \\ -15x-18z=-25 \end{cases} \xRightarrow{\times(-3)} \begin{cases} 15x+18z=9 \\ -15x-18z=-25 \end{cases}$$

$$0=-16$$

Adding the resulting equations together leads to a false statement, which indicates that the system is inconsistent. There is no simultaneous solution.

Answer: Ø

A system with no solutions is an inconsistent system. Given three planes, no simultaneous solution can occur in a number of ways.



Just as with linear systems with two variables, some linear systems with three variables have infinitely many solutions. Such systems are called dependent systems.

Example 6

Solve the system: $\begin{cases} 7x - 4y + z = -15 \\ 3x + 2y - z = -5 \\ 5x + 12y - 5z = -5 \end{cases}$.

Solution:

Eliminate z by adding the first and second equations together.

$$\begin{array}{lrl} (1) & \begin{cases} 7x - 4y + z = -15 \\ 3x + 2y - z = -5 \end{cases} \\ & \overline{\quad 10x - 2y = -20 \quad} & ✓ \end{array}$$

Next use equations 1 and 3 to eliminate z again.

$$\begin{array}{ll} (1) \\ (3) \end{array} \begin{cases} 7x - 4y + z = -15 \\ 5x + 12y - 5z = -5 \end{cases} \overset{\times 5}{\Rightarrow} \begin{cases} 35x - 20y + 5z = -75 \\ 5x + 12y - 5z = -5 \end{cases}$$
$$40x - 8y = -80 \quad ✓$$

This leaves a system of two equations with two variables x and y,

$$\begin{cases} 10x - 2y = -20 \\ 40x - 8y = -80 \end{cases}$$

Line up the variable y to eliminate by dividing the first equation by 2 and the second equation by -8.

$$\begin{cases} 10x - 2y = -20 \\ 40x - 8y = -80 \end{cases} \begin{matrix} \xrightarrow{\div 2} \\ \xrightarrow[\div(-8)]{} \end{matrix} \begin{cases} 5x - y = -10 \\ -5x + y = 10 \end{cases}$$

$$\underline{}$$

$$0 = 0 \quad \textit{True}$$

A true statement indicates that the system is dependent. To express the infinite number of solutions (x, y, z) in terms of one variable, we solve for y and z both in terms of x.

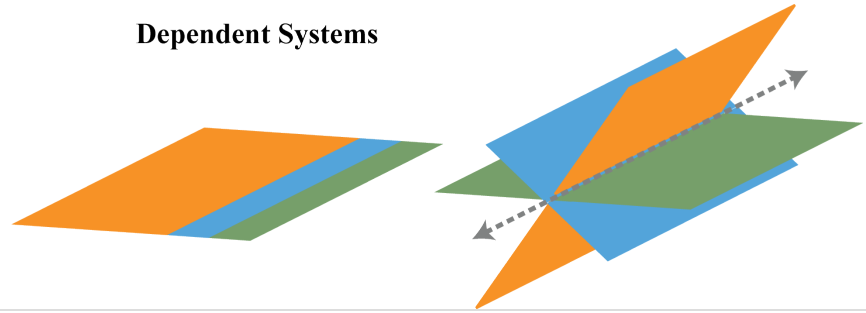
$$\begin{aligned} 10x - 2y &= -20 \\ -2y &= -10x - 20 \\ \frac{-2y}{-2} &= \frac{-10x - 20}{-2} \\ y &= 5x + 10 \end{aligned}$$

Once we have y in terms of x, we can solve for z in terms of x by back substituting into one of the original equations.

$$\begin{aligned} 7x - 4y + z &= -15 \\ 7x - 4\left(5x + 10\right) + z &= -15 \\ 7x - 20x - 40 + z &= -15 \\ -13x - 40 + z &= -15 \\ z &= 13x + 25 \end{aligned}$$

Answer: $(x, 5x + 10, 13x + 25)$.

A consistent system with infinitely many solutions is a dependent system. Given three planes, infinitely many simultaneous solutions can occur in a number of ways.



Try this! Solve: $\begin{cases} 7x + y - 2z &= -4 \\ -21x - 7y + 8z &= 4 \\ 7x + 3y - 3z &= 0 \end{cases}$.

Answer: $\left(x, \frac{7}{3}x + 4, \frac{14}{3}x + 4\right)$

(click to see video)

Applications Involving Three Unknowns

Many real-world applications involve more than two unknowns. When an application requires three variables, we look for relationships between the variables that allow us to write three equations.

Example 7

A community theater sold 63 tickets to the afternoon performance for a total of \$444. An adult ticket cost \$8, a child ticket cost \$4, and a senior ticket cost \$6. If twice as many tickets were sold to adults as to children and seniors combined, how many of each ticket were sold?

Solution:

Begin by identifying three variables.

Let x represent the number of adult tickets sold.

Let y represent the number of child tickets sold.

Let z represent the number of senior tickets sold.

The first equation comes from the statement that 63 tickets were sold.

$$(1) \qquad x + y + z = 63$$

The second equation comes from total ticket sales.

$$(2) \qquad 8x + 4y + 6z = 444$$

The third equation comes from the statement that twice as many adult tickets were sold as child and senior tickets combined.

$$x = 2\left(y + z\right)$$
$$x = 2y + 2z$$
$$(3) \quad x - 2y - 2z = 0$$

Therefore, the problem is modeled by the following linear system.

$$\begin{cases} x + y + z = 63 \\ 8x + 4y + 6z = 444 \\ x - 2y - 2z = 0 \end{cases}$$

Solving this system is left as an exercise. The solution is $(42, 9, 12)$.

Answer: The theater sold 42 adult tickets, 9 child tickets, and 12 senior tickets.

KEY TAKEAWAYS

- A simultaneous solution to a linear system with three equations and three variables is an ordered triple (x, y, z) that satisfies all of the equations. If it does not solve each equation, then it is not a solution.
- We can solve systems of three linear equations with three unknowns by elimination. Choose any two of the equations and eliminate a variable. Next choose any other two equations and eliminate the same variable. This will result in a system of two equations with two variables that can be solved by any method learned previously.
- If the process of solving a system leads to a false statement, then the system is inconsistent and has no solution.
- If the process of solving a system leads to a true statement, then the system is dependent and has infinitely many solutions.
- To solve applications that require three variables, look for relationships between the variables that allow you to write three linear equations.

TOPIC EXERCISES

PART A: LINEAR SYSTEMS WITH THREE VARIABLES

Determine whether or not the given ordered triple is a solution to the given system.

1. $(3, -2, -1)$;

$$\begin{cases} x + y - z = 2 \\ 2x - 3y + 2z = 10 \\ x + 2y + z = -3 \end{cases}$$

2. $(-8, -1, 5)$;

$$\begin{cases} x + 2y - z = -15 \\ 2x - 6y + 2z = 0 \\ 3x - 9y + 4z = 5 \end{cases}$$

3. $(1, -9, 2)$;

$$\begin{cases} 8x + y - z = -3 \\ 7x - 2y - 3z = 19 \\ x - y + 9z = 28 \end{cases}$$

4. $(-4, 1, -3)$;

$$\begin{cases} 3x + 2y - z = -7 \\ x - 5y + 2z = 3 \\ 2x + y + 3z = -16 \end{cases}$$

5. $\left(6, \frac{2}{3}, -\frac{1}{2}\right)$;

$$\begin{cases} x + 6y - 4z = 12 \\ -x + 3y - 2z = -3 \\ x - 9y + 8z = -4 \end{cases}$$

6. $\left(\frac{1}{4}, -1, -\frac{3}{4}\right)$;

$$\begin{cases} 2x - y - 2z = 3 \\ 4x + 5y - 8z = 2 \\ x - 2y - z = 3 \end{cases}$$

7. $(3, -2, 1)$;

$$\begin{cases} 4x - 5y = 22 \\ 2y - z = 8 \\ -5x + 2z = -13 \end{cases}$$

8. $\left(1, \frac{5}{2}, -\frac{1}{2}\right)$;

$$\begin{cases} 2y - 6z = 8 \\ 3x - 4z = 5 \\ 18z = -9 \end{cases}$$

9. $\left(\frac{1}{2}, -2, 6\right)$;

$$\begin{cases} a - b + c = 9 \\ 4a - 2b + c = 14 \\ 2a + b + \frac{1}{2}c = 3 \end{cases}$$

10. $\left(-1, 5, -7\right)$;

$$\begin{cases} 3a + b + \frac{1}{3}c = -\frac{1}{3} \\ 8a + 2b + \frac{1}{2}c = -\frac{3}{2} \\ 25a + 5b + c = -7 \end{cases}$$

PART B: SOLVING LINEAR SYSTEMS WITH THREE VARIABLES

Solve.

11. $$\begin{cases} 2x - 3y + z = 4 \\ 5x + 2y + 2z = 2 \\ x + 4y - 3z = 7 \end{cases}$$

12. $$\begin{cases} 5x - 2y + z = -9 \\ 2x + y - 3z = -5 \\ 7x + 3y + 2z = 6 \end{cases}$$

13. $\begin{cases} x + 5y - 2z = 15 \\ 3x - 7y + 4z = -7 \\ 2x + 4y - 3z = 21 \end{cases}$

14. $\begin{cases} x - 4y + 2z = 3 \\ 2x + 3y - 3z = 9 \\ 3x + 2y + 4z = -1 \end{cases}$

15. $\begin{cases} 5x + 4y - 2z = -5 \\ 4x - y + 3z = 14 \\ 6x + 3y - 5z = -12 \end{cases}$

16. $\begin{cases} 2x + 3y - 2z = -4 \\ 3x + 5y + 3z = 17 \\ 2x + y - 4z = -8 \end{cases}$

17. $\begin{cases} x + y - 4z = 1 \\ 9x - 3y + 6z = 2 \\ -6x + 2y - 4z = -2 \end{cases}$

18. $\begin{cases} 5x - 8y + z = 5 \\ -3x + 5y - z = -3 \\ -11x + 18y - 3z = -5 \end{cases}$

19. $\begin{cases} x - y + 2z = 3 \\ 2x - y + 3z = 2 \\ -x - 3y + 4z = 1 \end{cases}$

20. $\begin{cases} x + y + z = 8 \\ x - y + 4z = -7 \\ -x - y + 2z = 1 \end{cases}$

21. $\begin{cases} 4x - y + 2z = 3 \\ 6x + 3y - 4z = -1 \\ 3x - 2y + 3z = 4 \end{cases}$

22. $\begin{cases} x - 4y + 6z = -1 \\ 3x + 8y - 2z = 2 \\ 5x + 2y - 3z = -5 \end{cases}$

23. $$\begin{cases} 3x - 4y - z = 7 \\ 5x - 8y + 3z = 11 \\ 2x + 6y + z = 9 \end{cases}$$

24. $$\begin{cases} 3x + y - 4z = 6 \\ 6x - 5y + 3z = 1 \\ 9x + 3y - 4z = 10 \end{cases}$$

25. $$\begin{cases} 7x - 6y + z = 8 \\ -x + 2y - z = 4 \\ x + 2y - 2z = 14 \end{cases}$$

26. $$\begin{cases} -9x + 3y + z = 3 \\ 12x - 4y - z = 2 \\ -6x + 2y + z = 8 \end{cases}$$

27. $$\begin{cases} a - b + c = 9 \\ 4a - 2b + c = 14 \\ 2a + b + \frac{1}{2}c = 3 \end{cases}$$

28. $$\begin{cases} 3a + b + \frac{1}{3}c = -\frac{1}{3} \\ 8a + 2b + \frac{1}{2}c = -\frac{3}{2} \\ 25a + 5b + c = -7 \end{cases}$$

29. $$\begin{cases} 3x - 5y - 4z = -5 \\ 4x - 6y + 3z = -22 \\ 6x + 8y - 5z = 20 \end{cases}$$

30. $$\begin{cases} 7x + 4y - 2z = 8 \\ 2x + 2y + 3z = -4 \\ 3x - 6y - 7z = 8 \end{cases}$$

31. $$\begin{cases} 9x + 7y + 4z = 8 \\ 4x - 5y - 6z = -11 \\ -5x + 2y + 3z = 4 \end{cases}$$

32. $$\begin{cases} 3x + 7y + 2z = -7 \\ 5x + 4y + 3z = 5 \\ 2x - 3y + 5z = -4 \end{cases}$$

33. $\begin{cases} 4x - 3y = 1 \\ 2y - 3z = 2 \\ 3x + 2z = 3 \end{cases}$

34. $\begin{cases} 5y - 3z = -28 \\ 3x + 2y = 8 \\ 4y - 7z = -27 \end{cases}$

35. $\begin{cases} 2x + 3y + z = 1 \\ 6y + z = 4 \\ 2z = -4 \end{cases}$

36. $\begin{cases} x - 3y - 2z = 5 \\ 2y + 6z = -1 \\ 4z = -6 \end{cases}$

37. $\begin{cases} 2x = 10 \\ 6x - 5y = 30 \\ 3x - 4y - 2z = 3 \end{cases}$

38. $\begin{cases} 2x + 7z = 2 \\ -4y = 6 \\ 8y + 3z = 0 \end{cases}$

39. $\begin{cases} 5x + 7y + 2z = 4 \\ 12x + 16y + 4z = 15 \\ 10x + 13y + 3z = 14 \end{cases}$

40. $\begin{cases} 8x + 12y - 8z = 5 \\ 2x + 3y - 2z = 2 \\ 4x - 2y + 5z = -1 \end{cases}$

41. $\begin{cases} 17x - 4y - 3z = -2 \\ 5x + \frac{1}{2}y - 2z = -\frac{9}{2} \\ 2x + 5y - 4z = -13 \end{cases}$

42. $\begin{cases} 3x - 5y - \frac{1}{2}z = \frac{7}{2} \\ x - y - \frac{1}{2}z = -\frac{1}{2} \\ 3x - 8y + z = 11 \end{cases}$

43. $$\begin{cases} 4a - 2b + 3c = 9 \\ 3a + 3b - 5c = -6 \\ 10a - 6b + 5c = 13 \end{cases}$$

44. $$\begin{cases} 6a - 2b + 5c = -2 \\ 4a + 3b - 3c = -1 \\ 3a + 5b + 6c = 24 \end{cases}$$

PART C: APPLICATIONS

Set up a system of equations and use it to solve the following.

45. The sum of three integers is 38. Two less than 4 times the smaller integer is equal to the sum of the others. The sum of the smaller and larger integer is equal to 2 more than twice that of the other. Find the integers.

46. The sum of three integers is 40. Three times the smaller integer is equal to the sum of the others. Twice the larger is equal to 8 more than the sum of the others. Find the integers.

47. The sum of the angles A, B, and C of a triangle is 180°. The larger angle C is equal to twice the sum of the other two. Four times the smallest angle A is equal to the difference of angle C and B. Find the angles.

48. The sum of the angles A, B, and C of a triangle is 180°. Angle C is equal to the sum of the other two angles. Five times angle A is equal to the sum of angle C and B. Find the angles.

49. A total of $12,000 was invested in three interest earning accounts. The interest rates were 2%, 4%, and 5%. If the total simple interest for one year was $400 and the amount invested at 2% was equal to the sum of the amounts in the other two accounts, then how much was invested in each account?

50. Joe invested his $6,000 bonus in three accounts earning $4\frac{1}{2}$% interest. He invested twice as much in the account earning $4\frac{1}{2}$% as he did in the other two accounts combined. If the total simple interest for the year was $234, how much did Joe invest in each account?

51. A jar contains nickels, dimes, and quarters. There are 105 coins with a total value of $8.40. If there are 3 more than twice as many dimes as quarters, find how many of each coin are in the jar.

52. A billfold holds one-dollar, five-dollar, and ten-dollar bills and has a value of \$210. There are 50 bills total where the number of one-dollar bills is one less than twice the number of five-dollar bills. How many of each bill are there?

53. A nurse wishes to prepare a 15-ounce topical antiseptic solution containing 3% hydrogen peroxide. To obtain this mixture, purified water is to be added to the existing 1.5% and 10% hydrogen peroxide products. If only 3 ounces of the 10% hydrogen peroxide solution is available, how much of the 1.5% hydrogen peroxide solution and water is needed?

54. A chemist needs to produce a 32-ounce solution consisting of $8\frac{3}{4}$% acid. He has three concentrates with 5%, 10%, and 40% acid. If he is to use twice as much of the 5% acid solution as the 10% solution, then how many ounces of the 40% solution will he need?

55. A community theater sold 128 tickets to the evening performance for a total of \$1,132. An adult ticket cost \$10, a child ticket cost \$5, and a senior ticket cost \$6. If three times as many tickets were sold to adults as to children and seniors combined, how many of each ticket were sold?

56. James sold 82 items at the swap meet for a total of \$504. He sold packages of socks for \$6, printed t-shirts for \$12, and hats for \$5. If he sold 5 times as many hats as he did t-shirts, how many of each item did he sell?

57. A parabola passes through three points $(-1, 7)$, $(1, -1)$ and $(2, -2)$. Use these points and $y = ax^2 + bx + c$ to construct a system of three linear equations in terms of *a*, *b*, and *c* and then solve the system.

58. A parabola passes through three points $(-2, 11)$, $(-1, 4)$ and $(1, 2)$. Use these points and $y = ax^2 + bx + c$ to construct a system of three linear equations in terms of *a*, *b*, and *c* and solve it.

PART D: DISCUSSION BOARD

59. On a note card, write down the steps for solving a system of three linear equations with three variables using elimination. Use your notes to explain to a friend how to solve one of the exercises in this section.

60. Research and discuss curve fitting. Why is curve fitting an important topic?

ANSWERS

1. No
3. Yes
5. Yes
7. No
9. No
11. (2, −1, −3)
13. (4, 1, −3)
15. (1, −1, 3)
17. Ø
19. (5, −10, −6)
21. $\left(\frac{1}{2}, -2, -\frac{1}{2}\right)$
23. $\left(3, \frac{1}{2}, 0\right)$
25. $\left(x, \frac{3}{2}x - 3, 2x - 10\right)$
27. (1, −2, 6)
29. (−1, 2, −2)
31. (1,−3, 5)
33. (1, 1, 0)
35. (0, 1, −2)
37. (5, 0, 6)
39. $\emptyset$
41. $(x, 2x - 1, 3x + 2)$
43. (1, 2, 3)
45. 8, 12, 18

47. $A = 20°$, $B = 40°$, and $C = 120°$
49. The amount invested at 2% was $6,000, the amount invested at 4% was $2,000, and the amount invested at 5% was $4,000.
51. 72 nickels, 23 dimes, and 10 quarters
53. 10 ounces of the 1.5% hydrogen peroxide solution and 2 ounces of water
55. 96 adult tickets, 20 child tickets, and 12 senior tickets were sold.
57. $a = 1, b = -4$, and $c = 2$
59. Answer may vary

3.5 Matrices and Gaussian Elimination

LEARNING OBJECTIVES

1. Use back substitution to solve linear systems in upper triangular form.
2. Convert linear systems to equivalent augmented matrices.
3. Use matrices and Gaussian elimination to solve linear systems.

Back Substitution

Recall that a linear system of equations consists of a set of two or more linear equations with the same variables. A linear system consisting of three equations in standard form arranged so that the variable *x* does not appear in any equation after the first and the variable *y* does not appear in any equation after the second is said to be in **upper triangular form**[22]. For example,

$$\begin{cases} x - 6y + 2z = 16 \\ \quad\;\; 3y - 9z = 5 \\ \qquad\qquad\; z = -1 \end{cases}$$

Notice that the system forms a triangle where each successive equation contains one less variable. In general,

Linear Systems in Upper Triangular Form

$$\begin{cases} a_1x + b_1y = c_1 \\ \qquad\;\; b_2y = c_2 \end{cases} \qquad \begin{cases} a_1x + b_1y + c_1z = d_1 \\ \qquad\;\; b_2y + c_2z = d_2 \\ \qquad\qquad\quad\; c_3z = d_3 \end{cases}$$

If a linear system is in this form, we can easily solve for one of the variables and then back substitute to solve for the remaining variables.

22. A linear system consisting of equations with three variables in standard form arranged so that the variable *x* does not appear after the first equation and the variable *y* does not appear after the second equation.

Example 1

Solve: $\begin{cases} 3x - y = 7 \\ 2y = -2 \end{cases}$.

Solution:

Recall that solutions to linear systems with two variables, if they exist, are ordered pairs (x, y). We can determine the y-value easily using the second equation.

$$\begin{aligned} 2y &= -2 \\ y &= -1 \end{aligned}$$

Next, use the first equation $3x - y = 7$ and the fact that $y = -1$ to find x.

$$\begin{aligned} 3x - y &= 7 \\ 3x - \ (-1) &= 7 \\ 3x + 1 &= 7 \\ 3x &= 6 \\ x &= 2 \end{aligned}$$

Answer: $(2, -1)$

Example 2

Solve: $\begin{cases} x - 6y + 2z = 16 \\ \quad 3y - 9z = 5 \\ \qquad z = -1 \end{cases}$.

Solution:

Recall that solutions to linear systems with three variables, if they exist, are ordered triples (x, y, z). Use the second equation $3y - 9z = 5$ and the fact that $z = -1$ to find y.

$$\begin{aligned} 3y - 9z &= 5 \\ 3y - 9(-1) &= 5 \\ 3y + 9 &= 5 \\ 3y &= -4 \\ y &= -\frac{4}{3} \end{aligned}$$

Next substitute y and z into the first equation.

$$\begin{aligned} x - 6y + 2z &= 16 \\ x - 6\left(-\frac{4}{3}\right) + 2(-1) &= 16 \\ x + 8 - 2 &= 16 \\ x + 6 &= 16 \\ x &= 10 \end{aligned}$$

Answer: $\left(10, -\frac{4}{3}, -1\right)$

Try this! Solve: $\begin{cases} 4x - y + 3z = 1 \\ \quad 2y - 9z = -2. \\ \qquad\quad 3z = 2 \end{cases}$

Answer: $\left(\frac{1}{4}, 2, \frac{2}{3}\right)$

(click to see video)

Matrices and Gaussian Elimination

In this section the goal is to develop a technique that streamlines the process of solving linear systems. We begin by defining a **matrix**[23], which is a rectangular array of numbers consisting of rows and columns. Given a linear system in standard form, we create a **coefficient matrix**[24] by writing the coefficients as they appear lined up without the variables or operations as follows.

$$\begin{array}{ccc} \textit{Linear System} & & \textit{Coefficient Matrix} \\ \begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases} & \Rightarrow & \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \end{array}$$

23. A rectangular array of numbers consisting of rows and columns.

24. The matrix of coefficients of a linear system in standard form written as they appear lined up without the variables or operations.

25. The coefficient matrix with the column of constants included.

The rows represent the coefficients in the equations and the columns represent the coefficients of each variable. Furthermore, if we include a column that represents the constants we obtain what is called an **augmented matrix**[25]. For a linear system with two variables, *Linear System* *Augmented Matrix*

$$\begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases} \quad \Leftrightarrow \quad \left[\begin{array}{cc|c} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{array}\right]$$

And for a linear system with three variables we have

Linear System *Augmented Matrix*

$$\begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \\ a_3 x + b_3 y + c_3 z = d_3 \end{cases} \quad \Leftrightarrow \quad \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

Note: The dashed vertical line provides visual separation between the coefficient matrix and the column of constants. In other algebra resources that you may encounter, this is sometimes omitted.

Example 3

Construct the augmented matrix that corresponds to: $\begin{cases} 9x - 6y = 0 \\ -x + 2y = 1 \end{cases}$.

Solution:

This system consists of two linear equations in standard form; therefore, the coefficients in the matrix appear as they do in the system.

$$\begin{cases} 9x - 6y = 0 \\ -x + 2y = 1 \end{cases} \quad \Leftrightarrow \quad \left[\begin{array}{cc|c} 9 & -6 & 0 \\ -1 & 2 & 1 \end{array}\right]$$

Example 4

Construct the augmented matrix that corresponds to: $\begin{cases} x + 2y - 4z = 5 \\ 2x + y - 6z = 8 \\ 4x - y - 12z = 13 \end{cases}$.

Solution:

Since the equations are given in standard form, the coefficients appear in the matrix as they do in the system.

$$\begin{cases} x + 2y - 4z = 5 \\ 2x + y - 6z = 8 \\ 4x - y - 12z = 13 \end{cases} \quad \Leftrightarrow \quad \left[\begin{array}{ccc|c} 1 & 2 & -4 & 5 \\ 2 & 1 & -6 & 8 \\ 4 & -1 & -12 & 13 \end{array}\right]$$

A matrix is in upper triangular form if all elements below the leading nonzero element in each successive row are zero. For example,

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 5 \\ 0 & -3 & 2 & -2 \\ 0 & 0 & -2 & -1 \end{array}\right]$$

Notice that the elements below the main diagonal are zero and the coefficients above form a triangular shape. In general,

Upper Triangular Form

$$\begin{bmatrix} a_1 & b_1 \\ 0 & b_2 \end{bmatrix} \qquad \begin{bmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{bmatrix}$$

This is important because in this section we outline a process by which certain operations can be made to produce an equivalent linear system in upper triangular form so that it can be solved by using back substitution. An overview of the process is outlined below:

$$\begin{array}{ccc} \textit{Linear System} & & \textit{Augmented Matrix} \\ \begin{cases} x+2y-4z=5 \\ 2x+y-6z=8 \\ 4x-y-12z=13 \end{cases} & \Rightarrow & \left[\begin{array}{ccc|c} 1 & 2 & -4 & 5 \\ 2 & 1 & -6 & 8 \\ 4 & -1 & -12 & 13 \end{array}\right] \\ & & \Downarrow \\ \begin{cases} x+2y-4z=5 \\ -3y+2z=-2 \\ -2z=-1 \end{cases} & \Leftarrow & \left[\begin{array}{ccc|c} 1 & 2 & -4 & 5 \\ 0 & -3 & 2 & -2 \\ 0 & 0 & -2 & -1 \end{array}\right] \\ \textit{Equivalent System} & & \textit{Upper Triangular Form} \end{array}$$

Once the system is in upper triangular form, we can use back substitution to easily solve it. It is important to note that the augmented matrices presented here represent linear systems of equations in standard form.

The following **elementary row operations**[26] result in augmented matrices that represent equivalent linear systems:

1. Any two rows may be interchanged.
2. Each element in a row can be multiplied by a nonzero constant.
3. Any row can be replaced by the sum of that row and a multiple of another.

Note: These operations are consistent with the properties used in the elimination method.

To efficiently solve a system of linear equations first construct an augmented matrix. Then apply the appropriate elementary row operations to obtain an augmented matrix in upper triangular form. In this form, the equivalent linear system can easily be solved using back substitution. This process is called **Gaussian elimination**[27], named in honor of Carl Friedrich Gauss (1777–1855).

26. Operations that can be performed to obtain equivalent linear systems.

27. Steps used to obtain an equivalent linear system in upper triangular form so that it can be solved using back substitution.

Figure 3.1

Carl Friedrich Gauss (Wikipedia)

The steps for solving a linear equation with two variables using Gaussian elimination are listed in the following example.

Example 5

Solve using matrices and Gaussian elimination: $\begin{cases} 9x - 6y = 0 \\ -x + 2y = 1 \end{cases}$.

Solution:

Ensure that the equations in the system are in standard form before beginning this process.

Step 1: Construct the corresponding augmented matrix.

$$\begin{cases} 9x - 6y = 0 \\ -x + 2y = 1 \end{cases} \Leftrightarrow \left[\begin{array}{cc|c} 9 & -6 & 0 \\ -1 & 2 & 1 \end{array}\right]$$

Step 2: Apply the elementary row operations to obtain upper triangular form. In this case, we need only to eliminate the first element of the second row, -1. To do this, multiply the second row by 9 and add it to the first row.

$$\left[\begin{array}{cc|c} 9 & -6 & 0 \\ -1 & 2 & 1 \end{array}\right] \overset{\times 9}{\Rightarrow} \begin{array}{rrrr} & 9 & -6 & 0 \\ + & -9 & 18 & 9 \\ \hline & 0 & 12 & 9 \end{array}$$

Now use this to replace the second row.

$$\left[\begin{array}{cc|c} 9 & -6 & 0 \\ 0 & 12 & 9 \end{array}\right]$$

This results in an augmented matrix in upper triangular form.

Step 3: Convert back to a linear system and solve using back substitution. In this example, we have

$$\left[\begin{array}{cc|c} 9 & -6 & 0 \\ 0 & 12 & 9 \end{array}\right] \quad \Rightarrow \quad \begin{cases} 9x-6y=0 \\ 12y=9 \end{cases}$$

Solve the second equation for y,

$$\begin{aligned} 12y &= 9 \\ y &= \frac{9}{12} \\ y &= \frac{3}{4} \end{aligned}$$

Substitute this value for y into the first equation to find x,

$$\begin{aligned} 9x-6y &= 0 \\ 9x-6\left(\frac{3}{4}\right) &= 0 \\ 9x-\frac{9}{2} &= 0 \\ 9x &= \frac{9}{2} \\ x &= \frac{1}{2} \end{aligned}$$

Answer: $\left(\frac{1}{2},\ \frac{3}{4}\right)$

The steps for using Gaussian elimination to solve a linear equation with three variables are listed in the following example.

Example 6

Solve using matrices and Gaussian elimination: $\begin{cases} x+2y-4z=5 \\ 2x+y-6z=8 \\ 4x-y-12z=13 \end{cases}$.

Solution:

Ensure that the equations in the system are in standard form before beginning this process.

Step 1: Construct the corresponding augmented matrix.

$$\begin{cases} x+2y-4z=5 \\ 2x+y-6z=8 \\ 4x-y-12z=13 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -4 & 5 \\ 2 & 1 & -6 & 8 \\ 4 & -1 & -12 & 13 \end{array}\right]$$

Step 2: Apply the elementary row operations to obtain upper triangular form. We begin by eliminating the first element of the second row, 2 in this case. To do this multiply the first row by −2 and then add it to the second row.

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 5 \\ 2 & 1 & -6 & 8 \\ 4 & -1 & -12 & 13 \end{array}\right] \xrightarrow{\times(-2)} \begin{array}{rrrrr} & -2 & -4 & 8 & -10 \\ + & 2 & 1 & -6 & 8 \\ \hline & 0 & -3 & 2 & -2 \end{array}$$

Use this to replace the second row.

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 5 \\ 0 & -3 & 2 & -2 \\ 4 & -1 & -12 & 13 \end{array}\right]$$

Next, eliminate the first element of the third row, 4 in this case, by multiplying the first row by −4 and adding it to the third row.

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 5 \\ 0 & -3 & 2 & -2 \\ 4 & -1 & -12 & 13 \end{array}\right] \xRightarrow{\times(-4)} \begin{array}{rrrrr} & -4 & -8 & 16 & -20 \\ + & 4 & -1 & -12 & 13 \\ \hline & 0 & -9 & 4 & -7 \end{array}$$

Use this to replace the third row.

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 5 \\ 0 & -3 & 2 & -2 \\ 0 & -9 & 4 & -7 \end{array}\right]$$

This results in an augmented matrix where the elements below the first element of the first row are zero. Next eliminate the second element in the third row, in this case −9. Multiply the second row by −3 and add it to the third row.

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 5 \\ 0 & -3 & 2 & -2 \\ 0 & -9 & 4 & -7 \end{array}\right] \xRightarrow{\times(-3)} \begin{array}{rrrrr} + & 0 & 9 & -6 & 6 \\ & 0 & -9 & 4 & -7 \\ \hline & 0 & 0 & -2 & -1 \end{array}$$

Use this to replace the third row and we can see that we have obtained a matrix in upper triangular form.

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 5 \\ 0 & -3 & 2 & -2 \\ 0 & 0 & -2 & -1 \end{array}\right]$$

Step 3: Convert back to a linear system and solve using back substitution. In this example, we have

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 5 \\ 0 & -3 & 2 & -2 \\ 0 & 0 & -2 & -1 \end{array}\right] \Rightarrow \begin{cases} x + 2y - 4z = 5 \\ \quad -3y + 2z = -2 \\ \qquad -2z = -1 \end{cases}$$

Answer: It is left to the reader to verify that the solution is $\left(5, 1, \frac{1}{2}\right)$.

Note: Typically, the work involved in replacing a row by multiplying and adding is done on the side using scratch paper.

Example 7

Solve using matrices and Gaussian elimination: $\begin{cases} 2x - 9y + 3z = -18 \\ x - 2y - 3z = -8 \\ -4x + 23y + 12z = 47 \end{cases}$.

Solution:

We begin by converting the system to an augmented coefficient matrix.

$$\begin{cases} 2x - 9y + 3z = -18 \\ x - 2y - 3z = -8 \\ -4x + 23y + 12z = 47 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 2 & -9 & 3 & -18 \\ 1 & -2 & -3 & -8 \\ -4 & 23 & 12 & 47 \end{array}\right]$$

The elementary row operations are streamlined if the leading nonzero element in a row is 1. For this reason, begin by interchanging row one and two.

$$\begin{array}{r} (row2) \rightarrow \\ (row1) \rightarrow \\ \end{array} \left[\begin{array}{ccc|c} 1 & -2 & -3 & -8 \\ 2 & -9 & 3 & -18 \\ -4 & 23 & 12 & 47 \end{array}\right]$$

Replace row two with the sum of -2 times row one and row two.

$$\begin{array}{r} \\ -2(row1)+(row2) \rightarrow \\ \end{array} \left[\begin{array}{ccc|c} 1 & -2 & -3 & -8 \\ 0 & -5 & 9 & -2 \\ -4 & 23 & 12 & 47 \end{array}\right]$$

Replace row three with the sum of 4 times row one and row three.

$$\begin{array}{r} \\ \\ 4(row1)+(row3) \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & -2 & -3 & -8 \\ 0 & -5 & 9 & -2 \\ 0 & 15 & 0 & 15 \end{array}\right]$$

Next divide row 3 by 15.

$$\begin{array}{r} \\ \\ (row3)\div 15\rightarrow \end{array}\left[\begin{array}{rrr|r} 1 & -2 & -3 & -8 \\ 0 & -5 & 9 & -2 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

Interchange row three with row two.

$$\begin{array}{r} \\ (row2)\rightarrow \\ (row3)\rightarrow \end{array}\left[\begin{array}{rrr|r} 1 & -2 & -3 & -8 \\ 0 & 1 & 0 & 1 \\ 0 & -5 & 9 & -2 \end{array}\right]$$

Next replace row 3 with the sum of 5 times row two and row three.

$$\begin{array}{r} \\ \\ 5(row2)+(row3)\rightarrow \end{array}\left[\begin{array}{rrr|r} 1 & -2 & -3 & -8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 9 & 3 \end{array}\right]$$

This results in a matrix in upper triangular form. A matrix is in **row echelon form**[28] if it is in upper triangular form where the leading nonzero element of each row is 1. We can obtain this form by replacing row three with the results of dividing it by 9.

$$\begin{array}{r} \\ \\ (row3)\div 9\rightarrow \end{array}\left[\begin{array}{rrr|r} 1 & -2 & -3 & -8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right]$$

Convert to a system of linear equations and solve by back substitution.

$$\left[\begin{array}{rrr|r} 1 & -2 & -3 & -8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right] \Rightarrow \left\{\begin{aligned} x-2y-3z &= -8 \\ y &= 1 \\ z &= \frac{1}{3} \end{aligned}\right.$$

Here $y = 1$ and $z = \frac{1}{3}$. Substitute into the first equation to find x.

28. A matrix in triangular form where the leading nonzero element of each row is 1.

$$
\begin{aligned}
x-2y-3y&=-8\\
x-2(1)-3\left(\frac{1}{3}\right)&=-8\\
x-2-1&=-8\\
x-3&=-8\\
x&=-5
\end{aligned}
$$

Answer: Therefore the solution is $\left(-5,\ 1,\ \frac{1}{3}\right)$.

Technology note: Many modern calculators and computer algebra systems can perform Gaussian elimination. First you will need to find out how to enter a matrix. Then use the calculator's functions to find row echelon form. You are encouraged to conduct some web research on this topic for your particular calculator model.

Try this! Solve using Gaussian elimination: $\begin{cases} x-3y+2z=16 \\ 4x-11y-z=69 \\ 2x-5y-4z=36 \end{cases}$.

Answer: (6, -4, -1)

(click to see video)

Recall that some consistent linear systems are dependent, that is, they have infinitely many solutions. And some linear systems have no simultaneous solution; they are inconsistent systems.

Example 8

Solve using matrices and Gaussian elimination: $\begin{cases} x-2y+z=4 \\ 2x-3y+4z=7 \\ 4x-7y+6z=15 \end{cases}$.

Solution:

We begin by converting the system to an augmented coefficient matrix.

$$\begin{cases} x-2y+z=4 \\ 2x-3y+4z=7 \\ 4x-7y+6z=15 \end{cases} \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 2 & -3 & 4 & 7 \\ 4 & -7 & 6 & 15 \end{array}\right]$$

Replace row two with $-2\left(\text{row }1\right)+\left(\text{row }2\right)$ and replace row three with $-4\left(\text{row }1\right)+\left(\text{row }3\right)$.

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

Replace row three with $-1\left(\text{row }2\right)+\left(\text{row }3\right)$.

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last row indicates that this is a dependent system because converting the augmented matrix back to equations we have,

$$\begin{cases} x - 2y + z = 4 \\ y + 2z = -1 \\ 0x + 0y + 0z = 0 \end{cases}$$

Note that the row of zeros corresponds to the following identity,

$$\begin{aligned} 0x + 0y + 0z &= 0 \\ 0 &= 0 \quad \checkmark \end{aligned}$$

In this case, we can express the infinitely many solutions in terms of z. From the second row we have the following:

$$\begin{aligned} y + 2z &= -1 \\ y &= -2z - 1 \end{aligned}$$

And from the first equation,

$$\begin{aligned} x - 2y + z &= 4 \\ x - 2(-2z - 1) + z &= 4 \\ x + 5z + 2 &= 4 \\ x &= -5z + 2 \end{aligned}$$

The solutions take the form $(x,\ y,\ z) = (-5z + 2, -2z - 1,\ z)$ where z is any real number.

Answer: $(-5z + 2, -2z - 1, z)$

Dependent and inconsistent systems can be identified in an augmented coefficient matrix when the coefficients in one row are all zero.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \leftarrow \qquad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{array}\right] \leftarrow$$

$0 = 0$ ✓ *Dependent*

$0 = 2$ × *Inconsistent*

If a row of zeros has a corresponding constant of zero then the matrix represents a dependent system. If the constant is nonzero then the matrix represents an inconsistent system.

Try this! Solve using matrices and Gaussian elimination:

$$\begin{cases} 5x - 2y + z = -3 \\ 10x - y + 3z = 0 \\ -15x + 9y - 2z = 17 \end{cases}.$$

Answer: Ø

(click to see video)

KEY TAKEAWAYS

- A linear system in upper triangular form can easily be solved using back substitution.
- The augmented coefficient matrix and Gaussian elimination can be used to streamline the process of solving linear systems.
- To solve a system using matrices and Gaussian elimination, first use the coefficients to create an augmented matrix. Apply the elementary row operations as a means to obtain a matrix in upper triangular form. Convert the matrix back to an equivalent linear system and solve it using back substitution.

TOPIC EXERCISES

PART A: BACK SUBSTITUTION

Solve using back substitution.

1. $\begin{cases} 5x - 3y = 2 \\ y = -1 \end{cases}$

2. $\begin{cases} 3x + 2y = 1 \\ y = 3 \end{cases}$

3. $\begin{cases} x - 4y = 1 \\ 2y = -3 \end{cases}$

4. $\begin{cases} x - 5y = 3 \\ 10y = -6 \end{cases}$

5. $\begin{cases} 4x - 3y = -16 \\ 7y = 0 \end{cases}$

6. $\begin{cases} 3x - 5y = -10 \\ 4y = 8 \end{cases}$

7. $\begin{cases} 2x + 3y = -1 \\ 3y = 2 \end{cases}$

8. $\begin{cases} 6x - y = -3 \\ 4y = 3 \end{cases}$

9. $\begin{cases} x - y = 0 \\ 2y = 0 \end{cases}$

10. $\begin{cases} 2x + y = 2 \\ 3y = 0 \end{cases}$

11. $\begin{cases} x + 3y - 4z = 1 \\ y - 3z = -2 \\ z = 3 \end{cases}$

12. $\begin{cases} x - 5y + 4z = -1 \\ y - 7z = 10 \\ z = -2 \end{cases}$

13. $\begin{cases} x - 6y + 8z = 2 \\ 3y - 4z = -4 \\ 2z = -1 \end{cases}$

14. $\begin{cases} 2x - y + 3z = -9 \\ 2y + 6z = -2 \\ 3z = 2 \end{cases}$

15. $\begin{cases} 10x - 3y + z = 13 \\ 11y - 3z = 9 \\ 2z = -6 \end{cases}$

16. $\begin{cases} 3x - 2y + 5z = -24 \\ 4y + 5z = 3 \\ 4z = -12 \end{cases}$

17. $\begin{cases} x - y + 2z = 1 \\ 2y + z = 1 \\ 3z = -1 \end{cases}$

18. $\begin{cases} x + 2y - z = 2 \\ y - 3z = 1 \\ 6z = 1 \end{cases}$

19. $\begin{cases} x - 9y + 5z = -3 \\ 2y = 10 \\ 3z = 27 \end{cases}$

20. $\begin{cases} 4x - z = 3 \\ 3y - 2z = -1 \\ 2z = -8 \end{cases}$

PART B: MATRICES AND GAUSSIAN ELIMINATION

Construct the corresponding augmented matrix (do not solve).

21. $\begin{cases} x + 2y = 3 \\ 4x + 5y = 6 \end{cases}$

22. $\begin{cases} 6x + 5y = 4 \\ 3x + 2y = 1 \end{cases}$

23. $\begin{cases} x - 2y = 1 \\ 2x - y = 1 \end{cases}$

24. $\begin{cases} x - y = 2 \\ -x + y = -1 \end{cases}$

25. $\begin{cases} -x + 8y = 3 \\ 2y = 2 \end{cases}$

26. $\begin{cases} 3x - 2y = 4 \\ -y = 5 \end{cases}$

27. $\begin{cases} 3x - 2y + 7z = 8 \\ 4x - 5y - 10z = 6 \\ -x - 3y + 2z = -1 \end{cases}$

28. $\begin{cases} x - y - z = 0 \\ 2x - y + 3z = -1 \\ -x + 4y - 3z = -2 \end{cases}$

29. $\begin{cases} x - 9y + 5z = -3 \\ 2y = 10 \\ 3z = 27 \end{cases}$

30. $\begin{cases} 4x - z = 3 \\ 3y - 2z = -1 \\ 2z = -8 \end{cases}$

31. $\begin{cases} 8x + 2y = -13 \\ -2y + z = 1 \\ 12x - 5z = -18 \end{cases}$

32. $\begin{cases} x - 3z = 2 \\ y + 6z = 4 \\ 2x + 3y = 12 \end{cases}$

Solve using matrices and Gaussian elimination.

33. $\begin{cases} x - 5y = 2 \\ 2x - y = 1 \end{cases}$

34. $\begin{cases} x - 2y = -1 \\ x + y = 1 \end{cases}$

35. $\begin{cases} 10x - 7y = 15 \\ -2x + 3y = -3 \end{cases}$

36. $\begin{cases} 9x - 10y = 2 \\ 3x + 5y = -1 \end{cases}$

37. $\begin{cases} 3x + 5y = 8 \\ 2x - 3y = 18 \end{cases}$

38. $\begin{cases} 5x - 3y = -14 \\ 7x + 2y = -1 \end{cases}$

39. $\begin{cases} 9x + 15y = 5 \\ 3x + 5y = 7 \end{cases}$

40. $\begin{cases} 6x - 8y = 1 \\ -3x + 4y = -1 \end{cases}$

41. $\begin{cases} x + y = 0 \\ x - y = 0 \end{cases}$

42. $\begin{cases} 7x - 3y = 0 \\ 3x - 7y = 0 \end{cases}$

43. $\begin{cases} 2x - 3y = 4 \\ -10x + 15y = -20 \end{cases}$

44. $\begin{cases} 6x - 10y = 20 \\ -3x + 5y = -10 \end{cases}$

45. $\begin{cases} x + y - 2z = -1 \\ -x + 2y - z = 1 \\ x - y + z = 2 \end{cases}$

46. $\begin{cases} x - y + z = -2 \\ x + 2y - z = 6 \\ -x + y - 2z = 3 \end{cases}$

47. $\begin{cases} 2x - y + z = 2 \\ x - y + z = 2 \\ -2x + 2y - z = -1 \end{cases}$

48. $\begin{cases} 3x - y + 2z = 7 \\ -x + 2y + z = 6 \\ x + 3y - 2z = 1 \end{cases}$

49. $\begin{cases} x - 3y + z = 6 \\ -x - y + 2z = 4 \\ 2x + y + z = 3 \end{cases}$

50. $\begin{cases} 4x - y + 2z = 12 \\ x - 3y + 2z = 7 \\ -2x + 3y + 4z = -16 \end{cases}$

51. $\begin{cases} 2x - 4y + 6z = -4 \\ 3x - 2y + 5z = -2 \\ 5x - y + 2z = 1 \end{cases}$

52. $\begin{cases} 3x + 6y + 9z = 6 \\ 2x - 2y + 3z = 0 \\ -3x + 18y - 12z = 5 \end{cases}$

53. $\begin{cases} -x + y - z = -2 \\ 3x - 2y + 5z = 1 \\ 3x - 5y - z = 3 \end{cases}$

54. $\begin{cases} x + 2y + 3z = 4 \\ 3x + 8y + 13z = 21 \\ 2x + 5y + 8z = 16 \end{cases}$

55. $\begin{cases} 2x - 4y - 5z = 3 \\ -x + y + z = 1 \\ 3x - 4y - 5z = -4 \end{cases}$

56. $\begin{cases} 5x - 3y - 2z = 4 \\ 3x - 6y + 4z = -6 \\ -x + 2y - z = 2 \end{cases}$

57. $\begin{cases} -2x - 3y + 12z = 4 \\ 4x - 5y - 10z = -1 \\ -x - 3y + 2z = 0 \end{cases}$

58. $\begin{cases} 3x - 2y + 5z = 10 \\ 4x + 3y - 3z = -6 \\ x + y + z = 2 \end{cases}$

59. $\begin{cases} x + 2y + z = -3 \\ x + 6y + 3z = 7 \\ x + 4y + 2z = 2 \end{cases}$

60. $\begin{cases} 2x - y + z = 1 \\ 4x - y + 3z = 5 \\ 2x + y + 3z = 7 \end{cases}$

61. $\begin{cases} 2x + 3y - 4z = 0 \\ 3x - 5y + 3z = -10 \\ 5x - 2y + 5z = -4 \end{cases}$

62. $\begin{cases} 3x - 2y + 9z = 2 \\ -2x - 5y - 4z = 3 \\ 5x - 3y + 3z = 15 \end{cases}$

63. $\begin{cases} 8x + 2y = -13 \\ -2y + z = 1 \\ 12x - 5z = -18 \end{cases}$

64. $\begin{cases} x - 3z = 2 \\ y + 6z = 4 \\ 2x + 3y = 12 \end{cases}$

65. $\begin{cases} 9x + 3y - 11z = 6 \\ 2x + y - 3z = 1 \\ 7x + 2y - 8z = 3 \end{cases}$

66. $\begin{cases} 3x - y - z = 4 \\ -5x + y + 2z = -3 \\ 6x - 2y - 2z = 8 \end{cases}$

67. $\begin{cases} 2x - 4y + 3z = 15 \\ 3x - 5y + 2z = 18 \\ 5x + 2y - 6z = 0 \end{cases}$

68. $\begin{cases} 3x - 4y - 3z = -14 \\ 4x + 2y + 5z = 12 \\ -5x + 8y - 4z = -3 \end{cases}$

PART C: DISCUSSION BOARD

69. Research and discuss the history of Gaussian Elimination. Who is credited for first developing this process? Post something that you found interesting relating to this story.

70. Research and discuss the history of modern matrix notation. Who is credited for the development? In what fields are they used today? Post your findings on the discussion board.

ANSWERS

1. $\left(-\frac{1}{5}, -1\right)$

3. $\left(-5, -\frac{3}{2}\right)$

5. (−4, 0)

7. $\left(-\frac{3}{2}, \frac{2}{3}\right)$

9. (0, 0)

11. (−8, 7, 3)

13. $\left(-6, -2, -\frac{1}{2}\right)$

15. $\left(\frac{8}{5}, 0, -3\right)$

17. $\left(\frac{7}{3}, \frac{2}{3}, -\frac{1}{3}\right)$

19. (−3, 5, 9)

21. $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right]$

23. $\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 2 & -1 & 1 \end{array}\right]$

25. $\left[\begin{array}{cc|c} -1 & 8 & 3 \\ 0 & 2 & 2 \end{array}\right]$

27. $\left[\begin{array}{ccc|c} 3 & -2 & 7 & 8 \\ 4 & -5 & -10 & 6 \\ -1 & -3 & 2 & -1 \end{array}\right]$

29. $\left[\begin{array}{ccc|c} 1 & -9 & 5 & -3 \\ 0 & 2 & 0 & 10 \\ 0 & 0 & 3 & 27 \end{array}\right]$

31. $$\left[\begin{array}{rrr|r} 8 & 2 & 0 & -13 \\ 0 & -2 & 1 & 1 \\ 12 & 0 & -5 & -18 \end{array}\right]$$

33. $$\left(\frac{1}{3}, -\frac{1}{3}\right)$$

35. $$\left(\frac{3}{2}, 0\right)$$

37. (6, −2)

39. Ø

41. (0, 0)

43. $$\left(x, \frac{2}{3}x - \frac{4}{3}\right)$$

45. (2, 3, 3)

47. (0, 1, 3)

49. (1, −1, 2)

51. $$\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$$

53. Ø

55. (−7, −13, 7)

57. $$\left(1, 0, \frac{1}{2}\right)$$

59. $$\left(-8, -\frac{1}{2}z + \frac{5}{2}, z\right)$$

61. $(-1, 2, 1)$

63. $$\left(-\frac{3}{2}, -\frac{1}{2}, 0\right)$$

65. Ø

67. (2, −2, 1)

69. Answer may vary

3.6 Determinants and Cramer's Rule

LEARNING OBJECTIVES

1. Calculate the determinant of a 2×2 matrix.
2. Use Cramer's rule to solve systems of linear equations with two variables.
3. Calculate the determinant of a 3×3 matrix.
4. Use Cramer's rule to solve systems of linear equations with three variables.

Linear Systems of Two Variables and Cramer's Rule

Recall that a matrix is a rectangular array of numbers consisting of rows and columns. We classify matrices by the number of rows *n* and the number of columns *m*. For example, a 3×4 matrix, read "3 by 4 matrix," is one that consists of 3 rows and 4 columns. A **square matrix**[29] is a matrix where the number of rows is the same as the number of columns. In this section we outline another method for solving linear systems using special properties of square matrices. We begin by considering the following 2×2 coefficient matrix *A*,

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

The **determinant**[30] of a 2×2 matrix, denoted with vertical lines $|A|$, or more compactly as det(*A*), is defined as follows:

$$\det(A) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

The determinant is a real number that is obtained by subtracting the products of the values on the diagonal.

29. A matrix with the same number of rows and columns.

30. A real number associated with a square matrix.

Example 1

Calculate: $\begin{vmatrix} 3 & -5 \\ 2 & -2 \end{vmatrix}$.

Solution:

The vertical line on either side of the matrix indicates that we need to calculate the determinant.

$$\begin{aligned} \begin{vmatrix} 3 & -5 \\ 2 & -2 \end{vmatrix} &= 3\,(-2) - 2\,(-5) \\ &= -6 + 10 \\ &= 4 \end{aligned}$$

Answer: 4

Example 2

Calculate: $\begin{vmatrix} -6 & 4 \\ 0 & 3 \end{vmatrix}$.

Solution:

Notice that the matrix is given in upper triangular form.

$$\begin{aligned} \begin{vmatrix} -6 & 4 \\ 0 & 3 \end{vmatrix} &= -6\,(3) - 4\,(0) \\ &= -18 - 0 \\ &= -18 \end{aligned}$$

Answer: −18

We can solve linear systems with two variables using determinants. We begin with a general 2×2 linear system and solve for y. To eliminate the variable x, multiply the first equation by $-a_2$ and the second equation by a_1.

$$\left\{\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}\right. \quad \begin{aligned} &\xRightarrow{\times(-a_2)} \\ &\xRightarrow[\times a_1]{} \end{aligned} \quad \left\{\begin{aligned} -a_1a_2x - a_2b_1y &= -a_2c_1 \\ a_1a_2x + a_1b_2y &= a_1c_2 \end{aligned}\right.$$

This results in an equivalent linear system where the variable x is lined up to eliminate. Now adding the equations we have

$$\begin{array}{rl} & -a_1a_2x - a_2b_1y = -a_2c_1 \\ + & a_1a_2x + a_1b_2y = a_1c_2 \\ \hline & (a_1b_2 - a_2b_1)y = a_1c_2 - a_2c_1 \\ & y = \dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \end{array}$$

Both the numerator and denominator look very much like a determinant of a 2×2 matrix. In fact, this is the case. The denominator is the determinant of the coefficient matrix. And the numerator is the determinant of the matrix formed by replacing the column that represents the coefficients of *y* with the corresponding column of constants. This special matrix is denoted D_y.

$$y = \frac{D_y}{D} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

The value for *x* can be derived in a similar manner.

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}$$

In general, we can form the augmented matrix as follows:

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases} \quad \Leftrightarrow \quad \left[\begin{array}{cc|c} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{array}\right]$$

and then determine D, D_x and D_y by calculating the following determinants.

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \quad D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \quad D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

The solution to a system in terms of determinants described above, when $D \neq 0$, is called **Cramer's rule**[31].

Cramer's Rule

$$(x, y) = \left(\frac{D_x}{D}, \frac{D_y}{D} \right)$$

This theorem is named in honor of Gabriel Cramer (1704 - 1752).

31. The solution to an independent system of linear equations expressed in terms of determinants.

Figure 3.2

Gabriel Cramer

The steps for solving a linear system with two variables using determinants (Cramer's rule) are outlined in the following example.

Example 3

Solve using Cramer's rule: $\begin{cases} 2x + y = 7 \\ 3x - 2y = -7 \end{cases}$.

Solution:

Ensure the linear system is in standard form before beginning this process.

Step 1: Construct the augmented matrix and form the matrices used in Cramer's rule.

$$\begin{cases} 2x + y = 7 \\ 3x - 2y = -7 \end{cases} \Rightarrow \left[\begin{array}{cc|c} 2 & 1 & 7 \\ 3 & -2 & -7 \end{array}\right]$$

In the square matrix used to determine D_x, replace the first column of the coefficient matrix with the constants. In the square matrix used to determine D_y, replace the second column with the constants.

$$D = \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} \qquad D_x = \begin{vmatrix} 7 & 1 \\ -7 & -2 \end{vmatrix} \qquad D_y = \begin{vmatrix} 2 & 7 \\ 3 & -7 \end{vmatrix}$$

Step 2: Calculate the determinants.

$$D_x = \begin{vmatrix} 7 & 1 \\ -7 & -2 \end{vmatrix} = 7(-2) - (-7)(1) = -14 + 7 = -7$$

$$D_y = \begin{vmatrix} 2 & 7 \\ 3 & -7 \end{vmatrix} = 2(-7) - 3(7) = -14 - 21 = -35$$

$$D = \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} = 2(-2) - 3(1) = -4 - 3 = -7$$

Step 3: Use Cramer's rule to calculate x and y.

$$x = \frac{D_x}{D} = \frac{-7}{-7} = 1 \qquad \text{and} \qquad y = \frac{D_y}{D} = \frac{-35}{-7} = 5$$

Therefore the simultaneous solution $(x, y) = (1, 5)$.

Step 4: The check is optional; however, we do it here for the sake of completeness.

$\textit{Check}: (1, 5)$	
Equation 1	*Equation 2*
$\begin{aligned} 2x + y &= 7 \\ 2(1) + (5) &= 7 \\ 2 + 5 &= 7 \\ 7 &= 7 \ \checkmark \end{aligned}$	$\begin{aligned} 3x - 2y &= -7 \\ 3(1) - 2(5) &= -7 \\ 3 - 10 &= -7 \\ -7 &= -7 \ \checkmark \end{aligned}$

Answer: (1, 5)

Example 4

Solve using Cramer's rule: $\begin{cases} 3x - y = -2 \\ 6x + 4y = 2 \end{cases}$.

Solution:

The corresponding augmented coefficient matrix follows.

$$\begin{cases} 3x - y = -2 \\ 6x + 4y = 2 \end{cases} \quad \Rightarrow \quad \left[\begin{array}{cc|c} 3 & -1 & -2 \\ 6 & 4 & 2 \end{array}\right]$$

And we have,

$$D_x = \begin{vmatrix} -2 & -1 \\ 2 & 4 \end{vmatrix} = -8 - (-2) = -8 + 2 = -6$$
$$D_y = \begin{vmatrix} 3 & -2 \\ 6 & 2 \end{vmatrix} = 6 - (-12) = 6 + 12 = 18$$
$$D = \begin{vmatrix} 3 & -1 \\ 6 & 4 \end{vmatrix} = 12 - \left(-6\right) = 12 + 6 = 18$$

Use Cramer's rule to find the solution.

$$x = \frac{D_x}{D} = \frac{-6}{18} = -\frac{1}{3} \quad \text{and} \quad y = \frac{D_y}{D} = \frac{18}{18} = 1$$

Answer: $\left(-\frac{1}{3}, 1\right)$

Try this! Solve using Cramer's rule: $\begin{cases} 5x - 3y = -7 \\ -7x + 6y = 11 \end{cases}$.

Answer: $\left(-1, \ \frac{2}{3}\right)$

(click to see video)

When the determinant of the coefficient matrix *D* is zero, the formulas of Cramer's rule are undefined. In this case, the system is either dependent or inconsistent depending on the values of D_x and D_y. When $D = 0$ and both $D_x = 0$ and $D_y = 0$ the system is dependent. When $D = 0$ and either D_x or D_y is nonzero then the system is inconsistent.

$$\begin{aligned} &When\ D = 0, \\ &D_x = 0 \ \ and \ \ D_y = 0 \ \Rightarrow \ Dependent\ System \\ &D_x \neq 0 \ \ or \ \ D_y \neq 0 \ \ \ \Rightarrow \ Inconsistent\ System \end{aligned}$$

Example 5

Solve using Cramer's rule: $\begin{cases} x + \frac{1}{5}y = 3 \\ 5x + y = 15 \end{cases}$.

Solution:

The corresponding augmented matrix follows.

$$\begin{cases} x + \dfrac{1}{5}y = 3 \\ 5x + y = 15 \end{cases} \Rightarrow \left[\begin{array}{cc|c} 1 & \dfrac{1}{5} & 3 \\ 5 & 1 & 15 \end{array}\right]$$

And we have the following.

$$D_x = \begin{vmatrix} 3 & \dfrac{1}{5} \\ 15 & 1 \end{vmatrix} = 3 - 3 = 0$$
$$D_y = \begin{vmatrix} 1 & 3 \\ 5 & 15 \end{vmatrix} = 15 - 15 = 0$$
$$D = \begin{vmatrix} 1 & \dfrac{1}{5} \\ 5 & 1 \end{vmatrix} = 1 - 1 = 0$$

If we try to use Cramer's rule we have,

$$x = \frac{D_x}{D} = \frac{0}{0} \quad and \quad y = \frac{D_y}{D} = \frac{0}{0}$$

both of which are indeterminate quantities. Because $D = 0$ and both $D_x = 0$ and $D_y = 0$ we know this is a dependent system. In fact, we can see that both equations represent the same line if we solve for *y*.

$$\begin{cases} x + \frac{1}{5}y = 3 \\ 5x + y = 15 \end{cases} \Rightarrow \begin{cases} y = -5x + 15 \\ y = -5x + 15 \end{cases}$$

Therefore we can represent all solutions $(x, -5x + 15)$ where *x* is a real number.

Answer: $(x, -5x + 15)$

Try this! Solve using Cramer's rule: $\begin{cases} 3x - 2y = 10 \\ 6x - 4y = 12 \end{cases}$.

Answer: Ø

(click to see video)

Linear Systems of Three Variables and Cramer's Rule

Consider the following 3×3 coefficient matrix *A*,

$$A = \begin{bmatrix} a_1 \, b_1 \, c_1 \\ a_2 \, b_2 \, c_2 \\ a_3 \, b_3 \, c_3 \end{bmatrix}$$

The determinant of this matrix is defined as follows:

$$\begin{aligned} \det(A) &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ &= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \\ &= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \end{aligned}$$

Here each 2×2 determinant is called the **minor**[32] of the preceding factor. Notice that the factors are the elements in the first row of the matrix and that they alternate in sign (+ − +).

32. The determinant of the matrix that results after eliminating a row and column of a square matrix.

Example 6

Calculate: $\begin{vmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 0 & 5 & -1 \end{vmatrix}$.

Solution:

To easily determine the minor of each factor in the first row we line out the first row and the corresponding column. The determinant of the matrix of elements that remain determines the corresponding minor.

$$\begin{vmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 0 & 5 & -1 \end{vmatrix} \qquad \begin{vmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 0 & 5 & -1 \end{vmatrix} \qquad \begin{vmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 0 & 5 & -1 \end{vmatrix}$$

Take care to alternate the sign of the factors in the first row. The expansion by minors about the first row follows:

$$\begin{aligned}
\begin{vmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 0 & 5 & -1 \end{vmatrix} &= 1\begin{vmatrix} -1 & 3 \\ 5 & -1 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ 0 & 5 \end{vmatrix} \\
&= 1\left(1 - 15\right) - 3\left(-2 - 0\right) + 2\left(10 - 0\right) \\
&= 1\left(-14\right) - 3\left(-2\right) + 2\left(10\right) \\
&= -14 + 6 + 20 \\
&= 12
\end{aligned}$$

Answer: 12

Expansion by minors can be performed about any row or any column. The sign of the coefficients, determined by the chosen row or column, will alternate according the following sign array.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Therefore, to expand about the second row we will alternate the signs starting with the opposite of the first element. We can expand the previous example about the second row to show that the same answer for the determinant is obtained.

$$\begin{vmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 0 & 5 & -1 \end{vmatrix} \qquad \begin{vmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 0 & 5 & -1 \end{vmatrix} \qquad \begin{vmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 0 & 5 & -1 \end{vmatrix}$$

And we can write,

$$\begin{aligned} \begin{vmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 0 & 5 & -1 \end{vmatrix} &= -(2)\begin{vmatrix} 3 & 2 \\ 5 & -1 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} - (3)\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} \\ &= -2(-3-10) - 1(-1-0) - 3(5-0) \\ &= -2(-13) - 1(-1) - 3(5) \\ &= 26 + 1 - 15 \\ &= 12 \end{aligned}$$

Note that we obtain the same answer 12.

Example 7

Calculate: $\begin{vmatrix} 4 & 3 & 0 \\ 6 & \frac{1}{2} & 2 \\ 4 & 1 & 0 \end{vmatrix}$.

Solution:

The calculations are simplified if we expand about the third column because it contains two zeros.

$$\begin{vmatrix} 4 & 3 & 0 \\ 6 & \frac{1}{2} & 2 \\ 4 & 1 & 0 \end{vmatrix} \qquad \begin{vmatrix} 4 & 3 & 0 \\ 6 & \frac{1}{2} & 2 \\ 4 & 1 & 0 \end{vmatrix} \qquad \begin{vmatrix} 4 & 3 & 0 \\ 6 & \frac{1}{2} & 2 \\ 4 & 1 & 0 \end{vmatrix}$$

The expansion by minors about the third column follows:

$$\begin{aligned} \begin{vmatrix} 4 & 3 & 0 \\ 6 & \frac{1}{2} & 2 \\ 4 & 1 & 0 \end{vmatrix} &= 0 \begin{vmatrix} 6 & \frac{1}{2} \\ 4 & 1 \end{vmatrix} - 2 \begin{vmatrix} 4 & 3 \\ 4 & 1 \end{vmatrix} + 0 \begin{vmatrix} 4 & 3 \\ 6 & \frac{1}{2} \end{vmatrix} \\ &= 0 - 2\,(4 - 12) + 0 \\ &= -2\,(-8) \\ &= 16 \end{aligned}$$

Answer: 16

It should be noted that there are other techniques used for remembering how to calculate the determinant of a 3×3 matrix. In addition, many modern calculators and computer algebra systems can find the determinant of matrices. You are encouraged to research this rich topic.

We can solve linear systems with three variables using determinants. To do this, we begin with the augmented coefficient matrix,

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases} \quad \Leftrightarrow \quad \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

Let D represent the determinant of the coefficient matrix,

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Then determine D_x, D_y, and D_z by calculating the following determinants.

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

When $D \neq 0$, the solution to the system in terms of the determinants described above can be calculated using Cramer's rule:

Cramer's Rule

$$(x, y, z) = \left(\frac{D_x}{D}, \frac{D_y}{D}, \frac{D_z}{D}\right)$$

Use this to efficiently solve systems with three variables.

Example 8

Solve using Cramer's rule: $\begin{cases} 3x + 7y - 4z = 0 \\ 2x + 5y - 3z = 1 \\ -5x + 2y + 4z = 8 \end{cases}$.

Solution:

Begin by determining the corresponding augmented matrix.

$$\begin{cases} 3x + 7y - 4z = 0 \\ 2x + 5y - 3z = 1 \\ -5x + 2y + 4z = 8 \end{cases} \Leftrightarrow \left[\begin{array}{ccc|c} 3 & 7 & -4 & 0 \\ 2 & 5 & -3 & 1 \\ -5 & 2 & 4 & 8 \end{array}\right]$$

Next, calculate the determinant of the coefficient matrix.

$$\begin{aligned} D &= \begin{vmatrix} 3 & 7 & -4 \\ 2 & 5 & -3 \\ -5 & 2 & 4 \end{vmatrix} \\ &= 3\begin{vmatrix} 5 & -3 \\ 2 & 4 \end{vmatrix} - 7\begin{vmatrix} 2 & -3 \\ -5 & 4 \end{vmatrix} + (-4)\begin{vmatrix} 2 & 5 \\ -5 & 2 \end{vmatrix} \\ &= 3(20 - (-6)) - 7(8 - 15) - 4(4 - (-25)) \\ &= 3(26) - 7(-7) - 4(29) \\ &= 78 + 49 - 116 \\ &= 11 \end{aligned}$$

Similarly we can calculate D_x, D_y, and D_z. This is left as an exercise.

$$D_x = \begin{vmatrix} 0 & 7 & -4 \\ 1 & 5 & -3 \\ 8 & 2 & 4 \end{vmatrix} = -44$$
$$D_y = \begin{vmatrix} 3 & 0 & -4 \\ 2 & 1 & -3 \\ -5 & 8 & 4 \end{vmatrix} = 0$$
$$D_z = \begin{vmatrix} 3 & 7 & 0 \\ 2 & 5 & 1 \\ -5 & 2 & 8 \end{vmatrix} = -33$$

Using Cramer's rule we have,

$$x = \frac{D_x}{D} = \frac{-44}{11} = -4 \qquad y = \frac{D_y}{D} = \frac{0}{11} = 0 \qquad z = \frac{D_z}{D} = \frac{-33}{11} = -3$$

Answer: $(-4, 0, -3)$

If the determinant of the coefficient matrix $D = 0$, then the system is either dependent or inconsistent. This will depend on D_x, D_y, and D_z. If they are all zero, then the system is dependent. If at least one of these is nonzero, then it is inconsistent.

$$\begin{aligned} &When\ D = 0, \\ &D_x = 0 \ and \ D_y = 0 \ and \ D_z = 0 \ \Rightarrow \ Dependent\ System \\ &D_x \neq 0 \ or \ D_y \neq 0 \ or \ D_z \neq 0 \ \Rightarrow \ Inconsistent\ System \end{aligned}$$

Example 9

Solve using Cramer's rule: $\begin{cases} 4x - y + 3z = 5 \\ 21x - 4y + 18z = 7 \\ -9x + y - 9z = -8 \end{cases}$.

Solution:

Begin by determining the corresponding augmented matrix.

$$\begin{cases} 4x - y + 3z = 5 \\ 21x - 4y + 18z = 7 \\ -9x + y - 9z = -8 \end{cases} \quad \Leftrightarrow \quad \left[\begin{array}{rrr|r} 4 & -1 & 3 & 5 \\ 21 & -4 & 18 & 7 \\ -9 & 1 & -9 & -8 \end{array}\right]$$

Next, determine the determinant of the coefficient matrix.

$$\begin{aligned} D &= \begin{vmatrix} 4 & -1 & 3 \\ 21 & -4 & 18 \\ -9 & 1 & -9 \end{vmatrix} \\ &= 4\begin{vmatrix} -4 & 18 \\ 1 & -9 \end{vmatrix} - (-1)\begin{vmatrix} 21 & 18 \\ -9 & -9 \end{vmatrix} + 3\begin{vmatrix} 21 & -4 \\ -9 & 1 \end{vmatrix} \\ &= 4(36 - 18) + 1(-189 - (-162)) + 3(21 - 36) \\ &= 4(18) + 1(-27) + 3(-15) \\ &= 72 - 27 - 45 \\ &= 0 \end{aligned}$$

Since $D = 0$, the system is either dependent or inconsistent.

$$D_x = \begin{vmatrix} 5 & -1 & 3 \\ 7 & -4 & 18 \\ -8 & 1 & -9 \end{vmatrix} = 96$$

However, because D_x is nonzero we conclude the system is inconsistent. There is no simultaneous solution.

Answer: Ø

Try this! Solve using Cramer's rule: $\begin{cases} 2x + 6y + 7z = 4 \\ -3x - 4y + 5z = 12 \\ 5x + 10y - 3z = -13 \end{cases}$.

Answer: $\left(-3, \frac{1}{2}, 1\right)$

(click to see video)

KEY TAKEAWAYS

- The determinant of a matrix is a real number.
- The determinant of a 2×2 matrix is obtained by subtracting the product of the values on the diagonals.
- The determinant of a 3×3 matrix is obtained by expanding the matrix using minors about any row or column. When doing this, take care to use the sign array to help determine the sign of the coefficients.
- Use Cramer's rule to efficiently determine solutions to linear systems.
- When the determinant of the coefficient matrix is 0, Cramer's rule does not apply; the system will either be dependent or inconsistent.

TOPIC EXERCISES

PART A: LINEAR SYSTEMS WITH TWO VARIABLES

Calculate the determinant.

1. $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$
2. $\begin{vmatrix} 5 & 3 \\ 2 & 4 \end{vmatrix}$
3. $\begin{vmatrix} -1 & 3 \\ -3 & -2 \end{vmatrix}$
4. $\begin{vmatrix} 7 & 4 \\ 3 & -2 \end{vmatrix}$
5. $\begin{vmatrix} -4 & 1 \\ -3 & 0 \end{vmatrix}$
6. $\begin{vmatrix} 9 & 5 \\ -1 & 0 \end{vmatrix}$
7. $\begin{vmatrix} 1 & 0 \\ 5 & 0 \end{vmatrix}$
8. $\begin{vmatrix} 0 & 3 \\ 5 & 0 \end{vmatrix}$
9. $\begin{vmatrix} 0 & 4 \\ -1 & 3 \end{vmatrix}$
10. $\begin{vmatrix} 10 & 2 \\ 10 & 2 \end{vmatrix}$
11. $\begin{vmatrix} a_1 & b_1 \\ 0 & b_2 \end{vmatrix}$
12. $\begin{vmatrix} 0 & b_1 \\ a_2 & b_2 \end{vmatrix}$

Solve using Cramer's rule.

13. $\begin{cases} 3x - 5y = 8 \\ 2x - 7y = 9 \end{cases}$

14. $\begin{cases} 2x + 3y = -1 \\ 3x + 4y = -2 \end{cases}$

15. $\begin{cases} 2x - y = -3 \\ 4x + 3y = 4 \end{cases}$

16. $\begin{cases} x + 3y = 1 \\ 5x - 6y = -9 \end{cases}$

17. $\begin{cases} x + y = 1 \\ 6x + 3y = 2 \end{cases}$

18. $\begin{cases} x - y = -1 \\ 5x + 10y = 4 \end{cases}$

19. $\begin{cases} 5x - 7y = 14 \\ 4x - 3y = 6 \end{cases}$

20. $\begin{cases} 9x + 5y = -9 \\ 7x + 2y = -7 \end{cases}$

21. $\begin{cases} 6x - 9y = 3 \\ -2x + 3y = 1 \end{cases}$

22. $\begin{cases} 3x - 9y = 3 \\ 2x - 6y = 2 \end{cases}$

23. $\begin{cases} 4x - 5y = 20 \\ 3y = -9 \end{cases}$

24. $\begin{cases} x - y = 0 \\ 2x - 3y = 0 \end{cases}$

25. $\begin{cases} 2x + y = a \\ x + y = b \end{cases}$

26. $\begin{cases} ax + y = 0 \\ by = 1 \end{cases}$

PART B: LINEAR SYSTEMS WITH THREE VARIABLES

Calculate the determinant.

27. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{vmatrix}$

28. $\begin{vmatrix} 2 & 5 & 1 \\ 1 & 2 & 4 \\ 3 & 2 & 3 \end{vmatrix}$

29. $\begin{vmatrix} -3 & 1 & -1 \\ 3 & -1 & -2 \\ -2 & 5 & 1 \end{vmatrix}$

30. $\begin{vmatrix} 1 & -1 & 5 \\ -4 & 5 & -1 \\ -1 & 2 & -3 \end{vmatrix}$

31. $\begin{vmatrix} 3 & -1 & 2 \\ 2 & 3 & -1 \\ 5 & 2 & 1 \end{vmatrix}$

32. $\begin{vmatrix} 4 & 0 & -3 \\ 3 & -1 & 0 \\ 0 & -5 & 2 \end{vmatrix}$

33. $\begin{vmatrix} 0 & -3 & 4 \\ -3 & 0 & 6 \\ 0 & 2 & -3 \end{vmatrix}$

34. $\begin{vmatrix} 6 & -1 & -3 \\ 2 & 5 & 2 \\ 8 & 4 & -1 \end{vmatrix}$

35. $\begin{vmatrix} 2 & 5 & 7 \\ 0 & 3 & 5 \\ 0 & 0 & 4 \end{vmatrix}$

36. $\begin{vmatrix} 2 & 10 & 9 \\ 0 & 3 & 13 \\ 0 & 0 & 4 \end{vmatrix}$

37. $\begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{vmatrix}$

38. $\begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Solve using Cramer's rule.

39. $\begin{cases} x - y + 2z = -3 \\ 3x + 2y - z = 13 \\ -4x - 3y + z = -18 \end{cases}$

40. $\begin{cases} 3x + 4y - z = 10 \\ 4x + 6y + 7z = 9 \\ 2x + 3y + 5z = 3 \end{cases}$

41. $\begin{cases} 5x + y - z = 0 \\ 2x - 2y + z = -9 \\ -6x - 5y + 3z = -13 \end{cases}$

42. $\begin{cases} -4x + 5y + 2z = 12 \\ 3x - y - z = -2 \\ 5x + 3y - 2z = 5 \end{cases}$

43. $\begin{cases} x - y + z = -1 \\ -2x + 4y - 3z = 4 \\ 3x - 3y - 2z = 2 \end{cases}$

44. $\begin{cases} 2x + y - 4z = 7 \\ 2x - 3y + 2z = -4 \\ 4x - 5y + 2z = -5 \end{cases}$

45. $\begin{cases} 4x + 3y - 2z = 2 \\ 2x + 5y + 8z = -1 \\ x - y - 5z = 3 \end{cases}$

46. $\begin{cases} x - y + z = 7 \\ x + 2y + z = 1 \\ x - 2y - 2z = 9 \end{cases}$

47. $\begin{cases} 3x - 6y + 2z = 12 \\ -5x - 2y + 3z = 4 \\ 7x + 3y - 4z = -6 \end{cases}$

48. $\begin{cases} 2x - y - 5z = 2 \\ 3x + 2y - 4z = -3 \\ 5x + y - 9z = 4 \end{cases}$

49. $\begin{cases} 4x + 3y - 4z = -13 \\ 2x + 6y - 5z = -2 \\ -2x - 3y + 3z = 5 \end{cases}$

50. $\begin{cases} x - 2y + z = -1 \\ 4y - 3z = 0 \\ 3y - 2z = 1 \end{cases}$

51. $\begin{cases} 2x + 3y - z = -5 \\ x + 2y = 0 \\ 3x + 10y = 4 \end{cases}$

52. $\begin{cases} 2x - 3y - 2y = 9 \\ -3x + 4y + 4z = -13 \\ x - y - 2z = 4 \end{cases}$

53. $\begin{cases} 2x + y - 2z = -1 \\ x - y + 3z = 2 \\ 3x + y - z = 1 \end{cases}$

54. $\begin{cases} 3x - 8y + 9z = -2 \\ -x + 5y - 10z = 3 \\ x - 3y + 4z = -1 \end{cases}$

55. $\begin{cases} 5x - 6y + 3z = 2 \\ 3x - 4y + 2z = 0 \\ 2x - 2y + z = 0 \end{cases}$

56. $\begin{cases} 5x + 10y - 4z = 12 \\ 2x + 5y + 4z = 0 \\ x + 5y - 8z = 6 \end{cases}$

57. $\begin{cases} 5x + 6y + 7z = 2 \\ 2y + 3z = 3 \\ 4z = 4 \end{cases}$

58. $\begin{cases} x + 2z = -1 \\ -5y + 3z = 10 \\ 4x - 3y = 2 \end{cases}$

59. $\begin{cases} x + y + z = a \\ x + 2y + 2z = a + b \\ x + 2y + 3z = a + b + c \end{cases}$

60. $\begin{cases} x + y + z = a + b + c \\ x + 2y + 2z = a + 2b + 2c \\ x + y + 2z = a + b + 2c \end{cases}$

PART C: DISCUSSION BOARD

61. Research and discuss the history of the determinant. Who is credited for first introducing the notation of a determinant?

62. Research other ways in which we can calculate the determinant of a 3×3 matrix. Give an example.

ANSWERS

1. −2
3. 11
5. 3
7. 0
9. 4
11. $a_1 b_2$
13. (1, −1)
15. $\left(-\frac{1}{2}, 2\right)$
17. $\left(-\frac{1}{3}, \frac{4}{3}\right)$
19. (0, −2)
21. Ø
23. $\left(\frac{5}{4}, -3\right)$
25. $(a - b, 2b - a)$
27. 6
29. −39
31. 0
33. 3
35. 24
37. $a_1 b_2 c_3$
39. (2, 3, −1)
41. (−1, 2, −3)
43. $\left(\frac{1}{2}, \frac{1}{2}, -1\right)$
45. Ø

47. (0, −2, 0)

49. $\left(\frac{1}{2}z - 4, \frac{2}{3}z + 1, z\right)$

51. (−2, 1, 4)

53. $\left(-\frac{1}{2}, 5, \frac{5}{2}\right)$

55. Ø

57. (−1, 0, 1)

59. $\left(a - b, b - c, c\right)$

61. Answer may vary

3.7 Solving Systems of Inequalities with Two Variables

LEARNING OBJECTIVES

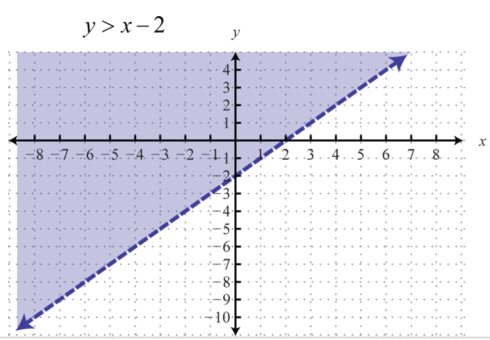
1. Check solutions to systems of inequalities with two variables.
2. Graph solution sets of systems of inequalities.

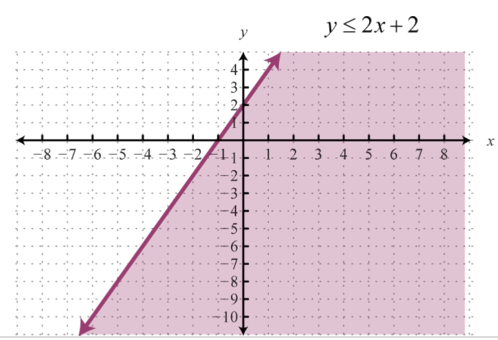
Solutions to Systems of Inequalities

A **system of inequalities**[33] consists of a set of two or more inequalities with the same variables. The inequalities define the conditions that are to be considered simultaneously. For example,

$$\begin{cases} y > x - 2 \\ y \leq 2x + 2 \end{cases}$$

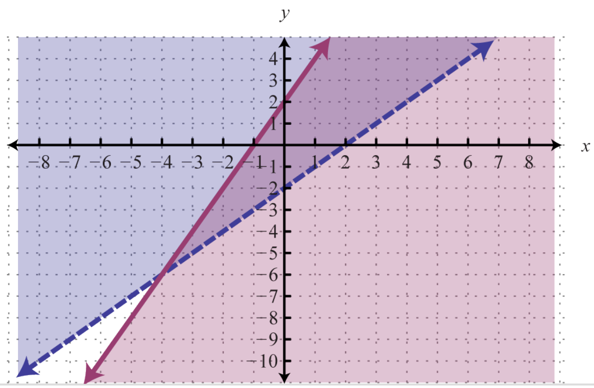
We know that each inequality in the set contains infinitely many ordered pair solutions defined by a region in a rectangular coordinate plane. When considering two of these inequalities together, the intersection of these sets will define the set of simultaneous ordered pair solutions. When we graph each of the above inequalities separately we have:



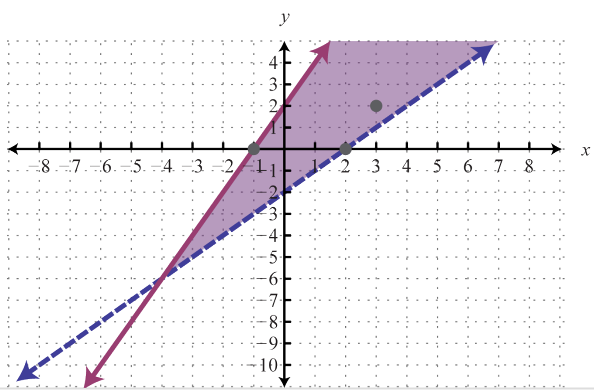


And when graphed on the same set of axes, the intersection can be determined.

33. A set of two or more inequalities with the same variables.



The intersection is shaded darker and the final graph of the solution set will be presented as follows:



The graph suggests that (3, 2) is a solution because it is in the intersection. To verify this, we can show that it solves both of the original inequalities as follows:

$Check: (3,2)$	
$Inequality\ 1:$ $y > x - 2$ $2 > 3 - 2$ $2 > 1$ ✓	$Inequality\ 2:$ $y \leq 2x + 2$ $2 \leq 2(3) + 2$ $2 \leq 8$ ✓

Points on the solid boundary are included in the set of simultaneous solutions and points on the dashed boundary are not. Consider the point (−1, 0) on the solid boundary defined by $y = 2x + 2$ and verify that it solves the original system:

$Check: (-1,0)$	
$Inequality\ 1:$ $y > x - 2$ $0 > -1 - 2$ $0 > -3$ ✓	$Inequality\ 2:$ $y \leq 2x + 2$ $0 \leq 2(-1) + 2$ $0 \leq 0$ ✓

Notice that this point satisfies both inequalities and thus is included in the solution set. Now consider the point (2, 0) on the dashed boundary defined by $y = x - 2$ and verify that it does not solve the original system:

$Check:\ (2,0)$	
$Inequality\,1:$ $y > x - 2$ $0 > 2 - 2$ $0 > 0$ ✗	$Inequality\,2:$ $y \leq 2x + 2$ $0 \leq 2(2) + 2$ $0 \leq 6$ ✓

This point does not satisfy both inequalities and thus is not included in the solution set.

Example 1

Determine whether or not (-3, 3) is a solution to the following system:

$$\begin{cases} 2x + 6y \leq 6 \\ -\frac{1}{3}x - y \leq 3 \end{cases}$$

Solution:

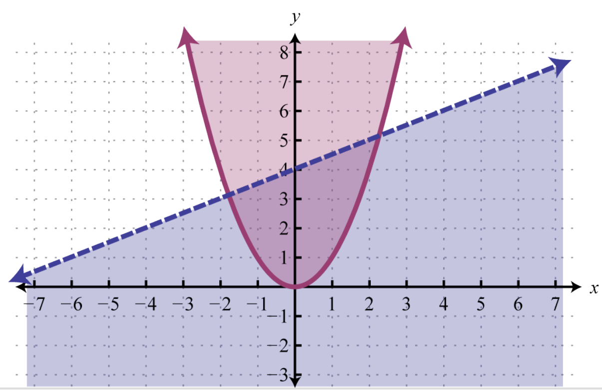
Substitute the coordinates of (x, y) = (-3, 3) into both inequalities.

$Check:\ (-3, 3)$	
$\begin{aligned} &Inequality\ 1: \\ 2x + 6y &\leq 6 \\ 2(-3) + 6(3) &\leq 6 \\ -6 + 18 &\leq 6 \\ 12 &\leq 6 \quad ✗ \end{aligned}$	$\begin{aligned} &Inequality\ 2: \\ -\tfrac{1}{3}x - y &\leq 3 \\ -\tfrac{1}{3}(-3) - (3) &\leq 3 \\ 1 - 3 &\leq 3 \\ -2 &\leq 3 \quad ✓ \end{aligned}$

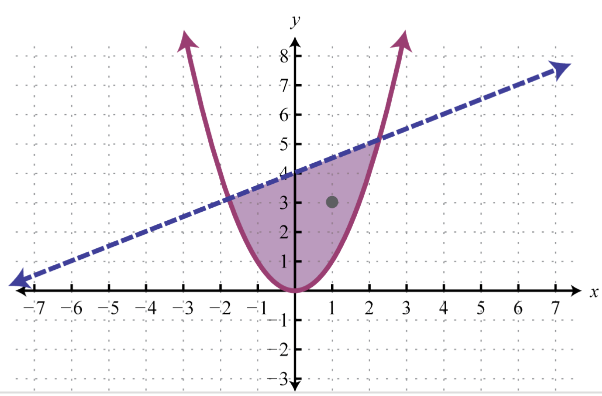
Answer: (-3, 3) is not a solution; it does not satisfy both inequalities.

We can graph the solutions of systems that contain nonlinear inequalities in a similar manner. For example, both solution sets of the following inequalities can be graphed on the same set of axes:

$$\begin{cases} y < \frac{1}{2}x + 4 \\ y \geq x^2 \end{cases}$$



And the intersection of both regions contains the region of simultaneous ordered pair solutions.



From the graph, we expect the ordered pair (1, 3) to solve both inequalities.

$Check: \ (1,3)$	
$Inequality\,1:$ $y < \frac{1}{2}x + 4$ $3 < \frac{1}{2}(1) + 4$ $3 < 4\frac{1}{2}$ ✓	$Inequality\,2:$ $y \geq x^2$ $3 \geq (1)^2$ $3 \geq 1$ ✓

Graphing Solutions to Systems of Inequalities

Solutions to a system of inequalities are the ordered pairs that solve all the inequalities in the system. Therefore, to solve these systems we graph the solution sets of the inequalities on the same set of axes and determine where they intersect. This intersection, or overlap, will define the region of common ordered pair solutions.

Example 2

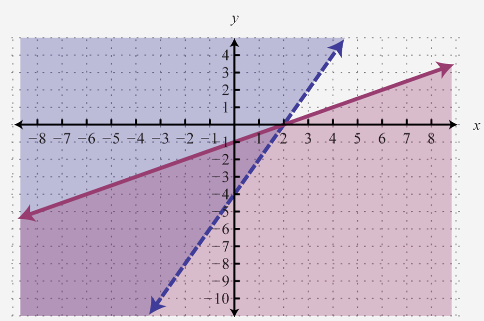
Graph the solution set: $\begin{cases} -2x + y > -4 \\ 3x - 6y \geq 6 \end{cases}$.

Solution:

To facilitate the graphing process, we first solve for y.

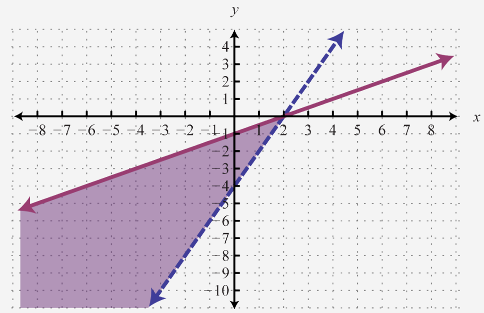
$$\begin{cases} -2x + y > -4 \\ 3x - 6y \geq 6 \end{cases} \quad \Rightarrow \quad \begin{cases} y > 2x - 4 \\ y \leq \dfrac{1}{2}x - 1 \end{cases}$$

For the first inequality, we use a dashed boundary defined by $y = 2x - 4$ and shade all points above the line. For the second inequality, we use a solid boundary defined by $y = \frac{1}{2}x - 1$ and shade all points below. The intersection is darkened.



Now we present our solution with only the intersection shaded.

Answer:


y
x
4
3
2
1
−8
−7
−6
−5
−4
−3
−2
−1
1
2
3
4
5
6
7
8
−1
−2
−3
−4
−5
−6
−7
−8
−9
−10

Example 3

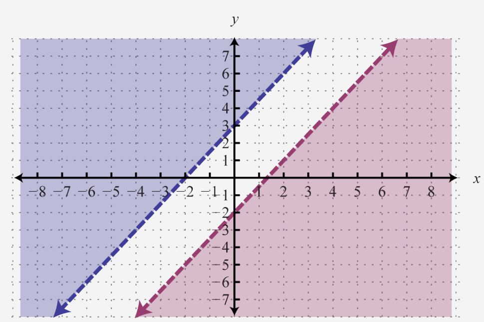
Graph the solution set: $\begin{cases} -3x + 2y > 6 \\ 6x - 4y > 8 \end{cases}$.

Solution:

We begin by solving both inequalities for y.

$$\begin{cases} -3x + 2y > 6 \\ 6x - 4y > 8 \end{cases} \quad \Rightarrow \quad \begin{cases} y > \frac{3}{2}x + 3 \\ y < \frac{3}{2}x - 2 \end{cases}$$

Because of the strict inequalities, we will use a dashed line for each boundary. For the first inequality shade all points above the boundary and for the second inequality shade all points below the boundary.



As we can see, there is no intersection of these two shaded regions. Therefore, there are no simultaneous solutions.

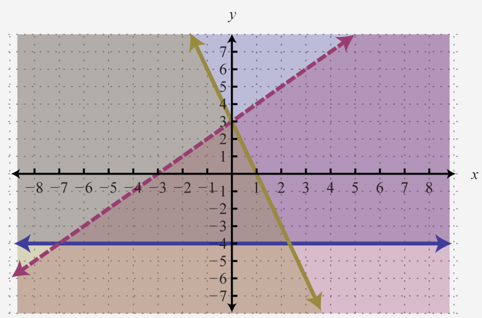
Answer: Ø

Example 4

Graph the solution set: $\begin{cases} y \geq -4 \\ y < x + 3 \\ y \leq -3x + 3 \end{cases}$.

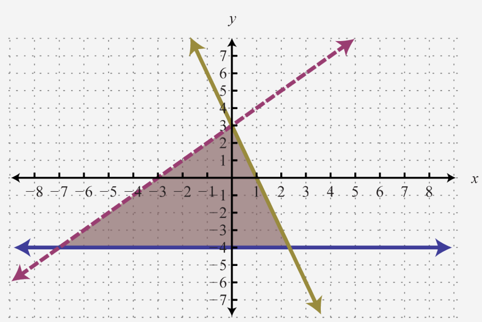
Solution:

Begin by graphing the solution sets to all three inequalities.



After graphing all three inequalities on the same set of axes, we determine that the intersection lies in the triangular region pictured below.

Answer:



The graph suggests that (−1, 1) is a simultaneous solution. As a check, we could substitute that point into the inequalities and verify that it solves all three conditions.

$Check: (-1, 1)$		
$Inequality\,1:$ $y \geq -4$ $1 \geq -4$ ✓	$Inequality\,2:$ $y < x + 3$ $1 < -1 + 3$ $1 < 2$ ✓	$Inequality\,3:$ $y \leq -3x +$ $1 \leq -3(-$ $1 \leq 3 + 3$ $1 \leq 6$ ✓

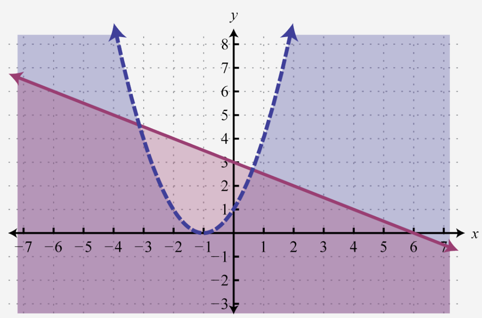
Use the same technique to graph the solution sets to systems of nonlinear inequalities.

Example 5

Graph the solution set: $\begin{cases} y < (x+1)^2 \\ y \leq -\frac{1}{2}x + 3 \end{cases}$.

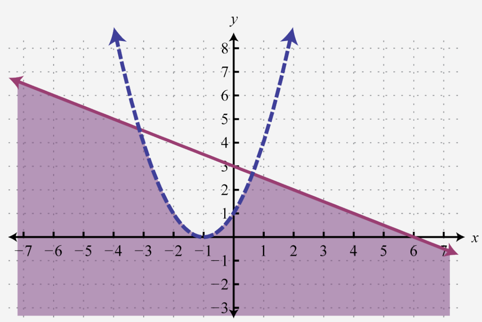
Solution:

The first inequality has a parabolic boundary. This boundary is a horizontal translation of the basic function $y = x^2$ to the left 1 unit. Because of the strict inequality, the boundary is dashed, indicating that it is not included in the solution set. The second inequality is linear and will be graphed with a solid boundary. Solution sets to both are graphed below.



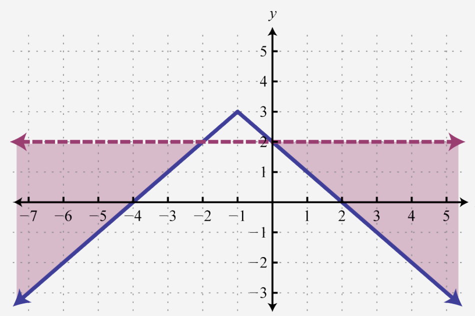
After graphing the inequalities on the same set of axes, we determine that the intersection lies in the region pictured below.

Answer:



Try this! Graph the solution set: $\begin{cases} y \geq -|x+1|+3 \\ y \leq 2 \end{cases}$.

Answer:



(click to see video)

KEY TAKEAWAYS

- To graph solutions to systems of inequalities, graph the solution sets of each inequality on the same set of axes and determine where they intersect.
- You can check your answer by choosing a few values inside and out of the shaded region to see if they satisfy the inequalities or not. While this is not a proof, doing so will give a good indication that you have graphed the correct region.

TOPIC EXERCISES

PART A: SOLUTIONS TO SYSTEMS OF INEQUALITIES

Determine whether or not the given point is a solution to the given system of inequalities.

1. (-2, 1);

$$\begin{cases} y > 3x + 5 \\ y \leq -x + 1 \end{cases}$$

2. (-1, -3);

$$\begin{cases} y \geq 3x - 1 \\ y < -2x \end{cases}$$

3. (-2, -1);

$$\begin{cases} x - 2y > -1 \\ 3x - y < -3 \end{cases}$$

4. (0, -5);

$$\begin{cases} 5x - y \geq 5 \\ 3x + 2y < -1 \end{cases}$$

5. $\left(-\frac{1}{2}, 0\right)$;

$$\begin{cases} -8x + 5y \geq 3 \\ 2x - 3y < 0 \end{cases}$$

6. $\left(-1, \frac{1}{3}\right)$;

$$\begin{cases} 2x - 9y < -1 \\ 3x - 6y > -2 \end{cases}$$

7. (-1, -2);

$$\begin{cases} 2x - y \geq -1 \\ x - 3y < 6 \\ 2x - 3y > -1 \end{cases}$$

8. (-5, 2);

$$\begin{cases} -x + 5y > 10 \\ 2x + y < 1 \\ x + 3y < -2 \end{cases}$$

9. (0, 3);

$$\begin{cases} y + 4 \geq 0 \\ \frac{1}{2}x + \frac{1}{3}y \leq 1 \\ -3x + 2y \leq 6 \end{cases}$$

10. (1, 1);

$$\begin{cases} y \leq -\frac{3}{4}x + 2 \\ y \geq -5x + 2 \\ y \geq \frac{1}{3}x - 1 \end{cases}$$

11. (-1, 2);

$$\begin{cases} y \geq x^2 + 1 \\ y < -2x + 3 \end{cases}$$

12. (4, 5);

$$\begin{cases} y < (x-1)^2 - 1 \\ y > \frac{1}{2}x - 1 \end{cases}$$

13. (-2, -3);

$$\begin{cases} y < 0 \\ y \geq -|x| + 4 \end{cases}$$

14. (1, 2);

$$\begin{cases} y < |x - 3| + 2 \\ y \geq 2 \end{cases}$$

15. $\left(-\frac{1}{2}, -5\right)$;

$$\begin{cases} y \leq -3x - 5 \\ y > (x-1)^2 - 10 \end{cases}$$

16. (−4, 1);

$$\begin{cases} x \geq -5 \\ y < (x+3)^2 - 2 \end{cases}$$

17. $\left(-\frac{3}{2}, \frac{1}{3}\right)$;

$$\begin{cases} x - 2y \leq 4 \\ y \leq |3x - 1| + 2 \end{cases}$$

18. $\left(-3, -\frac{3}{4}\right)$;

$$\begin{cases} 3x - 4y < 24 \\ y < (x+2)^2 - 1 \end{cases}$$

19. (4, 2);

$$\begin{cases} y < (x-3)^2 + 1 \\ y < -\frac{3}{4}x + 5 \end{cases}$$

20. $\left(\frac{5}{2}, 1\right)$;

$$\begin{cases} y \geq -1 \\ y < -(x-2)^2 + 3 \end{cases}$$

PART B: SOLVING SYSTEMS OF INEQUALITIES

Graph the solution set.

21. $\begin{cases} y \geq \frac{2}{3}x - 3 \\ y < -\frac{1}{3}x + 3 \end{cases}$

22. $\begin{cases} y \geq -\frac{1}{4}x + 1 \\ y < \frac{1}{2}x - 2 \end{cases}$

23. $\begin{cases} y > \dfrac{2}{3}x + 1 \\ y > \dfrac{4}{3}x - 5 \end{cases}$

24. $\begin{cases} y \leq -5x + 4 \\ y < \dfrac{4}{3}x - 2 \end{cases}$

25. $\begin{cases} x - y \geq -3 \\ x + y \geq 3 \end{cases}$

26. $\begin{cases} 3x + y < 4 \\ 2x - y \leq 1 \end{cases}$

27. $\begin{cases} -x + 2y \leq 0 \\ 3x + 5y < 15 \end{cases}$

28. $\begin{cases} 2x + 3y < 6 \\ -4x + 3y \geq -12 \end{cases}$

29. $\begin{cases} 3x + 2y > 1 \\ 4x - 2y > 3 \end{cases}$

30. $\begin{cases} x - 4y \geq 2 \\ 8x + 4y \leq 3 \end{cases}$

31. $\begin{cases} 5x - 2y \leq 6 \\ -5x + 2y < 2 \end{cases}$

32. $\begin{cases} 12x + 10y > 20 \\ 18x + 15y < -15 \end{cases}$

33. $\begin{cases} x + y < 0 \\ y + 4 > 0 \end{cases}$

34. $\begin{cases} x > -3 \\ y < 1 \end{cases}$

35. $\begin{cases} 2x - 2y < 0 \\ 3x - 3y > 3 \end{cases}$

36. $\begin{cases} y+1 \leq 0 \\ y+3 \geq 0 \end{cases}$

37. Construct a system of linear inequalities that describes all points in the first quadrant.

38. Construct a system of linear inequalities that describes all points in the second quadrant.

39. Construct a system of linear inequalities that describes all points in the third quadrant.

40. Construct a system of linear inequalities that describes all points in the fourth quadrant.

Graph the solution set.

41. $\begin{cases} y \geq -\frac{1}{2}x+3 \\ y \geq \frac{3}{2}x-3 \\ y \leq \frac{3}{2}x+1 \end{cases}$

42. $\begin{cases} y \leq -\frac{3}{4}x+2 \\ y \geq -5x+2 \\ y \geq \frac{1}{3}x-1 \end{cases}$

43. $\begin{cases} 3x-2y > 6 \\ 5x+2y > 8 \\ -3x+4y \leq 4 \end{cases}$

44. $\begin{cases} 3x-5y > -15 \\ 5x-2y \leq 8 \\ x+y < -1 \end{cases}$

45. $\begin{cases} 3x-2y < -1 \\ 5x+2y > 7 \\ y+1 > 0 \end{cases}$

46. $$\begin{cases} 3x - 2y < -1 \\ 5x + 2y < 7 \\ y + 1 > 0 \end{cases}$$

47. $$\begin{cases} 4x + 5y - 8 < 0 \\ y > 0 \\ x + 3 > 0 \end{cases}$$

48. $$\begin{cases} y - 2 < 0 \\ y + 2 > 0 \\ 2x - y \geq 0 \end{cases}$$

49. $$\begin{cases} \frac{1}{2}x + \frac{1}{2}y < 1 \\ x < 3 \\ -\frac{1}{2}x + \frac{1}{2}y \leq 1 \end{cases}$$

50. $$\begin{cases} \frac{1}{2}x + \frac{1}{3}y \leq 1 \\ y + 4 \geq 0 \\ -\frac{1}{2}x + \frac{1}{3}y \leq 1 \end{cases}$$

51. $$\begin{cases} y < x + 2 \\ y \geq x^2 - 3 \end{cases}$$

52. $$\begin{cases} y \geq x^2 + 1 \\ y > -\frac{3}{4}x + 3 \end{cases}$$

53. $$\begin{cases} y \leq (x + 2)^2 \\ y \leq \frac{1}{3}x + 4 \end{cases}$$

54. $$\begin{cases} y < (x - 3)^2 + 1 \\ y < -\frac{3}{4}x + 5 \end{cases}$$

55. $$\begin{cases} y \geq -1 \\ y < -(x - 2)^2 + 3 \end{cases}$$

56. $\begin{cases} y < -(x+1)^2 - 1 \\ y < \frac{3}{2}x - 2 \end{cases}$

57. $\begin{cases} y \leq \frac{1}{3}x + 3 \\ y \geq |x+3| - 2 \end{cases}$

58. $\begin{cases} y \leq -x + 5 \\ y > |x-1| + 2 \end{cases}$

59. $\begin{cases} y > -|x-2| + 5 \\ y > 2 \end{cases}$

60. $\begin{cases} y \leq -|x| + 3 \\ y < \frac{1}{4}x \end{cases}$

61. $\begin{cases} y > |x| + 1 \\ y \leq x - 1 \end{cases}$

62. $\begin{cases} y \leq |x| + 1 \\ y > x - 1 \end{cases}$

63. $\begin{cases} y \leq |x-3| + 1 \\ x \leq 2 \end{cases}$

64. $\begin{cases} y > |x+1| \\ y < x - 2 \end{cases}$

65. $\begin{cases} y < x^3 + 2 \\ y \leq x + 3 \end{cases}$

66. $\begin{cases} y \leq 4 \\ y \geq (x+3)^3 + 1 \end{cases}$

67. $\begin{cases} y \geq -2x + 6 \\ y > \sqrt{x} + 3 \end{cases}$

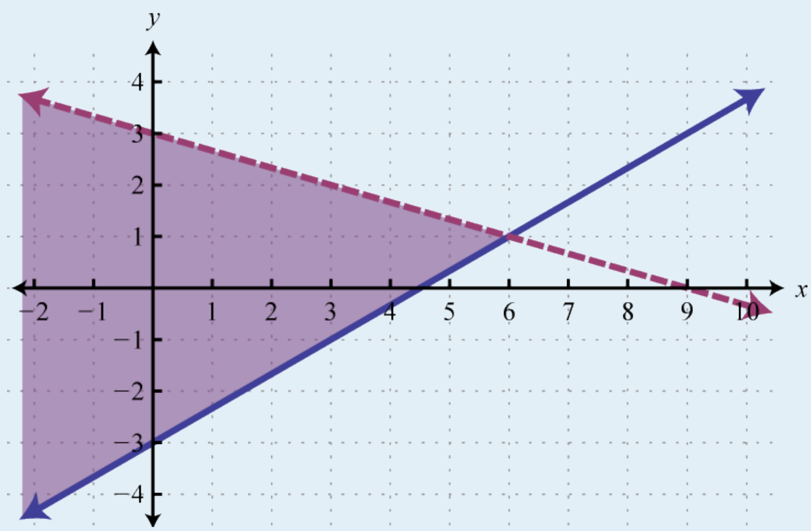
68. $\begin{cases} y \leq \sqrt{x+4} \\ x \leq -1 \end{cases}$

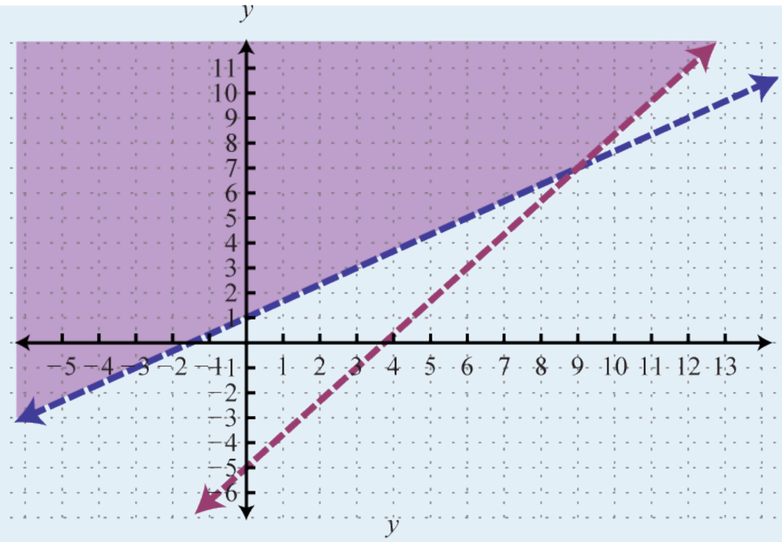
69. $\begin{cases} y \leq -x^2 + 4 \\ y \geq x^2 - 4 \end{cases}$

70. $\begin{cases} y \geq |x - 1| - 3 \\ y \leq -|x - 1| + 3 \end{cases}$

ANSWERS

1. Yes
3. Yes
5. Yes
7. Yes
9. Yes
11. Yes
13. No
15. Yes
17. Yes
19. No
21.

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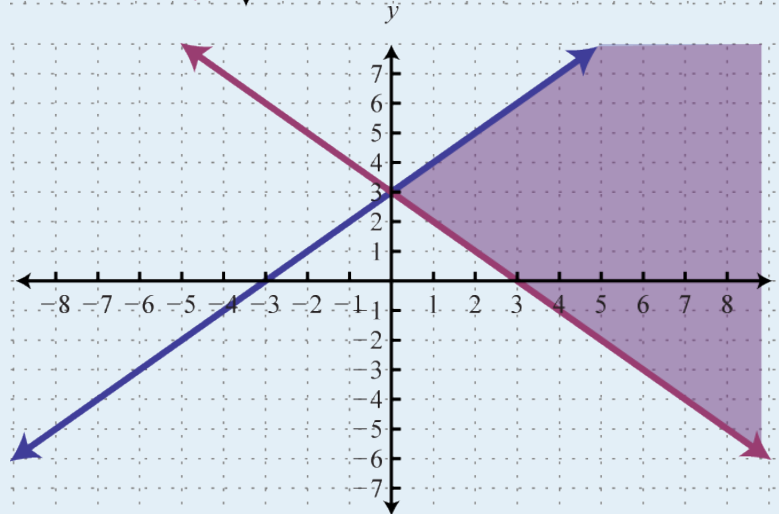
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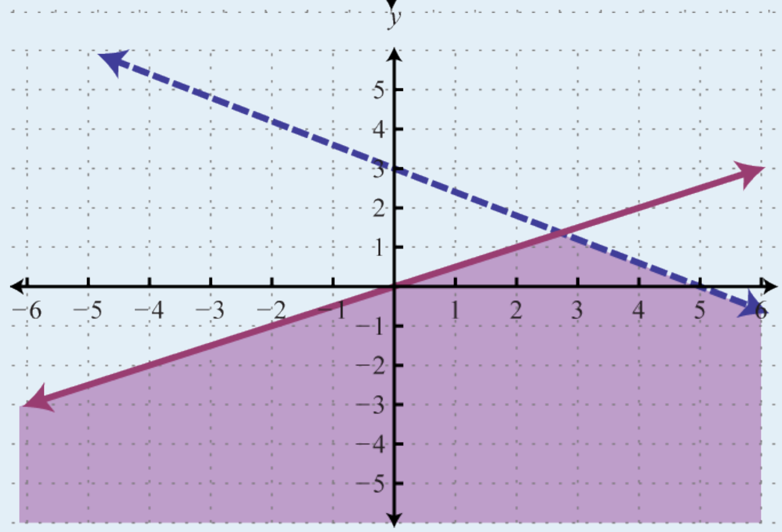
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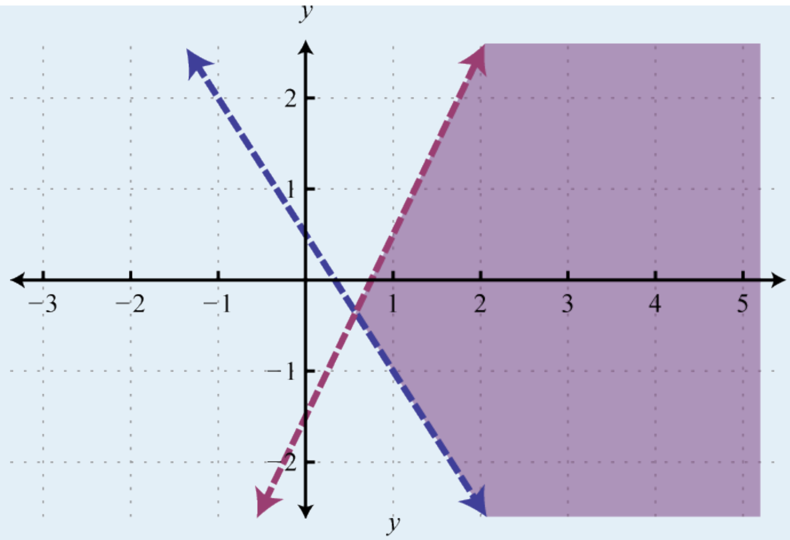

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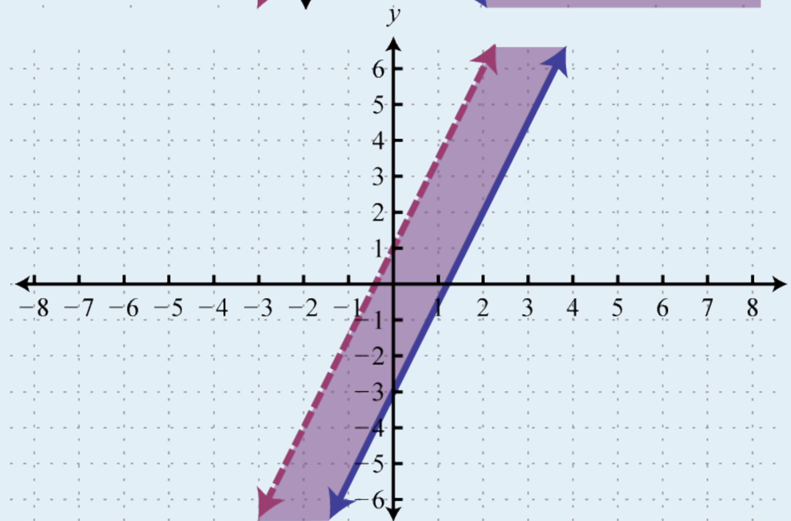

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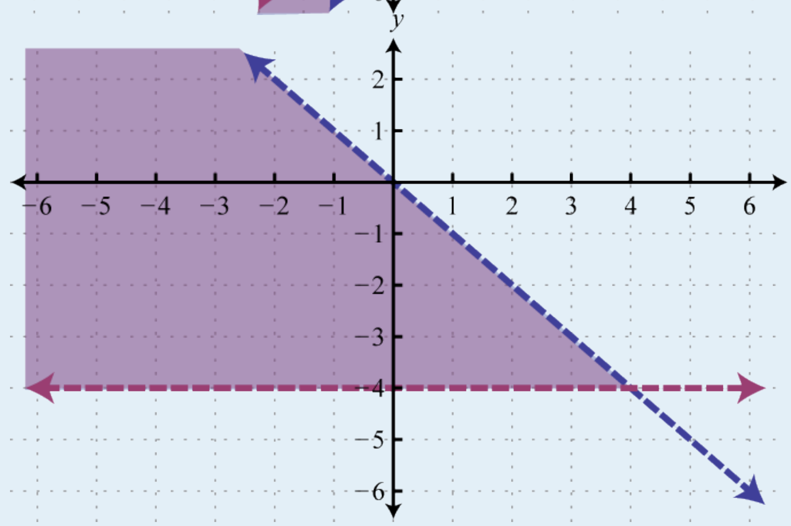
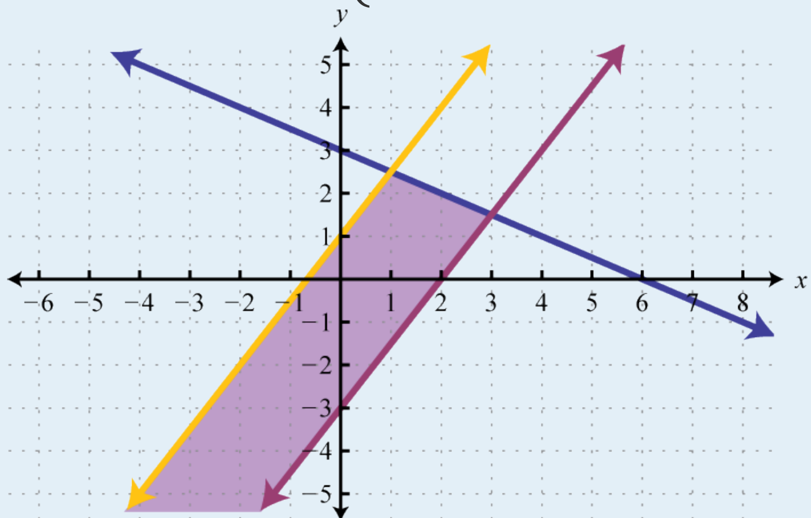
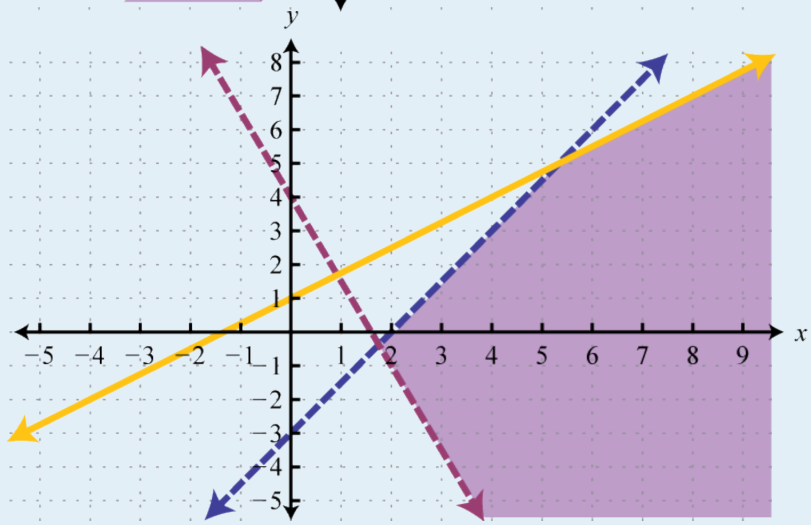


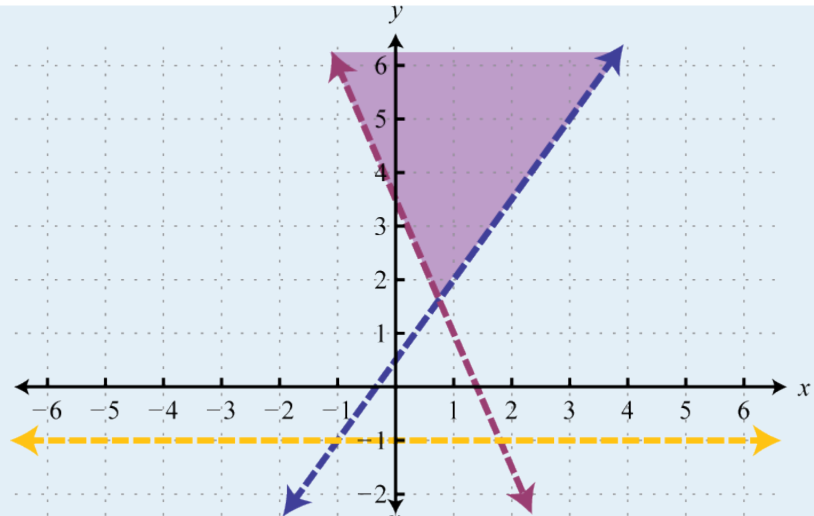
35. $\varnothing$

37. $\begin{cases} x > 0 \\ y > 0 \end{cases}$

39. $\begin{cases} x < 0 \\ y < 0 \end{cases}$

41. 

43. 

45. 
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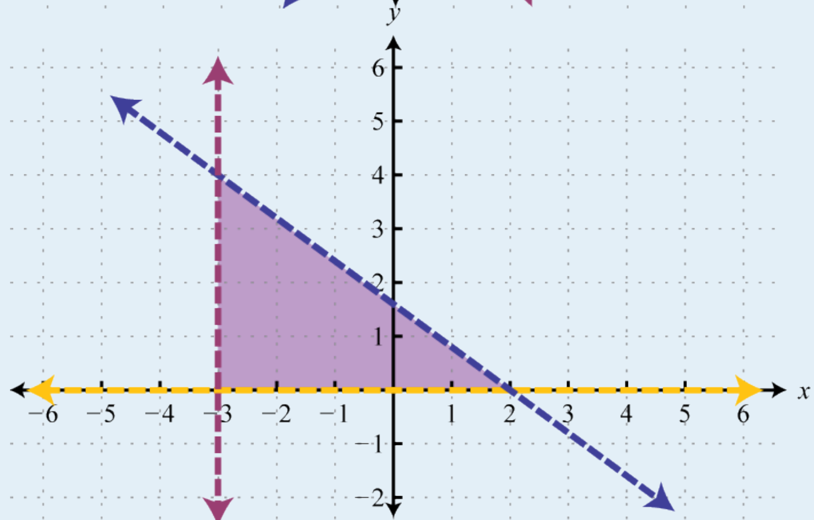
47. 
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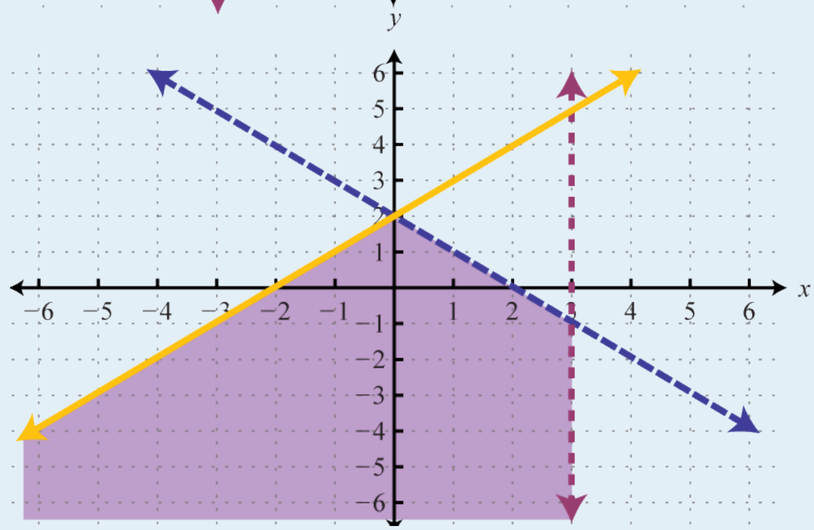
49. 
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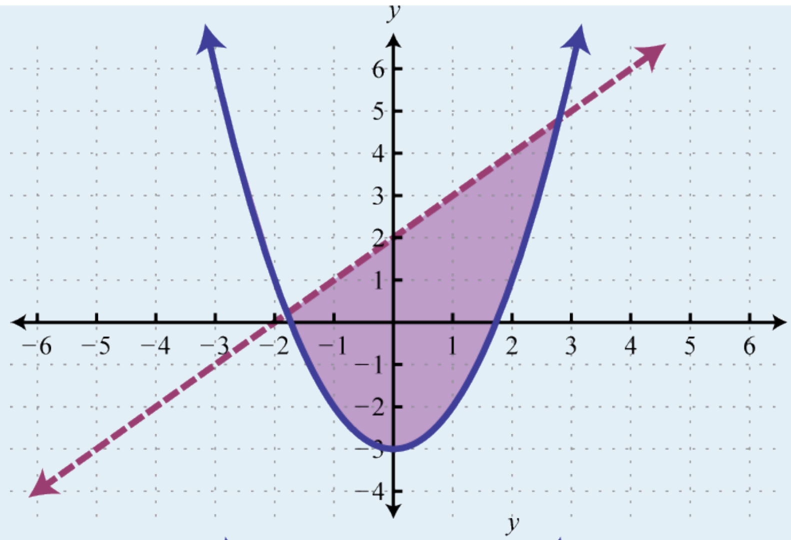

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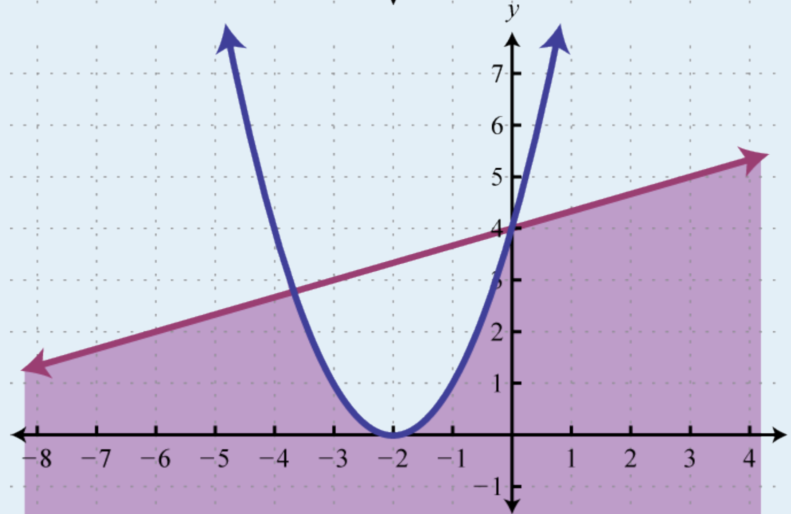

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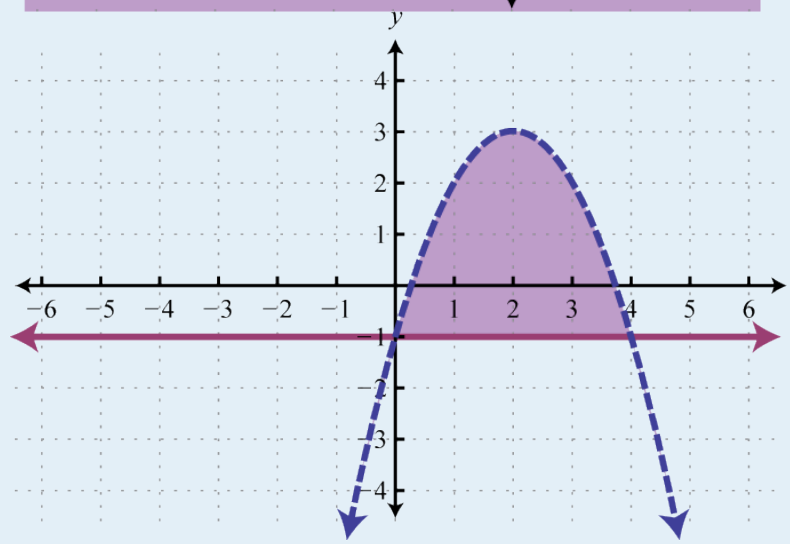

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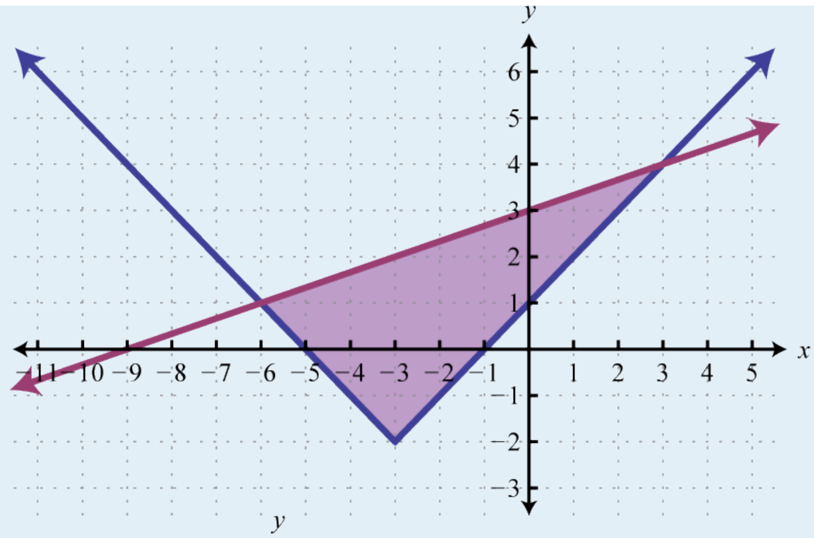
57. 
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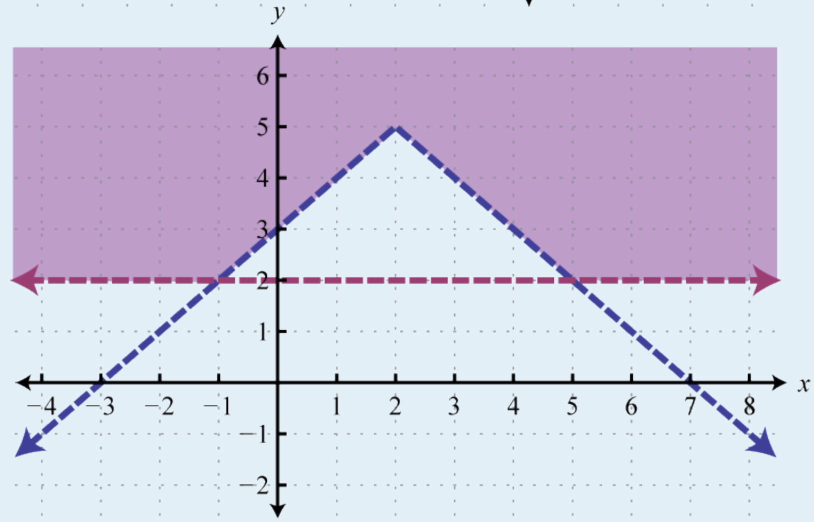
59. 
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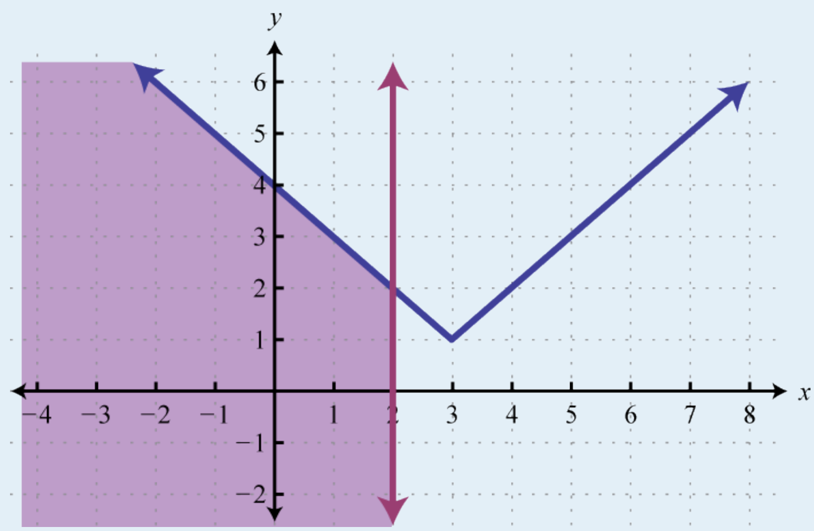
63. 
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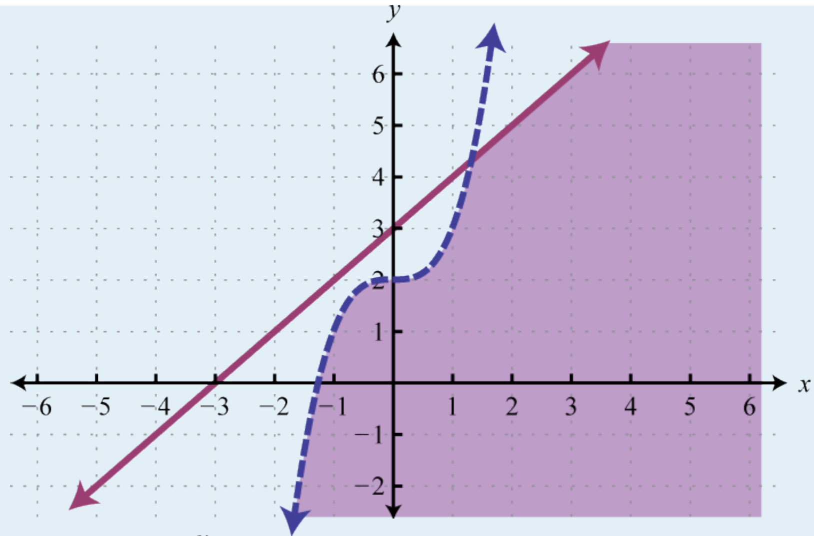
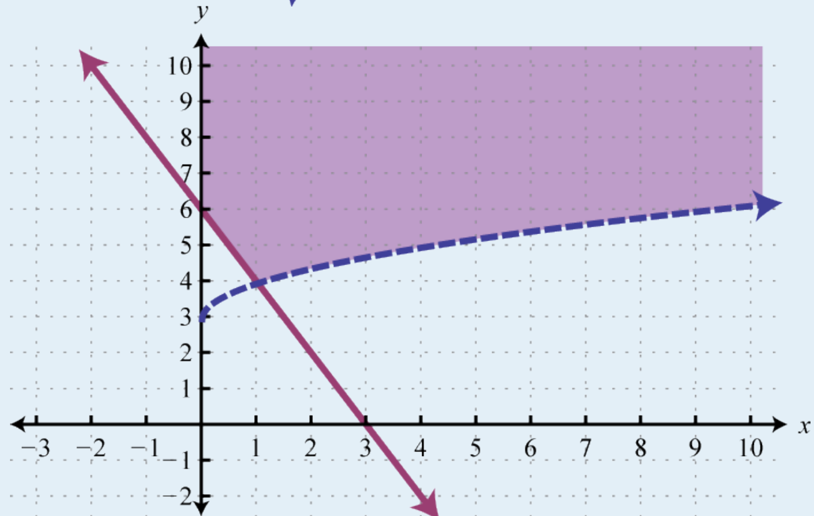
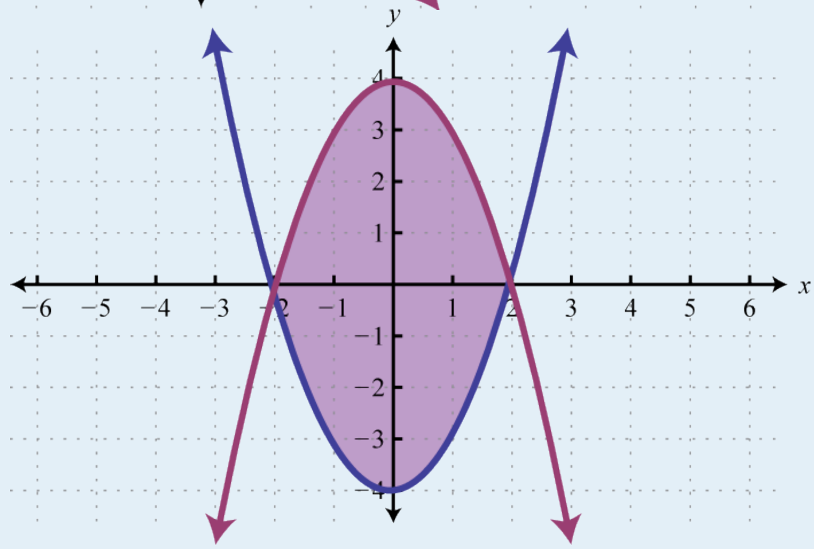
65. 

67. 

69. 

3.8 Review Exercises and Sample Exam

REVIEW EXERCISES

LINEAR SYSTEMS AND THEIR SOLUTIONS

Determine whether or not the given ordered pair is a solution to the given system.

1. $$\left(\frac{2}{3}, -4\right)$$

;

$$\begin{cases} 9x - y = 10 \\ 3x + 4y = -14 \end{cases}$$

2. $\left(-\frac{1}{2}, \frac{3}{4}\right)$;

$$\begin{cases} 6x - 8y = -9 \\ x + 2y = 1 \end{cases}$$

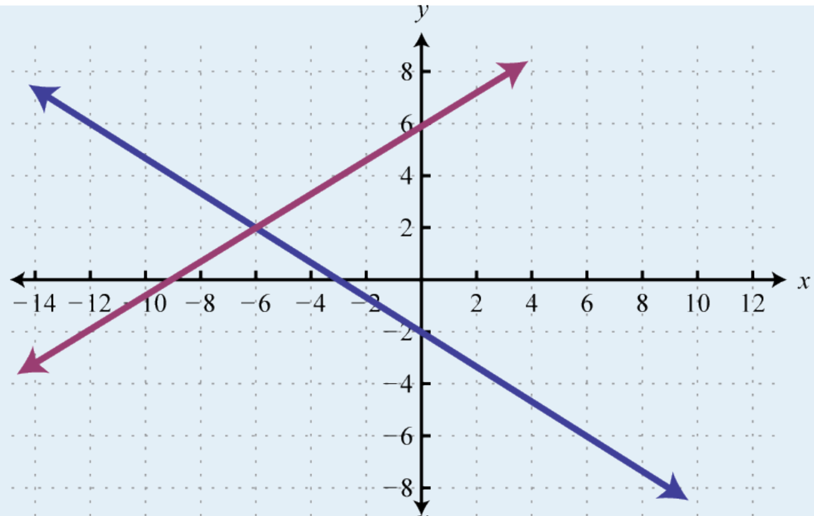
3. $\left(-5, -\frac{7}{8}\right)$;

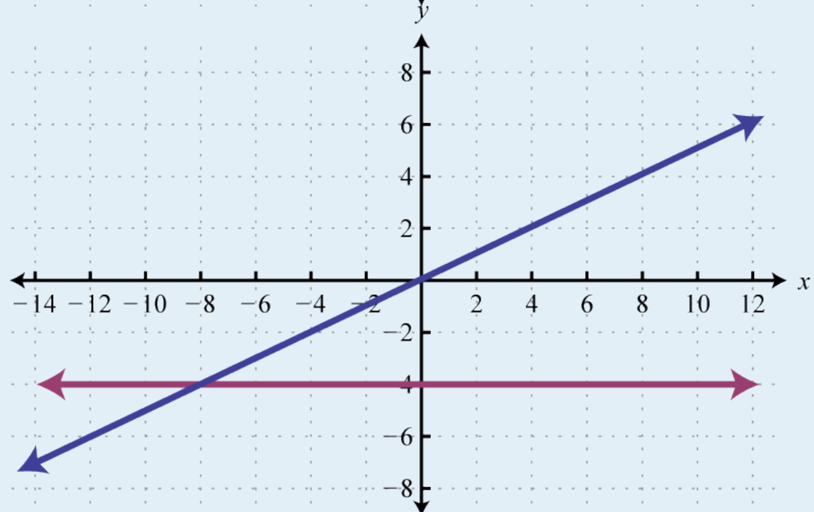
$$\begin{cases} x - 16y = 9 \\ 2x - 8y = -17 \end{cases}$$

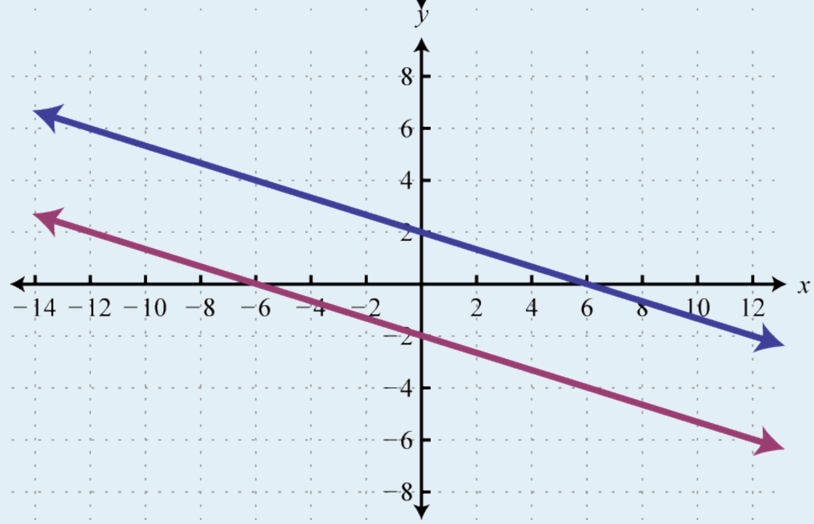
4. $\left(-1, \frac{4}{5}\right)$;

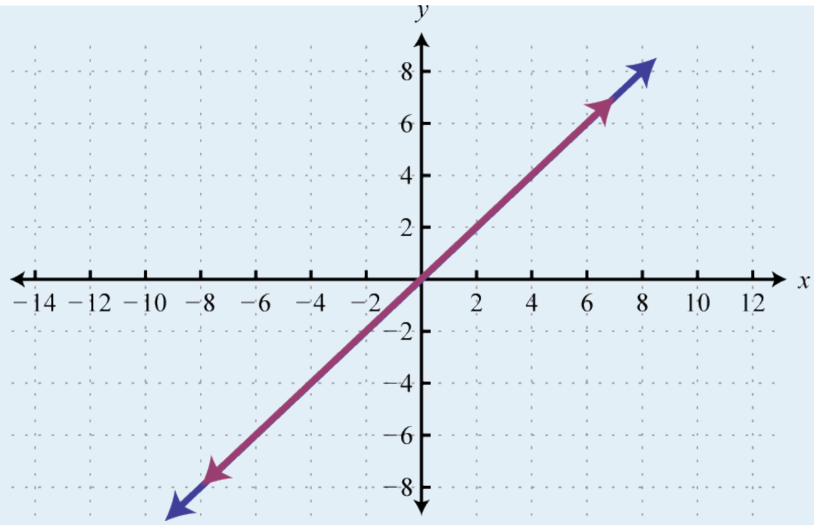
$$\begin{cases} 2x + 5y = 2 \\ 3x - 10y = -5 \end{cases}$$

Given the graphs, determine the simultaneous solution.

5. 

6. 

7. 

8. 

Solve by graphing.

9. $\begin{cases} 2x + y = 6 \\ x - 2y = 8 \end{cases}$

10. $\begin{cases} 5x - 2y = 0 \\ x - y = 3 \end{cases}$

11. $\begin{cases} 4x + 3y = -12 \\ -8x - 6y = 24 \end{cases}$

12. $\begin{cases} \frac{1}{2}x + 2y = 6 \\ x + 4y = -1 \end{cases}$

13. $\begin{cases} 5x + 2y = 30 \\ y - 5 = 0 \end{cases}$

14. $\begin{cases} 5x + 3y = -15 \\ x + 3 = 0 \end{cases}$

15. $\begin{cases} \frac{1}{3}x - \frac{1}{2}y = 2 \\ \frac{1}{2}x + \frac{3}{5}y = 3 \end{cases}$

16. $$\begin{cases} \frac{2}{5}x + \frac{1}{2}y = 1 \\ \frac{1}{15}x + \frac{1}{6}y = -\frac{1}{3} \end{cases}$$

SOLVING LINEAR SYSTEMS WITH TWO VARIABLES

Solve by substitution.

17. $$\begin{cases} 4x - y = 12 \\ x + 3y = -10 \end{cases}$$

18. $$\begin{cases} 9x - 2y = 3 \\ x - 3y = 17 \end{cases}$$

19. $$\begin{cases} 12x + y = 7 \\ 3x - 4y = 6 \end{cases}$$

20. $$\begin{cases} 3x - 2y = 1 \\ 2x + 3y = -1 \end{cases}$$

Solve by elimination.

21. $$\begin{cases} 5x - 2y = -12 \\ 4x + 6y = -21 \end{cases}$$

22. $$\begin{cases} 4x - 5y = 12 \\ 8x + 3y = -2 \end{cases}$$

23. $$\begin{cases} 5x - 3y = 11 \\ 2x - 4y = -4 \end{cases}$$

24. $$\begin{cases} 7x + 2y = 3 \\ 3x + 5y = -7 \end{cases}$$

Solve using any method.

25. $$\begin{cases} 4x - 8y = 4 \\ x + 2y = 9 \end{cases}$$

26. $\begin{cases} 6x - 9y = 8 \\ x - y = 1 \end{cases}$

27. $\begin{cases} 2x - 6y = -1 \\ 6x + 10y = -3 \end{cases}$

28. $\begin{cases} 2x - 3y = 36 \\ x - 3y = 9 \end{cases}$

29. $\begin{cases} 5x - 3y = 10 \\ -10x + 6y = 3 \end{cases}$

30. $\begin{cases} \frac{1}{2}x - y = 3 \\ 3x - 6y = 18 \end{cases}$

31. $\begin{cases} \frac{3}{5}x - \frac{1}{2}y = -1 \\ \frac{1}{10}x + \frac{3}{4}y = -1 \end{cases}$

32. $\begin{cases} \frac{4}{3}x - \frac{2}{5}y = -\frac{8}{15} \\ \frac{1}{2}x - \frac{2}{3}y = -\frac{11}{24} \end{cases}$

APPLICATIONS OF LINEAR SYSTEMS WITH TWO VARIABLES

Set up a linear system and solve.

33. The sum of two integers is 32. The larger is 4 less than twice the smaller. Find the integers.

34. The sum of 2 times a larger integer and 3 times a smaller integer is 54. When twice the smaller integer is subtracted from the larger, the result is −1. Find the integers.

35. The length of a rectangle is 2 centimeters less than three times its width and the perimeter measures 44 centimeters. Find the dimensions of the rectangle.

36. The width of a rectangle is one-third of its length. If the perimeter measures $53\frac{1}{3}$ centimeters, then find the dimensions of the rectangle.

37. The sum of a larger integer and 3 times a smaller is 61. When twice the smaller integer is subtracted from the larger, the result is 1. Find the integers.

38. A total of $8,600 was invested in two accounts. One account earned $4\frac{3}{4}\%$ annual interest and the other earned $6\frac{1}{2}\%$ annual interest. If the total interest for one year was $431.25, how much was invested in each account?

39. A jar consisting of only nickels and dimes contains 76 coins. If the total value is $6, how many of each coin are in the jar?

40. A nurse wishes to obtain 32 ounces of a 1.2% saline solution. How much of a 1% saline solution must she mix with a 2.6% saline solution to achieve the desired mixture?

41. A light aircraft flying with the wind can travel 330 miles in 2 hours. The aircraft can fly the same distance against the wind in 3 hours. Find the speed of the wind.

42. An executive was able to average 52 miles per hour to the airport in her car and then board an airplane that averaged 340 miles per hour. If the total 640-mile business trip took 4 hours, how long did she spend on the airplane?

SOLVING LINEAR SYSTEMS WITH THREE VARIABLES

Determine whether the given ordered triple is a solution to the given system.

43. $(-2, -1, 3)$;

$$\begin{cases} 4x - y + 2z = -1 \\ x - 4y + 3z = 11 \\ 3x + 5y - 4z = 1 \end{cases}$$

44. $(5, -3, -2)$;

$$\begin{cases} x - 4y + 6z = 5 \\ 2x + 5y - z = -3 \\ 3x - 4y + z = 25 \end{cases}$$

45. $\left(1, -\frac{3}{2}, -\frac{4}{3}\right)$;

$$\begin{cases} 5x - 4y + 3z = 7 \\ x + 2y - 6z = 6 \\ 12x - 6y + 6z = 13 \end{cases}$$

46. $\left(\frac{5}{4}, -\frac{1}{3}, 2\right)$;

$$\begin{cases} 8x + 9y + z = 9 \\ 4x + 12y - 4z = -7 \\ 12x - 6y - z = -5 \end{cases}$$

Solve.

47. $\begin{cases} 2x + 3y - z = 1 \\ 5y + 2z = 12 \\ 3z = 18 \end{cases}$

48. $\begin{cases} 3x - 5y - 2z = 21 \\ y - 7z = 18 \\ 4z = -12 \end{cases}$

49. $\begin{cases} 4x - 5y - z = -6 \\ 3x + 6y + 5z = 3 \\ 5x - 2y - 3z = -17 \end{cases}$

50. $\begin{cases} x - 6y + 3z = -2 \\ 5x + 4y - 2z = 24 \\ 6x - 8y - 5z = 25 \end{cases}$

51. $\begin{cases} x + 2y - 2z = 1 \\ 2x - y - z = -2 \\ 6x - 3y - 3z = 12 \end{cases}$

52. $\begin{cases} 3x + y + 2z = -1 \\ 9x + 3y + 6z = -3 \\ 4x + y + 4z = -3 \end{cases}$

53. $\begin{cases} 3a - 2b + 5c = -3 \\ 6a + 4b - c = -2 \\ -6a + 6b + 24c = 7 \end{cases}$

54. $$\begin{cases} 9a - 2b - 6c = 10 \\ 5a - 3b - 10c = 14 \\ -3a + 4b + 12c = -20 \end{cases}$$

Set up a linear system and solve.

55. The sum of three integers is 24. The larger is equal to the sum of the two smaller integers. Three times the smaller is equal to the larger. Find the integers.

56. The sports center sold 120 tickets to the Friday night basketball game for a total of $942. A general admission ticket cost $12, a student ticket cost $6, and a child ticket cost $4. If the sum of the general admission and student tickets totaled 105, then how many of each ticket were sold?

57. A 16-ounce mixed nut product containing 13.5% peanuts is to be packaged. The packager has a three-mixed nut product containing 6%, 10%, and 50% peanut concentrations in stock. If the amount of 50% peanut product is to be one-quarter that of the 10% peanut product, then how much of each will be needed to produce the desired peanut concentration?

58. Water is to be mixed with two acid solutions to produce a 25-ounce solution containing 6% acid. The acid mixtures on hand contain 10% and 25% acid. If the amount of 25% acid is to be one-half the amount of the 10% acid solution, how much water will be needed?

MATRICES AND GAUSSIAN ELIMINATION

Construct the corresponding augmented matrix.

59. $$\begin{cases} 9x - 7y = 4 \\ 3x - y = -1 \end{cases}$$

60. $$\begin{cases} x - 5y = 12 \\ 3y = -5 \end{cases}$$

61. $$\begin{cases} x - y + 2z = -6 \\ 3x - 6y - z = 3 \\ -x + y - 5z = 10 \end{cases}$$

62. $\begin{cases} 5x + 7y - z = 0 \\ -8y + z = -1 \\ -x + 3z = -9 \end{cases}$

Solve using matrices and Gaussian elimination.

63. $\begin{cases} 4x + 5y = 0 \\ 2x - 3y = 22 \end{cases}$

64. $\begin{cases} 3x - 8y = 20 \\ 2x + 5y = 3 \end{cases}$

65. $\begin{cases} x - y + 4z = 1 \\ -2x + 3y - 2z = 0 \\ x - 6y + 8z = 8 \end{cases}$

66. $\begin{cases} -x + 3y - z = 1 \\ 3x - 6y + 2z = -4 \\ 4x - 3y + 2z = -7 \end{cases}$

67. $\begin{cases} 5x - 3y - z = 2 \\ x - 6y + z = 7 \\ 2x + 6y - 2z = -8 \end{cases}$

68. $\begin{cases} x + 2y + 3z = 4 \\ x + 3y + z = 3 \\ 2x + 5y + 4z = 8 \end{cases}$

69. $\begin{cases} 2a + 5b - c = 4 \\ 2a + c = -2 \\ a + b + 3c = 6 \end{cases}$

70. $\begin{cases} a + 2b + 3c = -7 \\ 4b - 2c = 8 \\ 3a - c = -7 \end{cases}$

DETERMINANTS AND CRAMER'S RULE

Calculate the determinant.

71. $\begin{vmatrix} -9 & 5 \\ -1 & 3 \end{vmatrix}$

72. $\begin{vmatrix} -5 & 5 \\ -3 & 3 \end{vmatrix}$

73. $\begin{vmatrix} 0 & 7 \\ 2 & 3 \end{vmatrix}$

74. $\begin{vmatrix} 0 & b_1 \\ a_2 & b_2 \end{vmatrix}$

75. $\begin{vmatrix} 2 & -3 & 0 \\ 1 & -2 & -1 \\ 0 & 1 & 3 \end{vmatrix}$

76. $\begin{vmatrix} 3 & 2 & -1 \\ 1 & -1 & 0 \\ 5 & -2 & -4 \end{vmatrix}$

77. $\begin{vmatrix} 5 & -3 & -1 \\ 1 & -6 & 1 \\ 2 & 6 & -2 \end{vmatrix}$

78. $\begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Solve using Cramer's rule.

79. $\begin{cases} 2x - 3y = -4 \\ 3x + 5y = 1 \end{cases}$

80. $\begin{cases} 3x - y = 2 \\ -2x + 6y = 1 \end{cases}$

81. $\begin{cases} 3x + 5y = 6 \\ 6x + y = -6 \end{cases}$

82. $\begin{cases} 6x - 4y = -1 \\ -3x + 2y = 2 \end{cases}$

83. $\begin{cases} 5x + 2y + 4z = 4 \\ 4x + 3y + 2z = -5 \\ -5x - 3y - 5z = 0 \end{cases}$

84. $\begin{cases} 2x - y + 2z = 1 \\ x - 3y + z = 2 \\ 3x - y - 4z = -2 \end{cases}$

85. $\begin{cases} 4x - y - 2z = -7 \\ 2x + y + 6z = 0 \\ 2x + 2y + 4z = -1 \end{cases}$

86. $\begin{cases} x - y - z = 1 \\ 2x - y + 3z = 2 \\ x + y + z = -1 \end{cases}$

87. $\begin{cases} 4x - y + 2z = -1 \\ 2x + 3y - z = 3 \\ 6x + 2y + z = 2 \end{cases}$

88. $\begin{cases} x - y + 2z = 1 \\ 2x + 2y - z = 2 \\ 3x + y + z = 1 \end{cases}$

SYSTEMS OF INEQUALITIES WITH TWO VARIABLES

Determine whether or not the given point is a solution to the system of inequalities.

89. (-6, 1);

$$\begin{cases} -x + y > 2 \\ x - 2y \leq -1 \end{cases}$$

90. $\left(\frac{1}{2}, -3\right)$;

$$\begin{cases} 4x - 2y \geq 8 \\ 6x + 2y < -3 \end{cases}$$

91. (-4, -2);

$$\begin{cases} x - y > -3 \\ 2x + 3y \leq 0 \\ -3x + 4y \geq 4 \end{cases}$$

92. $\left(5, -\frac{1}{5}\right)$;

$$\begin{cases} y < x^2 - 25 \\ y > \dfrac{2}{3}x - 1 \end{cases}$$

93. (−3, −2);

$$\begin{cases} y < (x-1)^2 \\ y \leq |x+1| - 3 \end{cases}$$

94. $\left(2, -\frac{2}{3}\right)$;

$$\begin{cases} y < 0 \\ x^2 + y \geq 3 \end{cases}$$

Graph the solution set.

95. $\begin{cases} y \leq -4 \\ x - 2y > 8 \end{cases}$

96. $\begin{cases} x + 4y > 8 \\ 2x - y \leq 4 \end{cases}$

97. $\begin{cases} y - 3 < 0 \\ -2x + 3y > -9 \\ x + y \geq 1 \end{cases}$

98. $\begin{cases} y \leq 0 \\ 2x - 6y < 9 \\ -2x + 6y < 9 \end{cases}$

99. $\begin{cases} 2x + y < 3 \\ y > (x-2)^2 - 5 \end{cases}$

100. $\begin{cases} y > |x| \\ y \geq -x^2 + 6 \end{cases}$

101. $\begin{cases} x - 2y < 12 \\ y \leq (x-4)^3 \end{cases}$

102. $\begin{cases} y + 6 > 0 \\ y < \sqrt{x} \end{cases}$

ANSWERS

1. Yes
3. No
5. (−6, 2)
7. Ø
9. (4, −2)
11. $\left(x, -\frac{4}{3}x - 4\right)$
13. (4, 5)
15. (6, 0)
17. (2, −4)
19. $\left(\frac{2}{3}, -1\right)$
21. $\left(-3, -\frac{3}{2}\right)$
23. (4, 3)
25. (5, 2)
27. $\left(-\frac{1}{2}, 0\right)$
29. Ø
31. $\left(-\frac{5}{2}, -1\right)$
33. 12, 20
35. Length: 16 centimeters; width: 6 centimeters
37. 12, 25
39. The jar contains 32 nickels and 44 dimes.
41. 27.5 miles per hour
43. No

45. Yes

47. $\left(\frac{7}{2}, 0, 6\right)$

49. (−2, −1, 3)

51. Ø

53. $\left(-\frac{2}{3}, \frac{1}{2}, 0\right)$

55. 4, 8, 12

57. 6 oz of the 6% peanut stock, 8 oz of the 10% peanut stock, and 2 oz of the 50% peanut stock should be mixed.

59. $\left[\begin{array}{rr|r} 9 & -7 & 4 \\ 3 & -1 & -1 \end{array}\right]$

61. $\left[\begin{array}{rrr|r} 1 & -1 & 2 & -6 \\ 3 & -6 & -1 & 3 \\ -1 & 1 & -5 & 10 \end{array}\right]$

63. (5, −4)

65. $\left(-2, -1, \frac{1}{2}\right)$

67. $\left(x, \frac{2}{3}x - 1, 3x + 1\right)$

69. (−2, 2, 2)

71. −22

73. −14

75. −1

77. 0

79. $\left(-\frac{17}{19}, \frac{14}{19}\right)$

81. $\left(-\frac{4}{3}, 2\right)$

83. (2, −5, 1)

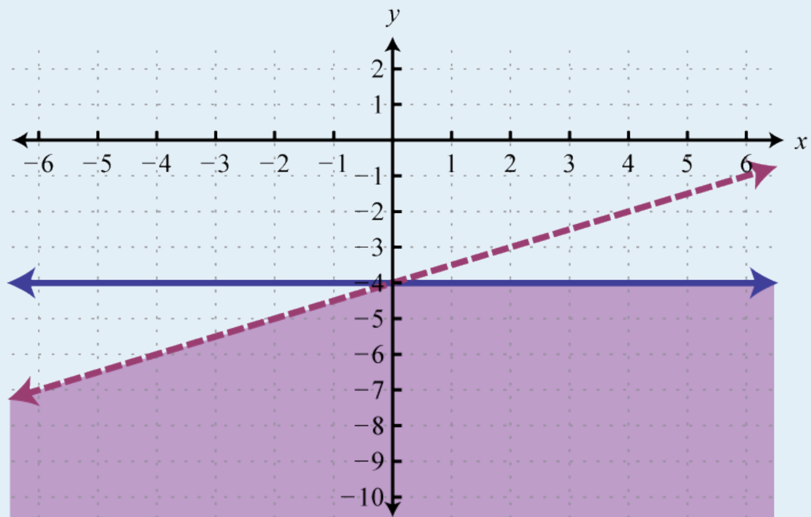
85. $\left(-\dfrac{3}{2}, 0, \dfrac{1}{2}\right)$

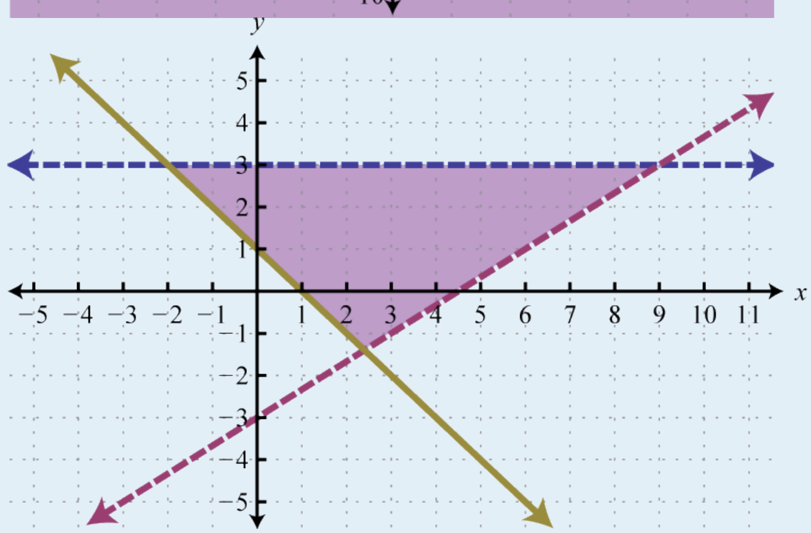
87. $\left(x, -\dfrac{8}{5}x+1, -\dfrac{14}{5}x\right)$

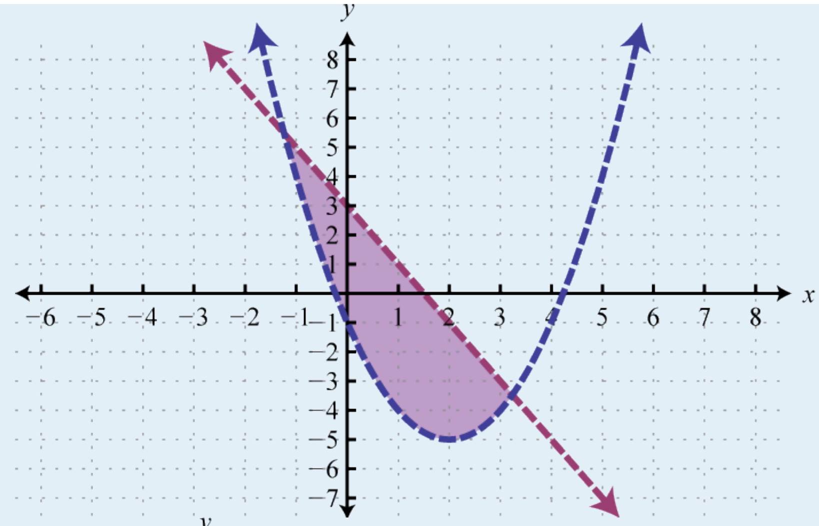
89. Yes

91. Yes

93. Yes

95. 

97. 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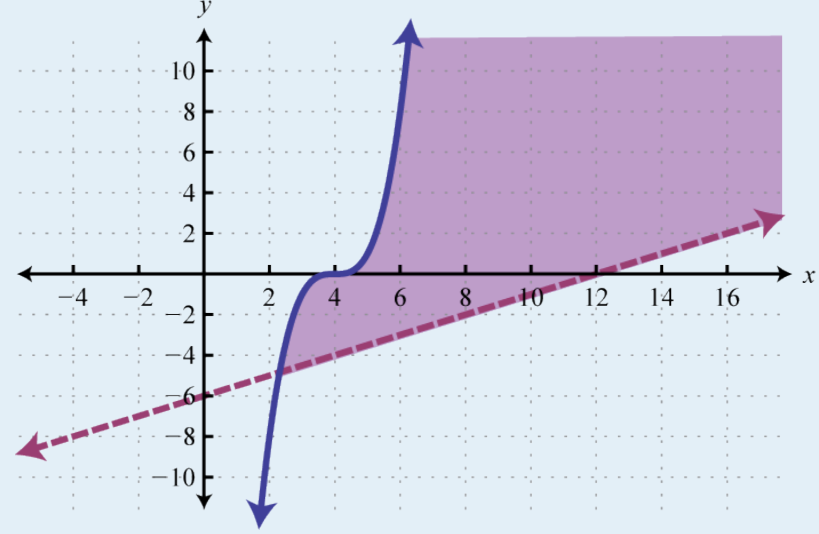
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101.

SAMPLE EXAM

1. Determine whether or not $\left(-2, \frac{3}{4}\right)$ is a solution to $\begin{cases} 2x - 8y = -10 \\ 3x + 4y = -3 \end{cases}$.

2. Determine whether or not $(-3, 2, -5)$ is a solution to
$\begin{cases} x - y + 2z = -15 \\ 2x - 3y + z = -17 \\ 3x + 5y - 2z = 10 \end{cases}$.

Solve by graphing.

3. $\begin{cases} x - y = -5 \\ x + y = -3 \end{cases}$

4. $\begin{cases} 6x - 8y = 48 \\ \frac{1}{2}x - \frac{2}{3}y = 1 \end{cases}$

5. $\begin{cases} \frac{1}{2}x + y = -6 \\ -2x - 4y = 24 \end{cases}$

Solve by substitution.

6. $\begin{cases} x - 8y = 10 \\ 3x + 2y = 17 \end{cases}$

7. $\begin{cases} \frac{3}{2}x - \frac{1}{6}y = -\frac{23}{2} \\ \frac{3}{8}x + \frac{5}{6}y = -\frac{11}{2} \end{cases}$

8. $\begin{cases} 5x - y = 15 \\ 2x - \frac{2}{5}y = 6 \end{cases}$

Solve.

9. $\begin{cases} 3x - 5y = 27 \\ 7x + 2y = 22 \end{cases}$

10. $\begin{cases} 12x + 3y = -3 \\ 5x + 2y = 1 \end{cases}$

11. $\begin{cases} 5x - 3y = -1 \\ -15x + 9y = 5 \end{cases}$

12. $\begin{cases} 6a - 3b + 2c = 11 \\ 2a - b - 4c = -15 \\ 4a - 5b + 3c = 23 \end{cases}$

13. $\begin{cases} 4x + y - 6z = 8 \\ 5x + 4y - 2z = 10 \\ 2x + y - 2z = 4 \end{cases}$

Solve using any method.

14. $\begin{cases} x - 5y + 8z = 1 \\ 2x + 9y - 4z = -8 \\ -3x + 11y + 12z = 15 \end{cases}$

15. $\begin{cases} 2x - y + z = 1 \\ x - y + 3z = 2 \\ 3x - 2y + 4z = 5 \end{cases}$

16. $\begin{cases} -5x + 3y = 2 \\ 4x + 2y = -1 \end{cases}$

17. $\begin{cases} 2x - 3y + 2z = 2 \\ x + 2y - 3z = 0 \\ -x - y + z = -2 \end{cases}$

Graph the solution set.

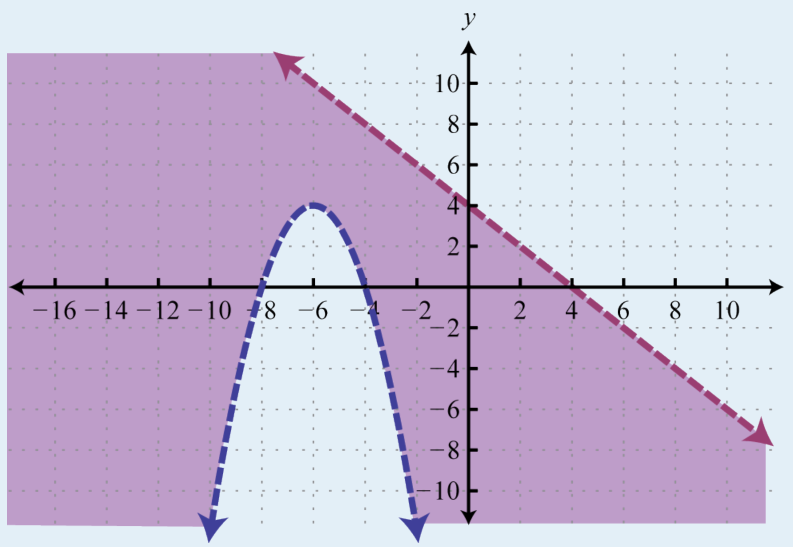
18. $\begin{cases} 3x + 4y < 24 \\ 2x - 3y \leq 3 \\ y + 1 > 0 \end{cases}$

19. $\begin{cases} x + y < 4 \\ y > -(x + 6)^2 + 4 \end{cases}$

Use algebra to solve the following.

20. The length of a rectangle is 1 inch less than twice that of its width. If the perimeter measures 49 inches, then find the dimensions of the rectangle.
21. Joe's $4,000 savings is in two accounts. One account earns 3.1% annual interest and the other earns 4.9% annual interest. His total interest for the year is $174.40. How much does he have in each account?
22. One solution contains 40% alcohol and another contains 72% alcohol. How much of each should be mixed together to obtain 16 ounces of a 62% alcohol solution?
23. Jerry took two buses on the 193-mile trip to visit his grandmother. The first bus averaged 46 miles per hour and the second bus was able to average 52 miles per hour. If the total trip took 4 hours, then how long was spent in each bus?
24. A total of $8,500 was invested in three interest earning accounts. The interest rates were 2%, 3%, and 6%. If the total simple interest for one year was $380 and the amount invested at 6% was equal to the sum of the amounts in the other two accounts, then how much was invested in each account?
25. A mechanic wishes to mix 6 gallons of a 22% antifreeze solution. In stock he has a 60% and an 80% antifreeze concentrate. Water is to be added in the amount that is equal to twice the amount of both concentrates combined. How much water is needed?

ANSWERS

1. Yes
3. (-4, 1)
5. $$\left(x, -\frac{1}{2}x - 6\right)$$
7. (-8, -3)
9. (4, -3)
11. Ø
13. $$\left(x, -x + 2, \frac{1}{2}x - 1\right)$$
15. Ø
17. (2, 2, 2)
19. 

21. Joe has $1,200 in the account earning 3.1% interest and $2,800 in the account earning 4.9% interest.
23. Jerry spent 2.5 hours in the first bus and 1.5 hours in the second.
25. 4 gallons of water is needed.

Chapter 4

Polynomial and Rational Functions

4.1 Algebra of Functions

LEARNING OBJECTIVES

1. Identify and evaluate polynomial functions.
2. Add and subtract functions.
3. Multiply and divide functions.
4. Add functions graphically.

Polynomial Functions

Any polynomial with one variable is a function and can be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

Here a_n represents any real number and n represents any whole number. The **degree of a polynomial** with one variable is the largest exponent of all the terms. Typically, we arrange terms of polynomials in descending order based on their degree and classify them as follows:

$$\begin{aligned}
f(x) &= 2 && \textit{Constant function (degree 0)} \\
g(x) &= 3x + 2 && \textit{Linear function (degree 1)} \\
h(x) &= 4x^2 + 3x + 2 && \textit{Quadratic function (degree 2)} \\
r(x) &= 5x^3 + 4x^2 + 3x + 2 && \textit{Cubic function (degree 3)}
\end{aligned}$$

In this textbook, we call any polynomial with degree higher than 3 an nth-degree polynomial. For example, if the degree is 4, we call it a fourth-degree polynomial; if the degree is 5, we call it a fifth-degree polynomial, and so on.

Example 1

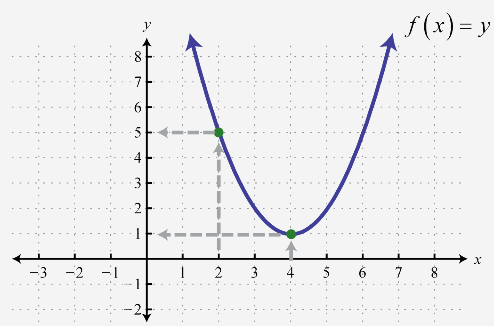
Given $f(x) = x^2 - 8x + 17$, find $f(2)$ and $f(4)$.

Solution:

Replace each instance of x with the value given inside the parentheses.

$f(2) = (2)^2 - 8(2) + 17$	$f(4) = (4)^2 - 8(4) + 17$
$= 4 - 16 + 17$	$= 16 - 32 + 17$
$= 4 + 1$	$= -16 + 17$
$= 5$	$= 1$

We can write $f(2) = 5$ and $f(4) = 1$. Remember that $f(x) = y$ and so we can interpret these results on the graph as follows:



Answer: $f(2) = 5; f(4) = 1$

Often we will be asked to evaluate polynomials for algebraic expressions.

Example 2

Given $g(x) = x^3 - x + 5$, find $g(-2u)$ and $g(x - 2)$.

Solution:

Replace x with the expressions given inside the parentheses.

$g(-2u) = (-2u)^3 - (-2u) + 5$ $= -8u^3 + 2u + 5$	$g(x-2) = (x-2)^3 - (x-2) + 5$ $= (x-2)(x-2)(x-2) -$ $= (x-2)\left(x^2 - 4x + 4\right) -$ $= x^3 - 4x^2 + 4x - 2x^2 + 8$ $= x^3 - 6x^2 + 11x - 1$

Answer: $g(-2u) = -8u^2 + 2u + 5$ and $g(x - 2) = x^3 - 6x^2 + 11x - 1$

The height of an object launched upward, ignoring the effects of air resistance, can be modeled with the following quadratic function:

$$h(t) = -\frac{1}{2}gt^2 + v_0 t + s_0$$

With this formula, the height $h(t)$ can be calculated at any given time t after the object is launched. The letter g represents acceleration due to gravity on the surface of the Earth, which is 32 feet per second squared (or, using metric units, g = 9.8 meters per second squared). The variable v_0, pronounced "*v-naught*," or sometimes "*v-zero*," represents the initial velocity of the object, and s_0 represents the initial height from which the object was launched.

Example 3

An object is launched from the ground at a speed of 64 feet per second. Write a function that models the height of the object and use it to calculate the objects height at 1 second and at 3.5 seconds.

Solution:

We know that the acceleration due to gravity is $g = 32$ feet per second squared and we are given the initial velocity $v_0 = 64$ feet per second. Since the object is launched from the ground, the initial height is $s_0 = 0$ feet. Create the mathematical model by substituting these coefficients into the following formula:

$$h(t) = -\frac{1}{2}gt^2 + v_0 t + s_0$$

$$h(t) = -\frac{1}{2}(32)t^2 + (64)t + 0$$
$$h(t) = -16t^2 + 64t$$

Use this model to calculate the height of the object at 1 second and 3.5 seconds.

$$h(1) = -16(1)^2 + 64(1) = -16 + 64 = 48$$
$$h(3.5) = -16(3.5)^2 + 64(3.5) = -196 + 224 = 28$$

Answer: $h(t) = -16t^2 + 64t$; At 1 second the object is at a height of 48 feet, and at 3.5 seconds it is at a height of 28 feet.

Try this! An object is dropped from a height of 6 meters. Write a function that models the height of the object and use it to calculate the object's height 1 second after it is dropped.

Answer: $h(t) = -4.9t^2 + 6$; At 1 second the object is at a height of 1.1 meters.

(click to see video)

Adding and Subtracting Functions

The notation used to indicate **addition**[1] and **subtraction**[2] of functions follows:

$$\begin{aligned} \textbf{Addition of functions:}\ \left(f+g\right)(x) &= f(x) + g(x) \\ \textbf{Subtraction of functions:}\ \left(f-g\right)(x) &= f(x) - g(x) \end{aligned}$$

When using function notation, be careful to group the entire function and add or subtract accordingly.

1. Add functions as indicated by the notation: $\left(f+g\right)(x) = f(x) + g(x)$.
2. Subtract functions as indicated by the notation: $\left(f-g\right)(x) = f(x) - g(x)$.

Example 4

Given $f(x) = x^3 - 5x - 7$ and $g(x) = 3x^2 + 7x - 2$, find $(f + g)(x)$ and $(f - g)(x)$.

Solution:

The notation $f + g$ indicates that we should add the given expressions.

$$\begin{aligned}(f+g)(x) &= f(x) + g(x)\\ &= \left(x^3 - 5x - 7\right) + \left(3x^2 + 7x - 2\right)\\ &= x^3 - 5x - 7 + 3x^2 + 7x - 2\\ &= x^3 + 3x^2 + 2x - 9\end{aligned}$$

The notation $f - g$ indicates that we should subtract the given expressions. When subtracting, the parentheses become very important. Recall that we can eliminate them after applying the distributive property.

$$\begin{aligned}(f-g)(x) &= f(x) - g(x)\\ &= \left(x^3 - 5x - 7\right) - \left(3x^2 + 7x - 2\right)\\ &= x^3 - 5x - 7 - 3x^2 - 7x + 2\\ &= x^3 - 3x^2 - 12x - 5\end{aligned}$$

Answer: $(f + g)(x) = x^3 + 3x^2 + 2x - 9$ and $(f - g)(x) = x^3 - 3x^2 - 12x - 5$

We may be asked to evaluate the sum or difference of two functions. We have the option to first find the sum or difference in general and then use the resulting function to evaluate for the given variable, or evaluate each first and then find the sum or difference.

Example 5

Evaluate $(f-g)\ (3)$ given $f(x) = 5x^2 - x + 4$ and $g(x) = x^2 + 2x - 3$.

Solution:

First, find $(f-g)\ (x)$.

$$\begin{aligned}(f-g)\ (x) &= f(x) - g(x)\\ &= \left(5x^2 - x + 4\right) - \left(x^2 + 2x - 3\right)\\ &= 5x^2 - x + 4 - x^2 - 2x + 3\\ &= 4x^2 - 3x + 7\end{aligned}$$

Therefore,

$$(f-g)\ (x) = 4x^2 - 3x + 7.$$

Next, substitute 3 in for the variable x.

$$\begin{aligned}(f-g)\ (3) &= 4(3)^2 - 3\,(3) + 7\\ &= 36 - 9 + 7\\ &= 34\end{aligned}$$

Hence $(f-g)\ (3) = 34$.

Alternate Solution: Since $(f - g)(3) = f(3) - g(3)$ we can find $f(3)$ and $g(3)$ and then subtract the results.

$\begin{aligned} f(x) &= 5x^2 - x + 4 \\ f(3) &= 5(3)^2 - (3) + 4 \\ &= 45 - 3 + 4 \\ &= 46 \end{aligned}$	$\begin{aligned} g(x) &= x^2 + 2x - 3 \\ g(3) &= (3)^2 + 2(3) - 3 \\ &= 9 + 6 - 3 \\ &= 12 \end{aligned}$

Therefore,

$$\begin{aligned}(f - g)(3) &= f(3) - g(3) \\ &= 46 - 12 \\ &= 34\end{aligned}$$

Notice that we obtain the same answer.

Answer: $(f - g)(3) = 34$

Note: If multiple values are to be evaluated, it is best to find the sum or difference in general first and then use it to evaluate.

Try this! Evaluate $(f+g)\ (-1)$ given $f\ (x)=x^3+x-8$ and $g\ (x)=2x^2-x+9$.

Answer: 2

(click to see video)

Multiplying and Dividing Functions

The notation used to indicate **multiplication**[3] and **division**[4] of functions follows:

Multiplication of functions:	$(f\cdot g)\ (x)=f\ (x)\cdot g(x)$
Division of functions:	$(f/g)\ (x)=\frac{f(x)}{g(x)}$, where $g\ (x)\neq 0$.

3. Multiply functions as indicated by the notation: $(f\cdot g)\ (x)=f\ (x)\cdot g\ (x)$.

4. Divide functions as indicated by the notation: $(f/g)\ (x)=\frac{f(x)}{g(x)}$, where $g\ (x)\neq 0$.

Example 6

Given $f(x) = 15x^4 - 9x^3 + 6x^2$ and $g(x) = 3x^2$, find $(f \cdot g)(x)$ and $(f/g)(x)$.

Solution:

The notation $f \cdot g$ indicates that we should multiply. Apply the distributive property and simplify.

$$\begin{aligned}(f \cdot g)(x) &= f(x) \cdot g(x) \\ &= \left(15x^4 - 9x^3 + 6x^2\right)\left(3x^2\right) \\ &= 15x^4 \cdot 3x^2 - 9x^3 \cdot 3x^2 + 6x^2 \cdot 3x^2 \\ &= 45x^6 - 27x^5 + 18x^4\end{aligned}$$

The notation f/g indicates that we should divide. For this quotient, assume $x \neq 0$.

$$\begin{aligned}(f/g)(x) &= \frac{f(x)}{g(x)} \\ &= \frac{15x^4 - 9x^3 + 6x^2}{3x^2} \\ &= \frac{15x^4}{3x^2} - \frac{9x^3}{3x^2} + \frac{6x^2}{3x^2} \\ &= 5x^2 - 3x + 2\end{aligned}$$

Answer: $(f \cdot g)(x) = 45x^6 - 27x^5 + 18x^4$ and $(f/g)(x) = 5x^2 - 3x + 2$ where $x \neq 0$.

Example 7

Given $f\left(x\right) = 6x - 5$ and $g\left(x\right) = 3x^2 - 2x - 1$, evaluate $\left(f \cdot g\right)\left(0\right)$ and $\left(f \cdot g\right)\left(-1\right)$

Solution:

Begin by finding $\left(f \cdot g\right)\left(x\right)$.

$$\begin{aligned}\left(f \cdot g\right)\left(x\right) &= f\left(x\right) \cdot g\left(x\right) \\ &= \left(6x - 5\right)\left(3x^2 - 2x - 1\right) \\ &= 18x^3 - 12x^2 - 6x - 15x^2 + 10x + 5 \\ &= 18x^3 - 27x^2 + 4x + 5\end{aligned}$$

Therefore $\left(f \cdot g\right)\left(x\right) = 18x^3 - 27x^2 + 4x + 5$, and we have,

$\begin{aligned}\left(f \cdot g\right)\left(0\right) &= 18(0)^3 - 27(0)^2 + 4\left(0\right) + 5 \\ &= 5\end{aligned}$	$\begin{aligned}\left(f \cdot g\right)\left(-1\right) &= 18(-1)^3 - 2 \\ &= -18 - 27 - \\ &= -44\end{aligned}$

Answer: $\left(f \cdot g\right)\left(0\right) = 5$ and $\left(f \cdot g\right)\left(-1\right) = -44$

Try this! Evaluate $(f \cdot g)\ (-1)$ given $f\ (x) = x^3 + x - 8$ and $g\ (x) = 2x^2 - x + 9$.

Answer: –120

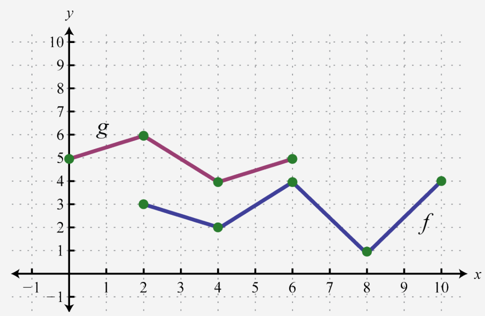
(click to see video)

Adding Functions Graphically

Here we explore the geometry of adding functions. One way to do this is to use the fact that $(f + g)\ (x) = f\ (x) + g\ (x)$. Add the functions together using *x*-values for which both f and g are defined.

Example 8

Use the graphs of f and g to graph $f + g$. Also, give the domain of $f + g$.

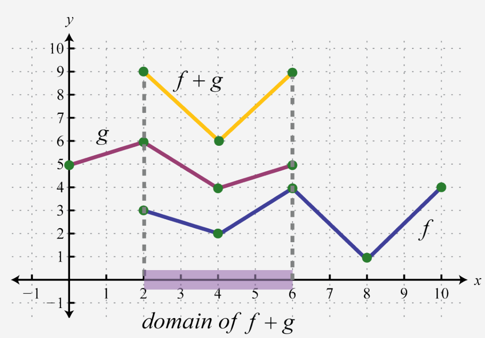


Solution:

In this case, both functions are defined for x-values between 2 and 6. We will use 2, 4, and 6 as representative values in the domain of $f + g$ to sketch its graph.

$$\left(f + g\right)\ (2) = f\ (2) + g\ (2) = 3 + 6 = 9$$
$$\left(f + g\right)\ (4) = f\ (4) + g\ (4) = 2 + 4 = 6$$
$$\left(f + g\right)\ \left(6\right) = f\ \left(6\right) + g\ \left(6\right) = 4 + 5 = 9$$

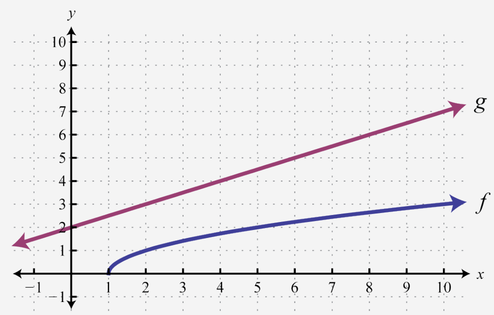
Sketch the graph of $f + g$ using the three ordered pair solutions $(2, 9)$, $\left(4, 6\right)$, and $\left(6, 9\right)$.



Answer: $f + g$ graphed above has domain $[2, 6]$.

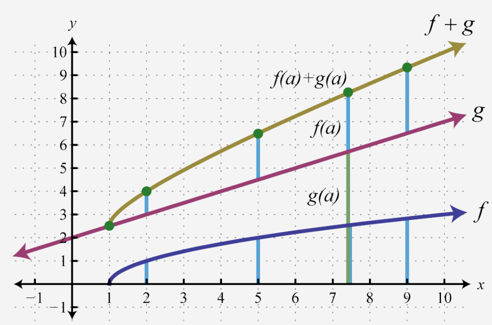
Example 9

Use the graphs of f and g to graph $f + g$. Also, give the domain of $f + g$.



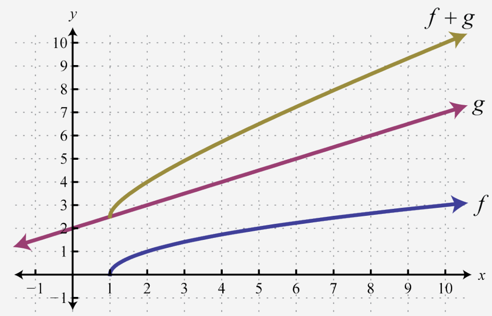
Solution:

Another way to add nonnegative functions graphically is to copy the line segment formed from the x-axis to one of the functions onto the other as illustrated below.



The line segment from the x-axis to the function f represents $f(a)$. Copy this line segment onto the other function over the same point; the endpoint represents $f(a) + g(a)$ .Doing this for a number of points allows us to obtain a quick sketch of the combined graph. In this example, the domain of $f + g$ is limited to the x-values for which f is defined.

Answer: Domain: $[1, \infty)$



In general, the domain of $f + g$ is the intersection of the domain of f with the domain of g. In fact, this is the case for all of the arithmetic operations with an extra consideration for division. When dividing functions, we take extra care to remove any values that make the denominator zero. This will be discussed in more detail as we progress in algebra.

KEY TAKEAWAYS

- Any polynomial with one variable is a function and can be written in the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$. The degree of the polynomial is the largest exponent of all the terms.
- Use function notation to streamline the evaluating process. Substitute the value or expression inside the parentheses for each instance of the variable.
- The notation $(f + g)(x)$ indicates that we should add $f(x) + g(x)$.
- The notation $(f - g)(x)$ indicates that we should subtract $f(x) - g(x)$.
- The notation $(f \cdot g)(x)$ indicates that we should multiply $f(x)\, g(x)$.
- The notation $(f/g)(x)$ indicates that we should divide $\frac{f(x)}{g(x)}$, where $g(x) \neq 0$.
- The domain of the function that results from these arithmetic operations is the intersection of the domain of each function. The domain of a quotient is further restricted to values that do not evaluate to zero in the denominator.

TOPIC EXERCISES

PART A: POLYNOMIAL FUNCTIONS

Evaluate.

1. Given $f(x) = x^2 - 10x + 3$, find $f(-3)$, $f(0)$, and $f(5)$.
2. Given $f(x) = 2x^2 - x + 9$, find $f(-1)$, $f(0)$, and $f(3)$.
3. Given $g(x) = x^3 - x^2 + x + 7$, find $g(-2)$, $g(0)$, and $g(3)$.
4. Given $g(x) = x^3 - 2x + 5$, find $g(-5)$, $g(0)$, and $g(3)$.
5. Given $s(t) = 5t^4 - t^2 + t - 3$, find $s(-1)$, $s(0)$, and $s(2)$.
6. Given $p(n) = n^4 - 10n^2 + 9$, find $p(-3)$, $p(-1)$, and $p(2)$.
7. Given $f(x) = x^6 - 64$, find $f(-2)$, $f(-1)$, and $f(0)$.
8. Given $f(x) = x^6 - x^3 + 3$, find $f(-2)$, $f(-1)$, and $f(0)$.
9. Given $f(x) = x^2 - 2x - 1$, find $f(2t)$ and $f(2t - 1)$.
10. Given $f(x) = x^2 - 2x + 4$, find $f(-3t)$ and $f(2 - 3t)$.
11. Given $g(x) = 2x^2 + 3x - 1$, find $g(-5a)$ and $g(5 - 2x)$.
12. Given $g(x) = 3x^2 - 5x + 4$, find $g(-4u)$ and $g(3x - 1)$.
13. Given $f(x) = x^3 - 1$, find $f(2a)$ and $f(x - 2)$.
14. Given $f(x) = x^3 - x + 1$, find $f(-3x)$ and $f(2x + 1)$.
15. Given $g(x) = x^3 + x^2 - 1$, find $g(x^2)$ and $g(x - 4)$.
16. Given $g(x) = 2x^3 - x + 1$, find $g(-2x^3)$ and $g(3x - 1)$.

Given the function calculate $f(x + h)$.

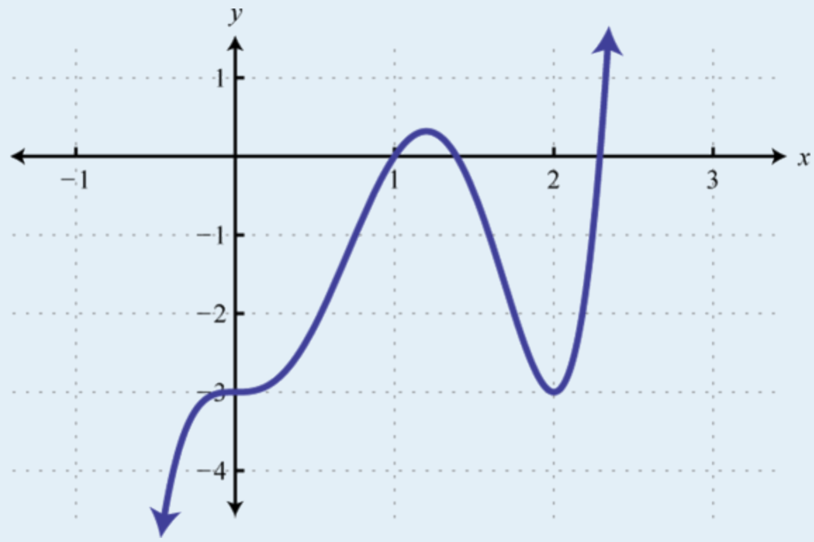
17. $f(x) = 5x - 3$
18. $f(x) = x^2 - 1$

19. $f(x) = x^3 - 8$

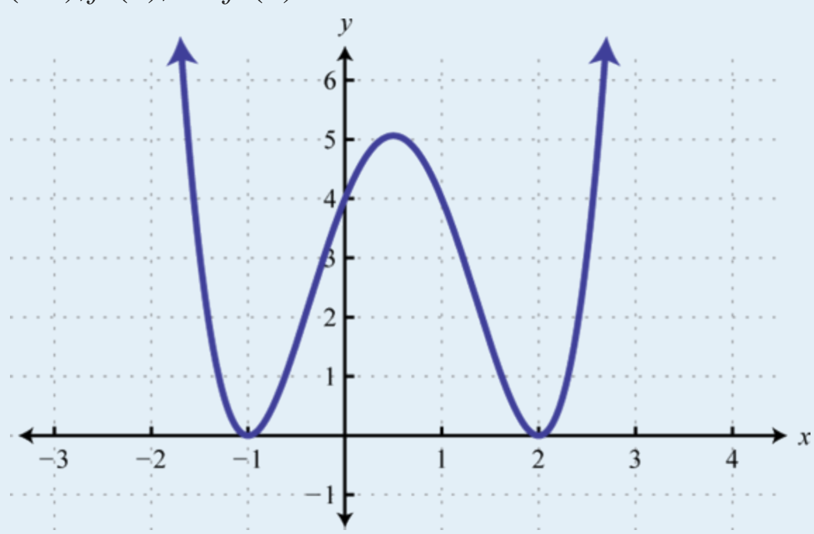
20. $f(x) = x^4$

Given the graph of the polynomial function f find the function values.

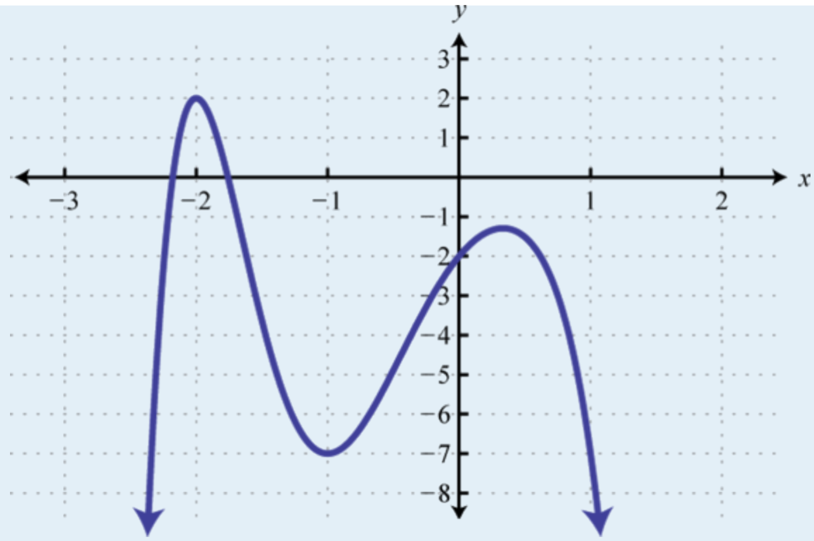
21. Find $f(0)$, $f(1)$, and $f(2)$.



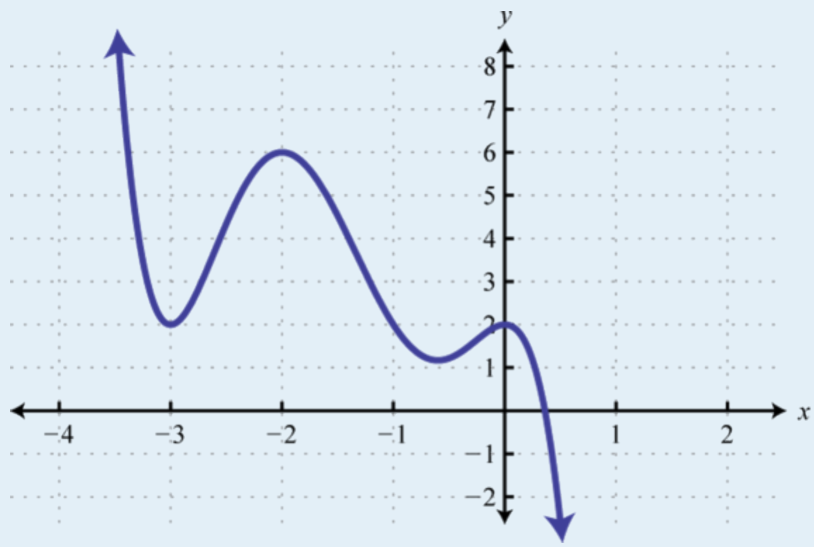
22. Find $f(-1)$, $f(0)$, and $f(1)$.



23. Find $f(-2)$, $f(-1)$, and $f(0)$.



24. Find $f(-3)$, $f(-2)$, and $f(0)$.



25. A projectile is launched upward from the ground at a speed of 48 feet per second. Write a function that models the height of the projectile and use it to calculate the height every 1/2 second after launch. Sketch a graph that shows the height of the projectile with respect to time.

26. An object is tossed upward from a 48-foot platform at a speed of 32 feet per second. Write a function that models the height of the object and use it to calculate the height every 1/2 second after the object is tossed. Sketch a graph that shows the height of the object with respect to time.

27. An object is dropped from a 128-foot bridge. Write a function that models the height of the object, and use it to calculate the height at 1 second and 2 seconds after it has been dropped.

28. An object is dropped from a 500-foot building. Write a function that models the height of the object, and use it to calculate the distance the object falls in the 1st second, 2nd second, and the 3rd second.

29. A bullet is fired straight up into the air at 320 meters per second. Ignoring the effects of air friction, write a function that models the height of the bullet, and use it to calculate the bullet's height 1 minute after it was fired into the air.

30. A book is dropped from a height of 10 meters. Write a function that gives the height of the book, and use it to determine how far it will fall in 1¼ seconds.

PART B: ADDING AND SUBTRACTING FUNCTIONS

Given functions f and g, find $(f+g)$ and $(f-g)$.

31. $f(x) = 5x - 3,\ g(x) = 4x - 1$

32. $f(x) = 3x + 2,\ g(x) = 7x - 5$

33. $f(x) = 2 - 3x,\ g(x) = 1 - x$

34. $f(x) = 8x - 5,\ g(x) = -7x + 4$

35. $f(x) = x^2 - 3x + 2,\ g(x) = x^2 + 4x - 7$

36. $f(x) = 2x^2 + x - 3,\ g(x) = x^2 - x + 4$

37. $f(x) = x^2 + 5x - 3,\ g(x) = 6x + 11$

38. $f(x) = 9x + 5,\ g(x) = 2x^2 - 5x + 4$

39. $f(x) = 9x^2 - 1,\ g(x) = x^2 + 5x$

40. $f(x) = 10x^2,\ g(x) = 5x^2 - 8$

41. $f(x) = 8x^3 + x - 4,\ g(x) = 4x^3 + x^2 - 1$

42. $f(x) = x^3 - x^2 + x + 1,\ g(x) = x^3 - x^2 - x - 1$

Given $f(x) = x^3 + 2x^2 - 8$ and $g(x) = 2x^2 - 3x + 5$, evaluate the following.

43. $(f+g)(-2)$

44. $(f+g)(3)$

45. $(f - g)\ (-2)$

46. $(f - g)\ (3)$

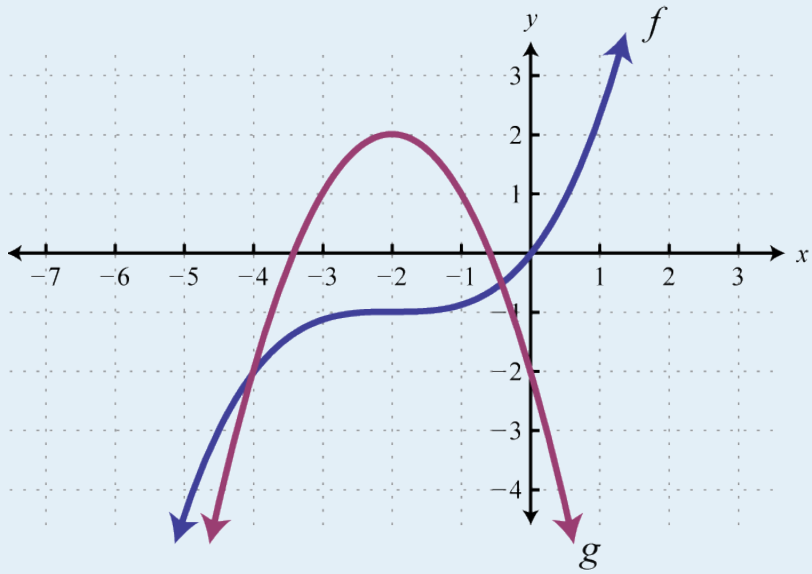
47. $(g - f)\ (-2)$

48. $(g - f)\ (3)$

49. $(f + f)\ (1)$

50. $(g + g)\ (-1)$

Given the graphs of f and g, evaluate the following.



51. $(f + g)\ (-4)$

52. $(f - g)\ (-4)$

53. $(f + g)\ (-2)$

54. $(f - g)\ (-2)$

55. $(f + g)\ (0)$

56. $(f - g)\ (0)$

PART C: MULTIPLYING AND DIVIDING FUNCTIONS

Given f and g, find $f \cdot g$.

57. $f(x) = 5x, g(x) = x - 3$
58. $f(x) = x - 4, g(x) = 6x$
59. $f(x) = 2x - 3, g(x) = 3x + 4$
60. $f(x) = 5x - 1, g(x) = 2x + 1$
61. $f(x) = 3x + 4, g(x) = 3x - 4$
62. $f(x) = x + 5, g(x) = x - 5$
63. $f(x) = x - 2, g(x) = x^2 - 3x + 2$
64. $f(x) = 2x - 3, g(x) = x^2 + 2x - 1$
65. $f(x) = 2x^2, g(x) = x^2 - 7x + 5$
66. $f(x) = 5x^3, g(x) = x^2 - 3x - 1$
67. $f(x) = x^2 - 3x - 2, g(x) = 2x^2 - x + 3$
68. $f(x) = x^2 + x - 1, g(x) = x^2 - x + 1$

Given f and g, find f/g. (Assume all expressions in the denominator are nonzero.)

69. $f(x) = 36x^3 - 16x^2 - 8x, g(x) = 4x$
70. $f(x) = 2x^3 - 6x^2 + 10x, g(x) = 2x$
71. $f(x) = 20x^7 - 15x^5 + 5x^3, g(x) = 5x^3$
72. $f(x) = 9x^6 + 12x^4 - 3x^2, g(x) = 3x^2$
73. $f(x) = x^3 + 4x^2 + 3x - 2, g(x) = x + 2$
74. $f(x) = x^3 - x^2 - 10x + 12, g(x) = x - 3$
75. $f(x) = 6x^3 - 13x^2 + 36x - 45, g(x) = 2x - 3$
76. $f(x) = 6x^3 - 11x^2 + 15x - 4, g(x) = 3x - 1$
77. $f(x) = 3x^3 - 13x^2 - x + 8, g(x) = 3x + 2$

78. $f(x) = 5x^3 - 16x^2 + 13x - 6, g(x) = 5x - 1$

Given $f(x) = 25x^4 + 10x^3 - 5x^2$ **and** $g(x) = 5x^2$ **evaluate the following.**

79. $(f \cdot g)(-1)$

80. $(f \cdot g)(1)$

81. $(f/g)(-2)$

82. $(f/g)(-3)$

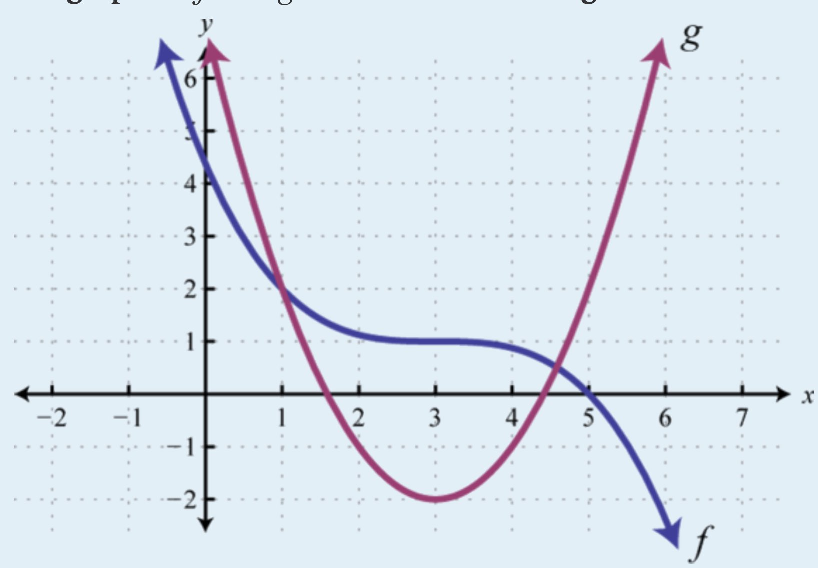
83. $(g \cdot f)(0)$

84. $(g/f)(1)$

85. $(g \cdot g)(-1)$

86. $(f \cdot f)(-1)$

Given the graphs of f **and** g **evaluate the following.**



87. $(f \cdot g)(3)$

88. $(f \cdot g)(5)$

89. $(f/g)\ (5)$

90. $(f/g)\ (3)$

91. $(f \cdot g)\ (1)$

92. $(f/g)\ (1)$

Given $f(x) = 5x^3 - 15x^2 + 10x$**,** $g(x) = x^2 - x + 3$**, and** $h(x) = -5x$ **, find the following. (Assume all expressions in the denominator are nonzero.)**

93. $(f - g)\ (x)$

94. $(g - f)\ (x)$

95. $(g \cdot h)\ (x)$

96. $(f/h)\ (x)$

97. $(h + g)\ (x)$

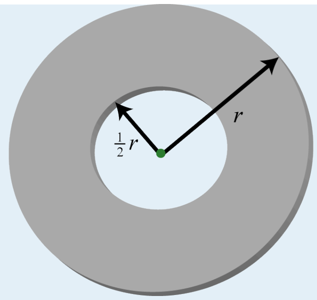
98. $(h \cdot f)\ (x)$

99. $(g/h)\ (2)$

100. $(g - h)\ (-3)$

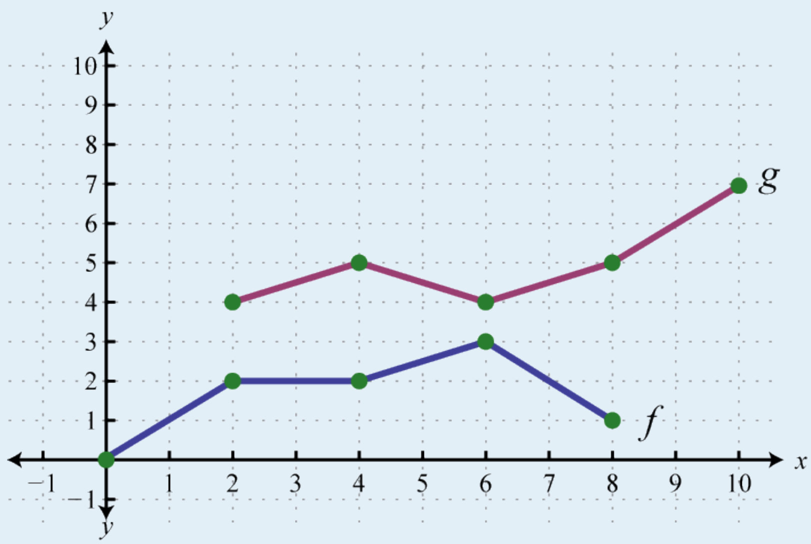
101. The revenue in dollars from selling MP3 players is given by the function $R(n) = 125n - 0.15n^2$, where n represents the number of units sold $(0 \le n < 833)$. The cost in dollars of producing the MP3 players is given by the formula $C(n) = 1200 + 42n$ where n represents the number of units produced. Write a function that models the profit of producing and selling n MP3 players. Use the function to determine the profit generated from producing and selling 225 MP3 players. Recall that profit equals revenues less costs.

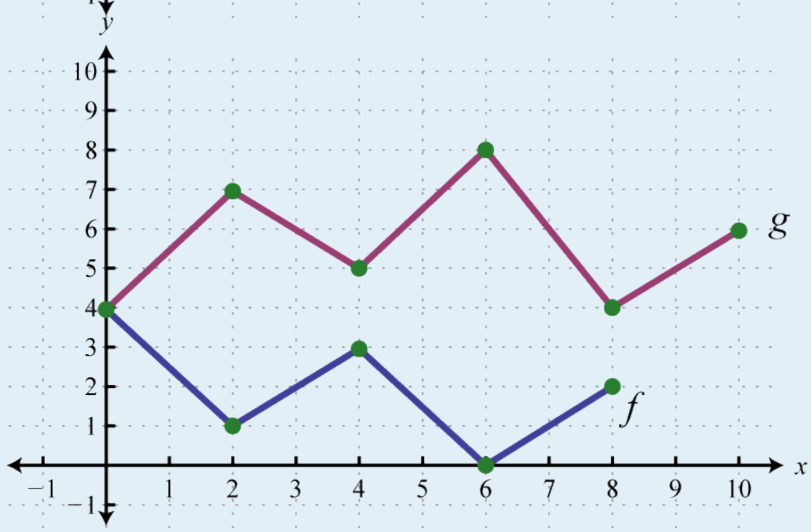
102. The inner radius of a washer is $\frac{1}{2}$ that of the outer radius.

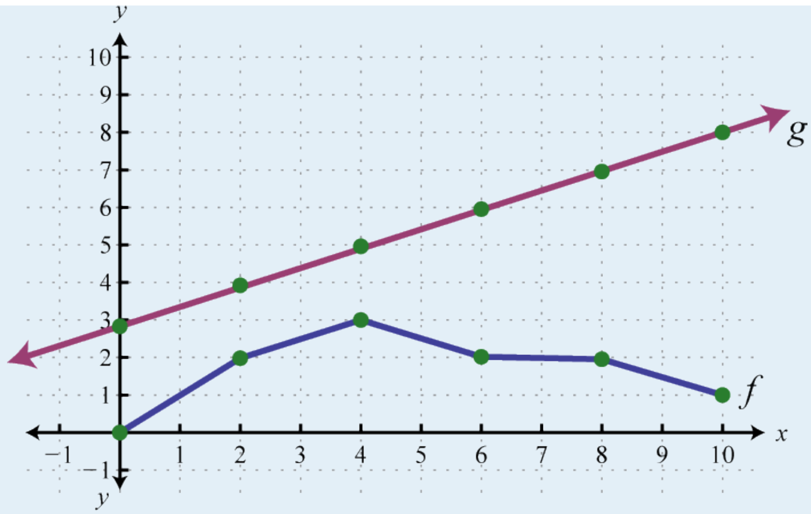


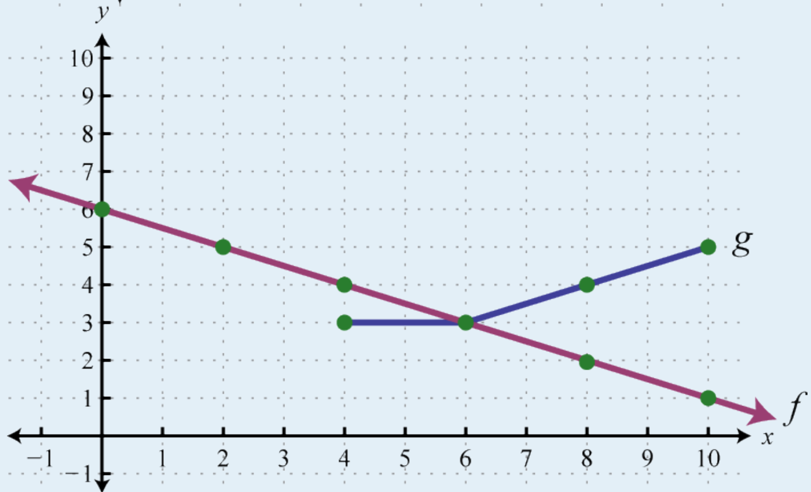
PART D: ADDING FUNCTIONS GEOMETRICALLY

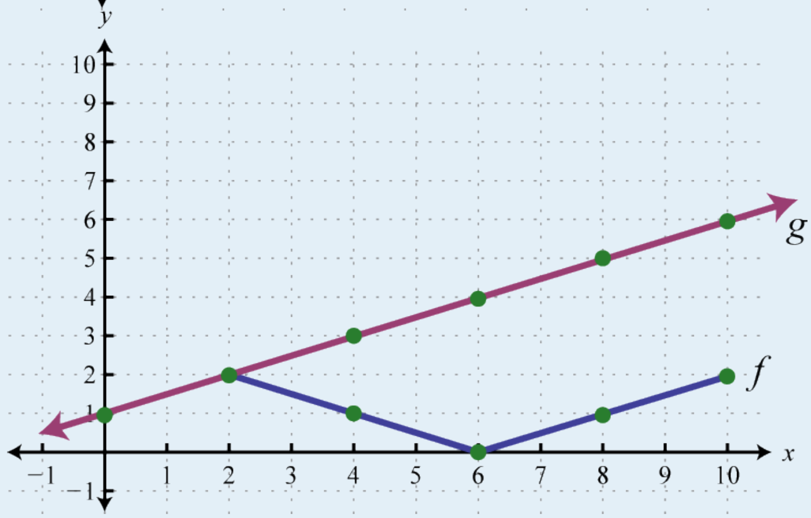
Use the graphs of f **and** g **to graph** $f + g$**. Also, give the domain of** $f + g$.

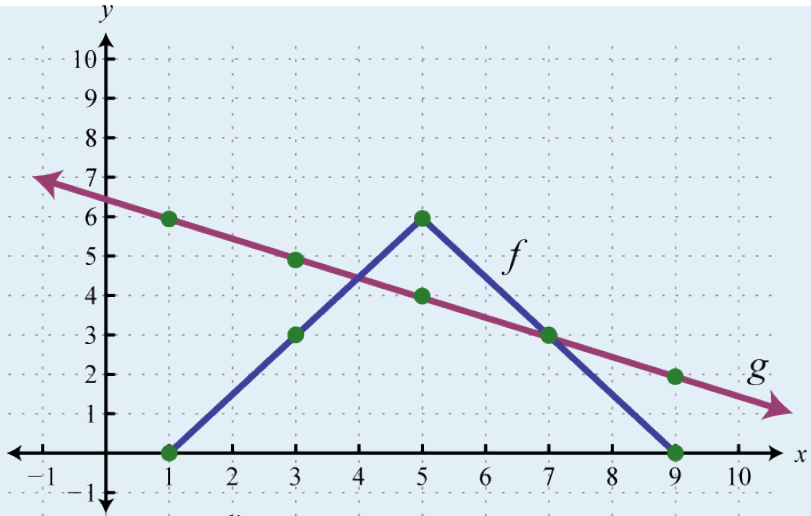
103. 

104. 

105. 

106. 

107. 

108. 
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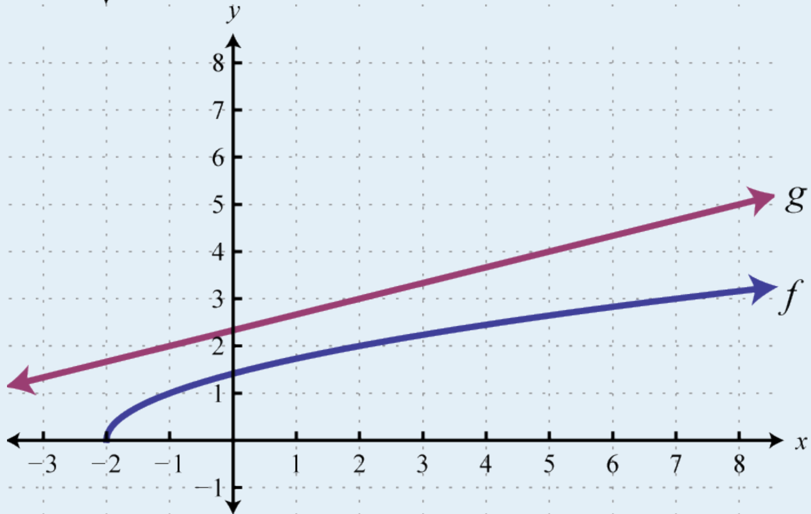
109. 
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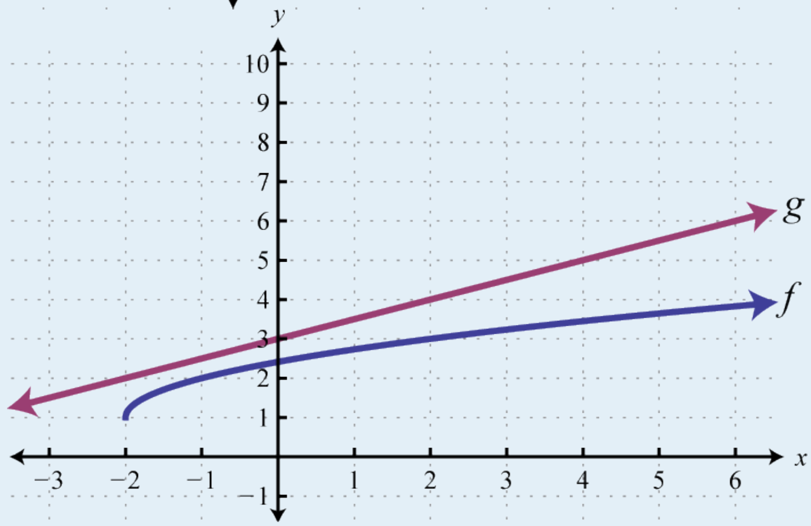
110. 
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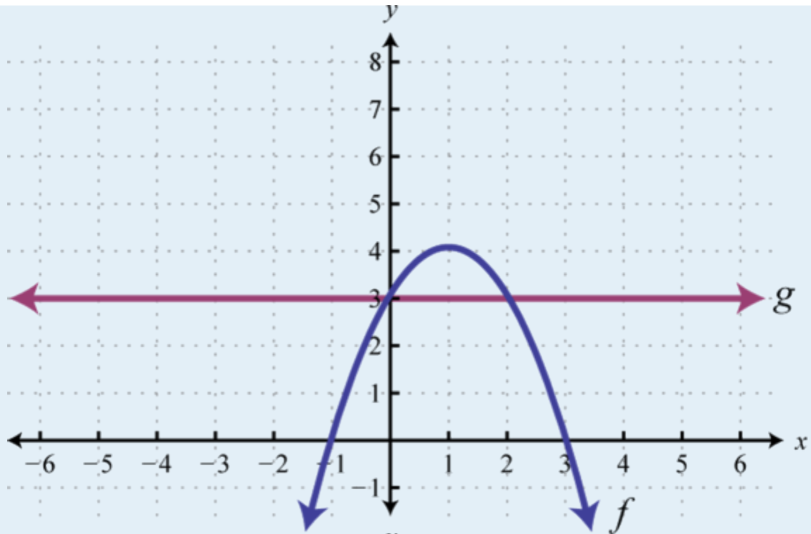
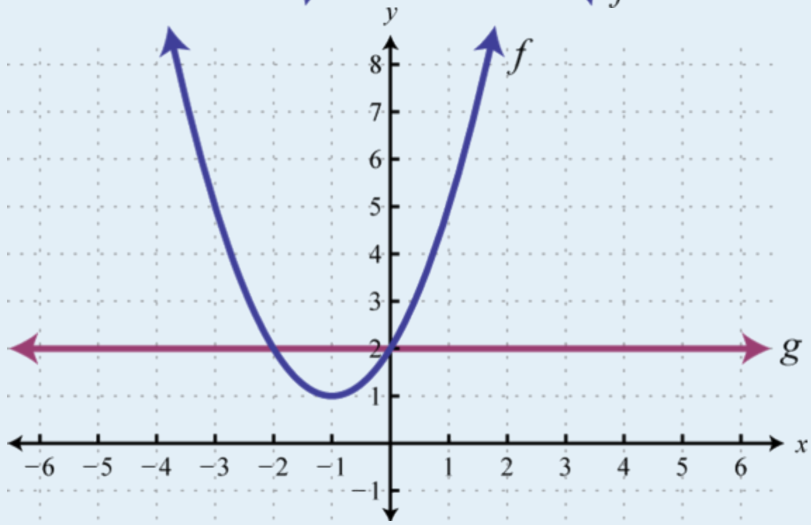
111. 

112. 

PART E: DISCUSSION BOARD

113. Which arithmetic operations on functions are commutative? Explain.

114. Explore ways we can add functions graphically if they happen to be negative.

ANSWERS

1. $f(-3) = 42; f(0) = 3; f\left(5\right) = -22$

3. $g(-2) = -7; g(0) = 7; g(3) = 28$

5. $s(-1) = 0; s(0) = -3; s(2) = 75$

7. $f(-2) = 0; f(-1) = -63; f(0) = -64$

9. $f(2t) = 4t^2 - 4t - 1; f(2t - 1) = 4t^2 - 8t + 2$

11. $g\left(-5a\right) = 50a^2 - 15a - 1; g\left(5 - 2x\right) = 8x^2 - 46x + 64$

13. $f(2a) = 8a^3 - 1; f(x - 2) = x^3 - 6x^2 + 12x - 9$

15. $g\left(x^2\right) = x^6 + x^4 - 1; g(x - 4) = x^3 - 11x^2 + 40x - 49$

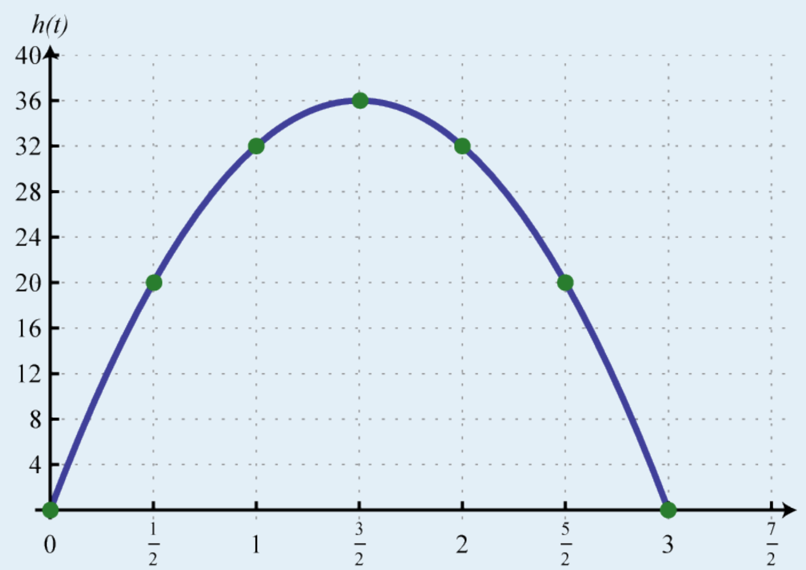
17. $f\left(x + h\right) = 5x + 5h - 3$

19. $f\left(x + h\right) = x^3 + 3hx^2 + 3h^2x + h^3 - 8$

21. $f(0) = -3; f(1) = 0; f(2) = -3$

23. $f(-2) = 2; f(-1) = -7; f(0) = -2$

25. $h(t) = -16t^2 + 48t;$



27. $h(t) = -16t^2 + 128;$ At 1 second the object's height is 112 feet and at 2 seconds its height is 64 feet.

29. $h(t) = -4.9t^2 + 320t$; 1,560 meters

31. $(f+g)(x) = 9x - 4$; $(f-g)(x) = x - 2$

33. $(f+g)(x) = -4x + 3$; $(f-g)(x) = -2x + 1$

35. $(f+g)(x) = 2x^2 + x - 5$; $(f-g)(x) = -7x + 9$

37. $(f+g)(x) = x^2 + 11x + 8$; $(f-g)(x) = x^2 - x - 14$

39. $(f+g)(x) = 10x^2 + 5x - 1$; $(f-g)(x) = 8x^2 - 5x - 1$

41. $(f+g)(x) = 12x^3 + x^2 + x - 5$;
$(f-g)(x) = 4x^3 - x^2 + x - 3$

43. 11

45. −27

47. 27

49. −10

51. −4

53. 1

55. −2

57. $(f \cdot g)(x) = 5x^2 - 15x$

59. $(f \cdot g)(x) = 6x^2 - x - 12$

61. $(f \cdot g)(x) = 9x^2 - 16$

63. $(f \cdot g)(x) = x^3 - 5x^2 + 8x - 4$

65. $(f \cdot g)(x) = 2x^4 - 14x^3 + 10x^2$

67. $(f \cdot g)(x) = 2x^4 - 7x^3 + 2x^2 - 7x - 6$

69. $(f/g)(x) = 9x^2 - 4x - 2$

71. $(f/g)(x) = 4x^4 - 3x^2 + 1$

73. $(f/g)(x) = x^2 + 2x - 1$

75. $\left(f/g\right)\ (x) = 3x^2 - 2x + 15$

77. $\left(f/g\right)\ (x) = x^2 - 5x + 3 + \frac{2}{3x+2}$

79. 50

81. 15

83. 0

85. 25

87. -2

89. 0

91. 4

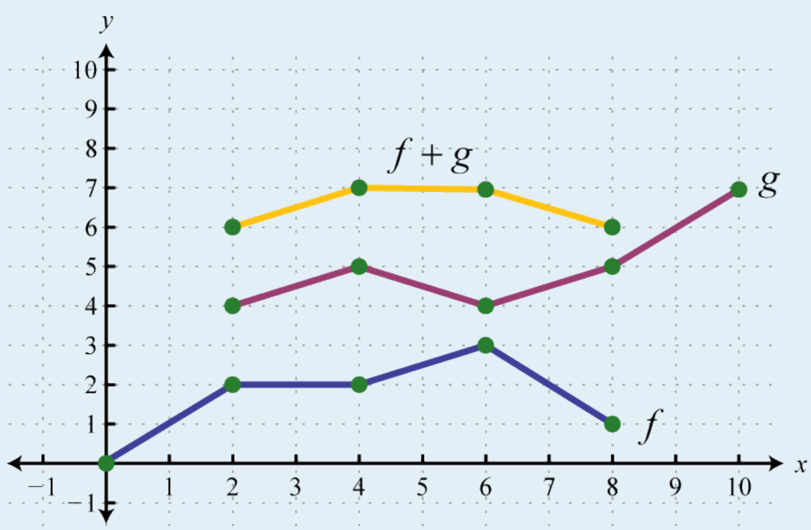
93. $\left(f - g\right)\ (x) = 5x^3 - 16x^2 + 11x - 3$

95. $\left(g \cdot h\right)\ (x) = -5x^3 + 5x^2 - 15x$

97. $\left(h + g\right)\ (x) = x^2 - 6x + 3$

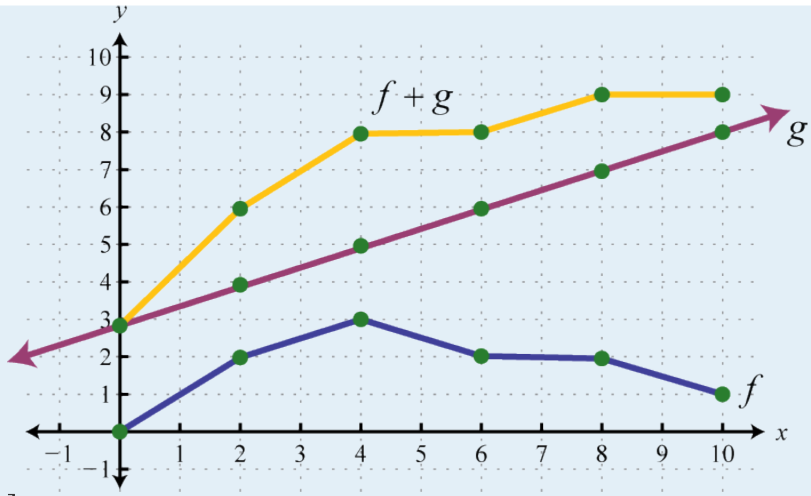
99. $\left(g/h\right)\ (2) = -\frac{1}{2}$

101. $P\left(n\right) = -0.15n^2 + 83n - 1200$; $9,881.25

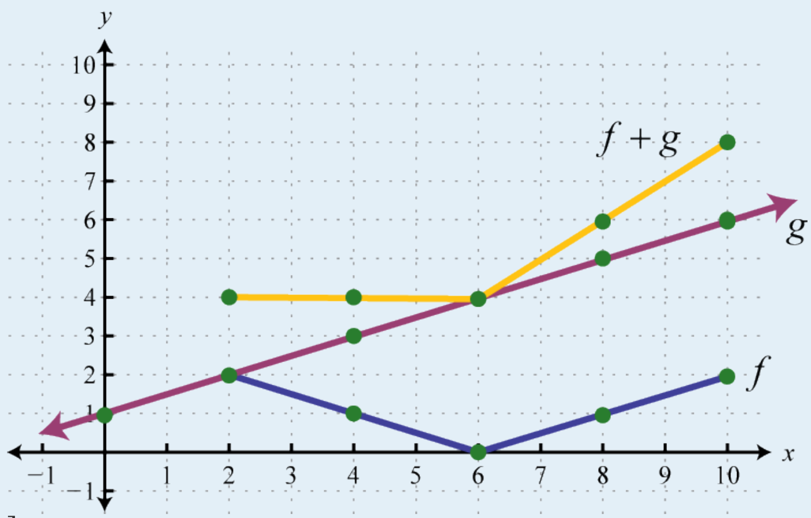


103. $\left[2, 8\right]$

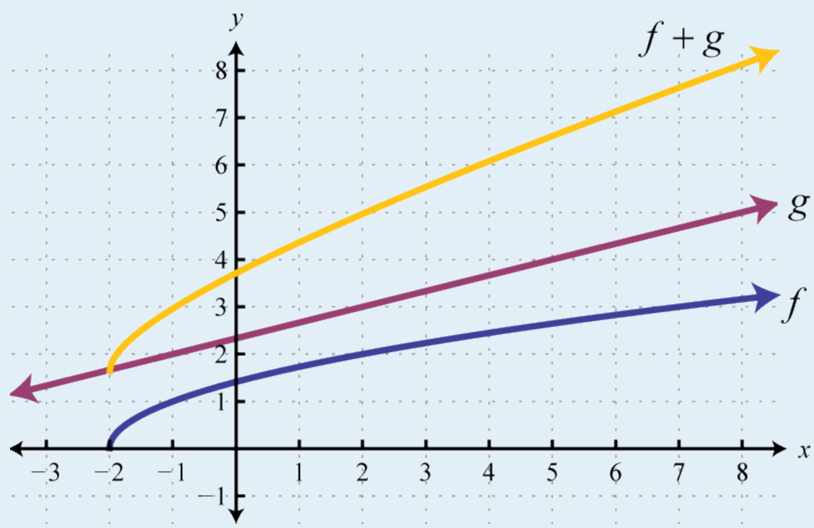
105.



$[0, 10]$

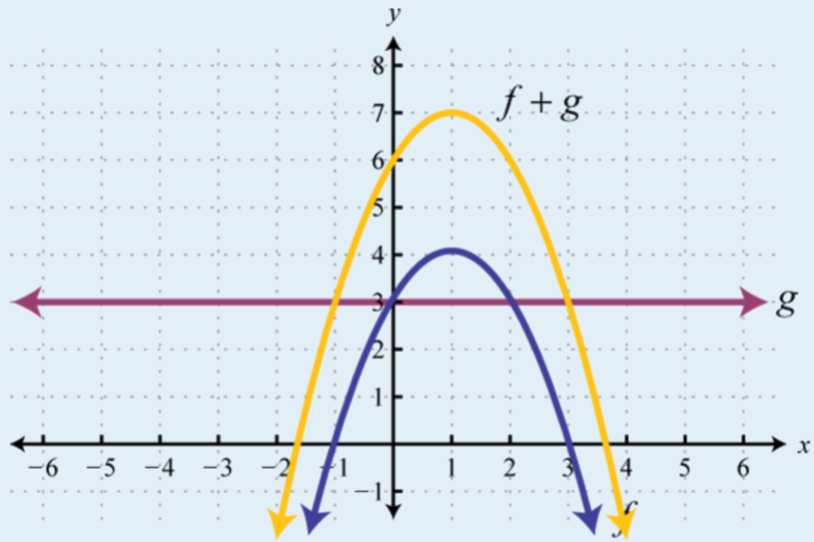


107. $[2, 10]$



109.

$[-2, \infty)$



111. $(-\infty, \infty)$

113. Answer may vary

4.2 Factoring Polynomials

LEARNING OBJECTIVES

1. Determine the greatest common factor (GCF) of monomials.
2. Factor out the GCF of a polynomial.
3. Factor a four-term polynomial by grouping.
4. Factor special binomials.

Determining the GCF of Monomials

The process of writing a number or expression as a product is called **factoring**[5]. If we write the monomial $8x^7 = 2x^5 \cdot 4x^2$, we say that the product $2x^5 \cdot 4x^2$ is a **factorization**[6] of $8x^7$ and that $2x^5$ and $4x^2$ are **factors**[7]. Typically, there are many ways to factor a monomial. Some factorizations of $8x^7$ follow:

$$\left.\begin{aligned} 8x^7 &= 2x^5 \cdot 4x^2 \\ 8x^7 &= 8x^6 \cdot x \\ 8x^7 &= 2x \cdot 2x^2 \cdot 2x^4 \end{aligned}\right\} \quad \textit{Factorizations of } 8x^7$$

Given two or more monomials, it will be useful to find the **greatest common monomial factor (GCF)**[8] of each. The GCF of the monomials is the product of the common variable factors and the GCF of the coefficients.

5. The process of writing a number or expression as a product.

6. Any combination of factors, multiplied together, resulting in the product.

7. Any of the numbers or expressions that form a product.

8. The product of the common variable factors and the GCF of the coefficients.

Example 1

Find the GCF of $25x^7y^2z$ and $15x^3y^4z^2$.

Solution:

Begin by finding the GCF of the coefficients. In this case, $25 = 5 \cdot 5$ and $15 = 3 \cdot 5$. It should be clear that

$$\text{GCF}\left(25, 15\right) = 5$$

Next determine the common variable factors with the smallest exponents.

$$25x^7y^2z \quad \text{and} \quad 15x^3y^4z^2$$

The common variable factors are x^3, y^2, and z. Therefore, given the two monomials,

$$\text{GCF} = 5x^3y^2z$$

Answer: $5x^3y^2z$

It is worth pointing out that the GCF divides both expressions evenly.

$$\frac{25x^7y^2z}{5x^3y^2z} = 5x^4 \qquad \text{and} \qquad \frac{15x^3y^4z^2}{5x^3y^2z} = 3y^2z$$

Furthermore, we can write the following:

$$25x^7y^2z = 5x^3y^2z \cdot 5x^4 \qquad \text{and} \qquad 15x^3y^4z^2 = 5x^3y^2z \cdot 3y^2z$$

The factors $5x^4$ and $3y^2z$ share no common monomial factors other than 1; they are **relatively prime**[9].

9. Expressions that share no common factors other than 1.

Example 2

Determine the GCF of the following three expressions: $12a^5b^2(a+b)^5$, $60a^4b^3c(a+b)^3$, and $24a^2b^7c^3(a+b)^2$.

Solution:

Begin by finding the GCF of the coefficients. To do this, determine the prime factorization of each and then multiply the common factors with the smallest exponents.

$$\begin{aligned}12 &= 2^2 \cdot 3 \\ 60 &= 2^2 \cdot 3 \cdot 5 \\ 24 &= 2^3 \cdot 3\end{aligned}$$

Therefore, the GCF of the coefficients of the three monomials is

$$\text{GCF}(12, 60, 24) = 2^2 \cdot 3 = 12$$

Next, determine the common factors of the variables.

$$12a^5b^2(a+b)^5 \quad \text{and} \quad 60a^4b^3c(a+b)^3 \quad \text{and} \quad 24a^2b^7c^3(a+$$

The variable factors in common are a^2, b^2, and $(a+b)^2$. Therefore,

$$\text{GCF} = 12 \cdot a^2 \cdot b^2 \cdot (a+b)^2$$

Note that the variable *c* is not common to all three expressions and thus is not included in the GCF.

Answer: $12a^2b^2(a+b)^2$

Factoring out the GCF

The application of the distributive property is the key to multiplying polynomials. For example,

$$\begin{aligned} 6xy^2(2xy+1) &= 6xy^2 \cdot 2xy + 6xy^2 \cdot 1 \qquad \textit{Multiplying} \\ &= 12x^2y^3 + 6xy^2 \end{aligned}$$

The process of factoring a polynomial involves applying the distributive property in reverse to write each polynomial as a product of polynomial factors.

$$\begin{aligned} a(b+c) &= ab + ac \qquad \textit{Multiplying} \\ ab + ac &= a(b+c) \qquad \textit{Factoring} \end{aligned}$$

Consider factoring the result of the opening example:

$$\begin{aligned} 12x^2y^3 + 6xy^2 &= 6xy^2 \cdot 2xy + 6xy^2 \cdot 1 \qquad \textit{Factoring} \\ &= 6xy^2 \left(\quad ? \quad \right) \\ &= 6xy^2 \left(2xy + 1\right) \end{aligned}$$

We see that the distributive property allows us to write the polynomial $12x^2y^3 + 6xy^2$ as a product of the two factors $6xy^2$ and $\left(2xy + 1\right)$. Note that in this case, $6x^2y$ is the GCF of the terms of the polynomial.

$$\text{GCF}\left(12x^2y^3,\ 6xy^2\right) = 6xy^2$$

Factoring out the greatest common factor (GCF)[10] of a polynomial involves rewriting it as a product where a factor is the GCF of all of its terms.

$$\left.\begin{aligned} 8x^3 + 4x^2 - 16x &= 4x\left(2x^2 + x - 4\right) \\ 9ab^2 - 18a^2b - 3ab &= 3ab\left(3b - 6a - 1\right) \end{aligned}\right\} \qquad \textit{Factoring out the GCF}$$

To factor out the GCF of a polynomial, we first determine the GCF of all of its terms. Then we can divide each term of the polynomial by this factor as a means to determine the remaining factor after applying the distributive property in reverse.

10. The process of rewriting a polynomial as a product using the GCF of all of its terms.

Example 3

Factor out the GCF: $18x^7 - 30x^5 + 6x^3$.

Solution:

In this case, the GCF(18, 30, 6) = 6, and the common variable factor with the smallest exponent is x^3. The GCF of the polynomial is $6x^3$.

$$18x^7 - 30x^5 + 6x^3 = 6x^3 (\qquad ? \qquad)$$

The missing factor can be found by dividing each term of the original expression by the GCF.

$$\frac{18x^7}{6x^3} = 3x^4 \qquad \frac{-30x^5}{6x^3} = -5x^2 \qquad \frac{+6x^3}{6x^3} = +1$$

Apply the distributive property (in reverse) using the terms found in the previous step.

$$18x^7 - 30x^5 + 6x^3 = 6x^3 \left(3x^4 - 5x^2 + 1\right)$$

If the GCF is the same as one of the terms, then, after the GCF is factored out, a constant term 1 will remain. The importance of remembering the constant term becomes clear when performing the check using the distributive property.

$$\begin{aligned} 6x^3\left(3x^4 - 5x^2 + 1\right) &= 6x^3 \cdot 3x^4 - 6x^3 \cdot 5x^2 + 6x^3 \cdot 1 \\ &= 18x^7 - 30x^5 + 6x^3 \quad ✓ \end{aligned}$$

Answer: $6x^3\left(3x^4 - 5x^2 + 1\right)$

Example 4

Factor out the GCF: $27x^5y^5z + 54x^5yz - 63x^3y^4$.

Solution:

The GCF of the terms is $9x^3y$. The last term does not have a variable factor of z, and thus z cannot be a part of the greatest common factor. If we divide each term by $9x^3y$, we obtain

$$\frac{27x^5y^5z}{9x^3y} = 3x^2y^4z \qquad \frac{54x^5yz}{9x^3y} = 6x^2z \qquad \frac{-63x^3y^4}{9x^3y} = -7y^3$$

and can write

$$\begin{aligned} 27x^5y^5z + 54x^5yz - 63x^3y^4 &= 9x^3y\,(\qquad ? \qquad) \\ &= 9x^3y\,\left(3x^2y^4z + 6x^2z - 7y^3\right) \end{aligned}$$

Answer: $9x^3y\,\left(3x^2y^4z + 6x^2z - 7y^3\right)$

Try this! Factor out the GCF: $12x^3y^4 - 6x^2y^3 - 3xy^2$

Answer: $3xy^2\,\left(4x^2y^2 - 2xy - 1\right)$

(click to see video)

Of course, not every polynomial with integer coefficients can be factored as a product of polynomials with integer coefficients other than 1 and itself. If this is the case, then we say that it is a **prime polynomial**[11]. For example, a linear factor such as $10x - 9$ is prime. However, it can be factored as follows:

$$10x - 9 = x\left(10 - \frac{9}{x}\right) \qquad \text{or} \qquad 10x - 9 = 5\left(2x - \frac{9}{5}\right)$$

If an x is factored out, the resulting factor is not a polynomial. If any constant is factored out, the resulting polynomial factor will not have integer coefficients. Furthermore, some linear factors are not prime. For example,

$$5x - 10 = 5\,(x - 2)$$

In general, any linear factor of the form $ax + b$, where a and b are relatively prime integers, is prime.

Factoring by Grouping

In this section, we outline a technique for factoring polynomials with four terms. First, review a preliminary example where the terms have a common binomial factor.

11. A polynomial with integer coefficients that cannot be factored as a product of polynomials with integer coefficients other than 1 and itself.

Example 5

Factor: $7x\,(3x - 2) - \ (3x - 2)$.

Solution:

Begin by rewriting the second term $-(3x - 2)$ as $-1\,(3x - 2)$. Next, consider $(3x - 2)$ as a common binomial factor and factor it out as follows:

$$\begin{aligned} 7x\,(3x - 2) - \ (3x - 2) &= 7x\,(3x - 2) - 1\,(3x - 2) \\ &= (3x - 2)\,\left(\ ? \ \right) \\ &= (3x - 2)\,(7x - 1) \end{aligned}$$

Answer: $(3x - 2)\,(7x - 1)$

Factoring by grouping[12] is a technique that enables us to factor polynomials with four terms into a product of binomials. This involves an intermediate step where a common binomial factor will be factored out. For example, we wish to factor

$$3x^3 - 12x^2 + 2x - 8$$

Begin by grouping the first two terms and the last two terms. Then factor out the GCF of each grouping:

$$\begin{aligned} &\underbrace{3x^3 - 12x^2}_{group} + \underbrace{2x - 8}_{group} \\ &= 3x^2\,(x - 4) + 2\,(x - 4) \end{aligned}$$

12. A technique for factoring polynomials with four terms.

In this form, the polynomial is a binomial with a common binomial factor, $(x - 4)$.

$$= (x - 4)\left(\quad ? \quad\right)$$
$$= (x - 4)\left(3x^2 + 2\right)$$

Therefore,

$$3x^3 - 12x^2 + 2x - 8 = (x - 4)\left(3x^2 + 2\right)$$

We can check by multiplying.

$$\begin{aligned}(x - 4)\left(3x^2 + 2\right) &= 3x^3 + 2x - 12x^2 - 8 \\ &= 3x^3 - 12x^2 + 2x - 8 \quad ✓\end{aligned}$$

Example 6

Factor by grouping: $24a^4 - 18a^3 - 20a + 15$.

Solution:

The GCF for the first group is $6a^3$. We have to choose 5 or −5 to factor out of the second group.

$$
\begin{aligned}
&\underbrace{24a^4 - 18a^3}_{group} - \underbrace{20a + 15}_{group} \\
&= 6a^3(4a-3) + 5(-4a+3) \quad \times \\
&= 6a^3(4a-3) - 5(4a-3) \quad \checkmark
\end{aligned}
$$

Factoring out +5 does not result in a common binomial factor. If we choose to factor out −5, then we obtain a common binomial factor and can proceed. Note that when factoring out a negative number, we change the signs of the factored terms.

$$
\begin{aligned}
24a^4 - 18a^3 - 20a + 15 &= \underbrace{24a^4 - 18a^3}_{group} - \underbrace{20a + 15}_{group} \\
&= 6a^3(\quad ? \quad) - 5(\quad ? \quad) \\
&= 6a^3(4a-3) - 5(4a-3) \\
&= (4a-3)(\quad ? \quad) \\
&= (4a-3)(6a^3-5)
\end{aligned}
$$

Answer: $(4a - 3)\left(6a^3 - 5\right)$.Check by multiplying; this is left to the reader as an exercise.

Sometimes we must first rearrange the terms in order to obtain a common factor.

Example 7

Factor: $ab - 2a^2b + a^3 - 2b^2$.

Solution:

Simply factoring the GCF out of the first group and last group does not yield a common binomial factor.

$$\underbrace{ab - 2a^2b}_{group} + \underbrace{a^3 - 2b^2}_{group}$$
$$= ab(1 - 2a) + 1(a^3 - 2b^2)$$

We must rearrange the terms, searching for a grouping that produces a common factor. In this example, we have a workable grouping if we switch the terms a^3 and ab.

$$\begin{aligned} ab - 2a^2b + a^3 - 2b^2 &= \underbrace{a^3 - 2a^2b}_{group} + \underbrace{ab - 2b^2}_{group} \\ &= a^2(a - 2b) + b(a - 2b) \\ &= (a - 2b)(a^2 + b) \end{aligned}$$

Answer: $(a - 2b)\ (a^2 + b)$

Try this! Factor: $x^3 - x^2y - xy + y^2$.

Answer: $(x - y)\ (x^2 - y)$

(click to see video)

Not all factorable four-term polynomials can be factored with this technique. For example,

$$3x^3 + 5x^2 - x + 2$$

This four-term polynomial cannot be grouped in any way to produce a common binomial factor. Despite this, the polynomial is not prime and can be written as a product of polynomials. It can be factored as follows:

$$3x^3 + 5x^2 - x + 2 = (x + 2)\left(3x^2 - x + 1\right)$$

Factoring such polynomials is something that we will learn to do as we move further along in our study of algebra. For now, we will limit our attempt to factor four-term polynomials to using the factor by grouping technique.

Factoring Special Binomials

A binomial is a polynomial with two terms. We begin with the special binomial called **difference of squares**[13]:

$$a^2 - b^2 = (a + b)(a - b)$$

To verify the above formula, multiply.

$$\begin{aligned}(a + b)(a - b) &= a^2 - ab + ba - b^2\\ &= a^2 - ab + ab - b^2\\ &= a^2 - b^2\end{aligned}$$

We use this formula to factor certain special binomials.

13. $a^2 - b^2 = (a + b)(a - b)$, where a and b represent algebraic expressions.

Example 8

Factor: $x^2 - 9y^2$.

Solution:

Identify the binomial as difference of squares and determine the square factors of each term.

$$\underset{\text{Difference}}{x^2 - 9y^2} \quad \text{of} \quad \underset{\text{Squares}}{x^2 - 9y^2}$$

Here we can write

$$x^2 - 9y^2 = (x)^2 - (3y)^2$$

Substitute into the difference of squares formula where $a = x$ and $b = 3y$.

$$\begin{aligned} a^2 - b^2 &= (a + b)(a - b) \\ &\quad\ \downarrow\ \ \downarrow\quad\ \downarrow\ \ \downarrow \\ x^2 - 9y^2 &= (x + 3y)(x - 3y) \end{aligned}$$

Multiply to check.

$$\begin{aligned}(x+3y)\,(x-3y) &= x^2 - 3xy + 3yx - 9y^2\\ &= x^2 - 3xy + 3xy - 9y^2\\ &= x^2 - 9y^2 \quad ✓\end{aligned}$$

Answer: $(x+3y)\,(x-3y)$

Example 9

Factor: $x^2 - (2x - 1)^2$.

Solution:

First, identify this expression as a difference of squares.

$$x^2 - (2x-1)^2 = (x)^2 - (2x-1)^2$$

Use $a = x$ and $b = 2x - 1$ in the formula for a difference of squares and then simplify.

$$a^2 - b^2 = (a+b)(a-b)$$

$$\begin{aligned} x^2 - (2x-1)^2 &= [x + (2x-1)]\,[x - (2x-1)] \\ &= (x + 2x - 1)\,(x - 2x + 1) \\ &= (3x-1)\,(-x+1) \end{aligned}$$

Answer: $(3x-1)\,(-x+1)$

Given any real number b, a polynomial of the form $x^2 + b^2$ is prime. Furthermore, the **sum of squares**[14] $a^2 + b^2$ does not have a general factored equivalent. Care should be taken not to confuse this with a perfect square trinomial.

14. $a^2 + b^2$, where a and b represent algebraic expressions. This does not have a general factored equivalent.

$$\begin{aligned}(a+b)^2 &= (a+b)\,(a+b)\\ &= a^2+ab+ba+b^2\\ &= a^2+2ab+b^2\end{aligned}$$

Therefore,

$$(a+b)^2 \neq a^2+b^2$$

For example, the sum of squares binomial x^2+9 is prime. Two other special binomials of interest are the **sum**[15] and **difference of cubes**[16]:

$$\begin{aligned}a^3+b^3 &= (a+b)\,(a^2-ab+b^2)\\ a^3-b^3 &= (a-b)\,(a^2+ab+b^2)\end{aligned}$$

We can verify these formulas by multiplying.

$$\begin{aligned}(a+b)\,(a^2-ab+b^2) &= a^3-a^2b+ab^2+a^2b-ab^2+b^3\\ &= a^3+b^3 \quad ✓\end{aligned}$$

$$\begin{aligned}(a-b)\,(a^2+ab+b^2) &= a^3+a^2b+ab^2-a^2b-ab^2-b^3\\ &= a^3-b^3 \quad ✓\end{aligned}$$

15. $a^3+b^3 = (a+b)(a^2-ab+b^2)$, where a and b represent algebraic expressions.

16. $a^3-b^3 = (a-b)(a^2+ab+b^2)$, where a and b represent algebraic expressions.

The process for factoring sums and differences of cubes is very similar to that of differences of squares. We first identify a and b and then substitute into the appropriate formula. The separate formulas for the sum and difference of cubes allow us to always choose a and b to be positive.

Example 10

Factor: $x^3 - 8y^3$.

Solution:

First, identify this binomial as a difference of cubes.

Difference *of* *Cubes*

$x^3 - 8y^3$ $\quad$ $x^3 - 8y^3$

Next, identify what is being cubed.

$$x^3 - 8y^3 = (x)^3 - (2y)^3$$

In this case, $a = x$ and $b = 2y$. Substitute into the difference of cubes formula.

$$\begin{aligned} a^3 + b^3 &= (a - b)(a^2 + a \cdot b + b^2) \\ &\quad\; \downarrow \quad \downarrow \quad\; \downarrow \quad \downarrow \quad \downarrow \quad\; \downarrow \\ x^3 - 8y^3 &= (x - 2y)((x)^2 + x \cdot 2y + (2y)^2) \\ &= (x - 2y)(x^2 + 2xy + 4y^2) \end{aligned}$$

We can check this factorization by multiplying.

$$
\begin{aligned}
(x - 2y)\ (x^2 + 2xy + 4y^2) &= x^3 + 2x^2y + 4xy^2 - 2x^2y - 4xy^2 - 8y^3 \\
&= x^3 + 2x^2y + 4xy^2 - 2x^2y - 4xy^2 - 8y^3 \\
&= x^3 - 8y^3 \quad \checkmark
\end{aligned}
$$

Answer: $(x - 2y)\ (x^2 + 2xy + 4y^2)$

It may be the case that the terms of the binomial have a common factor. If so, it will be difficult to identify it as a special binomial until we first factor out the GCF.

Example 11

Factor: $81x^4y + 3xy^4$.

Solution:

The terms are not perfect squares or perfect cubes. However, notice that they do have a common factor. First, factor out the GCF, $3xy$.

$$81x^4y + 3xy^4 = 3xy\left(27x^3 + y^3\right)$$

The resulting binomial factor is a sum of cubes with $a = 3x$ and $b = y$.

$$\begin{aligned} 81x^4y + 3xy^4 &= 3xy\left(27x^3 + y^3\right) \\ &= 3xy\left(3x + y\right)\left(9x^2 - 3xy + y^2\right) \end{aligned}$$

Answer: $3xy\left(3x + y\right)\left(9x^2 - 3xy + y^2\right)$

When the degree of the special binomial is greater than two, we may need to apply the formulas multiple times to obtain a complete factorization. A polynomial is **completely factored**[17] when it is prime or is written as a product of prime polynomials.

17. A polynomial that is prime or written as a product of prime polynomials.

Example 12

Factor completely: $x^4 - 81y^4$.

Solution:

First, identify what is being squared.

$$x^4 - 81y^4 = (\quad)^2 - (\quad)^2$$

To do this, recall the power rule for exponents, $(x^m)^n = x^{mn}$. When exponents are raised to a power, multiply them. With this in mind, we find

$$x^4 - 81y^4 = \left(x^2\right)^2 - \left(9y^2\right)^2$$

Therefore, $a = x^2$ and $b = 9y^2$. Substitute into the formula for difference of squares.

$$x^4 - 81y^4 = \left(x^2 + 9y^2\right)\left(x^2 - 9y^2\right)$$

At this point, notice that the factor $\left(x^2 - 9y^2\right)$ is itself a difference of two squares and thus can be further factored using $a = x^2$ and $b = 3y$. The factor $\left(x^2 + 9y^2\right)$ is prime and cannot be factored using real numbers.

$$\begin{aligned} x^4 - 81y^4 &= \left(x^2 + 9y^2\right)\left(x^2 - 9y^2\right) \\ &= \left(x^2 + 9y^2\right)\left(x + 3y\right)\left(x - 3y\right) \end{aligned}$$

Answer: $\left(x^2 + 9y^2\right)\left(x + 3y\right)\left(x - 3y\right)$

When factoring, always look for resulting factors to factor further.

Example 13

Factor completely: $64x^6 - y^6$.

Solution:

This binomial is both a difference of squares and difference of cubes.

$$64x^6 - y^6 = \left(4x^2\right)^3 - \left(y^2\right)^3 \quad \textit{Difference of cubes}$$
$$64x^6 - y^6 = \left(8x^3\right)^2 - \left(y^3\right)^2 \quad \textit{Difference of squares}$$

When confronted with a binomial that is a difference of both squares and cubes, as this is, make it a rule to factor using difference of squares first. Therefore, $a = 8x^3$ and $b = y^3$. Substitute into the difference of squares formula.

$$64x^6 - y^6 = \left(8x^3 + y^3\right)\left(8x^3 - y^3\right)$$

The resulting two binomial factors are sum and difference of cubes. Each can be factored further. Therefore, we have

$$\begin{aligned} 64x^6 - y^6 &= \left(8x^3 + y^3\right) \cdot \left(8x^3 - y^3\right) \\ &= \underbrace{(2x + y)\left(4x^2 - 2xy + y^2\right)} \cdot \underbrace{(2x - y)\left(4x^2 + 2xy + y^2\right)} \end{aligned}$$

The trinomial factors are prime and the expression is completely factored.

Answer: $(2x + y)\left(4x^2 - 2xy + y^2\right)(2x - y)\left(4x^2 + 2xy + y^2\right)$

As an exercise, factor the previous example as a difference of cubes first and then compare the results. Why do you think we make it a rule to factor using difference of squares first?

Try this! Factor: $a^6 b^6 - 1$

Answer: $\left(ab + 1\right) \left(a^2 b^2 - ab + 1\right) \left(ab - 1\right) \left(a^2 b^2 + ab + 1\right)$

(click to see video)

KEY TAKEAWAYS

- The GCF of two or more monomials is the product of the GCF of the coefficients and the common variable factors with the smallest power.
- If the terms of a polynomial have a greatest common factor, then factor out that GCF using the distributive property. Divide each term of the polynomial by the GCF to determine the terms of the remaining factor.
- Some four-term polynomials can be factored by grouping the first two terms and the last two terms. Factor out the GCF of each group and then factor out the common binomial factor.
- When factoring by grouping, you sometimes have to rearrange the terms to find a common binomial factor. After factoring out the GCF, the remaining binomial factors must be the same for the technique to work.
- When factoring special binomials, the first step is to identify it as a sum or difference. Once we identify the binomial, we then determine the values of *a* and *b* and then substitute into the appropriate formula.
- If a binomial is both a difference of squares and cubes, then first factor it as a difference of squares.

TOPIC EXERCISES

PART A: FACTORING OUT THE GCF

Determine the GCF of the given expressions.

1. $9x^5, 27x^2, 15x^7$
2. $20y^4, 12y^7, 16y^3$
3. $50x^2y^3, 35xy^3, 10x^3y^2$
4. $12x^7y^2, 36x^4y^2, 18x^3y$
5. $15a^7b^2c^5, 75a^7b^3c, 45ab^4c^3$
6. $12a^6b^3c^2, 48abc^3, 125a^2b^3c$
7. $60x^2(2x-1)^3, 42x(2x-1)^3, 6x^3(2x-1)$
8. $14y^5(y-8)^2, 28y^2(y-8), 35y(y-8)^3$
9. $10a^2b^3(a+b)^5, 48a^5b^2(a+b)^2, 26ab^5(a+b)^3$
10. $45ab^7(a-b)^7, 36a^2b^2(a-b)^3, 63a^4b^3(a-b)^2$

Determine the missing factor.

11. $18x^4 - 6x^3 + 2x^2 = 2x^2(\quad ? \quad)$
12. $6x^5 - 9x^3 - 3x = 3x(\quad ? \quad)$
13. $-10y^6 + 6y^4 - 4y^2 = -2y^2(\quad ? \quad)$
14. $-27y^9 - 9y^6 + 3y^3 = -3y^3(\quad ? \quad)$
15. $12x^3y^2 - 8x^2y^3 + 8xy = 4xy(\quad ? \quad)$
16. $10x^4y^3 - 50x^3y^2 + 15x^2y^2 = 5xy(\quad ? \quad)$
17. $14a^4b^5 - 21a^3b^4 - 7a^2b^3 = 7a^2b^3(\quad ? \quad)$

18. $15a^5b^4 + 9a^4b^2 - 3a^2b = 3a^2b\left(\quad ? \quad\right)$

19. $x^{3n} + x^{2n} + x^n = x^n\left(\quad ? \quad\right)$

20. $y^{4n} + y^{3n} - y^{2n} = y^{2n}\left(\quad ? \quad\right)$

Factor out the GCF.

21. $12x^4 - 16x^3 + 4x^2$

22. $15x^5 - 10x^4 - 5x^3$

23. $20y^8 + 28y^6 + 40y^3$

24. $18y^7 - 24y^5 - 30y^3$

25. $2a^4b^3 - 6a^3b^2 + 8a^2b$

26. $28a^3b^3 - 21a^2b^4 - 14ab^5$

27. $2x^3y^5 - 4x^4y^4 + x^2y^3$

28. $3x^5y - 2x^4y^2 + x^3y^3$

29. $5x^2(2x + 3) - 3(2x + 3)$

30. $y^2(y - 1) + 9(y - 1)$

31. $9x^2(3x - 1) + (3x - 1)$

32. $7y^2(5y + 2) - (5y + 2)$

33. $x^{5n} - x^{3n} + x^n$

34. $y^{6n} - y^{3n} - y^{2n}$

PART B: FACTORING BY GROUPING

Factor by grouping.

35. $2x^3 + 3x^2 + 2x + 3$

36. $5x^3 + 25x^2 + x + 5$

37. $6x^3 - 3x^2 + 4x - 2$

38. $3x^3 - 2x^2 - 15x + 10$
39. $x^3 - x^2 - 3x + 3$
40. $6x^3 - 15x^2 - 2x + 5$
41. $2x^3 + 7x^2 - 10x - 35$
42. $3x^3 - x^2 + 24x - 8$
43. $14y^4 + 10y^3 - 7y - 5$
44. $5y^4 + 2y^3 + 20y + 8$
45. $x^{4n} + x^{3n} + 2x^n + 2$
46. $x^{5n} + x^{3n} + 3x^{2n} + 3$
47. $x^3 - x^2y + xy^2 - y^3$
48. $x^3 + x^2y - 2xy^2 - 2y^3$
49. $3x^3y^2 + 9x^2y^3 - x - 3y$
50. $2x^3y^3 - x^2y^3 + 2x - y$
51. $a^2b - 4ab^2 - 3a + 12b$
52. $a^2b + 3ab^2 + 5a + 15b$
53. $a^4 + a^2b^3 + a^2b + b^4$
54. $a^3b + 2a^2 + 3ab^4 + 6b^3$
55. $3ax + 10by - 5ay - 6bx$
56. $a^2x - 5b^2y - 5a^2y + b^2x$
57. $x^4y^2 - x^3y^3 + x^2y^4 - xy^5$
58. $2x^5y^2 + 4x^4y^2 + 18x^3y + 36x^2y$
59. $a^5b^2 + a^4b^4 + a^3b^3 + a^2b^5$
60. $3a^6b + 3a^5b^2 + 9a^4b^2 + 9a^3b^3$

PART C: FACTORING SPECIAL BINOMIALS

Factor.

61. $x^2 - 64$
62. $x^2 - 100$
63. $9 - 4y^2$
64. $25 - y^2$
65. $x^2 - 81y^2$
66. $x^2 - 49y^2$
67. $a^2b^2 - 4$
68. $1 - 9a^2b^2$
69. $a^2b^2 - c^2$
70. $4a^2 - b^2c^2$
71. $x^4 - 64$
72. $36 - y^4$
73. $\left(2x + 5\right)^2 - x^2$
74. $\left(3x - 5\right)^2 - x^2$
75. $y^2 - \left(y - 3\right)^2$
76. $y^2 - \left(2y + 1\right)^2$
77. $\left(2x + 5\right)^2 - (x - 3)^2$
78. $(3x - 1)^2 - (2x - 3)^2$
79. $x^4 - 16$
80. $81x^4 - 1$
81. $x^4y^4 - 1$

82. $x^4 - y^4$
83. $x^8 - y^8$
84. $y^8 - 1$
85. $x^{2n} - y^{2n}$
86. $x^{2n}y^{2n} - 4$
87. $x^{4n} - y^{4n}$
88. $x^{4n}y^{4n} - 16$
89. $x^3 - 27$
90. $8x^3 - 125$
91. $8y^3 + 27$
92. $64x^3 + 343$
93. $x^3 - y^3$
94. $x^3 + y^3$
95. $8a^3b^3 + 1$
96. $27a^3 - 8b^3$
97. $x^3y^3 - 125$
98. $216x^3 + y^3$
99. $x^3 + (x+3)^3$
100. $y^3 - \left(2y - 1\right)^3$
101. $(2x+1)^3 - x^3$
102. $\left(3y - 5\right)^3 - y^3$
103. $x^{3n} - y^{3n}$
104. $x^{3n} + y^{3n}$
105. $a^6 + 64$

106. $64a^6 - 1$

107. $x^6 - y^6$

108. $x^6 + y^6$

109. $x^{6n} - y^{6n}$

110. $x^{6n} + y^{6n}$

111. Given $f(x) = 2x - 1$, show that $(f + f)(x) = 2f(x)$.

112. Given $f(x) = x^2 - 3x + 2$, show that $(f + f)(x) = 2f(x)$.

113. Given $f(x) = mx + b$, show that $(f + f)(x) = 2f(x)$.

114. Given $f(x) = ax^2 + bx + c$, show that $(f + f)(x) = 2f(x)$.

115. Given $f(x) = ax^2 + bx + c$, show that $(f - f)(x) = 0$.

116. Given $f(x) = mx + b$, show that $(f - f)(x) = 0$.

PART D: DISCUSSION BOARD

117. What can be said about the degree of a factor of a polynomial? Give an example.

118. If a binomial falls into both categories, difference of squares and difference of cubes, which would be best to use for factoring, and why? Create an example that illustrates this situation and factor it using both formulas.

119. Write your own examples for each of the three special types of binomial. Factor them and share your results.

ANSWERS

1. $3x^2$
3. $5xy^2$
5. $15ab^2c$
7. $6x(2x-1)$
9. $2ab^2(a+b)^2$
11. $(9x^2-3x+1)$
13. $(5y^4-3y^2+2)$
15. $(3x^2y-2xy^2+2)$
17. $(2a^2b^2-3ab-1)$
19. $(x^{2n}+x^n+1)$
21. $4x^2(3x^2-4x+1)$
23. $4y^3(5y^5+7y^3+10)$
25. $2a^2b(a^2b^2-3ab+4)$
27. $x^2y^3(2xy^2-4x^2y+1)$
29. $(2x+3)(5x^2-3)$
31. $(3x-1)(9x^2+1)$
33. $x^n(x^{4n}-x^{2n}+1)$
35. $(2x+3)(x^2+1)$
37. $(2x-1)(3x^2+2)$
39. $(x-1)(x^2-3)$
41. $(2x+7)(x^2-5)$

43. $(7y + 5)(2y^3 - 1)$

45. $(x^n + 1)(x^{3n} + 2)$

47. $(x - y)(x^2 + y^2)$

49. $(x + 3y)(3x^2y^2 - 1)$

51. $(a - 4b)(ab - 3)$

53. $(a^2 + b)(a^2 + b^3)$

55. $(a - 2b)(3x - 5y)$

57. $xy^2(x - y)(x^2 + y^2)$

59. $a^2b^2(a^2 + b)(a + b^2)$

61. $(x + 8)(x - 8)$

63. $(3 + 2y)(3 - 2y)$

65. $(x + 9y)(x - 9y)$

67. $(ab + 2)(ab - 2)$

69. $(ab + c)(ab - c)$

71. $(x^2 + 8)(x^2 - 8)$

73. $(3x + 5)(x + 5)$

75. $3(2y - 3)$

77. $(3x + 2)(x + 8)$

79. $(x^2 + 4)(x + 2)(x - 2)$

81. $(x^2y^2 + 1)(xy + 1)(xy - 1)$

83. $(x^4 + y^4)(x^2 + y^2)(x + y)(x - y)$

85. $(x^n + y^n)(x^n - y^n)$

87. $\left(x^{2n} + y^{2n}\right)\left(x^n + y^n\right)\left(x^n - y^n\right)$

89. $(x - 3)\left(x^2 + 3x + 9\right)$

91. $\left(2y + 3\right)\left(4y^2 - 6y + 9\right)$

93. $\left(x - y\right)\left(x^2 + xy + y^2\right)$

95. $\left(2ab + 1\right)\left(4a^2b^2 - 2ab + 1\right)$

97. $\left(xy - 5\right)\left(x^2y^2 + 5xy + 25\right)$

99. $(2x + 3)\left(x^2 + 3x + 9\right)$

101. $(x + 1)\left(7x^2 + 5x + 1\right)$

103. $\left(x^n - y^n\right)\left(x^{2n} + x^n y^n + y^{2n}\right)$

105. $\left(a^2 + 4\right)\left(a^4 - 4a^2 + 16\right)$

107. $\left(x + y\right)\left(x^2 - xy + y^2\right)\left(x - y\right)\left(x^2 + xy + y^2\right)$

109. $$\begin{array}{l}\left(x^n + y^n\right)\left(x^{2n} - x^n y^n + y^{2n}\right)\\ \times\left(x^n - y^n\right)\left(x^{2n} + x^n y^n + y^{2n}\right)\end{array}$$

111. Answer may vary

113. Answer may vary

115. Answer may vary

117. Answer may vary

119. Answer may vary

4.3 Factoring Trinomials

LEARNING OBJECTIVES

1. Factor trinomials of the form $x^2 + bx + c$.
2. Factor trinomials of higher degree.
3. Factor trinomials of the form $ax^2 + bx + c$.
4. Factor trinomials using the AC method.

Factoring Trinomials of the Form $x^2 + bx + c$

Some trinomials of the form $x^2 + bx + c$ can be factored as a product of binomials. If a trinomial of this type factors, then we have:

$$\begin{aligned} x^2 + bx + c &= (x + m)\ (x + n) \\ &= x^2 + nx + mx + mn \\ &= x^2 + (n + m)x + mn \end{aligned}$$

This gives us

$$b = n + m \quad \text{and} \quad c = mn$$

In short, if the leading coefficient of a factorable trinomial is 1, then the factors of the last term must add up to the coefficient of the middle term. This observation is the key to factoring trinomials using the technique known as the **trial and error (or guess and check) method**[18].

18. Describes the method of factoring a trinomial by systematically checking factors to see if their product is the original trinomial.

Example 1

Factor: $x^2 + 12x + 20$.

Solution:

We begin by writing two sets of blank parentheses. If a trinomial of this form factors, then it will factor into two linear binomial factors.

$$x^2 + 12x + 20 = (\quad\quad)(\quad\quad)$$

Write the factors of the first term in the first space of each set of parentheses. In this case, factor $x^2 = x \cdot x$.

$$x^2 + 12x + 20 = (x \quad\quad)(x \quad\quad)$$

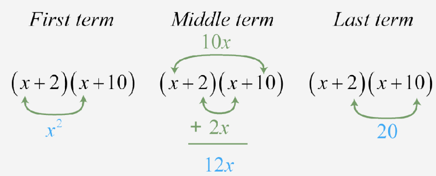
Determine the factors of the last term whose sum equals the coefficient of the middle term. To do this, list all of the factorizations of 20 and search for factors whose sum equals 12.

$$\begin{aligned} 20 &= 1 \cdot 20 \quad \rightarrow \quad 1 + 20 = 21 \\ &= 2 \cdot 10 \quad \rightarrow \quad 2 + 10 = 12 \\ &= 4 \cdot 5 \quad \rightarrow \quad 4 + 5 = 9 \end{aligned}$$

Choose $20 = 2 \cdot 10$ because $2 + 10 = 12$. Write in the last term of each binomial using the factors determined in the previous step.

$$x^2 + 12x + 20 = (x + 2)(x + 10)$$

This can be visually interpreted as follows:



Check by multiplying the two binomials.

$$\begin{aligned}(x+2)(x+10) &= x^2 + 10x + 2x + 20\\ &= x^2 + 12x + 20 \quad ✓\end{aligned}$$

Answer: $(x + 2)(x + 10)$

Since multiplication is commutative, the order of the factors does not matter.

$$\begin{aligned}x^2 + 12x + 20 &= (x + 2)(x + 10)\\ &= (x + 10)(x + 2)\end{aligned}$$

If the last term of the trinomial is positive, then either both of the constant factors must be negative or both must be positive.

Example 2

Factor: $x^2y^2 - 7xy + 12$.

Solution:

First, factor $x^2y^2 = xy \cdot xy$.

$$x^2y^2 - 7xy + 12 = (xy \quad ?)\ (xy \quad ?)$$

Next, search for factors of 12 whose sum is -7.

$$\begin{aligned} 12 &= 1 \cdot 12 & \rightarrow & \quad -1 + (-12) = -13 \\ &= 2 \cdot 6 & \rightarrow & \quad -2 + (-6) = -8 \\ &= 3 \cdot 4 & \rightarrow & \quad -3 + (-4) = -7 \end{aligned}$$

In this case, choose -3 and -4 because $(-3)\ (-4) = +12$and $-3 + \ (-4) = -7$.

$$\begin{aligned} x^2y^2 - 7xy + 12 &= (xy \quad ?)\ (xy \quad ?) \\ &= (xy - 3)\ (xy - 4) \end{aligned}$$

Check.

$$\begin{aligned}(xy-3)\ (xy-4) &= x^2y^2 - 4xy - 3xy + 12 \\ &= x^2y^2 - 7xy + 12 \quad ✓\end{aligned}$$

Answer: $(xy-3)\ (xy-4)$

If the last term of the trinomial is negative, then one of its factors must be negative.

Example 3

Factor: $x^2 - 4xy - 12y^2$.

Solution:

Begin by factoring the first term $x^2 = x \cdot x$.

$$x^2 - 4xy - 12y^2 = (x \quad ?)\ (x \quad ?)$$

The factors of 12 are listed below. In this example, we are looking for factors whose sum is -4.

$$\begin{aligned} 12 &= 1 \cdot 12 &\rightarrow\quad& 1 + (-12) = -11 \\ &= 2 \cdot 6 &\rightarrow\quad& 2 + (-6) = -4 \\ &= 3 \cdot 4 &\rightarrow\quad& 3 + (-4) = -1 \end{aligned}$$

Therefore, the coefficient of the last term can be factored as $-12 = 2\,(-6)$, where $2 + (-6) = -4$. Because the last term has a variable factor of y^2, use $-12y^2 = 2y\,(-6y)$ and factor the trinomial as follows:

$$\begin{aligned} x^2 - 4xy - 12y^2 &= (x \quad ?)\ (x \quad ?) \\ &= (x + 2y)\ (x - 6y) \end{aligned}$$

Multiply to check.

$$\begin{aligned}(x+2y)(x-6y) &= x^2 - 6xy + 2yx - 12y^2 \\ &= x^2 - 6xy + 2xy - 12y^2 \\ &= x^2 - 4xy - 12y^2 \quad ✓\end{aligned}$$

Answer: $(x+2y)(x-6y)$

Often our first guess will not produce a correct factorization. This process may require repeated trials. For this reason, the check is very important and is not optional.

Example 4

Factor: $a^2 + 10a - 24$.

Solution:

The first term of this trinomial, a^2, factors as $a \cdot a$.

$$a^2 + 10a - 24 = \left(a \quad ?\right)\left(a \quad ?\right)$$

Consider the factors of 24:

$$\begin{aligned} 24 &= 1 \cdot 24 \\ &= 2 \cdot 12 \\ &= 3 \cdot 8 \\ &= 4 \cdot 6 \end{aligned}$$

Suppose we choose the factors 4 and 6 because 4 + 6 = 10, the coefficient of the middle term. Then we have the following incorrect factorization:

$$a^2 + 10a - 24 \stackrel{?}{=} (a + 4)\left(a + 6\right) \quad \textit{Incorrect Factorization}$$

When we multiply to check, we find the error.

$$\begin{aligned}(a+4)\ (a+6) &= a^2+6a+4a+24\\ &= a^2+10a+24 \quad ✗\end{aligned}$$

In this case, the middle term is correct but the last term is not. Since the last term in the original expression is negative, we need to choose factors that are opposite in sign. Therefore, we must try again. This time we choose the factors −2 and 12 because $-2+12=10$.

$$a^2+10a-24 = (a-2)\ (a+12)$$

Now the check shows that this factorization is correct.

$$\begin{aligned}(a-2)\ (a+12) &= a^2+12a-2a-24\\ &= a^2+10a-24 \quad ✓\end{aligned}$$

Answer: $(a-2)\ (a+12)$

If we choose the factors wisely, then we can reduce much of the guesswork in this process. However, if a guess is not correct, do not get discouraged; just try a different set of factors. Keep in mind that some polynomials are prime. For example, consider the trinomial $x^2+3x+20$ and the factors of 20:

$$\begin{aligned}20 &= 1\cdot 20\\ &= 2\cdot 10\\ &= 4\cdot 5\end{aligned}$$

There are no factors of 20 whose sum is 3. Therefore, the original trinomial cannot be factored as a product of two binomials with integer coefficients. The trinomial is prime.

Factoring Trinomials of Higher Degree

We can use the trial and error technique to factor trinomials of higher degree.

Example 5

Factor: $x^4 + 6x^2 + 5$.

Solution:

Begin by factoring the first term $x^4 = x^2 \cdot x^2$.

$$x^4 + 6x^2 + 5 = \left(x^2 \quad ?\right)\left(x^2 \quad ?\right)$$

Since 5 is prime and the coefficient of the middle term is positive, choose +1 and +5 as the factors of the last term.

$$\begin{aligned} x^4 + 6x^2 + 5 &= \left(x^2 \quad ?\right)\left(x^2 \quad ?\right) \\ &= \left(x^2 + 1\right)\left(x^2 + 5\right) \end{aligned}$$

Notice that the variable part of the middle term is x^2 and the factorization checks out.

$$\begin{aligned} \left(x^2 + 1\right)\left(x^2 + 5\right) &= x^4 + 5x^2 + x^2 + 5 \\ &= x^4 + 6x^2 + 5 \quad \checkmark \end{aligned}$$

Answer: $\left(x^2 + 1\right)\left(x^2 + 5\right)$

Example 6

Factor: $x^{2n} + 4x^n - 21$ where n is a positive integer.

Solution:

Begin by factoring the first term $x^{2n} = x^n \cdot x^n$.

$$x^{2n} + 4x^n - 21 = \left(x^n \quad ?\right)\left(x^n \quad ?\right)$$

Factor $-21 = 7\,(-3)$ because $7 + \ (-3) = +4$ and write

$$\begin{aligned} x^{2n} + 4x^n - 21 &= \left(x^n \quad ?\right)\left(x^n \quad ?\right) \\ &= \left(x^n + 7\right)\left(x^n - 3\right) \end{aligned}$$

Answer: $(x^n + 7)\ (x^n - 3)$ The check is left to the reader.

Try this! Factor: $x^6 - x^3 - 42$.

Answer: $\left(x^3 + 6\right)\left(x^3 - 7\right)$

(click to see video)

Factoring Trinomials of the Form $ax^2 + bx + c$

Factoring trinomials of the form $ax^2 + bx + c$ can be challenging because the middle term is affected by the factors of both a and c. In general,

$$\begin{aligned} ax^2 + bx + c &= (px + m)\ (qx + n) \\ &= pqx^2 + pnx + qmx + mn \\ &= pqx^2 + \ (pn + qm)\, x + mn \end{aligned}$$

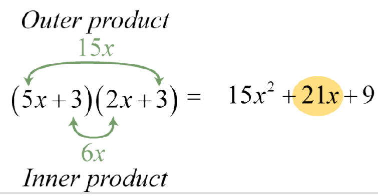
This gives us,

$$a = pq \quad \text{and} \quad b = pn + qm, \quad \text{where} \quad c = mn$$

In short, when the leading coefficient of a trinomial is something other than 1, there will be more to consider when determining the factors using the trial and error method. The key lies in the understanding of how the middle term is obtained. Multiply $(5x + 3)\ (2x + 3)$ and carefully follow the formation of the middle term.

$$\begin{aligned} (5x+3)(2x+3) &= 5x \cdot 2x \underbrace{+5x \cdot 3 + 3 \cdot 2x}_{\textit{middle term}} + 3 \cdot 3 \\ &= 10x^2 + 15x + 6x + 9 \\ &= 15x^2 + 21x + 9 \end{aligned}$$

As we have seen before, the product of the first terms of each binomial is equal to the first term of the trinomial. The middle term of the trinomial is the sum of the products of the outer and inner terms of the binomials. The product of the last terms of each binomial is equal to the last term of the trinomial. Visually, we have the following:



For this reason, we need to look for products of the factors of the first and last terms whose sum is equal to the coefficient of the middle term. For example, to factor $6x^2 + 29x + 35$, look at the factors of 6 and 35.

$$\begin{aligned} 6 &= 1 \cdot 6 & 35 &= 1 \cdot 35 \\ &= 2 \cdot 3 & &= 5 \cdot 7 \end{aligned}$$

The combination that produces the coefficient of the middle term is $2 \cdot 7 + 3 \cdot 5 = 14 + 15 = 29.$ Make sure that the outer terms have coefficients 2 and 7, and that the inner terms have coefficients 5 and 3. Use this information to factor the trinomial.

$$\begin{aligned} 6x^2 + 29x + 35 &= (2x \quad ?)\ (3x \quad ?) \\ &= (2x + 5)\ (3x + 7) \end{aligned}$$

We can always check by multiplying; this is left to the reader.

Example 7

Factor: $5x^2 + 16xy + 3y^2$.

Solution:

Since the leading coefficient and the last term are both prime, there is only one way to factor each.

$$5 = 1 \cdot 5 \quad \text{and} \quad 3 = 1 \cdot 3$$

Begin by writing the factors of the first term, $5x^2$, as follows:

$$5x^2 + 16xy + 3y^2 = (x \quad ?)\,(5x \quad ?)$$

The middle and last term are both positive; therefore, the factors of 3 are chosen as positive numbers. In this case, the only choice is in which grouping to place these factors.

$$(x + y)\;(5x + 3y) \qquad \text{or} \qquad (x + 3y)\;(5x + y)$$

Determine which grouping is correct by multiplying each expression.

$$\begin{aligned}(x+y)\ (5x+3y) &= 5x^2 + 3xy + 5xy + 3y^2 \\ &= 5x^2 + 8xy + 3y^2 \quad ✗ \\ (x+3y)\ (5x+y) &= 5x^2 + xy + 15xy + 3y^2 \\ &= 5x^2 + 16xy + 3y^2 \quad ✓\end{aligned}$$

Answer: $(x+3y)\ (5x+y)$

Example 8

Factor: $18a^2b^2 - ab - 4$.

Solution:

First, consider the factors of the coefficients of the first and last terms.

$$\begin{aligned} 18 &= 1 \cdot 18 & 4 &= 1 \cdot 4 \\ &= 2 \cdot 9 & &= 2 \cdot 2 \\ &= 3 \cdot 6 & & \end{aligned}$$

We are searching for products of factors whose sum equals the coefficient of the middle term, −1. After some thought, we can see that the sum of 8 and −9 is −1 and the combination that gives this follows:

$$2(4) + 9(-1) = 8 - 9 = -1$$

Factoring begins at this point with two sets of blank parentheses.

$$18a^2b^2 - ab - 4 = (\quad)(\quad)$$

Use $2ab$ and $9ab$ as factors of $18a^2b^2$.

$$18a^2b^2 - ab - 4 = (2ab \quad ?)(9ab \quad ?)$$

Next use the factors 1 and 4 in the correct order so that the inner and outer products are $-9ab$ and $8ab$ respectively.

$$18a^2b^2 - ab - 4 = (2ab - 1)(9ab + 4)$$

Answer: $(2ab - 1)(9ab + 4)$. The complete check is left to the reader.

It is a good practice to first factor out the GCF, if there is one. Doing this produces a trinomial factor with smaller coefficients. As we have seen, trinomials with smaller coefficients require much less effort to factor. This commonly overlooked step is worth identifying early.

Example 9

Factor: $12y^3 - 26y^2 - 10y$.

Solution:

Begin by factoring out the GCF.

$$12y^3 - 26y^2 - 10y = 2y\left(6y^2 - 13y - 5\right)$$

After factoring out $2y$, the coefficients of the resulting trinomial are smaller and have fewer factors. We can factor the resulting trinomial using $6 = 2\,(3)$ and $5 = \ \left(5\right)\ (1)$ .Notice that these factors can produce -13 in two ways:

$$\begin{aligned} 2\left(-5\right) + 3\,(-1) &= -10 - 3 = -13 \\ 2\,(1) + 3\left(-5\right) &= 2 - 15 = -13 \end{aligned}$$

Because the last term is -5, the correct combination requires the factors 1 and 5 to be opposite signs. Here we use 2(1) = 2 and 3(-5) = -15 because the sum is -13 and the product of (1)(-5) = -5.

$$\begin{aligned} 12y^3 - 26y^2 - 10y &= 2y\left(6y^2 - 13y - 5\right) \\ &= 2y\left(2y\ \ ?\right)\left(3y\ \ ?\right) \\ &= 2y\left(2y - 5\right)\left(3y + 1\right) \end{aligned}$$

Check.

$$\begin{aligned} 2y\left(2y-5\right)\left(3y+1\right) &= 2y\left(6y^2+2y-15y-5\right) \\ &= 2y\left(6y^2-13y-5\right) \\ &= 12y^3-26y^2-10y \quad \checkmark \end{aligned}$$

The factor $2y$ is part of the factored form of the original expression; be sure to include it in the answer.

Answer: $2y\left(2y-5\right)\left(3y+1\right)$

It is a good practice to consistently work with trinomials where the leading coefficient is positive. If the leading coefficient is negative, factor it out along with any GCF. Note that sometimes the factor will be −1.

Example 10

Factor: $-18x^6 - 69x^4 + 12x^2$.

Solution:

In this example, the GCF is $3x^2$. Because the leading coefficient is negative we begin by factoring out $-3x^2$.

$$-18x^6 - 69x^4 + 12x^2 = -3x^2\left(6x^4 + 23x^2 - 4\right)$$

At this point, factor the remaining trinomial as usual, remembering to write the $-3x^2$ as a factor in the final answer. Use 6 = 1(6) and −4 = 4(−1) because $1\left(-1\right) + 6\left(4\right) = 23$.Therefore,

$$\begin{aligned} -18x^6 - 69x^4 + 12x^2 &= -3x^2\left(6x^4 + 23x^2 - 4\right) \\ &= -3x^2\left(x^2 \quad\right)\left(6x^2 \quad\right) \\ &= -3x^2\left(x^2 + 4\right)\left(6x^2 - 1\right) \end{aligned}$$

Answer: $-3x^2\left(x^2 + 4\right)\left(6x^2 - 1\right)$.The check is left to the reader.

Try this! Factor: $-12a^5b + a^3b^3 + ab^5$.

Answer: $-ab\left(3a^2 - b^2\right)\left(4a^2 + b^2\right)$

(click to see video)

Factoring Using the AC Method

An alternate technique for factoring trinomials, called the **AC method**[19], makes use of the grouping method for factoring four-term polynomials. If a trinomial in the form $ax^2 + bx + c$ can be factored, then the middle term, bx, can be replaced with two terms with coefficients whose sum is b and product is ac. This substitution results in an equivalent expression with four terms that can be factored by grouping.

19. Method used for factoring trinomials by replacing the middle term with two terms that allow us to factor the resulting four-term polynomial by grouping.

Example 11

Factor using the AC method: $18x^2 - 31x + 6$.

Solution:

Here $a = 18$, $b = -31$, and $c = 6$.

$$\begin{aligned} ac &= 18(6) \\ &= 108 \end{aligned}$$

Factor 108, and search for factors whose sum is -31.

$$\begin{aligned} 108 &= -1(-108) \\ &= -2(-54) \\ &= -3(-36) \\ &= -4(-27) \ \checkmark \\ &= -6(-18) \\ &= -9(-12) \end{aligned}$$

In this case, the sum of the factors -27 and -4 equals the middle coefficient, -31. Therefore, $-31x = -27x - 4x$, and we can write

$$18x^2 - 31x + 6 = 18x^2 - 27x - 4x + 6$$

Factor the equivalent expression by grouping.

$$\begin{aligned} 18x^2 - 31x + 6 &= 18x^2 - 27x - 4x + 6 \\ &= 9x\,(2x - 3) - 2\,(2x - 3) \\ &= (2x - 3)\;(9x - 2) \end{aligned}$$

Answer: $(2x - 3)\;(9x - 2)$

Example 12

Factor using the AC method: $4x^2y^2 - 7xy - 15$.

Solution:

Here $a = 4$, $b = -7$, and $c = -15$.

$$\begin{aligned} ac &= 4(-15) \\ &= -60 \end{aligned}$$

Factor -60 and search for factors whose sum is -7.

$$\begin{aligned} -60 &= 1(-60) \\ &= 2(-30) \\ &= 3(-20) \\ &= 4(-15) \\ &= 5(-12) \quad ✓ \\ &= 6(-10) \end{aligned}$$

The sum of factors 5 and -12 equals the middle coefficient, -7. Replace $-7xy$ with $5xy - 12xy$.

$$\begin{aligned} 4x^2y^2 - 7xy - 15 &= 4x^2y^2 + 5xy - 12xy - 15 \qquad \textit{Factor by grouping.} \\ &= xy\left(4xy + 5\right) - 3\left(4xy + 5\right) \\ &= \left(4xy + 5\right)\left(xy - 3\right) \end{aligned}$$

Answer: $(4xy + 5)(xy - 3)$. The check is left to the reader.

If factors of *ac* cannot be found to add up to *b* then the trinomial is prime.

KEY TAKEAWAYS

- If a trinomial of the form $x^2 + bx + c$ factors into the product of two binomials, then the coefficient of the middle term is the sum of factors of the last term.
- If a trinomial of the form $ax^2 + bx + c$ factors into the product of two binomials, then the coefficient of the middle term will be the sum of certain products of factors of the first and last terms.
- If the trinomial has a greatest common factor, then it is a best practice to first factor out the GCF before attempting to factor it into a product of binomials.
- If the leading coefficient of a trinomial is negative, then it is a best practice to first factor that negative factor out before attempting to factor the trinomial.
- Factoring is one of the more important skills required in algebra. For this reason, you should practice working as many problems as it takes to become proficient.

TOPIC EXERCISES

PART A: FACTORING TRINOMIALS OF THE FORM $x^2 + bx + c$

Factor.

1. $x^2 + 5x - 6$
2. $x^2 + 5x + 6$
3. $x^2 + 4x - 12$
4. $x^2 + 3x - 18$
5. $x^2 - 14x + 48$
6. $x^2 - 15x + 54$
7. $x^2 + 11x - 30$
8. $x^2 - 2x + 24$
9. $x^2 - 18x + 81$
10. $x^2 - 22x + 121$
11. $x^2 - xy - 20y^2$
12. $x^2 + 10xy + 9y^2$
13. $x^2y^2 + 5xy - 50$
14. $x^2y^2 - 16xy + 48$
15. $a^2 - 6ab - 72b^2$
16. $a^2 - 21ab + 80b^2$
17. $u^2 + 14uv - 32v^2$
18. $m^2 + 7mn - 98n^2$
19. $(x + y)^2 - 2(x + y) - 8$

20. $(x - y)^2 - 2(x - y) - 15$
21. $x^4 - 7x^2 - 8$
22. $x^4 + 13x^2 + 30$
23. $x^4 - 8x^2 - 48$
24. $x^4 + 25x^2 + 24$
25. $y^4 - 20y^2 + 100$
26. $y^4 + 14y^2 + 49$
27. $x^4 + 3x^2y^2 + 2y^4$
28. $x^4 - 8x^2y^2 + 15y^4$
29. $a^4b^4 - 4a^2b^2 + 4$
30. $a^4 + 6a^2b^2 + 9b^4$
31. $x^6 - 18x^3 - 40$
32. $x^6 + 18x^3 + 45$
33. $x^6 - x^3y^3 - 6y^6$
34. $x^6 + x^3y^3 - 20y^6$
35. $x^6y^6 + 2x^3y^3 - 15$
36. $x^6y^6 + 16x^3y^3 + 48$
37. $x^{2n} + 12x^n + 32$
38. $x^{2n} + 41x^n + 40$
39. $x^{2n} + 2ax^n + a^2$
40. $x^{2n} - 2ax^n + a^2$

PART B: FACTORING TRINOMIALS OF THE FORM $ax^2 + bx + c$

Factor.

41. $3x^2 + 20x - 7$
42. $2x^2 - 9x - 5$
43. $6a^2 + 13a + 6$
44. $4a^2 + 11a + 6$
45. $6x^2 + 7x - 10$
46. $4x^2 - 25x + 6$
47. $24y^2 - 35y + 4$
48. $10y^2 - 23y + 12$
49. $14x^2 - 11x + 9$
50. $9x^2 + 6x + 8$
51. $4x^2 - 28x + 49$
52. $36x^2 - 60x + 25$
53. $27x^2 - 6x - 8$
54. $24x^2 + 17x - 20$
55. $6x^2 + 23xy - 4y^2$
56. $10x^2 - 21xy - 27y^2$
57. $8a^2b^2 - 18ab + 9$
58. $12a^2b^2 - ab - 20$
59. $8u^2 - 26uv + 15v^2$
60. $24m^2 - 26mn + 5n^2$
61. $4a^2 - 12ab + 9b^2$

62. $16a^2 + 40ab + 25b^2$
63. $5(x + y)^2 - 9(x + y) + 4$
64. $7(x - y)^2 + 15(x - y) - 18$
65. $7x^4 - 22x^2 + 3$
66. $5x^4 - 41x^2 + 8$
67. $4y^6 - 3y^3 - 10$
68. $12y^6 + 4y^3 - 5$
69. $5a^4b^4 - a^2b^2 - 18$
70. $21a^4b^4 + 5a^2b^2 - 4$
71. $6x^6y^6 + 17x^3y^3 + 10$
72. $16x^6y^6 + 46x^3y^3 + 15$
73. $8x^{2n} - 10x^n - 25$
74. $30x^{2n} - 11x^n - 6$
75. $36x^{2n} + 12ax^n + a^2$
76. $9x^{2n} - 12ax^n + 4a^2$
77. $-3x^2 + 14x + 5$
78. $-2x^2 + 13x - 20$
79. $-x^2 - 10x + 24$
80. $-x^2 + 8x + 48$
81. $54 - 12x - 2x^2$
82. $60 + 5x - 5x^2$
83. $4x^3 + 16x^2 + 20x$
84. $2x^4 - 12x^3 + 14x^2$
85. $2x^3 - 8x^2y - 24xy^2$

86. $6x^3 - 9x^2y - 6xy^2$
87. $4a^3b - 4a^2b^2 - 24ab^3$
88. $15a^4b - 33a^3b^2 + 6a^2b^3$
89. $3x^5y + 30x^3y^3 + 75xy^5$
90. $45x^5y^2 - 60x^3y^4 + 20xy^6$

Factor.

91. $4 - 25x^2$
92. $8x^3 - y^3$
93. $9x^2 - 12xy + 4y^2$
94. $30a^2 - 57ab - 6b^2$
95. $10a^2 - 5a - 6ab + 3b$
96. $3x^3 - 4x^2 + 9x - 12$
97. $x^2 + 4y^2$
98. $x^2 - x + 2$
99. $15a^3b^2 + 6a^2b^3 - 3ab^4$
100. $54x^2 - 63x$

PART D: DISCUSSION BOARD

101. Create your own trinomial of the form $ax^2 + bx + c$ that factors. Share it, along with the solution, on the discussion board.
102. Create a trinomial of the form $ax^2 + bx + c$ that does not factor and share it along with the reason why it does not factor.

ANSWERS

1. $(x-1)(x+6)$
3. $(x-2)(x+6)$
5. $(x-6)(x-8)$
7. Prime
9. $(x-9)^2$
11. $(x-5y)(x+4y)$
13. $(xy-5)(xy+10)$
15. $(a+6b)(a-12b)$
17. $(u-2v)(u+16v)$
19. $(x+y-4)(x+y+2)$
21. $(x^2-8)(x^2+1)$
23. $(x^2+4)(x^2-12)$
25. $(y^2-10)^2$
27. $(x^2+y^2)(x^2+2y^2)$
29. $(a^2b^2-2)^2$
31. $(x^3-20)(x^3+2)$
33. $(x^3+2y^3)(x^3-3y^3)$
35. $(x^3y^3-3)(x^3y^3+5)$
37. $(x^n+4)(x^n+8)$
39. $(x^n+a)^2$
41. $(3x-1)(x+7)$

43. $(2a+3)(3a+2)$

45. $(6x-5)(x+2)$

47. $(8y-1)(3y-4)$

49. Prime

51. $(2x-7)^2$

53. $(9x+4)(3x-2)$

55. $(6x-y)(x+4y)$

57. $(4ab-3)(2ab-3)$

59. $(2u-5v)(4u-3v)$

61. $(2a-3b)^2$

63. $(x+y-1)(5x+5y-4)$

65. $(x^2-3)(7x^2-1)$

67. $(y^3-2)(4y^3+5)$

69. $(a^2b^2-2)(5a^2b^2+9)$

71. $(6x^3y^3+5)(x^3y^3+2)$

73. $(2x^n-5)(4x^n+5)$

75. $(6x^n+a)^2$

77. $-(x-5)(3x+1)$

79. $-(x-2)(x+12)$

81. $-2(x-3)(x+9)$

83. $4x(x^2+4x+5)$

85. $2x(x+2y)(x-6y)$

87. $4ab(a-3b)(a+2b)$

89. $3xy\left(x^2 + 5y^2\right)^2$

91. $\left(2 - 5x\right)\left(2 + 5x\right)$

93. $\left(3x - 2y\right)^2$

95. $(2a - 1)\left(5a - 3b\right)$

97. Prime

99. $3ab^2\left(5a^2 + 2ab - b^2\right)$

101. Answer may vary

4.4 Solve Polynomial Equations by Factoring

LEARNING OBJECTIVES

1. Review general strategies for factoring.
2. Solve polynomial equations by factoring.
3. Find roots of a polynomial function.
4. Find polynomial equations given the solutions.

Reviewing General Factoring Strategies

We have learned various techniques for factoring polynomials with up to four terms. The challenge is to identify the type of polynomial and then decide which method to apply. The following outlines a general guideline for factoring polynomials:

1. Check for common factors. If the terms have common factors, then factor out the greatest common factor (GCF).
2. Determine the number of terms in the polynomial.
   1. Factor four-term polynomials by grouping.
   2. Factor trinomials (3 terms) using "trial and error" or the AC method.
   3. Factor binomials (2 terms) using the following special products:

$$\begin{array}{ll} \textbf{Difference of squares:} & a^2 - b^2 = (a+b)(a-b) \\ \textbf{Sum of squares:} & a^2 + b^2 \quad \textit{no general formula} \\ \textbf{Difference of cubes:} & a^3 - b^3 = (a-b)(a^2+ab+b^2) \\ \textbf{Sum of cubes:} & a^3 + b^3 = (a+b)(a^2-ab+b^2) \end{array}$$

3. Look for factors that can be factored further.
4. Check by multiplying.

Note: If a binomial is both a difference of squares and a difference cubes, then first factor it as difference of squares. This will result in a more complete factorization. In addition, not all polynomials with integer coefficients factor. When this is the case, we say that the polynomial is prime.

If an expression has a GCF, then factor this out first. Doing so is often overlooked and typically results in factors that are easier to work with. Furthermore, look for the resulting factors to factor further; many factoring problems require more than one step. A polynomial is completely factored when none of the factors can be factored further.

Example 1

Factor: $54x^4 - 36x^3 - 24x^2 + 16x$.

Solution:

This four-term polynomial has a GCF of $2x$. Factor this out first.

$$54x^4 - 36x^3 - 24x^2 + 16x = 2x\left(27x^3 - 18x^2 - 12x + 8\right)$$

Now factor the resulting four-term polynomial by grouping and look for resulting factors to factor further.

$$\begin{aligned} 54x^4 - 36x^3 - 24x^2 + 16x &= 2x\left(\underbrace{27x^3 - 18x^2}_{\text{group}} \underbrace{-12x + 8}_{\text{group}}\right) \\ &= 2x\left(9x^2(3x-2) - 4(3x-2)\right) \\ &= 2x(3x-2)\underbrace{\left(9x^2 - 4\right)}_{\substack{\text{difference} \\ \text{of squares}}} \\ &= 2x(3x-2)(3x-2)(3x+2) \end{aligned}$$

Answer: $2x(3x-2)^2\,(3x+2)$. The check is left to the reader.

Example 2

Factor: $x^4 - 3x^2 - 4$.

Solution:

This trinomial does not have a GCF.

$$\begin{aligned} x^4 - 3x^2 - 4 &= \left(x^2 \quad\right)\left(x^2 \quad\right) \\ &= \left(x^2 + 1\right)\left(x^2 - 4\right) \quad \textit{Difference of squares} \\ &= \left(x^2 + 1\right)(x + 2)(x - 2) \end{aligned}$$

The factor $\left(x^2 + 1\right)$ is prime and the trinomial is completely factored.

Answer: $\left(x^2 + 1\right)(x + 2)(x - 2)$

Example 3

Factor: $x^6 + 6x^3 - 16$.

Solution:

Begin by factoring $x^6 = x^3 \cdot x^3$ and look for the factors of 16 that add to 6.

$$\begin{aligned} x^6 + 6x^3 - 16 &= \left(x^3 \quad\right)\left(x^3 \quad\right) \\ &= \left(x^3 - 2\right)\left(x^3 + 8\right) \quad \color{blue}{\text{sum of cubes}} \\ &= \left(x^3 - 2\right)(x+2)\left(x^2 - 2x + 4\right) \end{aligned}$$

The factor $\left(x^3 - 2\right)$ cannot be factored any further using integers and the factorization is complete.

Answer: $\left(x^3 - 2\right)(x+2)\left(x^2 + 2x + 4\right)$

Try this! Factor: $9x^4 + 17x^2 - 2$

Answer: $(3x+1)(3x-1)\left(x^2+2\right)$

(click to see video)

Solving Polynomial Equations by Factoring

In this section, we will review a technique that can be used to solve certain polynomial equations. We begin with the **zero-product property**[20]:

$$a \cdot b = 0 \;\; \text{if and only if} \;\; a = 0 \;\text{ or }\; b = 0$$

20. A product is equal to zero if and only if at least one of the factors is zero.

The zero-product property is true for any number of factors that make up an equation. In other words, if any product is equal to zero, then at least one of the variable factors must be equal to zero. If an expression is equal to zero and can be factored into linear factors, then we will be able to set each factor equal to zero and solve for each equation.

Example 4

Solve: $2x\,(x-4)\,\left(5x+3\right)=0.$

Solution:

Set each variable factor equal to zero and solve.

$$\begin{array}{lclcl} 2x=0 & \text{or} & x-4=0 & \text{or} & 5x+3=0 \\ \frac{2x}{2}=\frac{0}{2} & & x=4 & & \frac{5x}{5}=\frac{-3}{5} \\ x=0 & & & & x=-\frac{3}{5} \end{array}$$

To check that these are solutions we can substitute back into the original equation to see if we obtain a true statement. Note that each solution produces a zero factor. This is left to the reader.

Answer: The solutions are 0, 4, and $-\frac{3}{5}$.

Of course, most equations will not be given in factored form.

Example 5

Solve: $4x^3 - x^2 - 100x + 25 = 0$.

Solution:

Begin by factoring the left side completely.

$$\begin{aligned} 4x^3 - x^2 - 100x + 25 &= 0 && \color{blue}{\textit{Factor by grouping.}} \\ x^2(4x-1) - 25(4x-1) &= 0 \\ (4x-1)(x^2-25) &= 0 && \color{blue}{\textit{Factor as a difference of squares.}} \\ (4x-1)(x+5)(x-5) &= 0 \end{aligned}$$

Set each factor equal to zero and solve.

$$\begin{aligned} 4x - 1 &= 0 \quad \text{or} \quad x + 5 = 0 \quad \text{or} \quad x - 5 = 0 \\ 4x &= 1 \qquad\qquad\quad x = -5 \qquad\qquad\quad x = 5 \\ x &= \frac{1}{4} \end{aligned}$$

Answer: The solutions are $\frac{1}{4}$, −5, and 5.

Using the zero-product property after factoring an equation that is equal to zero is the key to this technique. However, the equation may not be given equal to zero, and so there may be some preliminary steps before factoring. The steps required to **solve by factoring**[21] are outlined in the following example.

21. The process of solving an equation that is equal to zero by factoring it and then setting each variable factor equal to zero.

Example 6

Solve: $15x^2 + 3x - 8 = 5x - 7$.

Solution:

Step 1: Express the equation in standard form, equal to zero. In this example, subtract $5x$ from and add 7 to both sides.

$$\begin{aligned}15x^2 + 3x - 8 &= 5x - 7 \\ 15x^2 - 2x - 1 &= 0\end{aligned}$$

Step 2: Factor the expression.

$$(3x - 1)\left(5x + 1\right) = 0$$

Step 3: Apply the zero-product property and set each variable factor equal to zero.

$$3x - 1 = 0 \quad \text{or} \quad 5x + 1 = 0$$

Step 4: Solve the resulting linear equations.

$$\begin{aligned} 3x - 1 &= 0 \quad \text{or} \quad 5x + 1 = 0 \\ 3x &= 1 \qquad\qquad 5x = -1 \\ x &= \frac{1}{3} \qquad\qquad x = -\frac{1}{5} \end{aligned}$$

Answer: The solutions are $\frac{1}{3}$ and $-\frac{1}{5}$ .The check is optional.

Example 7

Solve: $(3x + 2)\ (x + 1) = 4$.

Solution:

This quadratic equation appears to be factored; hence it might be tempting to set each factor equal to 4. However, this would lead to incorrect results. We must rewrite the equation equal to zero, so that we can apply the zero-product property.

$$\begin{aligned} (3x + 2)\ (x + 1) &= 4 \\ 3x^2 + 3x + 2x + 2 &= 4 \\ 3x^2 + 5x + 2 &= 4 \\ 3x^2 + 5x - 2 &= 0 \end{aligned}$$

Once it is in standard form, we can factor and then set each factor equal to zero.

$$(3x - 1)\ (x + 2) = 0$$

$$\begin{aligned} 3x - 1 &= 0 \quad \text{or} \quad & x + 2 &= 0 \\ 3x &= 1 & x &= -2 \\ x &= \frac{1}{3} & & \end{aligned}$$

Answer: The solutions are $\frac{1}{3}$ and -2.

Finding Roots of Functions

Recall that any polynomial with one variable is a function and can be written in the form,

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

A **root**[22] of a function is a value in the domain that results in zero. In other words, the roots occur when the function is equal to zero, $f(x) = 0$.

22. A value in the domain of a function that results in zero.

Example 8

Find the roots: $f(x) = (x+2)^2 - 4$.

Solution:

To find roots we set the function equal to zero and solve.

$$\begin{aligned} f(x) &= 0 \\ (x+2)^2 - 4 &= 0 \\ x^2 + 4x + 4 - 4 &= 0 \\ x^2 + 4x &= 0 \\ x(x+4) &= 0 \end{aligned}$$

Next, set each factor equal to zero and solve.

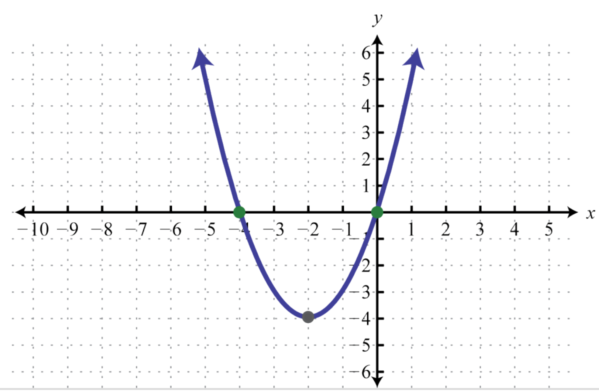
$$\begin{aligned} x = 0 \quad \text{or} \quad x + 4 &= 0 \\ x &= -4 \end{aligned}$$

We can show that these x-values are roots by evaluating.

$$\begin{aligned} f(0) &= (0+2)^2 - 4 \\ &= 4 - 4 \\ &= 0 \ \checkmark \end{aligned} \qquad \begin{aligned} f(-4) &= (-4+2)^2 - 4 \\ &= (-2)^2 - 4 \\ &= 4 - 4 \\ &= 0 \ \checkmark \end{aligned}$$

Answer: The roots are 0 and −4.

If we graph the function in the previous example we will see that the roots correspond to the x-intercepts of the function. Here the function f is a basic parabola shifted 2 units to the left and 4 units down.



Example 9

Find the roots: $f(x) = x^4 - 5x^2 + 4$.

Solution:

To find roots we set the function equal to zero and solve.

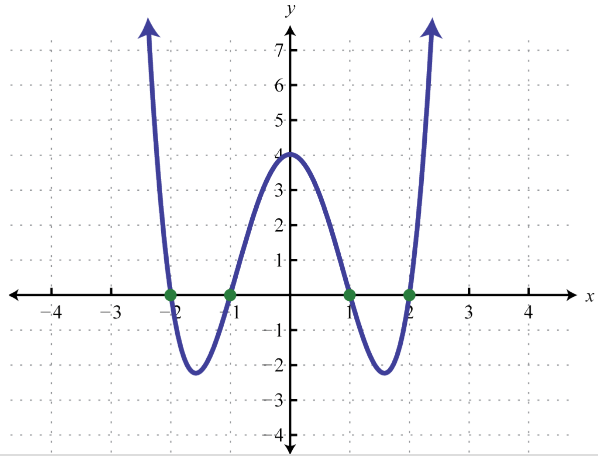
$$\begin{aligned} f(x) &= 0 \\ x^4 - 5x^2 + 4 &= 0 \\ \left(x^2 - 1\right)\left(x^2 - 4\right) &= 0 \\ (x + 1)(x - 1)(x + 2)(x - 2) &= 0 \end{aligned}$$

Next, set each factor equal to zero and solve.

$$\begin{array}{rcrcrcr} x + 1 = 0 & \text{or} & x - 1 = 0 & \text{or} & x + 2 = 0 & \text{or} & x - 2 = 0 \\ x = -1 & & x = 1 & & x = -2 & & x = 2 \end{array}$$

Answer: The roots are −1, 1, −2, and 2.

Graphing the previous function is not within the scope of this course. However, the graph is provided below:



Notice that the degree of the polynomial is 4 and we obtained four roots. In general, for any polynomial function with one variable of degree *n*, the **fundamental theorem of algebra**[23] guarantees *n* real roots or fewer. We have seen that many polynomials do not factor. This does not imply that functions involving these unfactorable polynomials do not have real roots. In fact, many polynomial functions that do not factor do have real solutions. We will learn how to find these types of roots as we continue in our study of algebra.

23. Guarantees that there will be as many (or fewer) roots to a polynomial function with one variable as its degree.

Example 10

Find the roots: $f(x) = -x^2 + 10x - 25$.

Solution:

To find roots we set the function equal to zero and solve.

$$\begin{aligned} f(x) &= 0 \\ -x^2 + 10x - 25 &= 0 \\ -\left(x^2 - 10x + 25\right) &= 0 \\ -(x-5)\ (x-5) &= 0 \end{aligned}$$

Next, set each variable factor equal to zero and solve.

$$\begin{aligned} x - 5 &= 0 \quad \text{or} \quad x - 5 = 0 \\ &= 5 \qquad\qquad\quad x = 5 \end{aligned}$$

A solution that is repeated twice is called a **double root**[24]. In this case, there is only one solution.

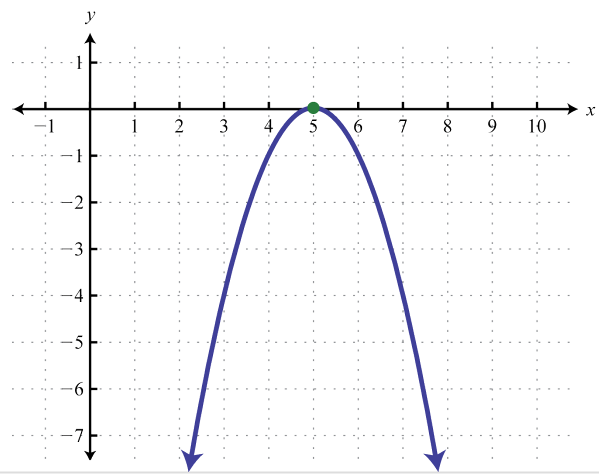
Answer: The root is 5.

The previous example shows that a function of degree 2 can have one root. From the factoring step, we see that the function can be written

24. A root that is repeated twice.

$$f(x) = -(x-5)^2$$

In this form, we can see a reflection about the *x*-axis and a shift to the right 5 units. The vertex is the *x*-intercept, illustrating the fact that there is only one root.



Try this! Find the roots of $f(x) = x^3 + 3x^2 - x - 3$.

Answer: ±1, −3

(click to see video)

Example 11

Assuming dry road conditions and average reaction times, the safe stopping distance in feet is given by $d(x) = \frac{1}{20}x^2 + x$, where x represents the speed of the car in miles per hour. Determine the safe speed of the car if you expect to stop in 40 feet.

Solution:

We are asked to find the speed x where the safe stopping distance $d(x) = 40$ feet.

$$\begin{aligned} d(x) &= 40 \\ \frac{1}{20}x^2 + x &= 40 \end{aligned}$$

To solve for x, rewrite the resulting equation in standard form. In this case, we will first multiply both sides by 20 to clear the fraction.

$$\begin{aligned} 20\left(\frac{1}{20}x^2 + x\right) &= 20(40) \\ x^2 + 20x &= 800 \\ x^2 + 20x - 800 &= 0 \end{aligned}$$

Next factor and then set each factor equal to zero.

$$\begin{aligned} x^2 + 20x - 800 &= 0 \\ (x+40)\,(x-20) &= 0 \\ x + 40 &= 0 \quad or\, x - 20 = 0 \\ x &= -40 \qquad\quad x = 20 \end{aligned}$$

The negative answer does not make sense in the context of this problem. Consider $x = 20$ miles per hour to be the only solution.

Answer: 20 miles per hour

Finding Equations with Given Solutions

We can use the zero-product property to find equations, given the solutions. To do this, the steps for solving by factoring are performed in reverse.

Example 12

Find a quadratic equation with integer coefficients, given solutions $-\frac{3}{2}$ and $\frac{1}{3}$.

Solution:

Given the solutions, we can determine two linear factors. To avoid fractional coefficients, we first clear the fractions by multiplying both sides by the denominator.

$$\begin{aligned} x &= -\frac{3}{2} \text{ or } & x &= \frac{1}{3} \\ 2x &= -3 & 3x &= 1 \\ 2x + 3 &= 0 & 3x - 1 &= 0 \end{aligned}$$

The product of these linear factors is equal to zero when $x = -\frac{3}{2}$ or $x = \frac{1}{3}$.

$$(2x + 3)\,(3x - 1) = 0$$

Multiply the binomials and present the equation in standard form.

$$\begin{aligned} 6x^2 - 2x + 9x - 3 &= 0 \\ 6x^2 + 7x - 3 &= 0 \end{aligned}$$

We may check our equation by substituting the given answers to see if we obtain a true statement. Also, the equation found above is not unique and so the check becomes essential when our equation looks different from someone else's. This is left as an exercise.

Answer: $6x^2 + 7x - 3 = 0$

Example 13

Find a polynomial function with real roots 1, -2, and 2.

Solution:

Given solutions to $f(x) = 0$ we can find linear factors.

$$\begin{aligned} x&=1 \text{ or} & x&=-2 \text{ or} & x&=2 \\ x-1&=0 & x+2&=0 & x-2&=0 \end{aligned}$$

Apply the zero-product property and multiply.

$$\begin{aligned} (x-1)(x+2)(x-2)&=0 \\ (x-1)\left(x^2-4\right)&=0 \\ x^3-4x-x^2+4&=0 \\ x^3-x^2-4x+4&=0 \end{aligned}$$

Answer: $f(x) = x^3 - x^2 - 4x + 4$

Try this! Find a polynomial equation with integer coefficients, given solutions $\frac{1}{2}$ and $-\frac{3}{4}$.

Answer: $8x^2 + 2x - 3 = 0$

(click to see video)

KEY TAKEAWAYS

- Factoring and the zero-product property allow us to solve equations.
- To solve a polynomial equation, first write it in standard form. Once it is equal to zero, factor it and then set each variable factor equal to zero. The solutions to the resulting equations are the solutions to the original.
- Not all polynomial equations can be solved by factoring. We will learn how to solve polynomial equations that do not factor later in the course.
- A polynomial function can have at most a number of real roots equal to its degree. To find roots of a function, set it equal to zero and solve.
- To find a polynomial equation with given solutions, perform the process of solving by factoring in reverse.

TOPIC EXERCISES

PART A: GENERAL FACTORING

Factor completely.

1. $50x^2 - 18$
2. $12x^3 - 3x$
3. $10x^3 + 65x^2 - 35x$
4. $15x^4 + 7x^3 - 4x^2$
5. $6a^4b - 15a^3b^2 - 9a^2b^3$
6. $8a^3b - 44a^2b^2 + 20ab^3$
7. $36x^4 - 72x^3 - 4x^2 + 8x$
8. $20x^4 + 60x^3 - 5x^2 - 15x$
9. $3x^5 + 2x^4 - 12x^3 - 8x^2$
10. $10x^5 - 4x^4 - 90x^3 + 36x^2$
11. $x^4 - 23x^2 - 50$
12. $2x^4 - 31x^2 - 16$
13. $-2x^5 - 6x^3 + 8x$
14. $-36x^5 + 69x^3 + 27x$
15. $54x^5 - 78x^3 + 24x$
16. $4x^6 - 65x^4 + 16x^2$
17. $x^6 - 7x^3 - 8$
18. $x^6 - 25x^3 - 54$
19. $3x^6 + 4x^3 + 1$
20. $27x^6 - 28x^3 + 1$

PART B: SOLVING POLYNOMIAL EQUATIONS BY FACTORING

Solve.

21. $(6x-5)(x+7)=0$
22. $(x+9)(3x-8)=0$
23. $5x(2x-5)(3x+1)=0$
24. $4x(5x-1)(2x+3)=0$
25. $(x-1)(2x+1)(3x-5)=0$
26. $(x+6)(5x-2)(2x+9)=0$
27. $(x+4)(x-2)=16$
28. $(x+1)(x-7)=9$
29. $(6x+1)(x+1)=6$
30. $(2x-1)(x-4)=39$
31. $x^2-15x+50=0$
32. $x^2+10x-24=0$
33. $3x^2+2x-5=0$
34. $2x^2+9x+7=0$
35. $\frac{1}{10}x^2-\frac{7}{15}x-\frac{1}{6}=0$
36. $\frac{1}{4}-\frac{4}{9}x^2=0$
37. $6x^2-5x-2=30x+4$
38. $6x^2-9x+15=20x-13$
39. $5x^2-23x+12=4(5x-3)$
40. $4x^2+5x-5=15(3-2x)$
41. $(x+6)(x-10)=4(x-18)$

42. $(x+4)\left(x-6\right)=2\left(x+4\right)$

43. $4x^3-14x^2-30x=0$

44. $9x^3+48x^2-36x=0$

45. $\frac{1}{3}x^3-\frac{3}{4}x=0$

46. $\frac{1}{2}x^3-\frac{1}{50}x=0$

47. $-10x^3-28x^2+48x=0$

48. $-2x^3+15x^2+50x=0$

49. $2x^3-x^2-72x+36=0$

50. $4x^3-32x^2-9x+72=0$

51. $45x^3-9x^2-5x+1=0$

52. $x^3-3x^2-x+3=0$

53. $x^4-5x^2+4=0$

54. $4x^4-37x^2+9=0$

Find the roots of the given functions.

55. $f\left(x\right)=x^2+10x-24$

56. $f\left(x\right)=x^2-14x+48$

57. $f\left(x\right)=-2x^2+7x+4$

58. $f\left(x\right)=-3x^2+14x+5$

59. $f\left(x\right)=16x^2-40x+25$

60. $f\left(x\right)=9x^2-12x+4$

61. $g\left(x\right)=8x^2+3x$

62. $g\left(x\right)=5x^2-30x$

63. $p\left(x\right)=64x^2-1$

64. $q\left(x\right)=4x^2-121$

65. $f(x) = \frac{1}{5}x^3 - 1x^2 - \frac{1}{20}x + \frac{1}{4}$

66. $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{4}{3}x - 2$

67. $g(x) = x^4 - 13x^2 + 36$

68. $g(x) = 4x^4 - 13x^2 + 9$

69. $f(x) = (x+5)^2 - 1$

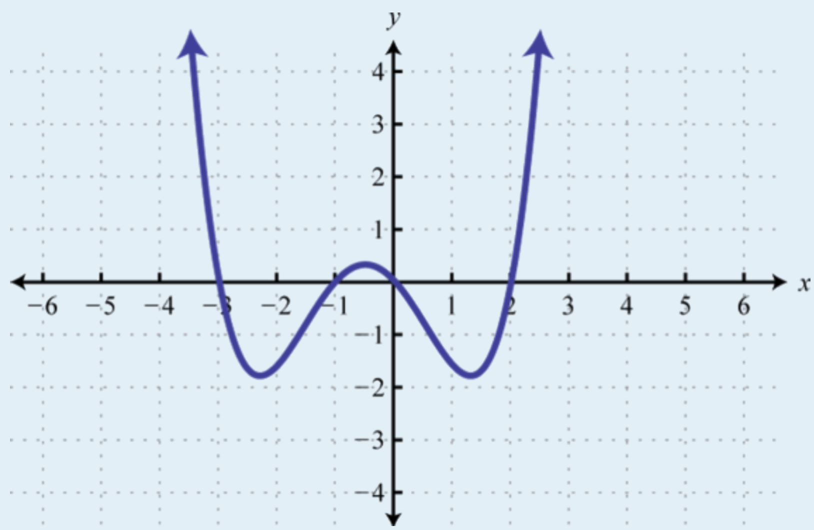
70. $g(x) = -(x+5)^2 + 9$

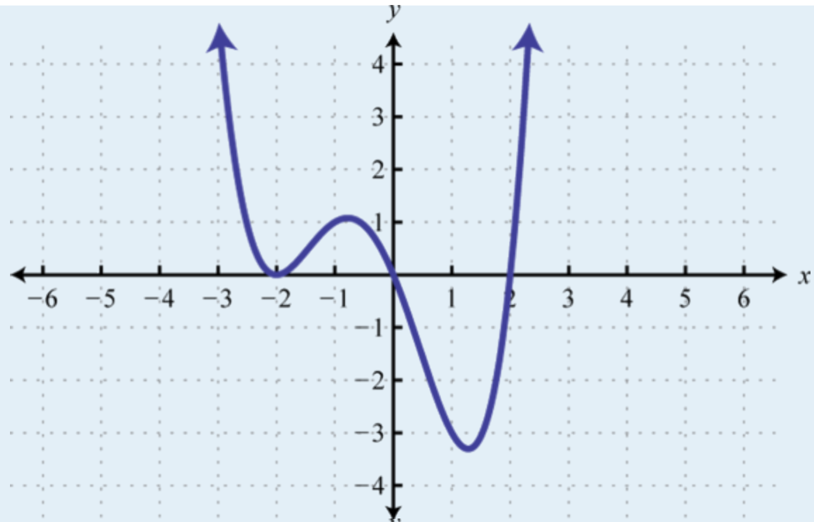
71. $f(x) = -(3x-5)^2$

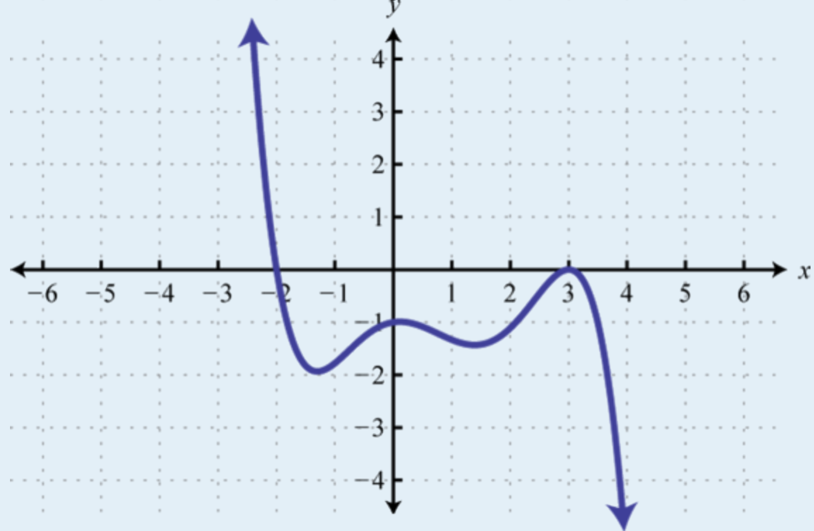
72. $g(x) = -(x+2)^2 + 4$

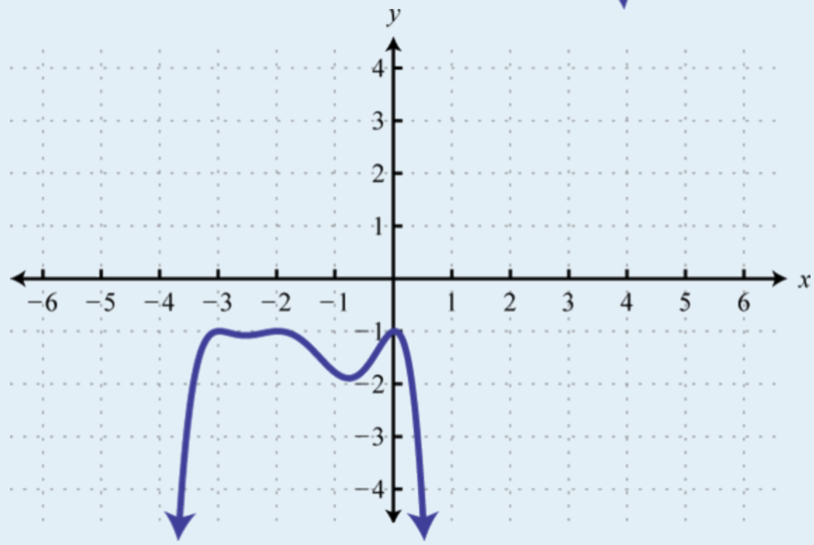
Given the graph of a function, determine the real roots.

73.



74. 

75. 

76. 

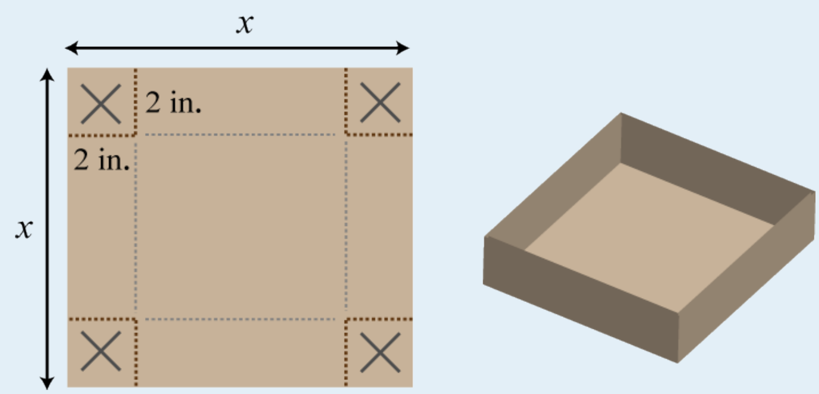
77. The sides of a square measure $x - 2$ units. If the area is 36 square units, then find x.

78. The sides of a right triangle have lengths that are consecutive even integers. Find the lengths of each side. (Hint: Apply the Pythagorean theorem)

79. The profit in dollars generated by producing and selling n bicycles per week is given by the formula $P(n) = -5n^2 + 400n - 6000$. How many bicycles must be produced and sold to break even?

80. The height in feet of an object dropped from the top of a 64-foot building is given by $h(t) = -16t^2 + 64$ where t represents the time in seconds after it is dropped. How long will it take to hit the ground?

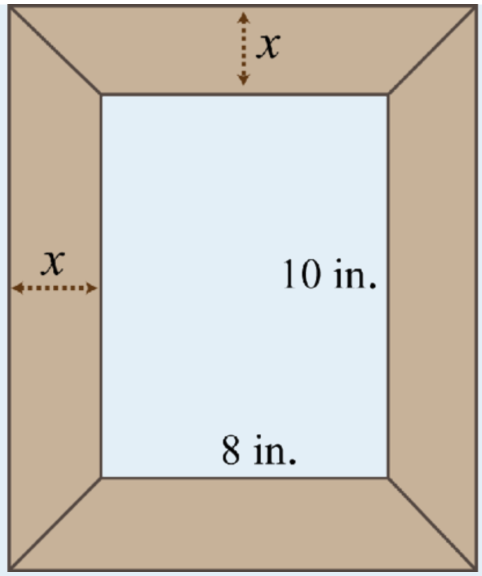
81. A box can be made by cutting out the corners and folding up the edges of a square sheet of cardboard. A template for a cardboard box of height 2 inches is given.



What is the length of each side of the cardboard sheet if the volume of the box is to be 98 cubic inches?

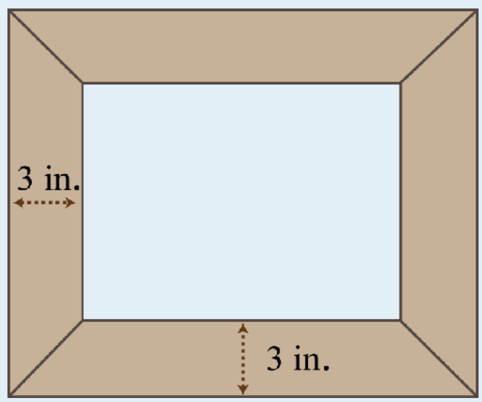
82. The height of a triangle is 4 centimeters less than twice the length of its base. If the total area of the triangle is 48 square centimeters, then find the lengths of the base and height.

83. A uniform border is to be placed around an 8×10 inch picture.



If the total area including the border must be 168 square inches, then how wide should the border be?

84. The area of a picture frame including a 3-inch wide border is 120 square inches.



If the width of the inner area is 2 inches less than its length, then find the dimensions of the inner area.

85. Assuming dry road conditions and average reaction times, the safe stopping distance in feet is given by $d\left(x\right)=\frac{1}{20}x^2+x$ where *x* represents the speed of the car in miles per hour. Determine the safe speed of the car if you expect to stop in 75 feet.

86. A manufacturing company has determined that the daily revenue in thousands of dollars is given by the formula $R\left(n\right)=12n-0.6n^2$ where *n* represents the number of palettes of product sold $\left(0\le n<20\right)$. Determine the number of palettes sold in a day if the revenue was 45 thousand dollars.

PART C: FINDING EQUATIONS WITH GIVEN SOLUTIONS

Find a polynomial equation with the given solutions.

87. −3, 5
88. −1, 8
89. $2, \frac{1}{3}$
90. $-\frac{3}{4}, 5$
91. 0, −4
92. 0,7
93. ±7
94. ±2
95. −3, 1, 3
96. −5, −1, 1

Find a function with the given roots.

97. $\frac{1}{2}, \frac{2}{3}$
98. $\frac{2}{5}, -\frac{1}{3}$
99. $\pm\frac{3}{4}$
100. $\pm\frac{5}{2}$
101. 5 double root
102. −3 double root
103. −1, 0, 3
104. −5, 0, 2

Recall that if $|X| = p$, then $X = -p$ or $X = p$. Use this to solve the following absolute value equations.

105. $|x^2 - 8| = 8$

106. $\left|2x^2 - 9\right| = 9$

107. $\left|x^2 - 2x - 1\right| = 2$

108. $\left|x^2 - 8x + 14\right| = 2$

109. $\left|2x^2 - 4x - 7\right| = 9$

110. $\left|x^2 - 3x - 9\right| = 9$

PART D: DISCUSSION BOARD

111. Explain to a beginning algebra student the difference between an equation and an expression.

112. What is the difference between a root and an x-intercept? Explain.

113. Create a function with three real roots of your choosing. Graph it with a graphing utility and verify your results. Share your function on the discussion board.

114. Research and discuss the fundamental theorem of algebra.

ANSWERS

1. $2(5x+3)(5x-3)$
3. $5x(x+7)(2x-1)$
5. $3a^2b(2a+b)(a-3b)$
7. $4x(x-2)(3x+1)(3x-1)$
9. $x^2(3x+2)(x+2)(x-2)$
11. $(x^2+2)(x+5)(x-5)$
13. $-2x(x^2+4)(x-1)(x+1)$
15. $6x(x+1)(x-1)(3x+2)(3x-2)$
17. $(x+1)(x^2-x+1)(x-2)(x^2+2x+4)$
19. $(3x^3+1)(x+1)(x^2-x+1)$
21. $-7, \frac{5}{6}$
23. $0, \frac{5}{2}, -\frac{1}{3}$
25. $-\frac{1}{2}, 1, \frac{5}{3}$
27. $-6, 4$
29. $-\frac{5}{3}, \frac{1}{2}$
31. $5, 10$
33. $-\frac{5}{3}, 1$
35. $-\frac{1}{3}, 5$
37. $-\frac{1}{6}, 6$
39. $\frac{3}{5}, 8$
41. $2, 6$
43. $0, -\frac{3}{2}, 5$

45. $0, \pm\frac{3}{2}$

47. $-4, 0, \frac{6}{5}$

49. $\pm 6, \frac{1}{2}$

51. $\pm\frac{1}{3}, \frac{1}{5}$

53. $\pm 1, \pm 2$

55. $2, -12$

57. $-\frac{1}{2}, 4$

59. $\frac{5}{4}$

61. $-\frac{3}{8}, 0$

63. $\pm\frac{1}{8}$

65. $\pm\frac{1}{2}, 5$

67. $\pm 2, \pm 3$

69. $-6, -4$

71. $\frac{5}{3}$

73. $-3, -1, 0, 2$

75. $-2, 3$

77. 8 units

79. 20 or 60 bicycles

81. 11 in

83. 2 inches

85. 30 miles per hour

87. $x^2 - 2x - 15 = 0$

89. $3x^2 - 7x + 2 = 0$

91. $x^2 + 4x = 0$

93. $x^2 - 49 = 0$

95. $x^3 - x^2 - 9x + 9 = 0$

97. $f(x) = 6x^2 - 7x + 2$

99. $f(x) = 16x^2 - 9$

101. $f(x) = x^2 - 10x + 25$

103. $f(x) = x^3 - 2x^2 - 3x$

105. ±4, 0

107. ±1, 3

109. −2, 1, 4

111. Answer may vary

113. Answer may vary

4.5 Rational Functions: Multiplication and Division

LEARNING OBJECTIVES

1. Identify restrictions to the domain of a rational function.
2. Simplify rational functions.
3. Multiply and divide rational functions.

Identifying Restrictions and Simplifying Rational Functions

Rational functions[25] have the form

$$r(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$. The **domain of a rational function**[26] consists of all real numbers *x* except those where the denominator $q(x) = 0$. **Restrictions**[27] are the real numbers for which the expression is not defined. We often express the domain of a rational function in terms of its restrictions. For example, consider the function

$$f(x) = \frac{x^2 - 4x + 3}{x^2 - 5x + 6}$$

which can be written in factored form

$$f(x) = \frac{(x-1)(x-3)}{(x-2)(x-3)}$$

25. Functions of the form $r(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$.

26. The set of real numbers for which the rational function is defined.

27. The set of real numbers for which a rational function is not defined.

Because rational expressions are undefined when the denominator is 0, we wish to find the values for *x* that make it 0. To do this, apply the zero-product property. Set each factor in the denominator equal to 0 and solve.

$$(x-2)\,(x-3)=0$$

$$\begin{aligned} x-2&=0 &\quad\text{or}\quad&& x-3&=0\\ x&=2 &&& x&=3 \end{aligned}$$

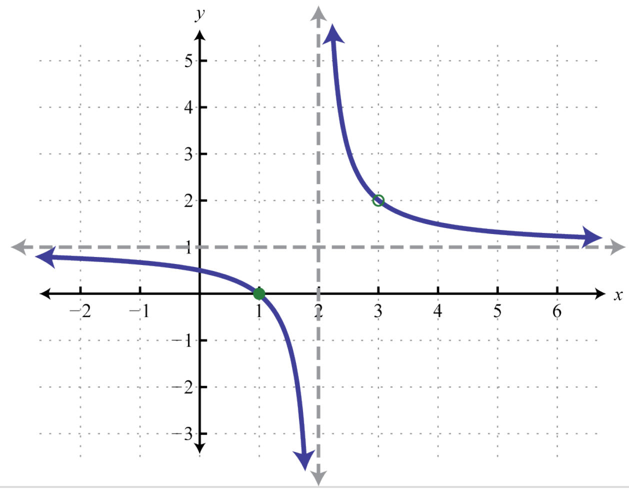
Therefore, the original function is defined for any real number except 2 and 3. We can express its domain using notation as follows:

Set-builder notation	*Interval notation*
$\{x \mid x \neq 2, 3\}$	$(-\infty, 2) \cup (2, 3) \cup (3, \infty)$

The restrictions to the domain of a rational function are determined by the denominator. Once the restrictions are determined we can cancel factors and obtain an equivalent function as follows:

$$f(x)=\frac{(x-1)\cancel{(x-3)}}{(x-2)\cancel{(x-3)}}=\frac{x-1}{x-2}$$

It is important to note that 1 is *not* a restriction to the domain because the expression is defined as 0 when the numerator is 0. In fact, $x=1$ is a root. This function is graphed below:



Notice that there is a vertical asymptote at the restriction $x = 2$ and the graph is left undefined at the restriction $x = 3$ as indicated by the open dot, or hole, in the graph. Graphing rational functions in general is beyond the scope of this textbook. However, it is useful at this point to know that the restrictions are an important part of the graph of rational functions.

Example 1

State the restrictions and simplify: $g(x) = \frac{24x^7}{6x^5}$.

Solution:

In this example, the function is undefined where x is 0.

$$g(0) = \frac{24(0)^7}{6(0)^5} = \frac{0}{0} \quad \textit{undefined}$$

Therefore, the domain consists of all real numbers x, where $x \neq 0$. With this understanding, we can simplify by reducing the rational expression to lowest terms. Cancel common factors.

$$g(x) = \frac{\overset{4}{\cancel{24}}\ \overset{x^2}{\cancel{x^7}}}{\cancel{6}\ \cancel{x^5}} = 4x^2$$

Answer: $g(x) = 4x^2$, where $x \neq 0$

Example 2

State the restrictions and simplify: $f(x) = \frac{2x^2+5x-3}{4x^2-1}$.

Solution:

First, factor the numerator and denominator.

$$f(x) = \frac{2x^2 + 5x - 3}{4x^2 - 1} = \frac{(2x - 1)(x + 3)}{(2x + 1)(2x - 1)}$$

Any x-value that makes the denominator zero is a restriction. To find the restrictions, first set the denominator equal to zero and then solve

$$(2x + 1)(2x - 1) = 0$$

$$\begin{aligned} 2x + 1 &= 0 & \text{or} \qquad 2x - 1 &= 0 \\ 2x &= -1 & 2x &= 1 \\ x &= -\frac{1}{2} & x &= \frac{1}{2} \end{aligned}$$

Therefore, $x \neq \pm\frac{1}{2}$ .With this understanding, we can cancel any common factors.

$$\begin{aligned} f(x) &= \frac{\cancel{(2x-1)}\ (x+3)}{(2x+1)\ \cancel{(2x-1)}} \\ &= \frac{x+3}{2x+1} \end{aligned}$$

Answer: $f(x) = \frac{x+3}{2x+1}$, where $x \neq \pm \frac{1}{2}$

We define the opposite of a polynomial *P* to be –*P*. Finding the opposite of a polynomial requires the application of the distributive property. For example, the opposite of the polynomial $(x - 3)$ is written as

$$\begin{aligned} -(x-3) &= -1 \cdot (x-3) \\ &= -x + 3 \\ &= 3 - x \end{aligned}$$

This leads us to the **opposite binomial property**[28], $-(a-b) = (b-a)$. Care should be taken not to confuse this with the fact that $(a+b) = (b+a)$. This is the case because addition is commutative. In general,

$$\begin{array}{c|c} -(a-b) = (b-a) & (a+b) = (b+a) \\ \text{or} & \text{or} \\ \dfrac{b-a}{a-b} = -1 & \dfrac{b+a}{a+b} = 1 \end{array}$$

Also, it is important to recall that

28. If given a binomial $a - b$, then the opposite is $-(a-b) = b - a$.

$$\frac{-a}{b} = -\frac{a}{b} = \frac{a}{-b}$$

In other words, a negative fraction is shown by placing the negative sign in either the numerator, in front of the fraction bar, or in the denominator. Generally, negative denominators are avoided.

Example 3

State the restrictions and simplify: $\frac{25-x^2}{x^2-10x+25}$.

Solution:

Begin by factoring the numerator and denominator.

$$\begin{aligned}
\frac{25-x^2}{x^2-10x+25} &= \frac{(5-x)(5+x)}{(x-5)(x-5)} \\
&= \frac{-1\cdot(x-5)(5+x)}{(x-5)(x-5)} && \textit{Opposite binomial property} \\
&= \frac{-1\cdot\cancel{(x-5)}(5+x)}{\cancel{(x-5)}(x-5)} && \textit{Cancel.} \\
&= -\frac{x+5}{x-5}
\end{aligned}$$

Answer: $-\frac{x+5}{x-5}$, where $x \neq 5$

It is important to remember that we can only cancel factors of a product. A common mistake is to cancel terms. For example,

$$\frac{\cancel{x^2}+7x-30}{\cancel{x^2}-7x+12} \quad \frac{\cancel{x}+10}{\cancel{x}-4} \quad \frac{2\cancel{x-1}}{\cancel{x-1}}$$

incorrect! *incorrect*! *incorrect*!

Try this! State the restrictions and simplify: $\frac{x-2x^2}{4x^4-x^2}$.

Answer: $-\frac{1}{x(2x+1)}$, where $x \neq 0, \pm\frac{1}{2}$

(click to see video)

In some examples, we will make a broad assumption that the denominator is nonzero. When we make that assumption, we do not need to determine the restrictions.

Example 4

Simplify: $\frac{x^3-2x^2y+4xy^2-8y^3}{x^4-16y^4}$.(*Assume all denominators are nonzero.*)

Solution:

Factor the numerator by grouping. Factor the denominator using the formula for a difference of squares.

$$\begin{aligned}\frac{x^3+4xy^2-2x^2y-8y^3}{x^4-16y^4} &= \frac{x\left(x^2+4y^2\right)-2y\left(x^2+4y^2\right)}{\left(x^2+4y^2\right)\left(x^2-4y^2\right)} \\ &= \frac{\left(x^2+4y^2\right)\left(x-2y\right)}{\left(x^2+4y^2\right)\left(x+2y\right)\left(x-2y\right)}\end{aligned}$$

Next, cancel common factors.

$$\begin{aligned}&= \frac{\overset{1}{\cancel{\left(x^2+4y^2\right)}}\,\overset{1}{\cancel{\left(x-2y\right)}}}{\cancel{\left(x^2+4y^2\right)}\left(x+2y\right)\cancel{\left(x-2y\right)}} \\ &= \frac{1}{x+2y}\end{aligned}$$

Note: When the entire numerator or denominator cancels out a factor of 1 always remains.

Answer: $\frac{1}{x+2y}$

Example 5

Given $f(x) = x^2 - 2x + 5$, simplify $\frac{f(x)-f(3)}{x-3}$.

Solution:

Begin by calculating $f(3)$.

$$\begin{aligned} f(3) &= (3)^2 - 2(3) + 5 \\ &= 9 - 6 + 5 \\ &= 3 + 5 \\ &= 8 \end{aligned}$$

Next, substitute into the quotient that is to be simplified.

$$\begin{aligned} \frac{f(x) - f(3)}{x - 3} &= \frac{x^2 - 2x + 5 - 8}{x - 3} \\ &= \frac{x^2 - 2x - 3}{x - 3} \\ &= \frac{(x + 1)(x - 3)}{(x - 3)} \\ &= x + 1 \end{aligned}$$

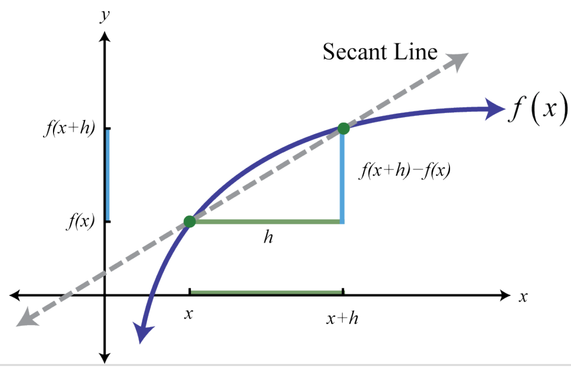
Answer: $x + 1$, where $x \neq 3$

An important quantity in higher level mathematics is the **difference quotient**[29]:

29. The mathematical quantity $\frac{f(x+h)-f(x)}{h}$, where $h \neq 0$, which represents the slope of a secant line through a function *f*.

$$\frac{f(x+h)-f(x)}{h}, \quad where \quad h \neq 0$$

This quantity represents the slope of the line connecting two points on the graph of a function. The line passing through the two points is called a **secant line**[30].



Calculating the difference quotient for many different functions is an important skill to learn in intermediate algebra. We will encounter this quantity often as we proceed in this textbook. When calculating the difference quotient we assume the denominator is nonzero.

30. Line that intersects two points on the graph of a function.

Example 6

Given $g(x) = -2x^2 + 1$, simplify $\frac{g(x+h)-g(x)}{h}$.

Solution:

$$
\begin{aligned}
\frac{g(x+h) - g(x)}{h} &= \frac{\left(-2(x+h)^2 + 1\right) - \left(-2x^2 + 1\right)}{h} \\
&= \frac{-2\left(x^2 + 2xh + h^2\right) + 1 + 2x^2 - 1}{h} \\
&= \frac{-2x^2 - 4xh - 2h^2 + 1 + 2x^2 - 1}{h} \\
&= \frac{-4xh - 2h^2}{h} \\
&= -4x - 2h
\end{aligned}
$$

Answer: $-4x - 2h$

Try this! Given $f(x) = x^2 - x - 1$, simplify $\frac{f(x+h)-f(x)}{h}$.

Answer: $2x - 1 + h$

(click to see video)

Multiplying and Dividing Rational Functions

When multiplying fractions, we can multiply the numerators and denominators together and then reduce. Multiplying rational expressions is performed in a similar manner. In general, given polynomials P, Q, R, and S, where $Q \neq 0$ and $S \neq 0$, we have

$$\frac{P}{Q} \cdot \frac{R}{S} = \frac{PR}{QS}$$

The restrictions to the domain of a product consist of the restrictions of each function.

Example 7

Given $f(x) = \frac{9x^2-25}{x-5}$ and $g(x) = \frac{x^2-2x-15}{3x+5}$, find $(f \cdot g)(x)$ and determine the restrictions to the domain.

Solution:

In this case, the domain of f consists of all real numbers except 5, and the domain of g consists of all real numbers except $-\frac{5}{3}$ .Therefore, the domain of the product consists of all real numbers except 5 and $-\frac{5}{3}$ .Multiply the functions and then simplify the result.

$$\begin{aligned}(f \cdot g)(x) &= f(x) \cdot g(x)\\ &= \frac{9x^2-25}{x-5} \cdot \frac{x^2-2x-15}{3x+5}\\ &= \frac{(3x+5)(3x-5)}{x-5} \cdot \frac{(x-5)(x+3)}{3x+5} & \textit{Factor.}\\ &= \frac{\cancel{(3x+5)}(3x-5)\cancel{(x-5)}(x+3)}{\cancel{(x-5)}\cancel{(3x+5)}} & \textit{Cancel.}\\ &= (3x-5)(x+3)\end{aligned}$$

Answer: $(f \cdot g)(x) = (3x-5)(x+3)$, where $x \neq 5, -\frac{5}{3}$

To divide two fractions, we multiply by the reciprocal of the divisor. Dividing rational expressions is performed in a similar manner. In general, given polynomials P, Q, R, and S, where $Q \neq 0$, $R \neq 0$, and $S \neq 0$, we have

$$\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \cdot \frac{S}{R} = \frac{PS}{QR}$$

The restrictions to the domain of a quotient will consist of the restrictions of each function as well as the restrictions on the reciprocal of the divisor.

Example 8

Given $f(x) = \frac{2x^2+13x-7}{x^2-4x-21}$ and $g(x) = \frac{2x^2+5x-3}{49-x^2}$, find $(f/g)(x)$ and determine the restrictions.

Solution:

$$
\begin{aligned}
(f/g)(x) &= f(x) \div g(x) \\
&= \frac{2x^2+13x-7}{x^2-4x-21} \div \frac{2x^2+5x-3}{49-x^2} \\
&= \frac{2x^2+13x-7}{x^2-4x-21} \cdot \frac{49-x^2}{2x^2+5x-3} && \textit{Multiply by the reciprocal of th} \\
&= \frac{(2x-1)(x+7)}{(x+3)(x-7)} \cdot \frac{(7+x)(7-x)}{(2x-1)(x+3)} && \textit{Factor.} \\
&= \frac{\cancel{(2x-1)}(x+7)(7+x)(-1)\cancel{(x-7)}}{(x+3)\cancel{(x-7)}\cancel{(2x-1)}(x+3)} && \textit{Cancel.} \\
&= -\frac{(x+7)^2}{(x+3)^2}
\end{aligned}
$$

In this case, the domain of f consists of all real numbers except −3 and 7, and the domain of g consists of all real numbers except 7 and −7. In addition, the reciprocal of $g(x)$ has a restriction of −3 and $\frac{1}{2}$ .Therefore, the domain of this quotient consists of all real numbers except −3, $\frac{1}{2}$, and ±7.

Answer: $(f/g)(x) = -\frac{(x+7)^2}{(x+3)^2}$, where $x \neq -3, \frac{1}{2}, \pm 7$

Recall that multiplication and division operations are to be performed from left to right.

Example 9

Simplify: $\frac{4x^2-1}{6x^2+3x} \div \frac{2x+1}{x^2+2x+1} \cdot \frac{27x^4}{2x^2+x-1}$ (*Assume all denominators are nonzero.*)

Solution:

Begin by replacing the factor that is to be divided by multiplication of its reciprocal.

$$\begin{aligned}
&\frac{4x^2-1}{6x^2+3x} \div \frac{2x+1}{x^2+2x+1} \cdot \frac{27x^4}{2x^2+x-1} \\
&= \frac{4x^2-1}{6x^2+3x} \cdot \frac{x^2+2x+1}{2x+1} \cdot \frac{27x^4}{2x^2+x-1} \\
&= \frac{(2x+1)(2x-1)}{3x(2x+1)} \cdot \frac{(x+1)(x+1)}{(2x+1)} \cdot \frac{27x^4}{(2x-1)(x+1)} \\
&= \frac{\cancel{(2x+1)}\,\cancel{(2x-1)}\,\cancel{(x+1)}\,(x+1)\cdot \overset{9}{\cancel{27}}\,\overset{x^3}{\cancel{x^4}}}{\cancel{3}\,\cancel{x}\,\cancel{(2x+1)}\,(2x+1)\,\cancel{(2x-1)}\,\cancel{(x+1)}} \\
&= \frac{9x^3(x+1)}{(2x+1)}
\end{aligned}$$

Answer: $\frac{9x^3(x+1)}{(2x+1)}$

Try this! Given $f(x) = \frac{2x+5}{3x^2+14x-5}$ and $g(x) = \frac{6x^2+13x-5}{x+5}$, calculate $(f/g)(x)$ and determine the restrictions.

Answer: $(f/g)(x) = \frac{1}{(3x-1)^2}$, where $x \neq -5, -\frac{5}{2}, \frac{1}{3}$

(click to see video)

If a cost function C represents the cost of producing *x* units, then the **average cost**[31] $\overline{C}$ is the cost divided by the number of units produced.

$$\overline{C}(x) = \frac{C(x)}{x}$$

31. The total cost divided by the number of units produced, which can be represented by $\overline{C}(x) = \frac{C(x)}{x}$, where $C(x)$ is a cost function.

Example 10

A manufacturer has determined that the cost in dollars of producing sweaters is given by $C(x) = 0.01x^2 - 3x + 1200$, where x represents the number of sweaters produced daily. Determine the average cost of producing 100, 200, and 300 sweaters per day.

Solution:

Set up a function representing the average cost.

$$\overline{C}(x) = \frac{C(x)}{x} = \frac{0.01x^2 - 3x + 1200}{x}$$

Next, calculate $\overline{C}(100)$, $\overline{C}(200)$, and $\overline{C}(300)$.

$$\overline{C}(100) = \frac{0.01(100)^2 - 3(100) + 1200}{(100)} = \frac{100 - 300 + 1200}{100} = \frac{1000}{100} = 10.00$$

$$\overline{C}(200) = \frac{0.01(200)^2 - 3(200) + 1200}{(200)} = \frac{400 - 600 + 1200}{200} = \frac{1000}{200} = 5.00$$

$$\overline{C}(300) = \frac{0.01(300)^2 - 3(300) + 1200}{(300)} = \frac{900 - 900 + 1200}{300} = \frac{1200}{300} = 4.00$$

Answer: The average cost of producing 100 sweaters per day is $10.00 per sweater. If 200 sweaters are produced, the average cost per sweater is $5.00. If 300 are produced, the average cost per sweater is $4.00.

KEY TAKEAWAYS

- Simplifying rational expressions is similar to simplifying fractions. First, factor the numerator and denominator and then cancel the common factors. Rational expressions are simplified if there are no common factors other than 1 in the numerator and the denominator.
- Simplified rational functions are equivalent for values in the domain of the original function. Be sure to state the restrictions unless the problem states that the denominators are assumed to be nonzero.
- After multiplying rational expressions, factor both the numerator and denominator and then cancel common factors. Make note of the restrictions to the domain. The values that give a value of 0 in the denominator for all expressions are the restrictions.
- To divide rational expressions, multiply the numerator by the reciprocal of the divisor.
- The restrictions to the domain of a product consist of the restrictions to the domain of each factor.

TOPIC EXERCISES

PART A: SIMPLIFYING RATIONAL FUNCTIONS

Simplify the function and state its domain using interval notation.

1. $f(x) = \dfrac{25x^9}{5x^5}$
2. $f(x) = \dfrac{64x^8}{16x^3}$
3. $f(x) = \dfrac{x^2 - 64}{x^2 + 16x + 64}$
4. $f(x) = \dfrac{x^2 + x - 20}{x^2 - 25}$
5. $g(x) = \dfrac{9 - 4x^2}{2x^2 - 5x + 3}$
6. $g(x) = \dfrac{x - 3x^2}{9x^2 - 6x + 1}$
7. $g(x) = \dfrac{2x^2 - 8x - 42}{2x^2 + 5x - 3}$
8. $g(x) = \dfrac{6x^2 + 5x - 4}{3x^2 + x - 4}$
9. $h(x) = \dfrac{x^3 + x^2 - x - 1}{x^2 + 2x + 1}$
10. $h(x) = \dfrac{2x^3 - 5x^2 - 8x + 20}{2x^2 - 9x + 10}$

State the restrictions and simplify the given rational expressions.

11. $\dfrac{66x(2x - 5)}{18x^3(2x - 5)^2}$
12. $\dfrac{26x^4(5x + 2)^3}{20x^5(5x + 2)}$
13. $\dfrac{x^2 + 5x + 6}{x^2 - 5x - 14}$

14. $\dfrac{x^2 - 8x + 12}{x^2 - 2x - 24}$

15. $\dfrac{1 - x^2}{5x^2 + x - 6}$

16. $\dfrac{4 - 9x^2}{3x^2 - 8x + 4}$

17. $\dfrac{4x^2 + 15x + 9}{9 - x^2}$

18. $\dfrac{6x^2 + 13x - 5}{25 - 4x^2}$

19. $\dfrac{x^2 - 5x + 4}{x^3 - x^2 - 16x + 16}$

20. $\dfrac{x^4 + 4x^2}{x^3 + 3x^2 + 4x + 12}$

Simplify the given rational expressions. Assume all variable expressions in the denominator are nonzero.

21. $\dfrac{50ab^3\left(a+b\right)^2}{200a^2b^3\left(a+b\right)^3}$

22. $\dfrac{36a^5b^7\left(a-b\right)^2}{9a^3b\left(a-b\right)}$

23. $\dfrac{a^2 - b^2}{a^2 + 2ab + b^2}$

24. $\dfrac{a^2 - 2ab + b^2}{a^2 - b^2}$

25. $\dfrac{6x^2 - xy}{6x^2 - 7xy + y^2}$

26. $\dfrac{y - x}{2x^3 - 4x^2y + 2xy^2}$

27. $\dfrac{x^2y^2 - 2xy^3}{x^2y^2 - xy^3 - 2y^4}$

28. $\dfrac{x^4y - x^2y^3}{x^3y + 2x^2y^2 + xy^3}$

29. $\dfrac{x^3 - x^2y + xy^2 - y^3}{x^4 - y^4}$

30. $\dfrac{y^4 - x^4}{x^3 + x^2y + xy^2 + y^3}$

31. $\dfrac{a^2 - (b+c)^2}{(a+b)^2 - c^2}$

32. $\dfrac{(a+b)^2 - c^2}{(a+c)^2 - b^2}$

33. $\dfrac{x^3 + y^3}{x^2 + 2xy + y^2}$

34. $\dfrac{x^3y + x^2y^2 + xy^3}{x^3 - y^3}$

Given the function, simplify the rational expression.

35. Given $f(x) = x^2 - 8$, simplify $\frac{f(x)-f(5)}{x-5}$.

36. Given $f(x) = x^2 + 4x - 1$, simplify $\frac{f(x)-f(2)}{x-2}$.

37. Given $g(x) = x^2 - 3x + 1$, simplify $\frac{g(x)-g(-1)}{x+1}$.

38. Given $g(x) = x^2 - 2x$, simplify $\frac{g(x)-g(-4)}{x+4}$.

39. Given $f(x) = 4x^2 + 6x + 1$, simplify $\frac{f(x)-f\left(\frac{1}{2}\right)}{2x-1}$.

40. Given $f(x) = 9x^2 + 1$, simplify $\frac{f(x)-f\left(-\frac{1}{3}\right)}{3x+1}$.

For the given function, simplify the difference quotient $\frac{f(x+h)-f(x)}{h}$, where $h \neq 0$.

41. $f(x) = 5x - 3$

42. $f(x) = 3 - 2x$

43. $f(x) = x^2 - 3$

44. $f(x) = x^2 + 8x$

45. $f(x) = x^2 - x + 5$

46. $f(x) = 4x^2 + 3x - 2$

47. $f(x) = ax^2 + bx + c$

48. $f(x) = ax^2 + bx$

49. $f(x) = x^3 + 1$

50. $f(x) = x^3 - x + 2$

PART B: MULTIPLYING AND DIVIDING RATIONAL FUNCTIONS

Simplify the product $f \cdot g$ and state its domain using interval notation.

51. $f(x) = \dfrac{52x^4}{(x-2)^2},\ g(x) = \dfrac{(x-2)^3}{12x^5}$

52. $f(x) = \dfrac{46(2x-1)^3}{15x^6},\ g(x) = \dfrac{25x^3}{23(2x-1)}$

53. $f(x) = \dfrac{10x^3}{x^2+4x+4},\ g(x) = \dfrac{x^2-4}{50x^4}$

54. $f(x) = \dfrac{25-x^2}{46x^5},\ g(x) = \dfrac{12x^3}{x^2+10x+25}$

55. $f(x) = \dfrac{5-3x}{x^2-10x+25},\ g(x) = \dfrac{x^2-6x+5}{3x^2-8x+5}$

56. $f(x) = \dfrac{1-4x^2}{6x^2+3x},\ g(x) = \dfrac{12x^2}{4x^2-4x+1}$

Simplify the quotient f/g and state its domain using interval notation.

57. $f(x) = \dfrac{12x^3}{5(5x-1)^3},\ g(x) = \dfrac{6x^2}{25(5x-1)^4}$

58. $f(x) = \dfrac{7x^2(x+9)}{(x-8)^2},\ g(x) = \dfrac{49x^3(x+9)}{(x-8)^4}$

59. $f(x) = \dfrac{25x^2-1}{3x^2-15x},\ g(x) = \dfrac{25x^2+10x+1}{x^3-5x^2}$

60. $f(x) = \dfrac{x^2-x-6}{2x^2+13x+15},\ g(x) = \dfrac{x^2-6x+9}{4x^2+12x+9}$

61. $f(x) = \dfrac{x^2 - 64}{x^2}, g(x) = 2x^2 + 19x + 24$
62. $f(x) = 2x^2 + 11x - 6, g(x) = 36 - x^2$

Multiply or divide as indicated, state the restrictions, and simplify.

63. $\dfrac{14(x+12)^2}{5x^3} \cdot \dfrac{45x^4}{2(x+12)^3}$
64. $\dfrac{27x^6}{20(x-7)^3} \cdot \dfrac{(x-7)^5}{54x^7}$
65. $\dfrac{x^2 - 64}{36x^4} \cdot \dfrac{12x^3}{x^2 + 4x - 32}$
66. $\dfrac{50x^5}{x^2 + 6x - 27} \cdot \dfrac{x^2 - 81}{125x^3}$
67. $\dfrac{2x^2 + 7x + 5}{3x^2} \cdot \dfrac{15x^3 - 30x^2}{2x^2 + x - 10}$
68. $\dfrac{3x^2 + 14x - 5}{2x^2 + 11x + 5} \cdot \dfrac{4x^2 + 4x + 1}{6x^2 + x - 1}$
69. $\dfrac{x^2 + 4x - 21}{5x^2 + 10x} \div \dfrac{x^2 - 6x + 9}{x^2 + 9x + 14}$
70. $\dfrac{x^2 - 49}{9x^2 - 24x + 16} \div \dfrac{2x^2 - 13x - 7}{6x^2 - 5x - 4}$
71. $\dfrac{5x^2 + x - 6}{4x^2 - 7x - 15} \div \dfrac{1 - x^2}{4x^2 + 9x + 5}$
72. $\dfrac{6x^2 - 8x - 8}{4 - 9x^2} \div \dfrac{3x^2 - 4x - 4}{9x^2 + 12x + 4}$
73. $\dfrac{x^2 + 4x - 12}{x^2 - 2x - 15} \div \dfrac{2x^2 - 13x + 18}{6x^2 - 31x + 5}$
74. $\dfrac{8x^2 + x - 9}{25x^2 - 1} \div \dfrac{2x^2 - x - 1}{10x^2 - 3x - 1}$

Perform the operations and simplify. Assume all variable expressions in the denominator are nonzero.

75. $\dfrac{1}{12ab} \cdot \dfrac{50a^2\left(a-b\right)^2}{a^2 - b^2} \cdot \dfrac{6b}{a\left(a-b\right)}$

76. $\frac{b^2 - a^2}{(a-b)^2} \cdot \frac{12a(a-b)}{36a^2 b} \cdot \frac{9ab(a-b)}{a+b}$

77. $\frac{x^3 + y^3}{5xy} \cdot \frac{x^2 - y^2}{x^2 - 2xy + y^2} \cdot \frac{25x^2 y}{(y+x)^2}$

78. $\frac{3xy^2}{(2y+x)^2} \cdot \frac{2x^2 + 5xy + 2y^2}{9x^2} \cdot \frac{x^3 + 8y^3}{6xy^2 + 3y^3}$

79. $\frac{2x+5}{x-3} \cdot \frac{x^2 - 9}{5x^4} \div \frac{2x^2 + 15x + 25}{25x^5}$

80. $\frac{5x^2 - 15x}{9x^2 - 4} \cdot \frac{3x-2}{20x^3} \div \frac{x-3}{3x^2 - x - 2}$

81. $\frac{x^2 + 5x - 50}{x^2 + 5x - 14} \div \frac{x^2 - 25}{x^2 - 49} \cdot \frac{x-2}{x^2 + 3x - 70}$

82. $\frac{x^2 - x - 56}{4x^2 - 4x - 3} \div \frac{2x^2 + 11x - 21}{25 - 9x^2} \cdot \frac{4x^2 - 12x + 9}{3x^2 - 19x - 40}$

83. $\frac{20x^2 - 8x - 1}{6x^2 + 13x + 6} \div \frac{1 - 100x^2}{3x^2 - x - 2} \cdot \frac{10x - 1}{2x^2 - 3x + 1}$

84. $\frac{12x^2 - 13x + 1}{x^2 + 18x + 81} \div (144x^2 - 1) \cdot \frac{x^2 + 14x + 45}{12x^2 - 11x - 1}$

85. A manufacturer has determined that the cost in dollars of producing bicycles is given by $C(x) = 0.5x^2 - x + 6200$, where *x* represents the number of bicycles produced weekly. Determine the average cost of producing 50, 100, and 150 bicycles per week.

86. The cost in dollars of producing custom lighting fixtures is given by the function $C(x) = x^2 - 20x + 1200$, where *x* represents the number of fixtures produced in a week. Determine the average cost per unit if 20, 40, and 50 units are produced in a week.

87. A manufacturer has determined that the cost in dollars of producing electric scooters is given by the function $C(x) = 3x(x - 100) + 32{,}000$, where *x* represents the number of scooters produced in a month. Determine the average cost per scooter if 50 are produced in a month.

88. The cost in dollars of producing a custom injected molded part is given by $C(n) = 1{,}900 + 0.01n$, where *n* represents the number of parts produced. Calculate the average cost of each part if 2,500 custom parts are ordered.

89. The cost in dollars of an environmental cleanup is given by the function $C\left(p\right) = \frac{25{,}000p}{1-p}$, where p represents the percentage of the area to be cleaned up $\left(0 \leq p < 1\right)$. Use the function to determine the cost of cleaning up 50% of an affected area and the cost of cleaning up 80% of the area.

90. The value of a new car is given by the function $V\left(t\right) = 16{,}500(t + 1)^{-1}$ where t represents the age of the car in years. Determine the value of the car when it is 6 years old.

PART D: DISCUSSION BOARD

91. Describe the restrictions to the rational expression $\frac{1}{x^2-y^2}$. Explain.

92. Describe the restrictions to the rational expression $\frac{1}{x^2+y^2}$. Explain.

93. Explain why $x = 5$ is a restriction to $\frac{1}{x+5} \div \frac{x-5}{x}$.

94. Explain to a beginning algebra student why we cannot cancel x in the rational expression $\frac{x+2}{x}$.

95. Research and discuss the importance of the difference quotient. What does it represent and in what subject does it appear?

ANSWERS

1. $f(x) = 5x^4$;

   Domain: $(-\infty, 0) \cup (0, \infty)$

3. $f(x) = \frac{x-8}{x+8}$; Domain: $(-\infty, -8) \cup (-8, \infty)$

5. $g(x) = -\frac{2x+3}{x-1}$; Domain: $(-\infty, 1) \cup \left(1, \frac{3}{2}\right) \cup \left(\frac{3}{2}, \infty\right)$

7. $g(x) = \frac{2(x-7)}{2x-1}$; Domain: $(-\infty, -3) \cup \left(-3, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$

9. $h(x) = x - 1$; Domain: $(-\infty, -1) \cup (-1, \infty)$

11. $\frac{11}{3x^2(2x-5)}; x \neq 0, \frac{5}{2}$

13. $\frac{x+3}{x-7}; x \neq -2, 7$

15. $-\frac{x+1}{5x+6}; x \neq -\frac{6}{5}, 1$

17. $-\frac{4x+3}{3-x}; x \neq \pm 3$

19. $\frac{1}{x+4}; x \neq 1, \pm 4$

21. $\frac{1}{4a(a+b)}$

23. $\frac{a-b}{a+b}$

25. $\frac{x}{x-y}$

27. $\frac{x}{x+y}$

29. $\frac{1}{x+y}$

31. $\frac{a-b-c}{a+b-c}$

33. $\frac{x^2-xy+y^2}{x+y}$

35. $x + 5$, where $x \neq 5$

37. $x - 4$, where $x \neq -1$

39. $2\left(x+2\right)$, where $x \neq \frac{1}{2}$

41. 5

43. $2x+h$

45. $2x-1+h$

47. $2ax+b+ah$

49. $3x^2+3xh+h^2$

51. $\left(f \cdot g\right)(x) = \frac{13(x-2)}{3x}$; Domain: $\left(-\infty, 0\right) \cup (0,2) \cup \left(2, \infty\right)$

53. $\left(f \cdot g\right)(x) = \frac{x-2}{5x(x+2)}$; Domain: $\left(-\infty, -2\right) \cup (-2,0) \cup \left(0, \infty\right)$

55. $\left(f \cdot g\right)(x) = -\frac{1}{x-5}$; Domain: $\left(-\infty, 1\right) \cup \left(1, \frac{5}{3}\right) \cup \left(\frac{5}{3}, 5\right) \cup \left(5, \infty\right)$

57. $\left(f/g\right)(x) = 10x\left(5x-1\right)$; Domain: $\left(-\infty, 0\right) \cup \left(0, \frac{1}{5}\right) \cup \left(\frac{1}{5}, \infty\right)$

59. $\left(f/g\right)(x) = \frac{x\left(5x-1\right)}{2\left(5x+1\right)}$; Domain: $\left(-\infty, -\frac{1}{5}\right) \cup \left(-\frac{1}{5}, 0\right) \cup \left(0, 5\right) \cup \left(5, \infty\right)$

61. $\left(f/g\right)(x) = \frac{x-8}{x^2(2x+3)}$; Domain: $\left(-\infty, -8\right) \cup \left(-8, -\frac{3}{2}\right) \cup \left(-\frac{3}{2}, 0\right) \cup \left(0, \infty\right)$

63. $\frac{63x}{x+12}; x \neq -12, 0$

65. $\frac{x-8}{3x(x-4)}; x \neq -8, 0, 4$

67. $5\left(x+1\right); x \neq -\frac{5}{2}, 0, 2$

69. $\frac{(x+7)^2}{5x(x-3)}; x \neq -7, -2, 0, 3$

71. $-\frac{5x+6}{x-3}; x \neq -\frac{5}{4}, -1, 1, 3$

73. $\frac{\left(x+6\right)\left(6x-1\right)}{(x+3)(2x-9)}; x \neq -3, \frac{1}{6}, 2, \frac{9}{2}, 5$

75. $\frac{25}{a+b}$

77. $\dfrac{5x\left(x^2 - xy + y^2\right)}{x - y}$

79. $\dfrac{5x\,(x + 3)}{x + 5}$

81. $\dfrac{1}{x + 5}$

83. $-\dfrac{1}{2x + 3}$

85. If 50 bicycles are produced, the average cost per bicycle is \$148. If 100 are produced, the average cost is \$111. If 150 bicycles are produced, the average cost is \$115.33.

87. If 50 scooters are produced, the average cost of each is \$490.

89. A 50% cleanup will cost \$25,000. An 80% cleanup will cost \$100,000.

91. Answer may vary

93. Answer may vary

95. Answer may vary

4.6 Rational Functions: Addition and Subtraction

LEARNING OBJECTIVES

1. Add and subtract rational functions.
2. Simplify complex rational expressions.

Adding and Subtracting Rational Functions

Adding and subtracting rational expressions is similar to adding and subtracting fractions. Recall that if the denominators are the same, we can add or subtract the numerators and write the result over the common denominator. When working with rational expressions, the common denominator will be a polynomial. In general, given polynomials P, Q, and R, where $Q \neq 0$, we have the following:

$$\frac{P}{Q} \pm \frac{R}{Q} = \frac{P \pm R}{Q}$$

The set of restrictions to the domain of a sum or difference of rational expressions consists of the restrictions to the domains of each expression.

Example 1

Subtract: $\frac{4x}{x^2-64} - \frac{3x+8}{x^2-64}$.

Solution:

The denominators are the same. Hence we can subtract the numerators and write the result over the common denominator. Take care to distribute the negative 1.

$$\begin{aligned}
\frac{4x}{x^2-64} - \frac{3x+8}{x^2-64} &= \frac{4x - (3x+8)}{x^2-64} && \textit{Subtract the numerators.} \\
&= \frac{4x-3x-8}{x^2-64} && \textit{Simplify.} \\
&= \frac{\overset{1}{\cancel{x-8}}}{(x+8)\cancel{(x-8)}} && \textit{Cancel.} \\
&= \frac{1}{x+8} && \textit{Restrictions } x \neq \pm 8
\end{aligned}$$

Answer: $\frac{1}{x+8}$, where $x \neq \pm 8$

To add rational expressions with unlike denominators, first find equivalent expressions with common denominators. Do this just as you have with fractions. If the denominators of fractions are relatively prime, then the least common denominator (LCD) is their product. For example,

$$\frac{1}{x} + \frac{1}{y} \quad \Rightarrow \quad \text{LCD} = x \cdot y = xy$$

Multiply each fraction by the appropriate form of 1 to obtain equivalent fractions with a common denominator.

$$\begin{aligned}\frac{1}{x}+\frac{1}{y}&=\frac{1\cdot y}{x\cdot y}+\frac{1\cdot x}{y\cdot x}\\&=\frac{y}{xy}+\frac{x}{xy} &&\textit{Equivalent fractions with a common denominator}\\&=\frac{y+x}{xy}\end{aligned}$$

In general, given polynomials *P*, *Q*, *R*, and *S*, where $Q \neq 0$ and $S \neq 0$, we have the following:

$$\frac{P}{Q}\pm\frac{R}{S}=\frac{PS\pm QR}{QS}$$

Example 2

Given $f\left(x\right) = \frac{5x}{3x+1}$ and $g\left(x\right) = \frac{2}{x+1}$, find $f + g$ and state the restrictions.

Solution:

Here the LCD is the product of the denominators $\left(3x + 1\right)\left(x + 1\right)$. Multiply by the appropriate factors to obtain rational expressions with a common denominator before adding.

$$\begin{aligned}(f + g)(x) &= f(x) + g(x) \\ &= \frac{5x}{3x+1} + \frac{2}{x+1} \\ &= \frac{5x}{(3x+1)} \cdot \frac{(x+1)}{(x+1)} + \frac{2}{(x+1)} \cdot \frac{(3x+1)}{(3x+1)} \\ &= \frac{5x(x+1)}{(3x+1)(x+1)} + \frac{2(3x+1)}{(x+1)(3x+1)} \\ &= \frac{5x(x+1) + 2(3x+1)}{(3x+1)(x+1)} \\ &= \frac{5x^2 + 5x + 6x + 2}{(3x+1)(x+1)} \\ &= \frac{5x^2 + 11x + 2}{(3x+1)(x+1)} \\ &= \frac{(5x+1)(x+2)}{(3x+1)(x+1)}\end{aligned}$$

The domain of *f* consists all real numbers except $-\frac{1}{3}$, and the domain of *g* consists of all real numbers except −1. Therefore, the domain of *f* + *g* consists of all real numbers except −1 and $-\frac{1}{3}$.

Answer: $\left(f + g\right)\left(x\right) = \frac{(5x+1)(x+2)}{(3x+1)(x+1)}$, where $x \neq -1, -\frac{1}{3}$

It is not always the case that the LCD is the product of the given denominators. Typically, the denominators are not relatively prime; thus determining the LCD requires some thought. Begin by factoring all denominators. The LCD is the product of all factors with the highest power.

Example 3

Given $f(x) = \frac{3x}{3x-1}$ and $g(x) = \frac{4-14x}{3x^2-4x+1}$, find $f - g$ and state the restrictions to the domain.

Solution:

To determine the LCD, factor the denominator of g.

$$\begin{aligned}(f-g)(x) &= f(x) - g(x)\\ &= \frac{3x}{3x-1} - \frac{4-14x}{3x^2-4x+1}\\ &= \frac{3x}{(3x-1)} - \frac{4-14x}{(3x-1)(x-1)}\end{aligned}$$

In this case the $\text{LCD} = (3x-1)(x-1)$. Multiply f by 1 in the form of $\frac{(x-1)}{(x-1)}$ to obtain equivalent algebraic fractions with a common denominator and then subtract.

$$\begin{aligned}&= \frac{3x}{(3x-1)} \cdot \frac{(x-1)}{(x-1)} - \frac{4-14x}{(3x-1)(x-1)}\\ &= \frac{3x(x-1) - 4 + 14x}{(3x-1)(x-1)}\\ &= \frac{3x^2 + 11x - 4}{(3x-1)(x-1)}\\ &= \frac{\cancel{(3x-1)}(x+4)}{\cancel{(3x-1)}(x-1)}\\ &= \frac{(x+4)}{(x-1)}\end{aligned}$$

The domain of f consists of all real numbers except $\frac{1}{3}$, and the domain of g consists of all real numbers except 1 and $\frac{1}{3}$ .Therefore, the domain of $f - g$ consists of all real numbers except 1 and $\frac{1}{3}$.

Answer: $\left(f - g\right)\ (x) = \frac{x+4}{x-1}$, where $x \neq \frac{1}{3}\ , 1$

Example 4

Simplify and state the restrictions: $\frac{-2x}{x+6} - \frac{3x}{6-x} - \frac{18(x-2)}{x^2-36}$.

Solution:

Begin by applying the opposite binomial property $6 - x = -\left(x - 6\right)$.

$$\begin{aligned}&\frac{-2x}{x+6} - \frac{3x}{6-x} - \frac{18\left(x-2\right)}{x^2-36}\\&= \frac{-2x}{\left(x+6\right)} - \frac{3x}{-1 \cdot \left(x-6\right)} - \frac{18\left(x-2\right)}{\left(x+6\right)\left(x-6\right)}\\&= \frac{-2x}{\left(x+6\right)} + \frac{3x}{\left(x-6\right)} - \frac{18\left(x-2\right)}{\left(x+6\right)\left(x-6\right)}\end{aligned}$$

Next, find equivalent fractions with the $LCD = \left(x+6\right)\left(x-6\right)$ and then simplify.

$$= \frac{-2x}{(x+6)} \cdot \frac{(x-6)}{(x-6)} + \frac{3x}{(x-6)} \cdot \frac{(x+6)}{(x+6)} - \frac{18(x-2)}{(x+6)(x-6)}$$

$$= \frac{-2x(x-6) + 3x(x+6) - 18(x-2)}{(x+6)(x-6)}$$

$$= \frac{-2x^2 + 12x + 3x^2 + 18x - 18x + 36}{(x+6)(x-6)}$$

$$= \frac{x^2 + 12x + 36}{(x+6)(x-6)}$$

$$= \frac{\cancel{(x+6)}(x+6)}{\cancel{(x+6)}(x-6)}$$

$$= \frac{x+6}{x-6}$$

Answer: $\frac{x+6}{x-6}$, where $x \neq \pm 6$

Try this! Simplify and state the restrictions: $\frac{x+1}{(x-1)^2} - \frac{2}{x^2-1} - \frac{4}{(x+1)(x-1)^2}$

Answer: $\frac{1}{x-1}$, where $x \neq \pm 1$

(click to see video)

Rational expressions are sometimes expressed using negative exponents. In this case, apply the rules for negative exponents before simplifying the expression.

Example 5

Simplify and state the restrictions: $5a^{-2} + (2a+5)^{-1}$.

Solution:

Recall that $x^{-n} = \frac{1}{x^n}$. Begin by rewriting the rational expressions with negative exponents as fractions.

$$5a^{-2} + (2a+5)^{-1} = \frac{5}{a^2} + \frac{1}{(2a+5)^1}$$

Then find the LCD and add.

$$\begin{aligned}
\frac{5}{a^2} + \frac{1}{(2a+5)^1} &= \frac{5}{a^2} \cdot \frac{(2a+5)}{(2a+5)} + \frac{1}{(2a+5)} \cdot \frac{a^2}{a^2} \\
&= \frac{5(2a+5)}{a^2(2a+5)} + \frac{a^2}{a^2(2a+5)} && \textit{Equivalent expressions with a com} \\
&= \frac{10a+25+a^2}{a^2(2a+5)} && \textit{Add.} \\
&= \frac{a^2+10a+25}{a^2(2a+5)} && \textit{Simplfiy.} \\
&= \frac{(a+5)(a+5)}{a^2(2a+5)}
\end{aligned}$$

Answer: $\frac{(a+5)^2}{a^2(2a+5)}$, where $a \neq -\frac{5}{2}, 0$

Simplifying Complex Rational Expressions

A **complex rational expression**[32] is defined as a rational expression that contains one or more rational expressions in the numerator or denominator or both. For example,

$$\frac{4 - \frac{12}{x} + \frac{9}{x^2}}{2 - \frac{5}{x} + \frac{3}{x^2}}$$

is a complex rational expression. We simplify a complex rational expression by finding an equivalent fraction where the numerator and denominator are polynomials. There are two methods for simplifying complex rational expressions, and we will outline the steps for both methods. For the sake of clarity, assume that variable expressions used as denominators are nonzero.

Method 1: Simplify Using Division

We begin our discussion on simplifying complex rational expressions using division. Before we can multiply by the reciprocal of the denominator, we must simplify the numerator and denominator separately. The goal is to first obtain single algebraic fractions in the numerator and the denominator. The steps for simplifying a complex algebraic fraction are illustrated in the following example.

32. A rational expression that contains one or more rational expressions in the numerator or denominator or both.

Example 6

Simplify: $\frac{4-\frac{12}{x}+\frac{9}{x^2}}{2-\frac{5}{x}+\frac{3}{x^2}}$.

Solution:

Step 1: Simplify the numerator and denominator to obtain a single algebraic fraction divided by another single algebraic fraction. In this example, find equivalent terms with a common denominator in both the numerator and denominator before adding and subtracting.

$$\frac{4-\frac{12}{x}+\frac{9}{x^2}}{2-\frac{5}{x}+\frac{3}{x^2}} = \frac{\frac{4}{1}\cdot\frac{x^2}{x^2}-\frac{12}{x}\cdot\frac{x}{x}+\frac{9}{x^2}}{\frac{2}{1}\cdot\frac{x^2}{x^2}-\frac{5}{x}\cdot\frac{x}{x}+\frac{3}{x^2}}$$

$$= \frac{\frac{4x^2}{x^2}-\frac{12x}{x^2}+\frac{9}{x^2}}{\frac{2x^2}{x^2}-\frac{5x}{x^2}+\frac{3}{x^2}} \quad \textit{Equivalent fractions with common denominat}$$

$$= \frac{\frac{4x^2-12x+9}{x^2}}{\frac{2x^2-5x+3}{x^2}} \quad \textit{Add the fractions in the numerator and denom}$$

At this point we have a single algebraic fraction divided by another single algebraic fraction.

Step 2: Multiply the numerator by the reciprocal of the denominator.

$$\frac{\frac{4x^2-12x+9}{x^2}}{\frac{2x^2-5x+3}{x^2}} = \frac{4x^2-12x+9}{x^2}\cdot\frac{x^2}{2x^2-5x+3}$$

Step 3: Factor all numerators and denominators completely.

$$= \frac{(2x-3)\,(2x-3)}{x^2} \cdot \frac{x^2}{(2x-3)\,(x-1)}$$

Step 4: Cancel all common factors.

$$\begin{aligned} &= \frac{\cancel{(2x-3)}\,(2x-3)}{\cancel{x^2}} \cdot \frac{\cancel{x^2}}{\cancel{(2x-3)}\,(x-1)} \\ &= \frac{2x-3}{x-1} \end{aligned}$$

Answer: $\frac{2x-3}{x-1}$

Example 7

Simplify: $\frac{\frac{2x}{x-1}+\frac{7}{x+3}}{\frac{2x}{x-1}-\frac{5}{x-3}}$.

Solution:

Obtain a single algebraic fraction in the numerator and in the denominator.

$$\begin{aligned}\frac{\frac{2x}{x-1}+\frac{7}{x+3}}{\frac{2x}{x-1}-\frac{5}{x-3}} &= \frac{\frac{2x}{x-1}\cdot\frac{(x+3)}{(x+3)}+\frac{7}{x+3}\cdot\frac{(x-1)}{(x-1)}}{\frac{2x}{x-1}\cdot\frac{(x-3)}{(x-3)}-\frac{5}{x-3}\cdot\frac{(x-1)}{(x-1)}} \\ &= \frac{\frac{2x(x+3)+7(x-1)}{(x-1)(x+3)}}{\frac{2x(x-3)-5(x-1)}{(x-1)(x-3)}} \\ &= \frac{\frac{2x^2+6x+7x-7}{(x-1)(x+3)}}{\frac{2x^2-6x-5x+5}{(x-1)(x-3)}} \\ &= \frac{\frac{2x^2+13x-7}{(x-1)(x+3)}}{\frac{2x^2-11x+5}{(x-1)(x-3)}}\end{aligned}$$

Next, multiply the numerator by the reciprocal of the denominator, factor, and then cancel.

$$
\begin{aligned}
&= \frac{2x^2 + 13x - 7}{(x-1)(x+3)} \cdot \frac{(x-1)(x-3)}{2x^2 - 11x + 5} \\
&= \frac{\cancel{(2x-1)}(x+7)}{\cancel{(x-1)}(x+3)} \cdot \frac{\cancel{(x-1)}(x-3)}{\cancel{(2x-1)}\left(x-5\right)} \\
&= \frac{(x+7)(x-3)}{(x+3)\left(x-5\right)}
\end{aligned}
$$

Answer: $\frac{(x+7)(x-3)}{(x+3)\left(x-5\right)}$

Try this! Simplify using division: $\frac{\frac{1}{y^2} - \frac{1}{x^2}}{\frac{1}{y} + \frac{1}{x}}$.

Answer: $\frac{x-y}{xy}$

(click to see video)

Sometimes complex rational expressions are expressed using negative exponents.

Example 8

Simplify: $\frac{2y^{-1}-x^{-1}}{x^{-2}-4y^{-2}}$.

Solution:

We begin by rewriting the expression without negative exponents.

$$\frac{2y^{-1}-x^{-1}}{x^{-2}-4y^{-2}}=\frac{\frac{2}{y}-\frac{1}{x}}{\frac{1}{x^2}-\frac{4}{y^2}}$$

Obtain single algebraic fractions in the numerator and denominator and then multiply by the reciprocal of the denominator.

$$\begin{aligned}\frac{\frac{2}{y}-\frac{1}{x}}{\frac{1}{x^2}-\frac{4}{y^2}}&=\frac{\frac{2x-y}{xy}}{\frac{y^2-4x^2}{x^2y^2}}\\&=\frac{2x-y}{xy}\cdot\frac{x^2y^2}{y^2-4x^2}\\&=\frac{2x-y}{xy}\cdot\frac{x^2y^2}{(y-2x)\,(y+2x)}\end{aligned}$$

Apply the opposite binomial property $(y-2x)=-(2x-y)$ and then cancel.

$$
\begin{aligned}
&= \frac{\cancel{(2x-y)}}{\cancel{x}\,\cancel{y}} \cdot \frac{\overset{x}{\cancel{x^2}}\,\overset{y}{\cancel{y^2}}}{-\cancel{(2x-y)}\,(y+2x)} \\
&= -\frac{xy}{y+2x}
\end{aligned}
$$

Answer: $-\frac{xy}{y+2x}$

Method 2: Simplify Using the LCD

An alternative method for simplifying complex rational expressions involves clearing the fractions by multiplying the expression by a special form of 1. In this method, multiply the numerator and denominator by the least common denominator (LCD) of all given fractions.

Example 9

Simplify: $\frac{4-\frac{12}{x}+\frac{9}{x^2}}{2-\frac{5}{x}+\frac{3}{x^2}}$.

Solution:

Step 1: Determine the LCD of all the fractions in the numerator and denominator. In this case, the denominators of the given fractions are 1, x, and x^2. Therefore, the LCD is x^2.

Step 2: Multiply the numerator and denominator by the LCD. This step should clear the fractions in both the numerator and denominator.

$$\begin{aligned}\frac{4-\frac{12}{x}+\frac{9}{x^2}}{2-\frac{5}{x}+\frac{3}{x^2}} &= \frac{\left(4-\frac{12}{x}+\frac{9}{x^2}\right)\cdot x^2}{\left(2-\frac{5}{x}+\frac{3}{x^2}\right)\cdot x^2} && \textit{Multiply numerator and denominator} \\ &= \frac{4\cdot x^2-\frac{12}{x}\cdot x^2+\frac{9}{x^2}\cdot x^2}{2\cdot x^2-\frac{5}{x}\cdot x^2+\frac{3}{x^2}\cdot x^2} && \textit{Distribute and then cancel.} \\ &= \frac{4x^2-12x+9}{2x^2-5x+3}\end{aligned}$$

This leaves us with a single algebraic fraction with a polynomial in the numerator and in the denominator.

Step 3: Factor the numerator and denominator completely.

$$= \frac{4x^2 - 12x + 9}{2x^2 - 5x + 3}$$
$$= \frac{(2x-3)\ (2x-3)}{(x-1)\ (2x-3)}$$

Step 4: Cancel all common factors.

$$= \frac{(2x-3)\ \cancel{(2x-3)}}{(x-1)\ \cancel{(2x-3)}}$$
$$= \frac{2x-3}{x-1}$$

Note: This was the same problem presented in Example 6 and the results here are the same. It is worth taking the time to compare the steps involved using both methods on the same problem.

Answer: $\frac{2x-3}{x-1}$

It is important to point out that multiplying the numerator and denominator by the same nonzero factor is equivalent to multiplying by 1 and does not change the problem.

Try this! Simplify using the LCD: $\frac{\frac{1}{y^2} - \frac{1}{x^2}}{\frac{1}{y} + \frac{1}{x}}$.

Answer: $\frac{x-y}{xy}$

(click to see video)

KEY TAKEAWAYS

- Adding and subtracting rational expressions is similar to adding and subtracting fractions. A common denominator is required. If the denominators are the same, then we can add or subtract the numerators and write the result over the common denominator.
- The set of restrictions to the domain of a sum or difference of rational functions consists of the restrictions to the domains of each function.
- Complex rational expressions can be simplified into equivalent expressions with a polynomial numerator and polynomial denominator. They are reduced to lowest terms if the numerator and denominator are polynomials that share no common factors other than 1.
- One method of simplifying a complex rational expression requires us to first write the numerator and denominator as a single algebraic fraction. Then multiply the numerator by the reciprocal of the denominator and simplify the result.
- Another method for simplifying a complex rational expression requires that we multiply it by a special form of 1. Multiply the numerator and denominator by the LCD of all the denominators as a means to clear the fractions. After doing this, simplify the remaining rational expression.

TOPIC EXERCISES

PART A: ADDING AND SUBTRACTING RATIONAL FUNCTIONS

State the restrictions and simplify.

1. $\frac{3x}{3x+4}+\frac{2}{3x+4}$
2. $\frac{3x}{2x-1}-\frac{2x+1}{2x-1}$
3. $\frac{x-2}{2x^2-11x-6}+\frac{x+3}{2x^2-11x-6}$
4. $\frac{4x-1}{3x^2+2x-5}-\frac{x-6}{3x^2+2x-5}$
5. $\frac{1}{x}-2x$
6. $\frac{4}{x^3}-\frac{1}{x}$
7. $\frac{1}{x-1}+5$
8. $\frac{1}{x+7}-1$
9. $\frac{1}{x-2}-\frac{1}{3x+4}$
10. $\frac{2}{5x-2}+\frac{x}{x+3}$
11. $\frac{1}{x^2}+\frac{1}{x-2}$
12. $\frac{2x}{x}+\frac{2}{x-2}$
13. $\frac{3x-7}{x(x-7)}+\frac{1}{7-x}$
14. $\frac{2}{8-x}+\frac{3x^2-1}{x^2(x-8)}$
15. $\frac{x-1}{x^2-25}-\frac{2}{x^2-10x+25}$
16. $\frac{x+1}{2x^2+5x-3}-\frac{x}{4x^2-1}$

17. $\frac{x}{x^2 + 4x} - \frac{2}{x^2 + 8x + 16}$

18. $\frac{2x - 1}{4x^2 + 8x - 5} - \frac{3}{4x^2 + 20x + 25}$

19. $\frac{5 - x}{7x + x^2} - \frac{x + 2}{49 - x^2}$

20. $\frac{2x}{4x^2 + x} - \frac{x + 1}{8x^2 + 6x + 1}$

21. $\frac{x - 1}{2x^2 - 7x - 4} + \frac{2x - 1}{x^2 - 5x + 4}$

22. $\frac{2(x + 3)}{3x^2 - 5x - 2} + \frac{4 - x}{3x^2 + 10x + 3}$

23. $\frac{x^2}{4 + 2x^2} - \frac{2}{x^4 + 2x^2}$

24. $\frac{3x}{4x^4 + 6x^3} - \frac{2x^2}{6x^3 + 9x^2}$

25. $\frac{3x^2 - 12}{x^4 - 8x^2 + 16} - \frac{x^2 + 2}{4 - x^2}$

26. $\frac{x^2}{2x^2 + 1} + \frac{6x^2 - 24}{2x^4 - 7x^2 - 4}$

Given f and g, simplify the sum $f + g$ and difference $f - g$. Also, state the domain using interval notation.

27. $f(x) = \frac{1}{x}, g(x) = \frac{5}{x^2}$

28. $f(x) = \frac{1}{x + 2}, g(x) = \frac{2}{x - 1}$

29. $f(x) = \frac{x - 2}{x + 2}, g(x) = \frac{x + 2}{x - 2}$

30. $f(x) = \frac{x}{2x - 1}, g(x) = \frac{2x}{2x + 1}$

31. $f(x) = \frac{6}{3x^2 + x}, g(x) = \frac{18}{9x^2 + 6x + 1}$

32. $f(x) = \frac{x - 1}{x^2 - 8x + 16}, g(x) = \frac{x - 2}{x^2 - 4x}$

33. $f(x) = \frac{x}{x^2 - 25}, g(x) = \frac{x - 1}{x^2 - 4x - 5}$

34. $f(x) = \frac{2x - 3}{x^2 - 4}, g(x) = \frac{x}{2x^2 + 3x - 2}$

35. $f(x) = \frac{1}{3x^2 - x - 2}, g(x) = -\frac{1}{4x^2 - 3x - 1}$

36. $f(x) = \frac{6}{6x^2 + 13x - 5}, g(x) = -\frac{2}{2x^2 + x - 10}$

State the restrictions and simplify.

37. $1 + \frac{3}{x} - \frac{5x - 1}{x^2}$

38. $4 + \frac{2}{x} - \frac{6x - 1}{x^2}$

39. $\frac{2x}{x - 8} - \frac{1}{3x + 1} - \frac{2x + 9}{3x^2 - 23x - 8}$

40. $\frac{4x}{x - 2} - \frac{10}{3x + 1} - \frac{19x + 18}{3x^2 - 5x - 2}$

41. $\frac{1}{x - 1} + \frac{1}{(x - 1)^2} - \frac{1}{x^2 - 1}$

42. $\frac{1}{x - 2} - \frac{1}{x^2 - 4} + \frac{1}{(x - 2)^2}$

43. $\frac{2x + 1}{x - 1} - \frac{3x}{2x^2 - 3x + 1} + \frac{x + 1}{x - 2x^2}$

44. $\frac{5x^2}{2x^2 + 2x} - \frac{x^2 - 4x}{x^2 - 2x} + \frac{4 + 2x^2}{4 + 2x - 2x^2}$

45. $\frac{x + 2}{2x(3x - 2)} + \frac{4x}{(x - 2)(3x - 2)} - \frac{3x + 2}{2x(x - 2)}$

46. $\frac{10x}{x\left(x - 5\right)} - \frac{2x^2}{\left(2x - 5\right)\left(x - 5\right)} - \frac{5x}{x\left(2x - 5\right)}$

Simplify the given algebraic expressions. Assume all variable expressions in the denominator are nonzero.

47. $x^{-2} + y^{-2}$

48. $x^{-2} + \left(2y\right)^{-2}$

49. $2x^{-1} + y^{-2}$

50. $x^{-2} - 4y^{-1}$

51. $16x^{-2} + y^2$

52. $xy^{-1} - yx^{-1}$

53. $3(x+y)^{-1} + x^{-2}$

54. $2(x-y)^{-2} - (x-y)^{-1}$

55. $a^{-2} - (a+b)^{-1}$

56. $(a-b)^{-1} - (a+b)^{-1}$

57. $x^{-n} + y^{-n}$

58. $xy^{-n} + yx^{-n}$

PART B: SIMPLIFYING COMPLEX RATIONAL EXPRESSIONS

Simplify. Assume all variable expressions in the denominators are nonzero.

59. $$\frac{\frac{75x^2}{(x-3)^2}}{\frac{25x^3}{x-3}}$$

60. $$\frac{\frac{x+5}{36x^3}}{\frac{(x+5)^3}{9x^2}}$$

61. $$\frac{\frac{x^2-36}{32x^5}}{\frac{x-6}{4x^3}}$$

62. $$\frac{\frac{x-8}{56x^2}}{\frac{x^2-64}{7x^3}}$$

63. $$\frac{\frac{5x+1}{2x^2+x-10}}{\frac{25x^2+10x+1}{4x^2-25}}$$

64. $$\frac{\frac{4x^2-27x-7}{4x^2-1}}{\frac{x-7}{6x^2-x-1}}$$

65. $\dfrac{\frac{x^2-4x-5}{2x^2+3x+1}}{\frac{x^2-10x+25}{2x^2+7x+3}}$

66. $\dfrac{\frac{5x^2+9x-2}{x^2+4x+4}}{\frac{10x^2+3x-1}{4x^2+7x-2}}$

67. $\dfrac{x^2}{\frac{1}{5}-\frac{3}{x}}$

68. $\dfrac{\frac{4}{x}-3}{2x^2}$

69. $\dfrac{\frac{1}{3}-\frac{1}{x}}{\frac{1}{9}-\frac{1}{x^2}}$

70. $\dfrac{\frac{2}{5}+\frac{1}{x}}{\frac{4}{25}-\frac{1}{x^2}}$

71. $\dfrac{\frac{1}{y^2}-36}{6-\frac{1}{y}}$

72. $\dfrac{\frac{1}{5}-\frac{1}{y}}{\frac{1}{y^2}-\frac{1}{25}}$

73. $\dfrac{1-\frac{6}{x}+\frac{8}{x^2}}{3-\frac{5}{x}-\frac{2}{x^2}}$

74. $\dfrac{2+\frac{13}{x}-\frac{7}{x^2}}{3+\frac{1}{x}-\frac{10}{x^2}}$

75. $\dfrac{9-\frac{12}{x}+\frac{4}{x^2}}{9-\frac{4}{x^2}}$

76. $\dfrac{4-\frac{25}{x^2}}{4-\frac{8}{x}-\frac{5}{x^2}}$

77. $\dfrac{\frac{1}{x}+\frac{5}{3x-1}}{\frac{2}{3x-1}-\frac{1}{x}}$

78. $\dfrac{\frac{2}{x-5} - \frac{1}{x}}{\frac{1}{x} - \frac{3}{x-5}}$

79. $\dfrac{\frac{1}{x+1} + \frac{2}{x-2}}{\frac{2}{x-3} - \frac{1}{x-2}}$

80. $\dfrac{\frac{4}{x+5} - \frac{1}{x-3}}{\frac{3}{x-3} + \frac{1}{2x-1}}$

81. $\dfrac{\frac{x-1}{3x-1} - \frac{1}{x+1}}{\frac{x-1}{x+1} - \frac{2}{x+1}}$

82. $\dfrac{\frac{x+1}{3x+5} - \frac{1}{x+3}}{\frac{2}{x+3} - \frac{x+1}{x+3}}$

83. $\dfrac{\frac{2x+3}{2x-3} + \frac{2x-3}{2x+3}}{\frac{2x+3}{2x-3} - \frac{2x-3}{2x+3}}$

84. $\dfrac{\frac{x-1}{x+1} - \frac{x+1}{x-1}}{\frac{x+1}{x-1} - \frac{x-1}{x+1}}$

85. $\dfrac{\frac{1}{2x+5} - \frac{1}{2x-5} + \frac{4x}{4x^2-25}}{\frac{1}{2x+5} + \frac{1}{2x-5} + \frac{4x}{4x^2-25}}$

86. $\dfrac{\frac{1}{3x-1} + \frac{1}{3x+1}}{\frac{3x}{3x-1} - \frac{1}{3x+1} - \frac{6x}{9x^2-1}}$

87. $\dfrac{1}{1 + \frac{1}{1+\frac{1}{x}}}$

88. $\dfrac{\frac{1}{x}}{1 - \frac{1}{1+\frac{1}{x}}}$

89. $\dfrac{\frac{1}{y} - \frac{1}{x}}{\frac{1}{y^2} - \frac{1}{x^2}}$

90. $\dfrac{\frac{2}{y} + \frac{1}{x}}{\frac{4}{y^2} - \frac{1}{x^2}}$

91. $\dfrac{\frac{1}{25y^2} - \frac{1}{x^2}}{\frac{1}{x} - \frac{1}{5y}}$

92. $\dfrac{16y^2 - \frac{1}{x^2}}{\frac{1}{x} - 4y}$

93. $\dfrac{\frac{1}{b} + \frac{1}{a}}{\frac{1}{b^3} + \frac{1}{a^3}}$

94. $\dfrac{\frac{1}{a} - \frac{1}{b}}{\frac{1}{b^3} - \frac{1}{a^3}}$

95. $\dfrac{\frac{x}{y} - \frac{y}{x}}{\frac{1}{y^2} - \frac{2}{xy} + \frac{1}{x^2}}$

96. $\dfrac{\frac{2}{y} - \frac{5}{x}}{4x - \frac{25y^2}{x}}$

97. $\dfrac{x^{-1} + y^{-1}}{y^{-2} - x^{-2}}$

98. $\dfrac{y^{-2} - 25x^{-2}}{5x^{-1} - y^{-1}}$

99. $\dfrac{1 - x^{-1}}{x - x^{-1}}$

100. $\dfrac{16 - x^{-2}}{x^{-1} - 4}$

101. $\dfrac{1 - 4x^{-1} - 21x^{-2}}{1 - 2x^{-1} - 15x^{-2}}$

102. $\dfrac{x^{-1} - 4\left(3x^2\right)^{-1}}{3 - 8x^{-1} + 16\left(3x^2\right)^{-1}}$

103. $\dfrac{(x-3)^{-1} + 2x^{-1}}{x^{-1} - 3(x-3)^{-1}}$

104. $\dfrac{\left(4x-5\right)^{-1} + x^{-2}}{x^{-2} + (3x-10)^{-1}}$

105. Given $f(x) = \frac{1}{x}$, simplify $\frac{f(b)-f(a)}{b-a}$.

106. Given $f(x) = \frac{1}{x-1}$, simplify $\frac{f(b)-f(a)}{b-a}$.

107. Given $f(x) = \frac{1}{x}$, simplify the difference quotient $\frac{f(x+h)-f(x)}{h}$.

108. Given $f(x) = \frac{1}{x} + 1$, simplify the difference quotient $\frac{f(x+h)-f(x)}{h}$.

PART C: DISCUSSION BOARD

109. Explain why the domain of a sum of rational functions is the same as the domain of the difference of those functions.

110. Two methods for simplifying complex rational expressions have been presented in this section. Which of the two methods do you feel is more efficient, and why?

ANSWERS

1. $\frac{3x+2}{3x+4}; x \neq -\frac{4}{3}$

3. $\frac{1}{x-6}; x \neq -\frac{1}{2}, 6$

5. $\frac{1-2x^2}{x}; x \neq 0$

7. $\frac{5x-4}{x-1}; x \neq 1$

9. $\frac{2(x+3)}{(x-2)(3x+4)}; x \neq -\frac{4}{3}, 2$

11. $\frac{(x-1)(x+2)}{x^2(x-2)}; x \neq 0, 2$

13. $\frac{2x-7}{x(x-7)}; x \neq 0, 7$

15. $\frac{x^2-8x-5}{(x+5)(x-5)^2}; x \neq \pm 5$

17. $\frac{x+2}{(x+4)^2}; x \neq 0, -4$

19. $\frac{7(5-2x)}{x(7+x)(7-x)}; x \neq -7, 0, 7$

21. $\frac{x(5x-2)}{(x-4)(x-1)(2x+1)}; x \neq -\frac{1}{2}, 1, 4$

23. $\frac{x^2-2}{2x^2}; x \neq 0$

25. $\frac{x^2+5}{(x+2)(x-2)}; x \neq \pm 2$

27. $(f+g)(x) = \frac{x+5}{x^2}$; $(f-g)(x) = \frac{x-5}{x^2}$; Domain: $(-\infty, 0) \cup (0, \infty)$

29. $(f+g)(x) = \frac{2(x^2+4)}{(x+2)(x-2)}$; $(f-g)(x) = -\frac{8x}{(x+2)(x-2)}$; Domain: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

31. $(f+g)(x) = \frac{6(6x+1)}{x(3x+1)^2}$; $(f-g)(x) = \frac{6}{x(3x+1)^2}$; Domain: $\left(-\infty, -\frac{1}{3}\right) \cup \left(-\frac{1}{3}, 0\right) \cup (0, \infty)$

33. $(f+g)(x) = \frac{2x^2+5x-5}{(x+1)(x+5)(x-5)}$; $(f-g)(x) = -\frac{3x-5}{(x+1)(x+5)(x-5)}$;
Domain: $(-\infty, -5) \cup (-5, -1) \cup (-1, 5) \cup (5, \infty)$

35. $(f+g)(x) = \frac{1}{(3x+2)(4x+1)}$; $(f-g)(x) = -\frac{7x+3}{(x-1)(3x+2)(4x+1)}$;
Domain: $\left(-\infty, -\frac{2}{3}\right) \cup \left(-\frac{2}{3}, \frac{1}{4}\right) \cup \left(\frac{1}{4}, 1\right) \cup (1, \infty)$

37. $\frac{(x-1)^2}{x^2}; x \neq 0$

39. $\frac{2x-1}{x-8}; x \neq -\frac{1}{3}, 8$

41. $\frac{x^2+1}{(x-1)^2(x+1)}; x \neq \pm 1$

43. $\frac{2x+1}{x}; x \neq 0, \frac{1}{2}, 1$

45. $0; x \neq 0, \frac{2}{3}, 2$

47. $\frac{y^2+x^2}{x^2y^2}$

49. $\frac{x+2y^2}{xy^2}$

51. $\frac{x^2y^2+16}{x^2}$

53. $\frac{3x^2+x+y}{x^2(x+y)}$

55. $\frac{a+b-a^2}{a^2(a+b)}$

57. $\frac{x^n+y^n}{x^ny^n}$

59. $\frac{3}{x(x-3)}$

61. $\frac{x+6}{8x^2}$

63. $\frac{2x-5}{(x-2)(5x+1)}$

65. $\frac{x+3}{x-5}$

67. $\dfrac{5x^3}{x-15}$

69. $\dfrac{3x}{x+3}$

71. $-\dfrac{6y+1}{y}$

73. $\dfrac{x-4}{3x+1}$

75. $\dfrac{3x-2}{3x+2}$

77. $-\dfrac{8x-1}{x-1}$

79. $\dfrac{3x(x-3)}{(x+1)(x-1)}$

81. $\dfrac{x}{3x-1}$

83. $\dfrac{4x^2+9}{12x}$

85. $\dfrac{2x-5}{4x}$

87. $\dfrac{x+1}{2x+1}$

89. $\dfrac{xy}{x+y}$

91. $-\dfrac{x+5y}{5xy}$

93. $\dfrac{a^2b^2}{a^2-ab+b^2}$

95. $\dfrac{xy(x+y)}{x-y}$

97. $\dfrac{xy}{x-y}$

99. $\dfrac{1}{x+1}$

101. $\dfrac{x-7}{x-5}$

103. $-\dfrac{3(x-2)}{2x+3}$

105. $-\frac{1}{ab}$

107. $-\frac{1}{x(x+h)}$

109. Answer may vary

4.7 Solving Rational Equations

LEARNING OBJECTIVES

1. Solve rational equations.
2. Solve literal equations, or formulas, involving rational expressions.
3. Solve applications involving the reciprocal of unknowns.

Solving Rational Equations

A **rational equation**[33] is an equation containing at least one rational expression. Rational expressions typically contain a variable in the denominator. For this reason, we will take care to ensure that the denominator is not 0 by making note of restrictions and checking our solutions. Solving rational equations involves clearing fractions by multiplying both sides of the equation by the least common denominator (LCD).

33. An equation containing at least one rational expression.

Example 1

Solve: $\frac{1}{x} + \frac{2}{x^2} = \frac{x+9}{2x^2}$.

Solution:

We first make a note of the restriction on x, $x \neq 0$. We then multiply both sides by the LCD, which in this case equals $2x^2$.

$$\begin{aligned} 2x^2 \cdot \left(\frac{1}{x} + \frac{2}{x^2}\right) &= 2x^2 \cdot \left(\frac{x+9}{2x^2}\right) && \textit{Multiply both sides by the LCD.} \\ 2x^2 \cdot \frac{1}{x} + 2x^2 \cdot \frac{2}{x^2} &= 2x^2 \cdot \frac{x+9}{2x^2} && \textit{Distribute.} \\ 2x + 4 &= x + 9 && \textit{Simplify and then solve.} \\ x &= 5 \end{aligned}$$

Check your answer. Substitute $x = 5$ into the original equation and see if you obtain a true statement.

$$\begin{aligned} \frac{1}{x} + \frac{2}{x^2} &= \frac{x+9}{2x^2} && \textit{Original equation} \\ \frac{1}{5} + \frac{2}{5^2} &= \frac{5+9}{2(5)^2} && \textit{Check } x = 5. \\ \frac{1}{5} + \frac{2}{25} &= \frac{14}{2 \cdot 25} \\ \frac{5}{25} + \frac{2}{25} &= \frac{7}{25} \\ \frac{7}{25} &= \frac{7}{25} \quad ✓ \end{aligned}$$

Answer: The solution is 5.

After multiplying both sides of the previous example by the LCD, we were left with a linear equation to solve. This is not always the case; sometimes we will be left with quadratic equation.

Example 2

Solve: $\frac{3(x+2)}{x-4} - \frac{x+4}{x-2} = \frac{x-2}{x-4}$.

Solution:

In this example, there are two restrictions, $x \neq 4$ and $x \neq 2$. Begin by multiplying both sides by the LCD, $(x-2)(x-4)$.

$$\begin{aligned}
(x-2)(x-4)\cdot\left(\frac{3(x+2)}{x-4} - \frac{x+4}{x-2}\right) &= (x-2)(x-4)\cdot\left(\frac{x-2}{x-4}\right.\\
(x-2)\cancel{(x-4)}\cdot\frac{3(x+2)}{\cancel{x-4}} - \cancel{(x-2)}(x-4)\cdot\frac{x+4}{\cancel{x-2}} &= (x-2)\cancel{(x-4)}\cdot\frac{x-2}{\cancel{x-4}}\\
3(x+2)(x-2) - (x+4)(x-4) &= (x-2)(x-2)\\
3\left(x^2-4\right) - \left(x^2-16\right) &= x^2-2x-2x+4\\
3x^2-12-x^2+16 &= x^2-4x+4\\
2x^2+4 &= x^2-4x+4
\end{aligned}$$

After distributing and simplifying both sides of the equation, a quadratic equation remains. To solve, rewrite the quadratic equation in standard form, factor, and then set each factor equal to 0.

$$\begin{aligned}
2x^2+4 &= x^2-4x+4\\
x^2+4x &= 0\\
x(x+4) &= 0
\end{aligned}$$

$$\begin{aligned}
x=0 \text{ or } x+4 &= 0\\
x &= -4
\end{aligned}$$

Check to see if these values solve the original equation.

$$\frac{3(x+2)}{x-4} - \frac{x+4}{x-2} = \frac{x-2}{x-4}$$

Check $x = 0$	*Check* $x = -4$
$\frac{3(0+2)}{0-4} - \frac{0+4}{0-2} = \frac{0-2}{0-4}$ $\frac{6}{-4} - \frac{4}{-2} = \frac{-2}{-4}$ $-\frac{3}{2} + 2 = \frac{1}{2}$ $-\frac{3}{2} + \frac{4}{2} = \frac{1}{2}$ $\frac{1}{2} = \frac{1}{2}$ ✓	$\frac{3(-4+2)}{-4-4} - \frac{-4+4}{-4-2} = \frac{-4-2}{-4-4}$ $\frac{3(-2)}{-8} - \frac{0}{-6} = \frac{-6}{-8}$ $\frac{-6}{-8} - 0 = \frac{3}{4}$ $\frac{3}{4} = \frac{3}{4}$ ✓

Answer: The solutions are 0 and -4.

Up to this point, all of the possible solutions have solved the original equation. However, this may not always be the case. Multiplying both sides of an equation by variable factors may lead to **extraneous solutions**[34], which are solutions that do not solve the original equation. A complete list of steps for solving a rational equation is outlined in the following example.

34. A solution that does not solve the original equation.

Example 3

Solve: $\frac{2x}{3x+1} = \frac{1}{x-5} - \frac{4(x-1)}{3x^2-14x-5}$.

Solution:

Step 1: Factor all denominators and determine the LCD.

$$\frac{2x}{3x+1} = \frac{1}{x-5} - \frac{4(x-1)}{3x^2-14x-5}$$
$$\frac{2x}{(3x+1)} = \frac{1}{(x-5)} - \frac{4(x-1)}{(3x+1)(x-5)}$$

The LCD is $(3x+1)(x-5)$.

Step 2: Identify the restrictions. In this case, $x \neq -\frac{1}{3}$ and $x \neq 5$.

Step 3: Multiply both sides of the equation by the LCD. Distribute carefully and then simplify.

$$(3x+1)(x-5) \cdot \frac{2x}{(3x+1)} = (3x+1)(x-5) \cdot \left(\frac{1}{(x-5)} - \frac{4(x-1)}{(3x+1)(x-5)}\right.$$
$$\cancel{(3x+1)}(x-5) \cdot \frac{2x}{\cancel{(3x+1)}} = (3x+1)\cancel{(x-5)} \cdot \frac{1}{\cancel{(x-5)}} - \cancel{(3x+1)}\cancel{(x-5}$$
$$2x(x-5) = (3x+1) - 4(x-1)$$

Step 4: Solve the resulting equation. Here the result is a quadratic equation. Rewrite it in standard form, factor, and then set each factor equal to 0.

$$\begin{aligned} 2x(x-5) &= (3x+1) - 4(x-1) \\ 2x^2 - 10x &= 3x + 1 - 4x + 4 \\ 2x^2 - 10x &= -x + 5 \\ 2x^2 - 9x - 5 &= 0 \\ (2x+1)(x-5) &= 0 \end{aligned}$$

$$\begin{aligned} 2x + 1 &= 0 \quad \text{or} \quad & x - 5 &= 0 \\ 2x &= -1 & x &= 5 \\ x &= -\frac{1}{2} & & \end{aligned}$$

Step 5: Check for extraneous solutions. Always substitute into the original equation, or the factored equivalent. In this case, choose the factored equivalent to check:

$$\frac{2x}{(3x+1)} = \frac{1}{\left(x-5\right)} - \frac{4\,(x-1)}{(3x+1)\,\left(x-5\right)}$$

Check $x = -\frac{1}{2}$	Check $x = 5$
$\frac{2\left(-\frac{1}{2}\right)}{\left(3\left(-\frac{1}{2}\right)+1\right)} = \frac{1}{\left(\left(-\frac{1}{2}\right)-5\right)} - \frac{4\left(\left(-\frac{1}{2}\right)-1\right)}{\left(3\left(-\frac{1}{2}\right)+1\right)\left(\left(-\frac{1}{2}\right)-5\right)}$ $\frac{-1}{\left(-\frac{1}{2}\right)} = \frac{1}{\left(-\frac{11}{2}\right)} - \frac{4\left(-\frac{3}{2}\right)}{\left(-\frac{1}{2}\right)\left(-\frac{11}{2}\right)}$ $2 = -\frac{2}{11} - \frac{-6}{\left(\frac{11}{4}\right)}$ $2 = -\frac{2}{11} + \frac{24}{11}$ $2 = \frac{22}{11}$ $2 = 2$ ✓	$\frac{2\cdot 5}{(3\cdot 5+1)} = \frac{}{(5}$ $\frac{10}{16} = \frac{1}{0}$ $\frac{10}{16} = \frac{1}{0}$
	Undefi

Here 5 is an extraneous solution and is not included in the solution set. It is important to note that 5 is a restriction.

Answer: The solution is $-\frac{1}{2}$.

If this process produces a solution that happens to be a restriction, then disregard it as a solution.

Try this! Solve: $\frac{4(x-3)}{36-x^2} = \frac{1}{6-x} + \frac{2x}{6+x}$

Answer: $-\frac{3}{2}$

(click to see video)

Sometimes all potential solutions are extraneous, in which case we say that there is no solution to the original equation. In the next two examples, we demonstrate two ways in which rational equation can have no solutions.

Example 4

Solve: $1 + \frac{5x+22}{x^2+3x-4} = \frac{x+4}{x-1}$.

Solution:

To identify the LCD, first factor the denominators.

$$\begin{aligned} 1 + \frac{5x+22}{x^2+3x-4} &= \frac{x+4}{x-1} \\ 1 + \frac{5x+22}{(x+4)\ (x-1)} &= \frac{x+4}{(x-1)} \end{aligned}$$

Multiply both sides by the LCD, $(x+4)\ (x-1)$, distributing carefully.

$$\begin{aligned} {\color{blue}(x+4)(x-1)} \cdot \left(1 + \frac{5x+22}{(x+4)(x-1)}\right) &= {\color{blue}(x+4)(x-1)} \cdot \frac{x+4}{(x-1)} \\ {\color{blue}(x+4)(x-1)} \cdot 1 + {\color{blue}(x+4)(x-1)} \cdot \frac{(5x+22)}{(x+4)(x-1)} &= {\color{blue}(x+4)(x-1)} \cdot \frac{(x+4)}{(x-1)} \\ (x+4)(x-1) + (5x+22) &= (x+4)(x+4) \\ x^2 - x + 4x - 4 + 5x + 22 &= x^2 + 4x + 4x + 16 \\ x^2 + 8x + 18 &= x^2 + 8x + 16 \\ 18 &= 16 \quad {\color{red}\mathit{False}} \end{aligned}$$

The equation is a contradiction and thus has no solution.

Answer: No solution, $\varnothing$

Example 5

Solve: $\frac{3x}{2x-3} - \frac{3(4x+3)}{4x^2-9} = \frac{x}{2x+3}$

Solution:

First, factor the denominators.

$$\frac{3x}{(2x-3)} - \frac{3\,(4x+3)}{(2x+3)\,(2x-3)} = \frac{x}{(2x+3)}$$

Take note that the restrictions on the domain are $x \neq \pm\,\frac{3}{2}$. To clear the fractions, multiply by the LCD, $(2x+3)\,(2x-3)$.

$$\begin{aligned}
\frac{3x \cdot \color{blue}{(2x+3)(2x-3)}}{(2x-3)} - \frac{3(4x+3) \cdot \color{blue}{(2x+3)(2x-3)}}{(2x+3)(2x-3)} &= \frac{x \cdot \color{blue}{(2x+3)(2x-3)}}{(2x+3)} \\
3x(2x+3) - 3(4x+3) &= x(2x-3) \\
6x^2 + 9x - 12x - 9 &= 2x^2 - 3x \\
6x^2 - 3x - 9 &= 2x^2 - 3x \\
4x^2 - 9 &= 0 \\
(2x+3)(2x-3) &= 0
\end{aligned}$$

$$\begin{aligned}
2x + 3 &= 0 \quad &\text{or } 2x - 3 &= 0 \\
2x &= -3 & 2x &= 3 \\
x &= -\frac{3}{2} & x &= \frac{3}{2}
\end{aligned}$$

Both of these values are restrictions of the original equation; hence both are extraneous.

Answer: No solution, Ø

It is important to point out that this technique for clearing algebraic fractions only works for equations. Do not try to clear algebraic fractions when simplifying expressions. As a reminder, an example of each is provided below.

Expression	Equation
$\frac{1}{x} + \frac{x}{2x+1}$	$\frac{1}{x} + \frac{x}{2x+1} = 0$

Expressions are to be simplified and equations are to be solved. If we multiply the expression by the LCD, $x\,(2x + 1)$, we obtain another expression that is not equivalent.

Incorrect	Correct
$\frac{1}{x} + \frac{x}{2x+1}$ $\neq x\,(2x + 1) \cdot \left(\frac{1}{x} + \frac{x}{2x+1}\right)$ $= 2x + 1 + x^2$ ✗	$\begin{aligned} \frac{1}{x} + \frac{x}{2x+1} &= 0 \\ x\,(2x + 1) \cdot \left(\frac{1}{x} + \frac{x}{2x+1}\right) &= x\,(2x + 1) \\ 2x + 1 + x^2 &= 0 \\ x^2 + 2x + 1 &= 0 \end{aligned}$ ✓

Rational equations are sometimes expressed using negative exponents.

Example 6

Solve: $6 + x^{-1} = x^{-2}$.

Solution:

Begin by removing the negative exponents.

$$\begin{aligned} 6 + x^{-1} &= x^{-2} \\ 6 + \frac{1}{x} &= \frac{1}{x^2} \end{aligned}$$

Here we can see the restriction, $x \neq 0$. Next, multiply both sides by the LCD, x^2.

$$\begin{aligned} x^2 \cdot \left(6 + \frac{1}{x}\right) &= x^2 \cdot \left(\frac{1}{x^2}\right) \\ x^2 \cdot 6 + x^2 \cdot \frac{1}{x} &= x^2 \cdot \frac{1}{x^2} \\ 6x^2 + x &= 1 \\ 6x^2 + x - 1 &= 0 \\ (3x - 1)(2x + 1) &= 0 \end{aligned}$$

$$\begin{aligned} 3x - 1 &= 0 & \quad or\ 2x + 1 &= 0 \\ 3x &= 1 & 2x &= -1 \\ x &= \frac{1}{3} & x &= -\frac{1}{2} \end{aligned}$$

Answer: $-\frac{1}{2}, \frac{1}{3}$

A **proportion**[35] is a statement of equality of two ratios.

$$\frac{a}{b} = \frac{c}{d}$$

This proportion is often read "*a* is to *b* as *c* is to *d*." Given any nonzero real numbers *a*, *b*, *c*, and *d* that satisfy a proportion, multiply both sides by the product of the denominators to obtain the following:

$$\begin{aligned} \frac{a}{b} &= \frac{c}{d} \\ bd \cdot \frac{a}{b} &= bd \cdot \frac{c}{d} \\ ad &= bc \end{aligned}$$

This shows that cross products are equal, and is commonly referred to as **cross multiplication**[36].

$$\text{If} \quad \frac{a}{b} = \frac{c}{d} \quad \text{then} \quad ad = bc$$

Cross multiply to solve proportions where terms are unknown.

35. A statement of equality of two ratios.

36. If $\frac{a}{b} = \frac{c}{d}$ then $ad = bc$.

Example 7

Solve: $\frac{5n-1}{5} = \frac{3n}{2}$.

Solution:

When cross multiplying, be sure to group $5n - 1$.

$$\frac{5n-1}{5} = \frac{3n}{2}$$

$$\left(5n - 1\right) \cdot 2 = 5 \cdot 3n$$

Apply the distributive property in the next step.

$$\begin{aligned} (5n - 1) \cdot 2 &= 5 \cdot 3n \\ 10n - 2 &= 15n \quad \textit{Distribute.} \\ -2 &= 5n \quad \textit{Solve.} \\ \frac{-2}{5} &= n \end{aligned}$$

Answer: $n = -\frac{2}{5}$

Cross multiplication can be used as an alternate method for solving rational equations. The idea is to simplify each side of the equation to a single algebraic fraction and then cross multiply.

Example 8

Solve: $\frac{1}{2} - \frac{4}{x} = -\frac{x}{8}$.

Solution:

Obtain a single algebraic fraction on the left side by subtracting the equivalent fractions with a common denominator.

$$\begin{aligned}\frac{1}{2} \cdot \frac{x}{x} - \frac{4}{x} \cdot \frac{2}{2} &= -\frac{x}{8} \\ \frac{x}{2x} - \frac{8}{2x} &= -\frac{x}{8} \\ \frac{x-8}{2x} &= -\frac{x}{8}\end{aligned}$$

Note that $x \neq 0$, cross multiply, and then solve for x.

$$\begin{aligned}\frac{x-8}{2x} &= \frac{-x}{8} \\ 8(x-8) &= -x \cdot 2x \\ 8x - 64 &= -2x^2 \\ 2x^2 + 8x - 64 &= 0 \\ 2\left(x^2 + 4x - 32\right) &= 0 \\ 2(x-4)(x+8) &= 0\end{aligned}$$

Next, set each variable factor equal to zero.

$$\begin{aligned} x-4=0 &\text{ or } x+8=0 \\ x=4 &\qquad\quad x=-8 \end{aligned}$$

The check is left to the reader.

Answer: -8, 4

Try this! Solve: $\frac{2(2x-5)}{x-1} = -\frac{x-4}{2x-5}$.

Answer: 2, 3

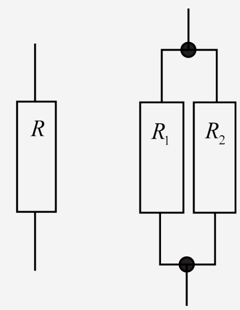
(click to see video)

Solving Literal Equations and Applications Involving Reciprocals

Literal equations, or formulas, are often rational equations. Hence the techniques described in this section can be used to solve for particular variables. Assume that all variable expressions in the denominator are nonzero.

Example 9

The reciprocal of the combined resistance R of two resistors R_1 and R_2 in parallel is given by the formula $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ Solve for R in terms of R_1 and R_2.



Solution:

The goal is to isolate R on one side of the equation. Begin by multiplying both sides of the equation by the LCD, RR_1R_2.

$$\begin{aligned} RR_1R_2 \cdot \frac{1}{R} &= RR_1R_2 \cdot \frac{1}{R_1} + RR_1R_2 \cdot \frac{1}{R_2} \\ R_1R_2 &= RR_2 + RR_1 \\ R_1R_2 &= R\left(R_2 + R_1\right) \\ \frac{R_1R_2}{R_2 + R_1} &= R \end{aligned}$$

Answer: $R = \frac{R_1R_2}{R_1+R_2}$

Try this! Solve for y: $x = \frac{2y+5}{y-3}$.

Answer: $y = \frac{3x+5}{x-2}$

(click to see video)

Recall that the reciprocal of a nonzero number n is $\frac{1}{n}$ .For example, the reciprocal of 5 is $\frac{1}{5}$ and $5 \cdot \frac{1}{5} = 1$.In this section, the applications will often involve the key word “reciprocal.” When this is the case, we will see that the algebraic setup results in a rational equation.

Example 10

A positive integer is 3 less than another. If the reciprocal of the smaller integer is subtracted from twice the reciprocal of the larger, then the result is $\frac{1}{20}$ .Find the two integers.

Solution:

Let n represent the larger positive integer.

Let n - 3 represent the smaller positive integer.

Set up an algebraic equation.

$$\overbrace{2\left(\frac{1}{n}\right)}^{\text{twice the reciprocal of the larger}} \overbrace{- \quad \frac{1}{n-3}}^{\text{the reciprocal of the smaller is subtracted}} \overbrace{=}^{\text{the result is}} \frac{1}{20}$$

Solve this rational expression by multiplying both sides by the LCD. The LCD is $20n(n-3)$.

$$\frac{2}{n} - \frac{1}{n-3} = \frac{1}{20}$$

$$20n\,(n-3)\cdot\left(\frac{2}{n} - \frac{1}{n-3}\right) = 20n\,(n-3)\cdot\left(\frac{1}{20}\right)$$

$$20n\,(n-3)\cdot\frac{2}{n} - 20n\,(n-3)\cdot\frac{1}{n-3} = 20n\,(n-3)\cdot\left(\frac{1}{20}\right)$$

$$\begin{aligned} 40(n-3)-20n &= n(n-3) \\ 40n-120-20n &= n^2-3n \\ 20n-120 &= n^2-3n \\ 0 &= n^2-23n+120 \\ 0 &= (n-8)\left(n-15\right) \end{aligned}$$

$$\begin{aligned} n-8 &= 0 \text{ or } n-15=0 \\ n &= 8 \qquad\qquad n=15 \end{aligned}$$

Here we have two viable possibilities for the larger integer n. For this reason, we will we have two solutions to this problem.

If $n = 8$, then $n - 3 = 8 - 3 = 5$.

If $n = 15$, then $n - 3 = 15 - 3 = 12$.

As a check, perform the operations indicated in the problem.

$$2\left(\frac{1}{n}\right) - \frac{1}{n-3} = \frac{1}{20}$$

Check 8 and 5.	Check 15 and 12.
$\begin{aligned} 2\left(\frac{1}{8}\right)-\frac{1}{5} &= \frac{1}{4}-\frac{1}{5} \\ &= \frac{5}{20}-\frac{4}{20} \\ &= \frac{1}{20} \quad ✓ \end{aligned}$	$\begin{aligned} 2\left(\frac{1}{15}\right)-\frac{1}{12} &= \frac{2}{15}-\frac{1}{12} \\ &= \frac{8}{60}-\frac{5}{60} \\ &= \frac{3}{60}=\frac{1}{20} \quad ✓ \end{aligned}$

Answer: Two sets of positive integers solve this problem: {5, 8} and {12, 15}.

Try this! When the reciprocal of the larger of two consecutive even integers is subtracted from 4 times the reciprocal of the smaller, the result is $\frac{5}{6}$ .Find the integers.

Answer: 4, 6

(click to see video)

KEY TAKEAWAYS

- Begin solving rational equations by multiplying both sides by the LCD. The resulting equivalent equation can be solved using the techniques learned up to this point.
- Multiplying both sides of a rational equation by a variable expression introduces the possibility of extraneous solutions. Therefore, we must check the solutions against the set of restrictions. If a solution is a restriction, then it is not part of the domain and is extraneous.
- When multiplying both sides of an equation by an expression, distribute carefully and multiply each term by that expression.
- If all of the resulting solutions are extraneous, then the original equation has no solutions.

TOPIC EXERCISES

PART A: SOLVING RATIONAL EQUATIONS

Solve.

1. $\frac{3}{x} + 2 = \frac{1}{3x}$
2. $5 - \frac{1}{2x} = -\frac{1}{x}$
3. $\frac{7}{x^2} + \frac{3}{2x} = \frac{1}{x^2}$
4. $\frac{4}{3x^2} + \frac{1}{2x} = \frac{1}{3x^2}$
5. $\frac{1}{6} + \frac{2}{3x} = \frac{7}{2x^2}$
6. $\frac{1}{12} - \frac{1}{3x} = \frac{1}{x^2}$
7. $2 + \frac{3}{x} + \frac{7}{x(x-3)} = 0$
8. $\frac{20}{x} - \frac{x+44}{x(x+2)} = 3$
9. $\frac{2x}{2x-3} + \frac{4}{x} = \frac{x-18}{x(2x-3)}$
10. $\frac{2x}{x-5} + \frac{2(4x+7)}{x\left(x-5\right)} = -\frac{1}{x}$
11. $\frac{4}{4x-1} - \frac{1}{x-1} = \frac{2}{4x-1}$
12. $\frac{5}{2x-3} - \frac{1}{x+3} = \frac{2}{2x-3}$
13. $\frac{4x}{x-3} + \frac{4}{x^2-2x-3} = -\frac{1}{x+1}$
14. $\frac{2x}{x-2} - \frac{15}{x+4} = \frac{24}{x^2+2x-8}$
15. $\frac{x}{x-8} - \frac{8}{x-1} = \frac{56}{x^2-9x+8}$
16. $\frac{2x}{x-1} + \frac{9}{3x-1} + \frac{11}{3x^2-4x+1} = 0$

17. $\dfrac{3x}{x-2} - \dfrac{14}{2x^2 - x - 6} = \dfrac{2}{2x+3}$

18. $\dfrac{x}{x-4} - \dfrac{4}{x-5} = -\dfrac{4}{x^2 - 9x + 20}$

19. $\dfrac{2x}{5+x} - \dfrac{1}{5-x} = \dfrac{2x}{x^2 - 25}$

20. $\dfrac{2x}{2x+3} - \dfrac{1}{2x-3} = \dfrac{6}{9 - 4x^2}$

21. $1 + \dfrac{1}{x+1} = \dfrac{8}{x-1} - \dfrac{16}{x^2 - 1}$

22. $1 - \dfrac{1}{3x+5} = \dfrac{2x}{3x-5} - \dfrac{2\left(6x+5\right)}{9x^2 - 25}$

23. $\dfrac{x}{x-2} - \dfrac{3}{x+8} = \dfrac{x+2}{x+8} + \dfrac{5\,(x+3)}{x^2 + 6x - 16}$

24. $\dfrac{2x}{x-10} + \dfrac{1}{x-3} = \dfrac{x+3}{x-10} + \dfrac{x^2 - 5x + 5}{x^2 - 13x + 30}$

25. $\dfrac{5}{x^2 + 9x + 18} + \dfrac{x+3}{x^2 + 7x + 6} = \dfrac{5}{x^2 + 4x + 3}$

26. $\dfrac{1}{x^2 + 4x - 60} + \dfrac{x-6}{x^2 + 16x + 60} = \dfrac{1}{x^2 - 36}$

27. $\dfrac{4}{x^2 + 10x + 21} + \dfrac{2\,(x+3)}{x^2 + 6x - 7} = \dfrac{x+7}{x^2 + 2x - 3}$

28. $\dfrac{x-1}{x^2 - 11x + 28} + \dfrac{x-1}{x^2 - 5x + 4} = \dfrac{x-4}{x^2 - 8x + 7}$

29. $\dfrac{5}{x^2 + 5x + 4} + \dfrac{x+1}{x^2 + 3x - 4} = \dfrac{5}{x^2 - 1}$

30. $\dfrac{1}{x^2 - 2x - 63} + \dfrac{x-9}{x^2 + 10x + 21} = \dfrac{1}{x^2 - 6x - 27}$

31. $\dfrac{4}{x^2 - 4} + \dfrac{2\,(x-2)}{x^2 - 4x - 12} = \dfrac{x+2}{x^2 - 8x + 12}$

32. $\dfrac{x+2}{x^2 - 5x + 4} + \dfrac{x+2}{x^2 + x - 2} = \dfrac{x-1}{x^2 - 2x - 8}$

Solve the following equations involving negative exponents.

33. $2x^{-1} = 2x^{-2} - x^{-1}$

34. $3 + x(x+1)^{-1} = 2(x+1)^{-1}$

35. $x^{-2} - 64 = 0$

36. $1 - 4x^{-2} = 0$

37. $x - (x + 2)^{-1} = -2$

38. $2x - 9(2x - 1)^{-1} = 1$

39. $2x^{-2} + (x - 12)^{-1} = 0$

40. $-2x^{-2} + 3(x + 4)^{-1} = 0$

Solve by cross multiplying.

41. $\frac{5}{n} = -\frac{3}{n - 2}$

42. $\frac{2n - 1}{2n} = -\frac{1}{2}$

43. $-3 = \frac{5n + 2}{3n}$

44. $\frac{n + 1}{2n - 1} = \frac{1}{3}$

45. $\frac{x + 2}{x - 5} = \frac{x + 4}{x - 2}$

46. $\frac{x + 1}{x + 5} = \frac{x - 5}{x}$

47. $\frac{2x + 1}{6x - 1} = \frac{x + 5}{3x - 2}$

48. $\frac{6(2x + 3)}{4x - 1} = \frac{3x}{x + 2}$

49. $\frac{3(x + 1)}{1 - x} = \frac{x + 3}{x + 1}$

50. $\frac{8(x - 2)}{x + 1} = \frac{5 - x}{x - 2}$

51. $\frac{x + 3}{x + 7} = \frac{x + 3}{3(5 - x)}$

52. $\frac{x + 1}{x + 4} = \frac{-8(x + 4)}{x + 7}$

Simplify or solve, whichever is appropriate.

53. $\frac{1}{x} + \frac{2}{x - 3} = -\frac{2}{3}$

54. $\dfrac{1}{x-3} - \dfrac{3}{4} = \dfrac{1}{x}$

55. $\dfrac{x-2}{3x-1} - \dfrac{2-x}{x}$

56. $\dfrac{5}{2} + \dfrac{x}{2x-1} - \dfrac{1}{2x}$

57. $\dfrac{x-1}{3x} + \dfrac{2}{x+1} - \dfrac{5}{6}$

58. $\dfrac{x-1}{3x} + \dfrac{2}{x+1} = \dfrac{5}{6}$

59. $\dfrac{2x+1}{2x-3} + 2 = \dfrac{1}{2x}$

60. $5 - \dfrac{3x+1}{2x} + \dfrac{1}{x+1}$

Find the roots of the given function.

61. $f(x) = \dfrac{2x-1}{x-1}$

62. $f(x) = \dfrac{3x+1}{x+2}$

63. $g(x) = \dfrac{x^2-81}{x^2-5x}$

64. $g(x) = \dfrac{x^2-x-20}{x^2-9}$

65. $f(x) = \dfrac{4x^2-9}{2x-3}$

66. $f(x) = \dfrac{3x^2-2x-1}{x^2-1}$

67. Given $f(x) = \frac{1}{x} + 5$, find x when $f(x) = 2$.

68. Given $f(x) = \frac{1}{x-4}$, find x when $f(x) = \frac{1}{2}$.

69. Given $f(x) = \frac{1}{x+3} + 2$, find x when $f(x) = 1$.

70. Given $f(x) = \frac{1}{x-2} + 5$, find x when $f(x) = 3$.

Find the *x*- and *y*-intercepts.

71. $f(x) = \frac{1}{x+1} + 4$

72. $f(x) = \frac{1}{x-2} - 6$

73. $f(x) = \frac{1}{x-3} + 2$

74. $f(x) = \frac{1}{x+1} - 1$

75. $f(x) = \frac{1}{x} - 3$

76. $f(x) = \frac{1}{x+5}$

Find the points where the given functions coincide. (Hint: Find the points where $f(x) = g(x)$.)

77. $f(x) = \frac{1}{x}, g(x) = x$

78. $f(x) = -\frac{1}{x}, g(x) = -x$

79. $f(x) = \frac{1}{x-2} + 3, g(x) = x + 1$

80. $f(x) = \frac{1}{x+3} - 1, g(x) = x + 2$

Recall that if $|X| = p$, then $X = -p$ or $X = p$. Use this to solve the following absolute value equations.

81. $\left|\frac{1}{x+1}\right| = 2$

82. $\left|\frac{2x}{x+2}\right| = 1$

83. $\left|\frac{3x-2}{x-3}\right| = 4$

84. $\left|\frac{5x-3}{2x+1}\right| = 3$

85. $\left|\frac{x^2}{5x+6}\right| = 1$

86. $\left|\frac{x^2-48}{x}\right| = 2$

PART B: SOLVING LITERAL EQUATIONS

Solve for the given variable.

87. Solve for P: $w = \frac{P-2l}{2}$

88. Solve for A: $t = \frac{A-P}{Pr}$

89. Solve for t: $\frac{1}{t_1} + \frac{1}{t_2} = \frac{1}{t}$

90. Solve for n: $P = 1 + \frac{r}{n}$

91. Solve for y: $m = \frac{y-y_0}{x-x_0}$

92. Solve for m_1: $F = G\frac{m_1 m_2}{r^2}$

93. Solve for y: $x = \frac{2y-1}{y-1}$

94. Solve for y: $x = \frac{3y+2}{y+3}$

95. Solve for y: $x = \frac{2y}{2y+5}$

96. Solve for y: $x = \frac{5y+1}{3y}$

97. Solve for x: $\frac{a}{x} + \frac{c}{b} = \frac{a}{c}$

98. Solve for y: $\frac{a}{y} - \frac{1}{a} = b$

Use algebra to solve the following applications.

99. The value in dollars of a tablet computer is given by the function $V(t) = 460(t+1)^{-1}$, where t represents the age of the tablet. Determine the age of the tablet if it is now worth $100.

100. The value in dollars of a car is given by the function $V(t) = 24{,}000\left(0.5t+1\right)^{-1}$, where t represents the age of the car. Determine the age of the car if it is now worth $6,000.

Solve for the unknowns.

101. When 2 is added to 5 times the reciprocal of a number, the result is 12. Find the number.

102. When 1 is subtracted from 4 times the reciprocal of a number, the result is 11. Find the number.

103. The sum of the reciprocals of two consecutive odd integers is $\frac{12}{35}$. Find the integers.

104. The sum of the reciprocals of two consecutive even integers is $\frac{9}{40}$. Find the integers.

105. An integer is 4 more than another. If 2 times the reciprocal of the larger is subtracted from 3 times the reciprocal of the smaller, then the result is $\frac{1}{8}$. Find the integers.

106. An integer is 2 more than twice another. If 2 times the reciprocal of the larger is subtracted from 3 times the reciprocal of the smaller, then the result is $\frac{5}{14}$. Find the integers.

107. If 3 times the reciprocal of the larger of two consecutive integers is subtracted from 2 times the reciprocal of the smaller, then the result is $\frac{1}{2}$. Find the two integers.

108. If 3 times the reciprocal of the smaller of two consecutive integers is subtracted from 7 times the reciprocal of the larger, then the result is $\frac{1}{2}$. Find the two integers.

109. A positive integer is 5 less than another. If the reciprocal of the smaller integer is subtracted from 3 times the reciprocal of the larger, then the result is $\frac{1}{12}$. Find the two integers.

110. A positive integer is 6 less than another. If the reciprocal of the smaller integer is subtracted from 10 times the reciprocal of the larger, then the result is $\frac{3}{7}$. Find the two integers.

PART C: DISCUSSION BOARD

111. Explain how we can tell the difference between a rational expression and a rational equation. How do we treat them differently? Give an example of each.

112. Research and discuss reasons why multiplying both sides of a rational equation by the LCD sometimes produces extraneous solutions.

ANSWERS

1. $-\frac{4}{3}$

3. −4

5. −7, 3

7. $-\frac{1}{2}$, 2

9. −2, $-\frac{3}{2}$

11. $-\frac{1}{2}$

13. $-\frac{1}{4}$

15. Ø

17. −2, $\frac{5}{6}$

19. $\frac{1}{2}$

21. 6

23. Ø

25. −8, 2

27. 5

29. −6, 4

31. 10

33. $\frac{2}{3}$

35. $\pm\frac{1}{8}$

37. −3, −1

39. −6, 4

41. $\frac{5}{4}$

43. $-\frac{1}{7}$

45. −16

47. $\frac{1}{10}$

49. −2, 0

51. −3, 2

53. Solve; −3, $\frac{3}{2}$

55. Simplify; $\frac{(4x-1)(x-2)}{x(3x-1)}$

57. Simplify; $-\frac{(x-2)(3x-1)}{6x(x+1)}$

59. Solve; $\frac{1}{2}$

61. $\frac{1}{2}$

63. ±9

65. $-\frac{3}{2}$

67. $x = -\frac{1}{3}$

69. $x = -4$

71. x-intercept: $\left(-\frac{5}{4}, 0\right)$; y-intercept: (0, 5)

73. x-intercept: $\left(\frac{5}{2}, 0\right)$; y-intercept: $\left(0, \frac{5}{3}\right)$

75. x-intercept: $\left(\frac{1}{3}, 0\right)$; y-intercept: none

77. (−1, −1) and (1, 1)

79. (1, 2) and (3, 4)

81. $-\frac{3}{2}, -\frac{1}{2}$

83. 2, 10

85. −3, −2, −1, 6

87. $P = 2l + 2w$

89. $t = \frac{t_1 t_2}{t_1 + t_2}$

91. $y = m(x - x_0) + y_0$

93. $y = \dfrac{x-1}{x-2}$

95. $y = -\dfrac{5x}{2x-2}$

97. $x = \dfrac{abc}{ab - c^2}$

99. 3.6 years old

101. $\frac{1}{2}$

103. 5, 7

105. {-8, -4} and {12, 16}

107. {1, 2} or {-4, -3}

109. {4, 9} or {15, 20}

111. Answer may vary

4.8 Applications and Variation

LEARNING OBJECTIVES

1. Solve applications involving uniform motion (distance problems).
2. Solve work-rate applications.
3. Set up and solve applications involving direct, inverse, and joint variation.

Solving Uniform Motion Problems

Uniform motion (or distance)[37] problems involve the formula $D = rt$, where the distance D is given as the product of the average rate r and the time t traveled at that rate. If we divide both sides by the average rate r, then we obtain the formula

$$t = \frac{D}{r}$$

For this reason, when the unknown quantity is time, the algebraic setup for distance problems often results in a rational equation. We begin any uniform motion problem by first organizing our data with a chart. Use this information to set up an algebraic equation that models the application.

37. Described by the formula $D = rt$, where the distance D is given as the product of the average rate r and the time t traveled at that rate.

Example 1

Sally traveled 15 miles on the bus and then another 72 miles on a train. The train was 18 miles per hour faster than the bus, and the total trip took 2 hours. What was the average speed of the train?

Solution:

First, identify the unknown quantity and organize the data.

Let x represent the average speed (in miles per hour) of the bus.

Let $x + 18$ represent the average speed of the train.

	Distance =	***Rate*** ×	***Time***
Bus trip	15 mi	x	
Train trip	72 mi	$x + 18$	
Total			2 hours

To avoid introducing two more variables for the time column, use the formula $t = \frac{D}{r}$ .The time for each leg of the trip is calculated as follows:

$$\textit{Time spent on the bus}: t = \frac{D}{r} = \frac{15}{x}$$
$$\textit{Time spent on the train}: t = \frac{D}{r} = \frac{72}{x+18}$$

Use these expressions to complete the chart.

	Distance =	*Rate* ×	*Time*
Bus trip	15 mi	x	$\frac{15}{x}$
Train trip	72 mi	$x + 18$	$\frac{72}{x+18}$
Total			2 hours

$$\frac{15}{x} + \frac{72}{x+18} = 2$$

The algebraic setup is defined by the time column. Add the time spent on each leg of the trip to obtain a total of 2 hours:

$$\overbrace{\frac{15}{x}}^{\text{time spent on the bus}} + \overbrace{\frac{72}{x+18}}^{\text{time spent on the train}} = \overbrace{2}^{\text{total time of trip}}$$

We begin solving this equation by first multiplying both sides by the LCD, $x\,(x + 18)$.

$$\begin{aligned}
\frac{15}{x} + \frac{72}{x+18} &= 2 \\
x(x+18) \cdot \left(\frac{15}{x} + \frac{72}{x+18}\right) &= x(x+18) \cdot 2 \\
x(x+18) \cdot \frac{15}{x} + x(x+18) \cdot \frac{72}{x+18} &= x(x+18) \cdot 2 \\
15(x+18) + 72x &= 2x(x+18) \\
15x + 270 + 72x &= 2x^2 + 36x \\
87x + 270 &= 2x^2 + 36x \\
0 &= 2x^2 - 51x - 270
\end{aligned}$$

Solve the resulting quadratic equation by factoring.

$$\begin{aligned} 0 &= 2x^2 - 51x - 270 \\ 0 &= (2x + 9)\ (x - 30) \end{aligned}$$

$$\begin{aligned} 2x + 9 &= 0 \quad &\text{or } x - 30 &= 0 \\ x &= -\frac{9}{2} & x &= 30 \end{aligned}$$

Since we are looking for an average speed we will disregard the negative answer and conclude the bus averaged 30 mph. Substitute $x = 30$ in the expression identified as the speed of the train.

$$x + 18 = 30 + 18 = 48$$

Answer: The speed of the train was 48 mph.

Example 2

A boat can average 12 miles per hour in still water. On a trip downriver the boat was able to travel 29 miles with the current. On the return trip the boat was only able to travel 19 miles in the same amount of time against the current. What was the speed of the current?

Solution:

First, identify the unknown quantities and organize the data.

Let c represent the speed of the river current.

Next, organize the given data in a chart. Traveling downstream, the current will increase the speed of the boat, so it adds to the average speed of the boat. Traveling upstream, the current slows the boat, so it will subtract from the average speed of the boat.

	Distance =	*Rate* ×	*Time*
trip downriver	29 mi	$12+c$	
trip upriver	19 mi	$12-c$	
Total			

Use the formula $t=\frac{D}{r}$ to fill in the time column.

$$\textit{trip downriver}: t=\frac{D}{r}=\frac{29}{12+c}$$
$$\textit{trip upriver}: t=\frac{D}{r}=\frac{19}{12-c}$$

	Distance =	Rate ×	Time
trip downriver	29 mi	$12 + c$	$\frac{29}{12+c}$
trip upriver	19 mi	$12 - c$	$\frac{19}{12-c}$
Total			

Because the boat traveled the same amount of time downriver as it did upriver, finish the algebraic setup by setting the expressions that represent the times equal to each other.

$$\frac{29}{12+c} = \frac{19}{12-c}$$

Since there is a single algebraic fraction on each side, we can solve this equation using cross multiplication.

$$\begin{aligned}\frac{29}{12+c} &= \frac{19}{12-c}\\ 29\,(12-c) &= 19\,(12+c)\\ 348 - 29c &= 228 + 19c\\ 120 &= 48c\\ \frac{120}{48} &= c\\ \frac{5}{2} &= c\end{aligned}$$

Answer: The speed of the current was $2\,\frac{1}{2}$ miles per hour.

Try this! A jet aircraft can average 160 miles per hour in calm air. On a trip, the aircraft traveled 600 miles with a tailwind and returned the 600 miles against a headwind of the same speed. If the total round trip took 8 hours, then what was the speed of the wind?

Answer: 40 miles per hour

(click to see video)

Solving Work-Rate Problems

The rate at which a task can be performed is called a **work rate**[38]. For example, if a painter can paint a room in 6 hours, then the task is to paint the room, and we can write

$$\frac{1 \text{ task}}{6 \text{ hours}} \quad \text{work rate}$$

In other words, the painter can complete $\frac{1}{6}$ of the task per hour. If he works for less than 6 hours, then he will perform a fraction of the task. If he works for more than 6 hours, then he can complete more than one task. For example,

$$\begin{array}{lcccl} \textit{work-rate} & \times & \textit{time} & = & \textit{amount of task completed} \\ \frac{1}{6} & \times & 3\ hrs & = & \frac{1}{2} \quad \textit{one-half of the room painted} \\ \frac{1}{6} & \times & 6\ hrs & = & 1 \quad \textit{one whole room painted} \\ \frac{1}{6} & \times & 12\ hrs & = & 2 \quad \textit{two whole rooms painted} \end{array}$$

38. The rate at which a task can be performed.

Obtain the amount of the task completed by multiplying the work rate by the amount of time the painter works. Typically, work-rate problems involve people or machines working together to complete tasks. In general, if t represents the time two people work together, then we have the following **work-rate formula**[39]:

$$\frac{1}{t_1}t + \frac{1}{t_2}t = \textit{amount of task completed together}$$

Here $\frac{1}{t_1}$ and $\frac{1}{t_2}$ are the individual work rates.

39. $\frac{1}{t_1} \cdot t + \frac{1}{t_2} \cdot t = 1$, where $\frac{1}{t_1}$ and $\frac{1}{t_2}$ are the individual work rates and t is the time it takes to complete the task working together.

Example 3

Joe can paint a typical room in 2 hours less time than Mark. If Joe and Mark can paint 5 rooms working together in a 12 hour shift, how long does it take each to paint a single room?

Solution:

Let x represent the time it takes Mark to paint a typical room.

Let x – 2 represent the time it takes Joe to paint a typical room.

Therefore, Mark's individual work-rate is $\frac{1}{x}$ rooms per hour and Joe's is $\frac{1}{x-2}$ rooms per hour. Both men worked for 12 hours. We can organize the data in a chart, just as we did with distance problems.

	Amount of Task Completed =	*Work-Rate* ×	*Time*
Joe	$\frac{1}{x-2}\cdot 12$	$\frac{1}{x-2}$	12
Mark	$\frac{1}{x}\cdot 12$	$\frac{1}{x}$	12
Total	5		

$$\frac{1}{x-2}\cdot 12+\frac{1}{x}\cdot 12=5$$

Working together, they can paint 5 total rooms in 12 hours. This leads us to the following algebraic setup:

$$\frac{12}{x-2}+\frac{12}{x}=5$$

Multiply both sides by the LCD, $x\,(x-2)$.

$$\begin{aligned} x(x-2)\cdot\left(\frac{12}{x-2}+\frac{12}{x}\right) &= x(x-2)\cdot 5 \\ x(x-2)\cdot\frac{12}{x-2}+x(x-2)\cdot\frac{12}{x} &= x(x-2)\cdot 5 \\ 12x+12(x-2) &= 5x(x-2) \\ 12x+12x-24 &= 5x^2-10x \\ 0 &= 5x^2-34x+24 \end{aligned}$$

Solve the resulting quadratic equation by factoring.

$$\begin{aligned} 0 &= 5x^2-34x+24 \\ 0 &= (5x-4)\ (x-6) \end{aligned}$$

$$\begin{aligned} 5x-4 &= 0 \quad or\ x-6=0 \\ 5x &= 4 \qquad x \quad =6 \\ x &= \frac{4}{5} \end{aligned}$$

We can disregard $\frac{4}{5}$ because back substituting into x – 2 would yield a negative time to paint a room. Take x = 6 to be the only solution and use it to find the time it takes Joe to paint a typical room.

$$x-2=6-2=4$$

Answer: Joe can paint a typical room in 4 hours and Mark can paint a typical room in 6 hours. As a check we can multiply both work rates by 12 hours to see that together they can paint 5 rooms.

$$\left.\begin{array}{ll} Joe & \dfrac{1\ room}{4\ hrs} \cdot 12\ hrs = 3\ rooms \\ Mark & \dfrac{1\ room}{6\ hrs} \cdot 12\ hrs = 2\ rooms \end{array}\right\} Total\ 5\ rooms\ \checkmark$$

Example 4

It takes Bill twice as long to lay a tile floor by himself as it does Manny. After working together with Bill for 4 hours, Manny was able to complete the job in 2 additional hours. How long would it have taken Manny working alone?

Solution:

Let x represent the time it takes Manny to lay the floor alone.

Let $2x$ represent the time it takes Bill to lay the floor alone.

Manny's work rate is $\frac{1}{x}$ of the floor per hour and Bill's work rate is $\frac{1}{2x}$ .Bill worked on the job for 4 hours and Manny worked on the job for 6 hours.

	Amount of Task Completed =	***Work - Rate*** ×	***Time***
Manny	$\frac{1}{x}\cdot 6$	$\frac{1}{x}$	6 hours
Bill	$\frac{1}{2x}\cdot 4$	$\frac{1}{2x}$	4 hours
Total	1		

This leads us to the following algebraic setup:

$$\frac{1}{x}\cdot 6+\frac{1}{2x}\cdot 4=1$$

Solve.

$$
\begin{aligned}
\frac{6}{x}+\frac{4}{2x} &= 1 \\
{\color{blue}x\cdot}\left(\frac{6}{x}+\frac{2}{x}\right) &= {\color{blue}x\cdot}1 \\
6+2 &= x \\
8 &= x
\end{aligned}
$$

Answer: It would have taken Manny 8 hours to complete the floor by himself.

Consider the work-rate formula where one task is to be completed.

$$\frac{1}{t_1}t+\frac{1}{t_2}t=1$$

Factor out the time t and then divide both sides by t. This will result in equivalent specialized work-rate formulas:

$$
\begin{aligned}
t\left(\frac{1}{t_1}+\frac{1}{t_2}\right) &= 1 \\
\frac{1}{t_1}+\frac{1}{t_2} &= \frac{1}{t}
\end{aligned}
$$

In summary, we have the following equivalent work-rate formulas:

$$\color{blue}{Work\ rate\ formulas}$$

$$\frac{1}{t_1}t+\frac{1}{t_2}t=1 \quad or \quad \frac{t}{t_1}+\frac{t}{t_2}=1 \quad or \quad \frac{1}{t_1}+\frac{1}{t_2}=\frac{1}{t}$$

Try this! Matt can tile a countertop in 2 hours, and his assistant can do the same job in 3 hours. If Matt starts the job and his assistant joins him 1 hour later, then how long will it take to tile the countertop?

Answer: $1\frac{3}{5}$ hours

(click to see video)

Solving Problems involving Direct, Inverse, and Joint variation

Many real-world problems encountered in the sciences involve two types of functional relationships. The first type can be explored using the fact that the distance *s* in feet an object falls from rest, without regard to air resistance, can be approximated using the following formula:

$$s = 16t^2$$

Here *t* represents the time in seconds the object has been falling. For example, after 2 seconds the object will have fallen $s = 16(2)^2 = 16 \cdot 4 = 64$ feet.

Time *t* in seconds	Distance $s = 16t^2$ in feet
0	0
1	16

Time t in seconds	Distance $s = 16t^2$ in feet
2	64
3	144
4	256

In this example, we can see that the distance varies over time as the product of a constant 16 and the square of the time t. This relationship is described as **direct variation**[40] and 16 is called the **constant of variation**[41]. Furthermore, if we divide both sides of $s = 16t^2$ by t^2 we have

$$\frac{s}{t^2} = 16$$

In this form, it is reasonable to say that s is proportional to t^2, and 16 is called the **constant of proportionality**[42]. In general, we have

Key words	*Translation*
"y varies directly as x"	$y = kx$

40. Describes two quantities x and y that are constant multiples of each other: $y = kx$.
41. The nonzero multiple k, when quantities vary directly or inversely.
42. Used when referring to the constant of variation.

Key words	*Translation*
"y is **directly proportional** to x"	
"y is proportional to x"	

[43]

Here k is nonzero and is called the constant of variation or the constant of proportionality. Typically, we will be given information from which we can determine this constant.

43. Used when referring to direct variation.

Example 5

An object's weight on Earth varies directly to its weight on the Moon. If a man weighs 180 pounds on Earth, then he will weigh 30 pounds on the Moon. Set up an algebraic equation that expresses the weight on Earth in terms of the weight on the Moon and use it to determine the weight of a woman on the Moon if she weighs 120 pounds on Earth.

Solution:

Let y represent weight on Earth.

Let x represent weight on the Moon.

We are given that the "weight on Earth varies directly to the weight on the Moon."

$$y = kx$$

To find the constant of variation k, use the given information. A 180-lb man on Earth weighs 30 pounds on the Moon, or $y = 180$ when $x = 30$.

$$180 = k \cdot 30$$

Solve for k.

$$\frac{180}{30} = k$$
$$6 = k$$

Next, set up a formula that models the given information.

$$y = 6x$$

This implies that a person's weight on Earth is 6 times his weight on the Moon. To answer the question, use the woman's weight on Earth, y = 120 lbs, and solve for x.

$$120 = 6x$$
$$\frac{120}{6} = x$$
$$20 = x$$

Answer: The woman weighs 20 pounds on the Moon.

The second functional relationship can be explored using the formula that relates the intensity of light I to the distance from its source d.

$$I = \frac{k}{d^2}$$

Here k represents some constant. A foot-candle is a measurement of the intensity of light. One foot-candle is defined to be equal to the amount of illumination produced by a standard candle measured one foot away. For example, a 125-Watt fluorescent growing light is advertised to produce 525 foot-candles of illumination. This means that at a distance $d = 1$ foot, $I = 525$ foot-candles and we have:

$$\begin{aligned} 525 &= \frac{k}{(1)^2} \\ 525 &= k \end{aligned}$$

Using $k = 525$ we can construct a formula which gives the light intensity produced by the bulb:

$$I = \frac{525}{d^2}$$

Here d represents the distance the growing light is from the plants. In the following chart, we can see that the amount of illumination fades quickly as the distance from the plants increases.

distance t in feet	Light Intensity $I = \frac{525}{d^2}$
1	525

distance t in feet	Light Intensity $I = \frac{525}{d^2}$
2	131.25
3	58.33
4	32.81
5	21

This type of relationship is described as an **inverse variation**[44]. We say that I is **inversely proportional**[45] to the square of the distance d, where 525 is the constant of proportionality. In general, we have

Key words	*Translation*
"y varies inversely as x"	$y = \frac{k}{x}$
"y is inversely proportional to x"	

Again, k is nonzero and is called the constant of variation or the constant of proportionality.

44. Describes two quantities x and y, where one variable is directly proportional to the reciprocal of the other: $y = \frac{k}{x}$.

45. Used when referring to inverse variation.

Example 6

The weight of an object varies inversely as the square of its distance from the center of Earth. If an object weighs 100 pounds on the surface of Earth (approximately 4,000 miles from the center), how much will it weigh at 1,000 miles above Earth's surface?

Solution:

Let w represent the weight of the object.

Let d represent the object's distance from the center of Earth.

Since "w varies inversely as the square of d," we can write

$$w = \frac{k}{d^2}$$

Use the given information to find k. An object weighs 100 pounds on the surface of Earth, approximately 4,000 miles from the center. In other words, $w = 100$ when $d = 4{,}000$:

$$100 = \frac{k}{(4{,}000)^2}$$

Solve for k.

$$
\begin{aligned}
(4{,}000)^2 \cdot 100 &= (4{,}000)^2 \cdot \frac{k}{(4{,}000)^2} \\
1{,}600{,}000{,}000 &= k \\
1.6 \times 10^9 &= k
\end{aligned}
$$

Therefore, we can model the problem with the following formula:

$$w = \frac{1.6 \times 10^9}{d^2}$$

To use the formula to find the weight, we need the distance from the center of Earth. Since the object is 1,000 miles above the surface, find the distance from the center of Earth by adding 4,000 miles:

$$d = 4{,}000 + 1{,}000 = 5{,}000 \text{ miles}$$

To answer the question, use the formula with $d = 5{,}000$.

$$
\begin{aligned}
y &= \frac{1.6 \times 10^9}{(5{,}000)^2} \\
&= \frac{1.6 \times 10^9}{25{,}000{,}000} \\
&= \frac{1.6 \times 10^9}{2.5 \times 10^7} \\
&= 0.64 \times 10^2 \\
&= 64
\end{aligned}
$$

Answer: The object will weigh 64 pounds at a distance 1,000 miles above the surface of Earth.

Lastly, we define relationships between multiple variables, described as **joint variation**[46]. In general, we have

Key Words	*Translation*
"y varies jointly as x and z"	$y = kxz$
"y is **jointly proportional** to x and z"	

47

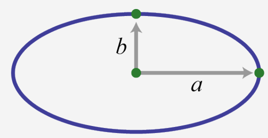
Here k is nonzero and is called the constant of variation or the constant of proportionality.

46. Describes a quantity y that varies directly as the product of two other quantities x and z: $y = kxz$.

47. Used when referring to joint variation.

Example 7

The area of an ellipse varies jointly as a, half of the ellipse's major axis, and b, half of the ellipse's minor axis as pictured. If the area of an ellipse is $300\pi\ \text{cm}^2$, where $a = 10\ \text{cm}$ and $b = 30\ \text{cm}$, what is the constant of proportionality? Give a formula for the area of an ellipse.



Solution:

If we let A represent the area of an ellipse, then we can use the statement "area varies jointly as a and b" to write

$$A = kab$$

To find the constant of variation k, use the fact that the area is 300π when $a = 10$ and $b = 30$.

$$\begin{aligned} 300\pi &= k(10)(30) \\ 300\pi &= 300k \\ \pi &= k \end{aligned}$$

Therefore, the formula for the area of an ellipse is

$$A = \pi ab$$

Answer: The constant of proportionality is π and the formula for the area of an ellipse is $A = ab\pi$.

Try this! Given that *y* varies directly as the square of *x* and inversely with *z*, where *y* = 2 when *x* = 3 and *z* = 27, find *y* when *x* = 2 and *z* = 16.

Answer: $\frac{3}{2}$

(click to see video)

KEY TAKEAWAYS

- When solving distance problems where the time element is unknown, use the equivalent form of the uniform motion formula, $t = \frac{D}{r}$, to avoid introducing more variables.
- When solving work-rate problems, multiply the individual work rate by the time to obtain the portion of the task completed. The sum of the portions of the task results in the total amount of work completed.
- The setup of variation problems usually requires multiple steps. First, identify the key words to set up an equation and then use the given information to find the constant of variation *k*. After determining the constant of variation, write a formula that models the problem. Once a formula is found, use it to answer the question.

TOPIC EXERCISES

PART A: SOLVING UNIFORM MOTION PROBLEMS

Use algebra to solve the following applications.

1. Every morning Jim spends 1 hour exercising. He runs 2 miles and then he bikes 16 miles. If Jim can bike twice as fast as he can run, at what speed does he average on his bike?
2. Sally runs 3 times as fast as she walks. She ran for $\frac{3}{4}$ of a mile and then walked another $3\frac{1}{2}$ miles. The total workout took $1\frac{1}{2}$ hours. What was Sally's average walking speed?
3. On a business trip, an executive traveled 720 miles by jet and then another 80 miles by helicopter. If the jet averaged 3 times the speed of the helicopter, and the total trip took 4 hours, what was the average speed of the jet?
4. A triathlete can run 3 times as fast as she can swim and bike 6 times as fast as she can swim. The race consists of a $\frac{1}{4}$ mile swim, 3 mile run, and a 12 mile bike race. If she can complete all of these events in $1\frac{5}{8}$ hour, then how fast can she swim, run and bike?
5. On a road trip, Marty was able to drive an average 4 miles per hour faster than George. If Marty was able to drive 39 miles in the same amount of time George drove 36 miles, what was Marty's average speed?
6. The bus is 8 miles per hour faster than the trolley. If the bus travels 9 miles in the same amount of time the trolley can travel 7 miles, what is the average speed of each?
7. Terry decided to jog the 5 miles to town. On the return trip, she walked the 5 miles home at half of the speed that she was able to jog. If the total trip took 3 hours, what was her average jogging speed?
8. James drove the 24 miles to town and back in 1 hour. On the return trip, he was able to average 20 miles per hour faster than he averaged on the trip to town. What was his average speed on the trip to town?
9. A light aircraft was able to travel 189 miles with a 14 mile per hour tailwind in the same time it was able to travel 147 miles against it. What was the speed of the aircraft in calm air?

10. A jet flew 875 miles with a 30 mile per hour tailwind. On the return trip, against a 30 mile per hour headwind, it was able to cover only 725 miles in the same amount of time. How fast was the jet in calm air?

11. A helicopter averaged 90 miles per hour in calm air. Flying with the wind it was able to travel 250 miles in the same amount of time it took to travel 200 miles against it. What is the speed of the wind?

12. Mary and Joe took a road-trip on separate motorcycles. Mary's average speed was 12 miles per hour less than Joe's average speed. If Mary drove 115 miles in the same time it took Joe to drive 145 miles, what was Mary's average speed?

13. A boat averaged 12 miles per hour in still water. On a trip downstream, with the current, the boat was able to travel 26 miles. The boat then turned around and returned upstream 33 miles. How fast was the current if the total trip took 5 hours?

14. If the river current flows at an average 3 miles per hour, a tour boat can make an 18-mile tour downstream with the current and back the 18 miles against the current in $4\frac{1}{2}$ hours. What is the average speed of the boat in still water?

15. Jose drove 10 miles to his grandmother's house for dinner and back that same evening. Because of traffic, he averaged 20 miles per hour less on the return trip. If it took $\frac{1}{4}$ hour longer to get home, what was his average speed driving to his grandmother's house?

16. Jerry paddled his kayak, upstream against a 1 mph current, for 12 miles. The return trip, downstream with the 1 mph current, took one hour less time. How fast did Jerry paddle the kayak in still water?

17. James and Mildred left the same location in separate cars and met in Los Angeles 300 miles away. James was able to average 10 miles an hour faster than Mildred on the trip. If James arrived 1 hour earlier than Mildred, what was Mildred's average speed?

18. A bus is 20 miles per hour faster than a bicycle. If Bill boards a bus at the same time and place that Mary departs on her bicycle, Bill will arrive downtown 5 miles away $\frac{1}{3}$ hour earlier than Mary. What is the average speed of the bus?

PART B: SOLVING WORK-RATE PROBLEMS

Use algebra to solve the following applications.

19. Mike can paint the office by himself in $4\ \frac{1}{2}$ hours. Jordan can paint the office in 6 hours. How long will it take them to paint the office working together?

20. Barry can lay a brick driveway by himself in $3\ \frac{1}{2}$ days. Robert does the same job in 5 days. How long will it take them to lay the brick driveway working together?

21. A larger pipe fills a water tank twice as fast as a smaller pipe. When both pipes are used, they fill the tank in 10 hours. If the larger pipe is left off, how long would it take the smaller pipe to fill the tank?

22. A newer printer can print twice as fast as an older printer. If both printers working together can print a batch of flyers in 45 minutes, then how long would it take the older printer to print the batch working alone?

23. Mary can assemble a bicycle for display in 2 hours. It takes Jane 3 hours to assemble a bicycle. How long will it take Mary and Jane, working together, to assemble 5 bicycles?

24. Working alone, James takes twice as long to assemble a computer as it takes Bill. In one 8-hour shift, working together, James and Bill can assemble 6 computers. How long would it take James to assemble a computer if he were working alone?

25. Working alone, it takes Harry one hour longer than Mike to install a fountain. Together they can install 10 fountains in 12 hours. How long would it take Mike to install 10 fountains by himself?

26. Working alone, it takes Henry 2 hours longer than Bill to paint a room. Working together they painted $2\ \frac{1}{2}$ rooms in 6 hours. How long would it have taken Henry to paint the same amount if he were working alone?

27. Manny, working alone, can install a custom cabinet in 3 hours less time than his assistant. Working together they can install the cabinet in 2 hours. How long would it take Manny to install the cabinet working alone?

28. Working alone, Garret can assemble a garden shed in 5 hours less time than his brother. Working together, they need 6 hours to build the garden shed. How long would it take Garret to build the shed working alone?

29. Working alone, the assistant-manager takes 2 more hours than the manager to record the inventory of the entire shop. After working together for 2 hours, it took the assistant-manager 1 additional hour to complete the inventory. How long would it have taken the manager to complete the inventory working alone?

30. An older printer can print a batch of sales brochures in 16 minutes. A newer printer can print the same batch in 10 minutes. After working together for some time, the newer printer was shut down and it took the older printer 3 more minutes to complete the job. How long was the newer printer operating?

PART C: SOLVING VARIATION PROBLEMS

Translate each of the following sentences into a mathematical formula.

31. The distance D an automobile can travel is directly proportional to the time t that it travels at a constant speed.

32. The extension of a hanging spring d is directly proportional to the weight w attached to it.

33. An automobile's braking distance d is directly proportional to the square of the automobile's speed v.

34. The volume V of a sphere varies directly as the cube of its radius r.

35. The volume V of a given mass of gas is inversely proportional to the pressure p exerted on it.

36. Every particle of matter in the universe attracts every other particle with a force F that is directly proportional to the product of the masses m_1 and m_2 of the particles, and it is inversely proportional to the square of the distance d between them.

37. Simple interest I is jointly proportional to the annual interest rate r and the time t in years a fixed amount of money is invested.

38. The time t it takes an object to fall is directly proportional to the square root of the distance d it falls.

Construct a mathematical model given the following:

39. y varies directly as x, and $y = 30$ when $x = 6$.

40. y varies directly as x, and $y = 52$ when $x = 4$.

41. y is directly proportional to x, and $y = 12$ when $x = 3$.

42. y is directly proportional to x, and $y = 120$ when $x = 20$.

43. y is inversely proportional to x, and $y = 3$ when $x = 9$.

44. y is inversely proportional to x, and $y = 21$ when $x = 3$.

45. y varies inversely as x, and $y = 2$ when $x = \frac{1}{8}$.

46. y varies inversely as x, and $y = \frac{3}{2}$ when $x = \frac{1}{9}$.

47. y is jointly proportional to x and z, where $y = 2$ when $x = 1$ and $z = 3$.

48. y is jointly proportional to x and z, where $y = 15$ when $x = 3$ and $z = 7$.

49. y varies jointly as x and z, where $y = \frac{2}{3}$ when $x = \frac{1}{2}$ and $z = 12$.

50. y varies jointly as x and z, where $y = 5$ when $x = \frac{3}{2}$ and $z = \frac{2}{9}$.

51. y varies directly as the square of x, where $y = 45$ when $x = 3$.

52. y varies directly as the square of x, where $y = 3$ when $x = \frac{1}{2}$.

53. y is inversely proportional to the square of x, where $y = 27$ when $x = \frac{1}{3}$.

54. y is inversely proportional to the square of x, where $y = 9$ when $x = \frac{2}{3}$.

55. y varies jointly as x and the square of z, where $y = 6$ when $x = \frac{1}{4}$ and $z = \frac{2}{3}$.

56. y varies jointly as x and z and inversely as the square of w, where $y = 5$ when $x = 1$, $z = 3$, and $w = \frac{1}{2}$.

57. y varies directly as the square root of x and inversely as the square of z, where $y = 15$ when $x = 25$ and $z = 2$.

58. y varies directly as the square of x and inversely as z and the square of w, where $y = 14$ when $x = 4$, $w = 2$, and $z = 2$.

Solve applications involving variation.

59. Revenue in dollars is directly proportional to the number of branded sweatshirts sold. The revenue earned from selling 25 sweatshirts is $318.75. Determine the revenue if 30 sweatshirts are sold.

60. The sales tax on the purchase of a new car varies directly as the price of the car. If an $18,000 new car is purchased, then the sales tax is $1,350. How much sales tax is charged if the new car is priced at $22,000?

61. The price of a share of common stock in a company is directly proportional to the earnings per share (EPS) of the previous 12 months. If the price of a share of common stock in a company is $22.55, and the EPS is published to be $1.10, determine the value of the stock if the EPS increases by $0.20.

62. The distance traveled on a road trip varies directly with the time spent on the road. If a 126-mile trip can be made in 3 hours, then what distance can be traveled in 4 hours?
63. The circumference of a circle is directly proportional to its radius. The circumference of a circle with radius 7 centimeters is measured as 14π centimeters. What is the constant of proportionality?
64. The area of circle varies directly as the square of its radius. The area of a circle with radius 7 centimeters is determined to be 49π square centimeters. What is the constant of proportionality?
65. The surface area of a sphere varies directly as the square of its radius. When the radius of a sphere measures 2 meters, the surface area measures 16π square meters. Find the surface area of a sphere with radius 3 meters.
66. The volume of a sphere varies directly as the cube of its radius. When the radius of a sphere measures 3 meters, the volume is 36π cubic meters. Find the volume of a sphere with radius 1 meter.
67. With a fixed height, the volume of a cone is directly proportional to the square of the radius at the base. When the radius at the base measures 10 centimeters, the volume is 200 cubic centimeters. Determine the volume of the cone if the radius of the base is halved.
68. The distance *d* an object in free fall drops varies directly with the square of the time *t* that it has been falling. If an object in free fall drops 36 feet in 1.5 seconds, then how far will it have fallen in 3 seconds?

Hooke's law suggests that the extension of a hanging spring is directly proportional to the weight attached to it. The constant of variation is called the spring constant.

Figure 4.1

Robert Hooke (1635—1703)

69. A hanging spring is stretched 5 inches when a 20-pound weight is attached to it. Determine its spring constant.

70. A hanging spring is stretched 3 centimeters when a 2-kilogram weight is attached to it. Determine the spring constant.

71. If a hanging spring is stretched 3 inches when a 2-pound weight is attached, how far will it stretch with a 5-pound weight attached?

72. If a hanging spring is stretched 6 centimeters when a 4-kilogram weight is attached to it, how far will it stretch with a 2-kilogram weight attached?

The braking distance of an automobile is directly proportional to the square of its speed.

73. It takes 36 feet to stop a particular automobile moving at a speed of 30 miles per hour. How much breaking distance is required if the speed is 35 miles per hour?

74. After an accident, it was determined that it took a driver 80 feet to stop his car. In an experiment under similar conditions, it takes 45 feet to stop the car moving at a speed of 30 miles per hour. Estimate how fast the driver was moving before the accident.

Figure 4.2

Robert Boyle (1627—1691)

Boyle's law states that if the temperature remains constant, the volume *V* of a given mass of gas is inversely proportional to the pressure *p* exerted on it.

75. A balloon is filled to a volume of 216 cubic inches on a diving boat under 1 atmosphere of pressure. If the balloon is taken underwater approximately 33

feet, where the pressure measures 2 atmospheres, then what is the volume of the balloon?

76. A balloon is filled to 216 cubic inches under a pressure of 3 atmospheres at a depth of 66 feet. What would the volume be at the surface, where the pressure is 1 atmosphere?

77. To balance a seesaw, the distance from the fulcrum that a person must sit is inversely proportional to his weight. If a 72-pound boy is sitting 3 feet from the fulcrum, how far from the fulcrum must a 54-pound boy sit to balance the seesaw?

78. The current I in an electrical conductor is inversely proportional to its resistance R. If the current is $\frac{1}{4}$ ampere when the resistance is 100 ohms, what is the current when the resistance is 150 ohms?

79. The amount of illumination I is inversely proportional to the square of the distance d from a light source. If 70 foot-candles of illumination is measured 2 feet away from a lamp, what level of illumination might we expect $\frac{1}{2}$ foot away from the lamp?

80. The amount of illumination I is inversely proportional to the square of the distance d from a light source. If 40 foot-candles of illumination is measured 3 feet away from a lamp, at what distance can we expect 10 foot-candles of illumination?

81. The number of men, represented by y, needed to lay a cobblestone driveway is directly proportional to the area A of the driveway and inversely proportional to the amount of time t allowed to complete the job. Typically, 3 men can lay 1,200 square feet of cobblestone in 4 hours. How many men will be required to lay 2,400 square feet of cobblestone in 6 hours?

82. The volume of a right circular cylinder varies jointly as the square of its radius and its height. A right circular cylinder with a 3-centimeter radius and a height of 4 centimeters has a volume of 36π cubic centimeters. Find a formula for the volume of a right circular cylinder in terms of its radius and height.

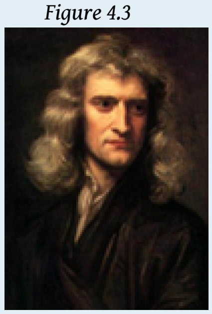
83. The period T of a pendulum is directly proportional to the square root of its length L. If the length of a pendulum is 1 meter, then the period is approximately 2 seconds. Approximate the period of a pendulum that is 0.5 meter in length.

84. The time t it takes an object to fall is directly proportional to the square root of the distance d it falls. An object dropped from 4 feet will take $\frac{1}{2}$ second to hit

the ground. How long will it take an object dropped from 16 feet to hit the ground?

Newton's universal law of gravitation states that every particle of matter in the universe attracts every other particle with a force *F* that is directly proportional to the product of the masses m_1 and m_2 of the particles and inversely proportional to the square of the distance *d* between them. The constant of proportionality is called the gravitational constant.

Figure 4.3



Sir Isaac Newton (1643—1727)

Source: Portrait of Isaac Newton by Sir Godfrey Kneller, from http://commons.wikimedia.org/wiki/File:GodfreyKneller-IsaacNewton-1689.

*http://commons.wikimedia.org/wiki/File:Frans_Hals_-_Portret_*

_van_Ren%C3%A9_Descartes.jpg.

85. If two objects with masses 50 kilograms and 100 kilograms are $\frac{1}{2}$ meter apart, then they produce approximately 1.34×10^{-6} newtons (N) of force. Calculate the gravitational constant.

86. Use the gravitational constant from the previous exercise to write a formula that approximates the force *F* in newtons between two masses m_1 and m_2, expressed in kilograms, given the distance *d* between them in meters.

87. Calculate the force in newtons between Earth and the Moon, given that the mass of the Moon is approximately 7.3×10^{22} kilograms, the mass of Earth is approximately 6.0×10^{24} kilograms, and the distance between them is on average 1.5×10^{11} meters.

88. Calculate the force in newtons between Earth and the Sun, given that the mass of the Sun is approximately 2.0×10^{30} kilograms, the mass of Earth is approximately 6.0×10^{24} kilograms, and the distance between them is on average 3.85×10^{8} meters.

89. If *y* varies directly as the square of *x*, then how does *y* change if *x* is doubled?

90. If *y* varies inversely as square of *t*, then how does *y* change if *t* is doubled?

91. If *y* varies directly as the square of *x* and inversely as the square of *t*, then how does *y* change if both *x* and *t* are doubled?

ANSWERS

1. 20 miles per hour
3. 240 miles per hour
5. 52 miles per hour
7. 5 miles per hour
9. 112 miles per hour
11. 10 miles per hour
13. 1 mile per hour
15. 40 miles per hour
17. 50 miles per hour
19. $2\frac{4}{7}$ hours
21. 30 hours
23. 6 hours
25. 20 hours
27. 3 hours
29. 4 hours
31. $D = kt$
33. $d = kv^2$
35. $V = \frac{k}{p}$
37. $I = krt$
39. $y = 5x$
41. $y = 4x$
43. $y = \frac{27}{x}$
45. $y = \frac{1}{4x}$
47. $y = \frac{2}{3}xz$

49. $y = \frac{1}{9} xz$

51. $y = 5x^2$

53. $y = \frac{3}{x^2}$

55. $y = 54xz^2$

57. $y = \frac{12\sqrt{x}}{z^2}$

59. $382.50

61. $26.65

63. 2π

65. 36π square meters

67. 50 cubic centimeters

69. $\frac{1}{4}$

71. 7.5 inches

73. 49 feet

75. 108 cubic inches

77. 4 feet

79. 1,120 foot-candles

81. 4 men

83. 1.4 seconds

85. 6.7×10^{-11} N m^2/kg^2

87. 1.98×10^{20} N

89. y changes by a factor of 4

91. y remains unchanged

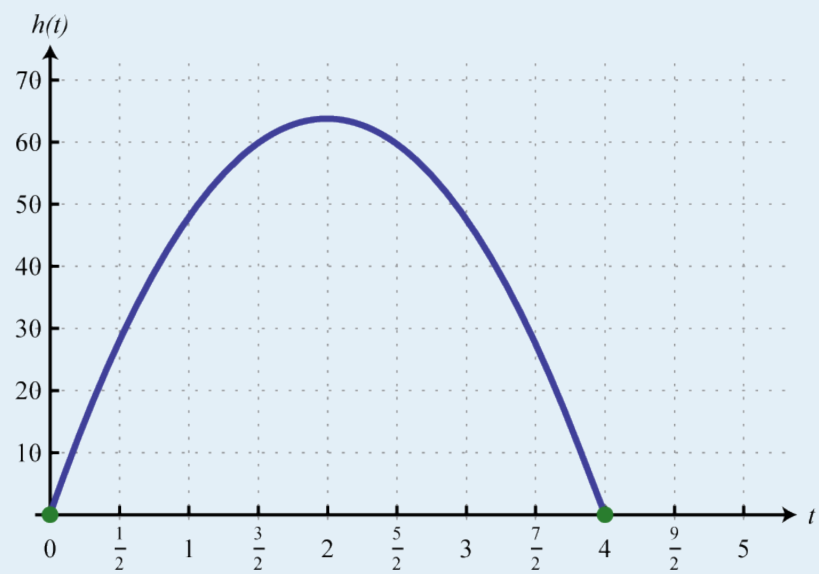
4.9 Review Exercises and Sample Exam

REVIEW EXERCISES

ALGEBRA OF FUNCTIONS

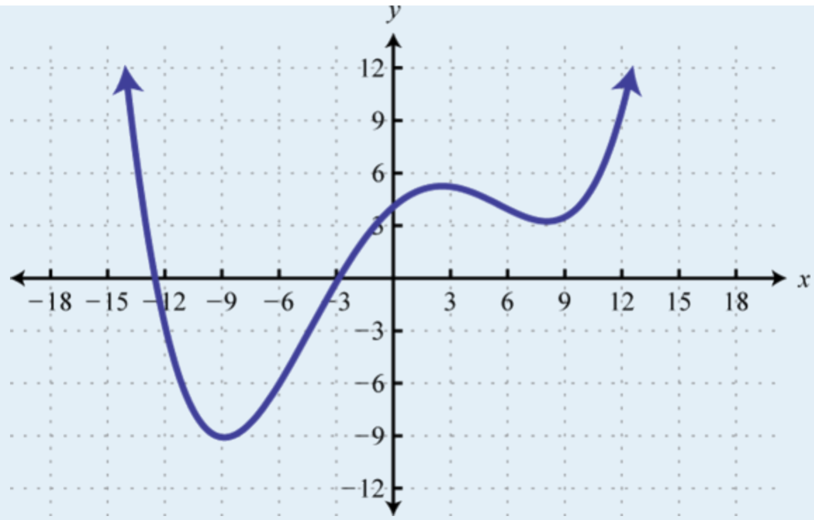
Evaluate

1. Given $f(x) = 2x^2 - x + 6$, find $f(-3)$, $f(0)$, and $f(10)$.
2. Given $g(x) = -x^2 + 4x - 1$, find $g(-1)$, $g(0)$, and $g(3)$.
3. Given $h(t) = -t^3 - 2t^2 + 3$, find $h(-3)$, $h(0)$, and $h(2)$.
4. Given $p(x) = x^4 - 2x^2 + x$, find $p(-1)$, $p(0)$, and $p(2)$.
5. The following graph gives the height $h(t)$ in feet of a projectile over time t in seconds.



   a. Use the graph to determine the height of the projectile at 2.5 seconds.
   b. At what time does the projectile reach its maximum height?
   c. How long does it take the projectile to return to the ground?

6. Given the graph of the function f, find $f(-9)$, $f(-3)$, and $f(12)$.



7. From the ground, a bullet is fired straight up into the air at 340 meters per second. Ignoring the effects of air friction, write a function that models the height of the bullet and use it to calculate the bullet's height after one-quarter of a second. (Round off to the nearest meter.)

8. An object is tossed into the air at an initial speed of 30 feet per second from a rooftop 10 feet high. Write a function that models the height of the object and use it to calculate the height of the object after 1 second.

Perform the operations.

9. Given $f(x) = 5x^2 - 3x + 1$ and $g(x) = 2x^2 - x - 1$, find $(f + g)(x)$.

10. Given $f(x) = x^2 + 3x - 8$ and $g(x) = x^2 - 5x - 7$, find $(f - g)(x)$.

11. Given $f(x) = 3x^2 - x + 2$ and $g(x) = 2x - 3$, find $(f \cdot g)(x)$.

12. Given $f(x) = 27x^5 - 15x^3 - 3x^2$ and $g(x) = 3x^2$, find $(f/g)(x)$.

13. Given $g(x) = x^2 - x + 1$, find $g(-3u)$.

14. Given $g(x) = x^3 - 1$, find $g(x - 1)$.

Given $f(x) = 16x^3 - 12x^2 + 4x$, $g(x) = x^2 - x + 1$, **and** $h(x) = 4x$, **find the following:**

15. $(g \cdot h)(x)$

16. $(f-g)\;(x)$
17. $(g+f)\;(x)$
18. $(f/h)\;(x)$
19. $(f \cdot h)\;(-1)$
20. $(g+h)\;(-3)$
21. $(g-f)\;(2)$
22. $(f/h)\;\left(\frac{3}{2}\right)$

FACTORING POLYNOMIALS

Factor out the greatest common factor (GCF).

23. $2x^4 - 12x^3 - 2x^2$
24. $18a^3b - 3a^2b^2 + 3ab^3$
25. $x^4y^3 - 3x^3y + x^2y$
26. $x^{3n} - x^{2n} - x^n$

Factor by grouping.

27. $2x^3 - x^2 + 2x - 1$
28. $3x^3 - x^2 - 6x + 2$
29. $x^3 - 5x^2y + xy^2 - 5y^3$
30. $a^2b - a + ab^3 - b^2$
31. $2x^4 - 4xy^3 + 2x^2y^2 - 4x^3y$
32. $x^4y^2 - xy^5 + x^3y^4 - x^2y^3$

Factor the special binomials.

33. $64x^2 - 1$

34. $9 - 100y^2$
35. $x^2 - 36y^2$
36. $4 - (2x - 1)^2$
37. $a^3b^3 + 125$
38. $64x^3 - y^3$
39. $81x^4 - y^4$
40. $x^8 - 1$
41. $x^6 - 64y^6$
42. $1 - a^6b^6$

FACTORING TRINOMIALS

Factor.

43. $x^2 - 8x - 48$
44. $x^2 - 15x + 54$
45. $x^2 - 4x - 6$
46. $x^2 - 12xy + 36y^2$
47. $x^2 + 20xy + 75y^2$
48. $-x^2 + 5x + 150$
49. $-2y^2 + 20y + 48$
50. $28x^2 + 20x + 3$
51. $150x^2 - 100x + 6$
52. $24a^2 - 38ab + 3b^2$
53. $27u^2 - 3uv - 4v^2$
54. $16x^2y^2 - 78xy + 27$

55. $16m^2 + 72mn + 81n^2$
56. $4x^2 - 5x + 20$
57. $25x^4 - 35x^2 + 6$
58. $2x^4 + 7x^2 + 3$
59. $x^6 + 3x^3y^3 - 10y^6$
60. $a^6 - 8a^3b^3 + 15b^6$
61. $x^{2n} - 2x^n + 1$
62. $6x^{2n} - x^n - 2$

SOLVE POLYNOMIAL EQUATIONS BY FACTORING

Factor completely.

63. $45x^3 - 20x$
64. $12x^4 - 70x^3 + 50x^2$
65. $-20x^2 + 32x - 3$
66. $-x^3y + 9xy^3$
67. $24a^4b^2 + 3ab^5$
68. $64a^6b^6 - 1$
69. $64x^2 + 1$
70. $x^3 + x^2y - xy^2 - y^3$

Solve by factoring.

71. $9x^2 + 8x = 0$
72. $x^2 - 1 = 0$
73. $x^2 - 12x + 20 = 0$
74. $x^2 - 2x - 48 = 0$

75. $(2x + 1)\,(x - 2) = 3$

76. $2 - (x - 4)^2 = -7$

77. $\left(x - 6\right)\,(x + 3) = -18$

78. $\left(x + 5\right)\,(2x - 1) = 3\,(2x - 1)$

79. $\frac{1}{2}x^2 + \frac{2}{3}x - \frac{1}{8} = 0$

80. $\frac{1}{4}x^2 - \frac{19}{12}x + \frac{1}{2} = 0$

81. $x^3 - 2x^2 - 24x = 0$

82. $x^4 - 5x^2 + 4 = 0$

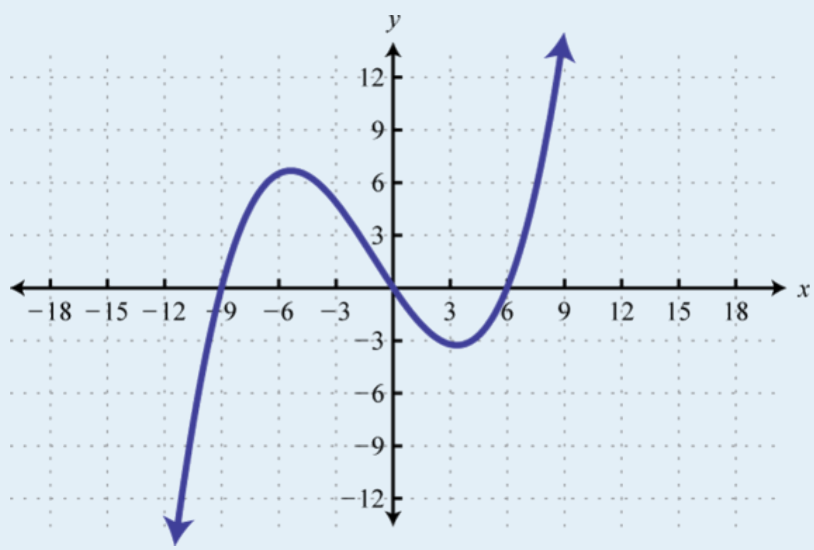
Find the roots of the given functions.

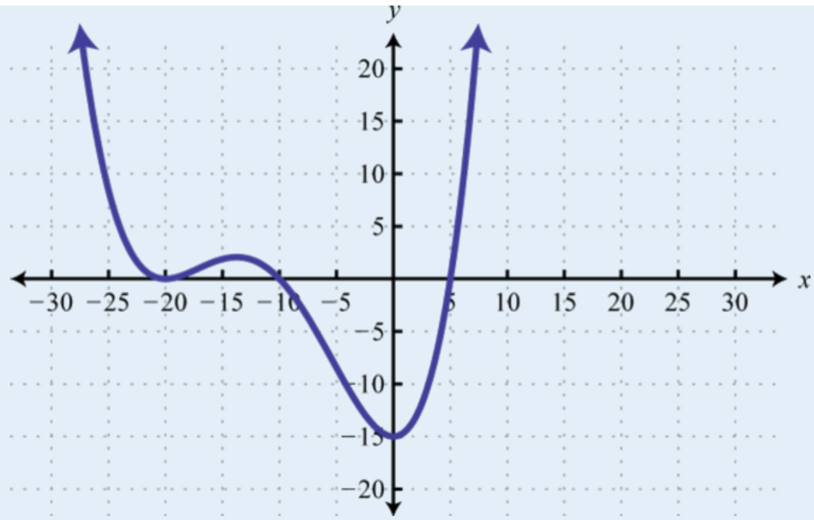
83. $f\,(x) = 12x^2 - 8x$

84. $g\,(x) = 2x^3 - 18x$

85. $h\,(t) = -16t^2 + 64$

86. $p\,(x) = 5x^2 - 21x + 4$

87. 

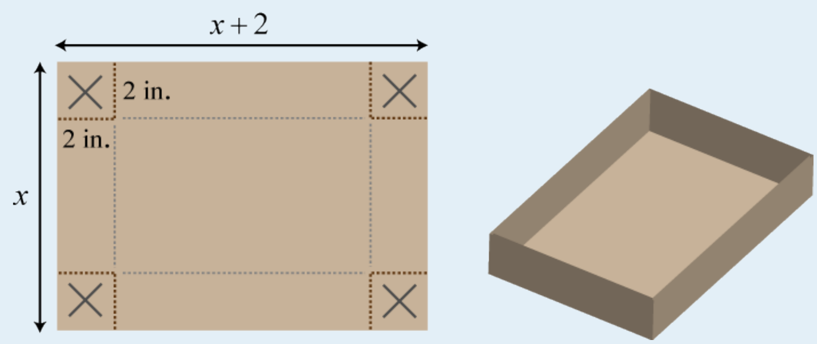
88. 

89. The height in feet of an object dropped from the top of a 16-foot ladder is given by $h(t) = -16t^2 + 16$, where t represents the time in seconds after the object has been dropped. How long will it take to hit the ground?

90. The length of a rectangle is 2 centimeters less than twice its width. If the area of the rectangle is 112 square centimeters, find its dimensions.

91. A triangle whose base is equal in measure to its height has an area of 72 square inches. Find the length of the base.

92. A box can be made by cutting out the corners and folding up the edges of a sheet of cardboard. A template for a rectangular cardboard box of height 2 inches is given.



What are the dimensions of a cardboard sheet that will make a rectangular box with volume 240 cubic inches?

Solve or factor.

93. $x^2 - 25$

94. $x^2 - 121 = 0$

95. $16x^2 - 22x - 3 = 0$

96. $3x^2 - 14x - 5$

97. $x^3 - x^2 - 2x - 2$

98. $3x^2 = -15x$

Find a polynomial equation with integer coefficients, given the solutions.

99. 5, −2

100. $\frac{2}{3}, \ -\frac{1}{2}$

101. $\pm \frac{4}{5}$

102. ±10

103. −4, 0, 3

104. −8 double root

RATIONAL FUNCTIONS: MULTIPLICATION AND DIVISION

State the restrictions and simplify.

105. $\dfrac{108x^3}{12x^2}$

106. $\dfrac{56x^2(x-2)^2}{8x(x-2)^3}$

107. $\dfrac{64 - x^2}{2x^2 - 15x - 8}$

108. $\dfrac{3x^2 + 28x + 9}{81 - x^2}$

109. $\dfrac{x^2 - 25}{5x^2} \cdot \dfrac{10x^2 - 15x}{2x^2 + 7x - 15}$

110. $\dfrac{7x^2 - 41x - 6}{(x-7)^2} \cdot \dfrac{49 - x^2}{x^2 + x - 42}$

111. $\dfrac{28x^2(2x-3)}{4x^2 - 9} \div \dfrac{7x}{4x^2 - 12x + 9}$

112. $\frac{x^2 - 10x + 24}{x^2 - 8x + 16} \div \frac{2x^2 - 13x + 6}{x^2 + 2x - 24}$

Perform the operations and simplify. Assume all variable expressions in the denominator are nonzero.

113. $\frac{a^2 - b^2}{4a^2b^2 + 4ab^3} \cdot \frac{2ab}{a^2 - 2ab + b^2}$

114. $\frac{a^2 - 5ab + 6b^2}{a^2 - 4ab + 4b^2} \div \frac{9b^2 - a^2}{3a^3b - 6a^2b^2}$

115. $\frac{x^2 + xy + y^2}{4x^2 + 3xy - y^2} \cdot \frac{x^2 - y^2}{x^3 - y^3} \div \frac{x + y}{12x^2y - 3xy^2}$

116. $\frac{x^4 - y^4}{x^2 - 2xy + y^2} \div \frac{x^2 - 4xy - 5y^2}{10x^3} \cdot \frac{2x^2 - 11xy + 5y^2}{2x^3y + 2xy^3}$

Perform the operations and state the restrictions.

117. Given $f(x) = \frac{4x^2+39x-10}{x^2+3x-10}$ and $g(x) = \frac{2x^2+7x-15}{x^2+13x+30}$, find $(f \cdot g)(x)$.

118. Given $f(x) = \frac{25-x^2}{3+x}$ and $g(x) = \frac{9-x^2}{5-x}$, find $(f \cdot g)(x)$.

119. Given $f(x) = \frac{42x^2}{2x^2+3x-2}$ and $g(x) = \frac{14x}{4x^2-4x+1}$, find $(f/g)(x)$.

120. Given $f(x) = \frac{x^2-20x+100}{x^2-1}$ and $g(x) = \frac{x^2-100}{x^2+2x+1}$, find $(f/g)(x)$.

121. The daily cost in dollars of running a small business is given by $C(x) = 150 + 45x$ where x represents the number of hours the business is in operation. Determine the average cost per hour if the business is in operation for 8 hours in a day.

122. An electric bicycle manufacturer has determined that the cost of producing its product in dollars is given by the function $C(n) = 2n^2 + 100n + 2{,}500$ where n represents the number of electric bicycles produced in a day. Determine the average cost per bicycle if 10 and 20 are produced in a day.

123. Given $f(x) = 3x - 5$, simplify $\frac{f(x+h)-f(x)}{h}$.

124. Given $g(x) = 2x^2 - x + 1$, simplify $\frac{g(x+h)-g(x)}{h}$.

RATIONAL FUNCTIONS: ADDITION AND SUBTRACTION

State the restrictions and simplify.

125. $\frac{5x-6}{x^2-36} - \frac{4x}{x^2-36}$
126. $\frac{2}{x} + 5x$
127. $\frac{5}{x-5} + \frac{1}{2x}$
128. $\frac{x}{x-2} + \frac{3}{x+3}$
129. $\frac{7(x-1)}{4x^2-17x+15} - \frac{2}{x-3}$
130. $\frac{5}{x} - \frac{19x+25}{2x^2+5x}$
131. $\frac{x}{x-5} - \frac{2}{x-3} - \frac{5(x-3)}{x^2-8x+15}$
132. $\frac{3x}{2x-1} - \frac{x-4}{x+4} + \frac{12(2-x)}{2x^2+7x-4}$
133. $\frac{1}{t-1} + \frac{1}{(t-1)^2} - \frac{1}{t^2-1}$
134. $\frac{1}{t-1} - \frac{2t-5}{t^2-2t+1} - \frac{5t^2-3t-2}{(t-1)^3}$
135. $2x^{-1} + x^{-2}$
136. $(x-4)^{-1} - 2x^{-2}$

Simplify. Assume that all variable expressions used as denominators are nonzero.

137. $\frac{\frac{1}{7}+\frac{1}{x}}{\frac{1}{49}-\frac{1}{x^2}}$
138. $\frac{\frac{1}{100}-\frac{1}{x^2}}{\frac{1}{x}-\frac{1}{10}}$
139. $\frac{\frac{3}{x}-\frac{1}{x-5}}{\frac{5}{x+2}-\frac{2}{x}}$

140. $$\frac{1-\frac{12}{x}+\frac{35}{x^2}}{1-\frac{25}{x^2}}$$

141. $$\frac{x-4x^{-1}}{2-5x^{-1}+2x^{-2}}$$

142. $$\frac{8x^{-1}+y^{-1}}{y^{-2}-64x^{-2}}$$

Perform the operations and state the restrictions.

143. Given $f(x)=\frac{3}{x-3}$ and $g(x)=\frac{x-2}{x+2}$, find $(f+g)(x)$.

144. Given $f(x)=\frac{1}{x^2+x}$ and $g(x)=\frac{2x}{x^2-1}$, find $(f+g)(x)$.

145. Given $f(x)=\frac{x-3}{x-5}$ and $g(x)=\frac{x^2-x}{x^2-25}$, find $(f-g)(x)$.

146. Given $f(x)=\frac{11x+4}{x^2-2x-8}$ and $g(x)=\frac{2x}{x-4}$, find $(f-g)(x)$.

SOLVING RATIONAL EQUATIONS

Solve.

147. $$\frac{3}{x}=\frac{1}{2x+15}$$

148. $$\frac{x}{x-4}=\frac{x+8}{x-8}$$

149. $$\frac{x+5}{2(x+2)}+\frac{x-2}{x+4}=1$$

150. $$\frac{2x}{x-5}+\frac{1}{x+1}=0$$

151. $$\frac{x+1}{x-4}+\frac{4}{x+6}=-\frac{10}{x^2+2x-24}$$

152. $$\frac{2}{x}-\frac{12}{2x+3}=\frac{2-3x^2}{2x^2+3x}$$

153. $$\frac{x+7}{x-2}-\frac{9}{x+7}=\frac{81}{x^2+5x-14}$$

154. $$\frac{x}{x+5}+\frac{1}{x-4}=\frac{4x-7}{x^2+x-20}$$

155. $$\frac{2}{3x-1}+\frac{x}{2x+1}=\frac{2\,(3-4x)}{6x^2+x-1}$$

156. $$\frac{x}{x-1}+\frac{1}{x+1}=\frac{2x}{x^2-1}$$

157. $$\frac{2x}{x+5}-\frac{1}{2x-3}=\frac{4-7x}{2x^2+7x-15}$$

158. $$\frac{x}{x+4}+\frac{1}{2x+7}=\frac{x+8}{2x^2+15x+28}$$

159. $$\frac{1}{t-1}-\frac{2}{2t+1}=\frac{1}{t-2}-\frac{2}{2t-1}$$

160. $$\frac{t-1}{t-2}-\frac{t-2}{t-3}=\frac{t-3}{t-4}-\frac{t-4}{t-5}$$

161. Solve for a: $\frac{1}{a}=\frac{1}{b}-\frac{1}{c}$

162. Solve for y: $x=\frac{3y-1}{y-5}$

163. A positive integer is 4 less than another. If the reciprocal of the larger integer is subtracted from twice the reciprocal of the smaller, the result is $\frac{1}{6}$. Find the two integers.

164. If 3 times the reciprocal of the larger of two consecutive odd integers is added to 7 times the reciprocal of the smaller, the result is $\frac{4}{3}$. Find the integers.

165. If the reciprocal of the smaller of two consecutive integers is subtracted from three times the reciprocal of the larger, the result is $\frac{3}{10}$. Find the integers.

166. A positive integer is twice that of another. The sum of the reciprocals of the two positive integers is $\frac{1}{4}$. Find the two integers.

APPLICATIONS AND VARIATION

Use algebra to solve the following applications.

167. Manuel traveled 8 miles on the bus and another 84 miles on a train. If the train was 16 miles per hour faster than the bus, and the total trip took 2 hours, what was the average speed of the train?

168. A boat can average 10 miles per hour in still water. On a trip downriver, the boat was able to travel 7.5 miles with the current. On the return trip, the boat was only able to travel 4.5 miles in the same amount of time against the current. What was the speed of the current?

169. Susan can jog, on average, $1\frac{1}{2}$ miles per hour faster than her husband Bill. Bill can jog 10 miles in the same amount of time it takes Susan to jog 13 miles. How fast, on average, can Susan jog?

170. In the morning, Raul drove 8 miles to visit his grandmother and then returned later that evening. Because of traffic, his average speed on the return trip was $\frac{1}{2}$ that of his average speed that morning. If the total driving time was $\frac{3}{4}$ of an hour, what was his average speed on the return trip?

171. One pipe can completely fill a water tank in 6 hours while another smaller pipe takes 8 hours to fill the same tank. How long will it take to fill the tank to $\frac{3}{4}$ capacity if both pipes are turned on?

172. It takes Bill 3 minutes longer than Jerry to fill an order. Working together they can fill 15 orders in 30 minutes. How long does it take Bill to fill an order by himself?

173. Manny takes twice as long as John to assemble a skateboard. If they work together, they can assemble a skateboard in 6 minutes. How long would it take Manny to assemble the skateboard without John's help?

174. Working alone, Joe can complete the yard work in 30 minutes. It takes Mike 45 minutes to complete work on the same yard. How long would it take them working together?

Construct a mathematical model given the following:

175. y varies directly as x, where $y = 30$ when $x = 5$.

176. y varies inversely as x, where $y = 3$ when $x = -2$.

177. y is jointly proportional to x and z, where $y = -50$ when $x = -2$ and $z = 5$.

178. y is directly proportional to the square of x and inversely proportional to z, where $y = -6$ when $x = 2$ and $z = -8$.

179. The distance an object in free fall varies directly with the square of the time that it has been falling. It is observed that an object falls 36 feet in $1\frac{1}{2}$ seconds. Find an equation that models the distance an object will fall, and use it to determine how far it will fall in $2\frac{1}{2}$ seconds.

180. After the brakes are applied, the stopping distance d of an automobile varies directly with the square of the speed s of the car. If a car traveling 55 miles per hour takes 181.5 feet to stop, how many feet will it take to stop if it is moving 65 miles per hour?

181. The weight of an object varies inversely as the square of its distance from the center of the Earth. If an object weighs 180 lbs on the surface of the Earth (approximately 4,000 miles from the center), then how much will it weigh at 2,000 miles above the Earth's surface?

182. The cost per person of renting a limousine varies inversely with the number of people renting it. If 5 people go in on the rental, the limousine will cost $112 per person. How much will the rental cost per person if 8 people go in on the rental?

183. To balance a seesaw, the distance from the fulcrum that a person must sit is inversely proportional to his weight. If a 52-pound boy is sitting 3 feet away from the fulcrum, then how far from the fulcrum must a 44-pound boy sit? Round to the nearest tenth of a foot.

ANSWERS

1. $f(-3) = 27; f(0) = 6; f(10) = 196$
3. $h(-3) = 12; h(0) = 3; h(2) = -13$
5. a. 60 feet; b. 2 seconds; c. 4 seconds
7. $h(t) = -4.9t^2 + 340t$; at 0.25 second, the bullet's height is about 85 meters.
9. $(f + g)(x) = 7x^2 - 4x$
11. $(f \cdot g)(x) = 6x^3 - 11x^2 + 7x - 6$
13. $g(-3u) = 9u^2 + 3u + 1$
15. $(g \cdot h)(x) = 4x^3 - 4x^2 + 4x$
17. $(g + f)(x) = 16x^3 - 11x^2 + 3x + 1$
19. $(f \cdot h)(-1) = 128$
21. $(g - f)(2) = -85$
23. $2x^2(x^2 - 6x - 1)$
25. $x^2y(x^2y^2 - 3x + 1)$
27. $(x^2 + 1)(2x - 1)$
29. $(x^2 + y^2)(x - 5y)$
31. $2x(x - 2y)(x^2 + y^2)$
33. $(8x + 1)(8x - 1)$
35. $(x + 6y)(x - 6y)$
37. $(ab + 5)(a^2b^2 - 5ab + 25)$
39. $(9x^2 + y^2)(3x + y)(3x - y)$
41. $(x + 2y)(x^2 - 2xy + 4y^2)(x - 2y)(x^2 + 2xy + 4y^2)$

43. $(x-12)(x+4)$

45. Prime

47. $(x+5y)(x+15y)$

49. $-2(y-12)(y+2)$

51. $2(15x-1)(5x-3)$

53. $(3u+v)(9u-4v)$

55. $(4m+9n)^2$

57. $(5x^2-6)(5x^2-1)$

59. $(x^3+5y^3)(x^3-2y^3)$

61. $(x^n-1)^2$

63. $5x(3x+2)(3x-2)$

65. $-(10x-1)(2x-3)$

67. $3ab^2(2a+b)(4a^2-2ab+b^2)$

69. Prime

71. $-\frac{8}{9}, 0$

73. 2, 10

75. $-1, \frac{5}{2}$

77. 0, 3

79. $-\frac{3}{2}, \frac{1}{6}$

81. −4, 0, 6

83. $0, \frac{2}{3}$

85. ±2

87. −9, 0, 6

89. 1 second

91. 12 inches

93. Factor; $(x+5)\ (x-5)$

95. Solve; $-\frac{1}{8}\ ,\ \frac{3}{2}$

97. Factor; $(x-1)\ (x^2-2)$

99. $x^2-3x-10=0$

101. $25x^2-16=0$

103. $x^3+x^2-12x=0$

105. $9x\,;x\neq 0$

107. $-\frac{x+8}{2x+1};x\neq -\frac{1}{2}\,,8$

109. $\frac{x-5}{x};x\neq -5,0,\frac{3}{2}$

111. $\frac{4x(2x-3)^2}{2x+3};x\neq \pm\frac{3}{2}\,,0$

113. $\frac{1}{2b(a-b)}$

115. $\frac{3xy}{x+y}$

117. $(f\cdot g)\ (x)=\frac{(4x-1)(2x-3)}{(x-2)(x+3)};x\neq -10,-5,-3,2$

119. $(f/g)\ (x)=\frac{3x(2x-1)}{x+2};x\neq -2,0,\frac{1}{2}$

121. \$63.75 per hour

123. 3

125. $\frac{1}{x+6};x\neq \pm 6$

127. $\frac{11x-5}{2x(x-5)};x\neq 0,5$

129. $-\frac{1}{4x-5};x\neq \frac{5}{4}\,,3$

131. $\frac{x-5}{x-3};x\neq 3,5$

133. $\frac{t^2+1}{(t+1)(t-1)^2}\ ;\ t\neq \pm 1$

135. $\frac{2x+1}{x^2}; x \neq 0$

137. $\frac{7x}{x-7}$

139. $\frac{(x+2)(2x-15)}{(x-5)(3x-4)}$

141. $\frac{x(x+2)}{2x-1}$

143. $(f+g)(x) = \frac{x^2-2x+12}{(x-3)(x+2)}; x \neq -2, 3$

145. $(f-g)(x) = \frac{3}{x+5}; x \neq \pm 5$

147. −9

149. −1, 4

151. −11, 0

153. Ø

155. −4

157. $-\frac{3}{2}$

159. $\frac{3}{4}$

161. $a = \frac{bc}{c-b}$

163. {8, 12}

165. {5, 6}

167. 48 miles per hour

169. 6.5 miles per hour

171. Approximately 2.6 hours

173. 18 minutes

175. $y = 6x$

177. $y = 5xz$

179. $d = 16t^2$; 100 feet

181. 80 lbs

183. Approximately 3.5 feet

SAMPLE EXAM

Given $f(x) = x^2 - x + 4, g(x) = 5x - 1$**, and** $h(x) = 2x^2 + x - 3$**, find the following:**

1. $(g \cdot h)(x)$
2. $(h - f)(x)$
3. $(f + g)(-1)$

Factor.

4. $x^3 + 16x - 2x^2 - 32$
5. $x^3 - 8y^3$
6. $x^4 - 81$
7. $25x^2y^2 - 40xy + 16$
8. $16x^3y + 12x^2y^2 - 18xy^3$

Solve.

9. $6x^2 + 24x = 0$
10. $(2x + 1)(3x + 2) = 12$
11. $(2x + 1)^2 = 23x + 6$
12. Find a quadratic equation with integer coefficients given the solutions $\left\{\frac{1}{2}, -3\right\}$.
13. Given $f(x) = 5x^2 - x + 4$, simplify $\frac{f(x+h)-f(x)}{h}$, where $h \neq 0$.

Simplify and state the restrictions.

14. $$\frac{4x^2 - 33x + 8}{x^2 - 10x + 16} \div \frac{16x^2 - 1}{x^2 - 4x + 4}$$
15. $$\frac{x - 1}{x - 7} + \frac{1}{1 - x} - \frac{2(x + 11)}{x^2 - 8x + 7}$$

Assume all variable expressions in the denominator are nonzero and simplify.

16. $$\frac{\frac{3}{x}+\frac{1}{y}}{\frac{1}{y^2}-\frac{9}{x^2}}$$

Solve.

17. $$\frac{6x-5}{3x+2}=\frac{2x}{x+1}$$

18. $$\frac{2x}{x+5}-\frac{1}{5-x}=\frac{2x}{x^2-25}$$

19. Find the root of the function defined by $f(x)=\frac{1}{x+3}-4.$

20. Solve for y: $x=\frac{4y}{3y-1}$

Use algebra to solve.

21. The height of an object dropped from a 64-foot building is given by the function $h(t)=-16t^2+64$, where t represents time in seconds after it was dropped.

    a. Determine the height of the object at $\frac{3}{4}$ of a second.
    b. How long will it take the object to hit the ground?

22. One positive integer is 3 units more than another. When the reciprocal of the larger is subtracted from twice the reciprocal of the smaller, the result is $\frac{2}{9}$. Find the two positive integers.

23. A light airplane can average 126 miles per hour in still air. On a trip, the airplane traveled 222 miles with a tailwind. On the return trip, against a headwind of the same speed, the plane was only able to travel 156 miles in the same amount of time. What was the speed of the wind?

24. On the production line, it takes John 2 minutes less time than Mark to assemble a watch. Working together they can assemble 5 watches in 12 minutes. How long does it take John to assemble a watch working alone?

25. Write an equation that relates x and y, given that y varies inversely with the square of x, where $y=-\frac{1}{3}$ when $x=3$. Use it to find y when $x=\frac{1}{2}$.

ANSWERS

1. $(g \cdot h)\ (x) = 10x^3 + 3x^2 - 16x + 3$
3. $(f + g)\ (-1) = 0$
5. $(x - 2y)\ (x^2 + 2xy + 4y^2)$
7. $(5xy - 4)^2$
9. −4, 0
11. $-\frac{1}{4}$, 5
13. $10x + 5h - 1$
15. $\frac{x+2}{x-1}; x \neq 1, 7$
17. $-\frac{5}{3}$
19. $-\frac{11}{4}$
21. a. 55 feet; b. 2 seconds
23. 22 miles per hour
25. $y = -\frac{3}{x^2}; y = -12$

Chapter 5

Radical Functions and Equations

5.1 Roots and Radicals

LEARNING OBJECTIVES

1. Identify and evaluate square and cube roots.
2. Determine the domain of functions involving square and cube roots.
3. Evaluate *n*th roots.
4. Simplify radicals using the product and quotient rules for radicals.

Square and Cube Roots

Recall that a **square root**[1] of a number is a number that when multiplied by itself yields the original number. For example, 5 is a square root of 25, because $5^2 = 25$. Since $(-5)^2 = 25$, we can say that −5 is a square root of 25 as well. Every positive real number has two square roots, one positive and one negative. For this reason, we use the radical sign $\sqrt{\ }$ to denote the **principal (nonnegative) square root**[2] and a negative sign in front of the radical $-\sqrt{\ }$ to denote the negative square root.

$$\begin{aligned} \sqrt{25} &= 5 \quad \textit{Positive square root of 25} \\ -\sqrt{25} &= -5 \quad \textit{Negative square root of 25} \end{aligned}$$

Zero is the only real number with one square root.

$$\sqrt{0} = 0 \quad \text{because} \quad 0^2 = 0$$

1. A number that when multiplied by itself yields the original number.
2. The positive square root of a positive real number, denoted with the symbol $\sqrt{\ }$.

Example 1

Evaluate.

a. $\sqrt{121}$
b. $-\sqrt{81}$

Solution:

a. $\sqrt{121} = \sqrt{11^2} = 11$
b. $-\sqrt{81} = -\sqrt{9^2} = -9$

If the **radicand**[3], the number inside the radical sign, can be factored as the square of another number, then the square root of the number is apparent. In this case, we have the following property:

$$\sqrt{a^2} = a \quad \text{if} \quad a \geq 0$$

Or more generally,

$$\sqrt{a^2} = |a| \quad \text{if} \quad a \in \mathbb{R}$$

The absolute value is important because *a* may be a negative number and the radical sign denotes the principal square root. For example,

$$\sqrt{(-8)^2} = |-8| = 8$$

3. The expression A within a radical sign, $\sqrt[n]{A}$.

Make use of the absolute value to ensure a positive result.

Example 2

Simplify: $\sqrt{(x-2)^2}$.

Solution:

Here the variable expression $x-2$ could be negative, zero, or positive. Since the sign depends on the unknown quantity *x*, we must ensure that we obtain the principal square root by making use of the absolute value.

$$\sqrt{(x-2)^2} = |x-2|$$

Answer: $|x-2|$

The importance of the use of the absolute value in the previous example is apparent when we evaluate using values that make the radicand negative. For example, when $x=1$,

$$\begin{aligned}\sqrt{(x-2)^2} &= |x-2| \\ &= |1-2| \\ &= |-1| \\ &= 1\end{aligned}$$

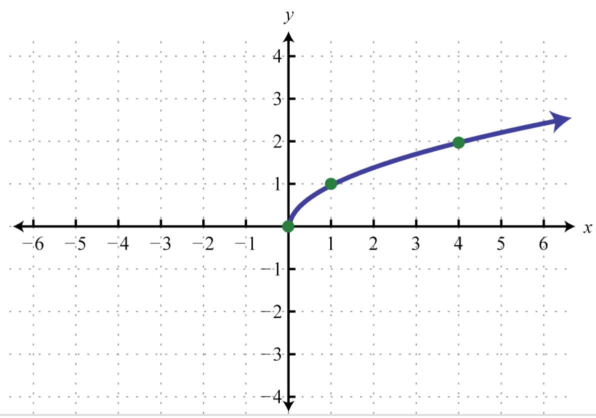
Next, consider the square root of a negative number. To determine the square root of −25, you must find a number that when squared results in −25:

$$\sqrt{-25} = ? \quad \text{or} \quad (\,?\,)^2 = -25$$

However, any real number squared always results in a positive number. The square root of a negative number is currently left undefined. For now, we will state that $\sqrt{-25}$ is not a real number. Therefore, the **square root function**[4] given by $f(x) = \sqrt{x}$ is not defined to be a real number if the *x*-values are negative. The smallest value in the domain is zero. For example, $f(0) = \sqrt{0} = 0$ and $f(4) = \sqrt{4} = 2$. Recall the graph of the square root function.

$f(x) = \sqrt{x}$

x	$f(x)$
0	0
1	1
4	2
9	3



The domain and range both consist of real numbers greater than or equal to zero: $[0, \infty)$. To determine the domain of a function involving a square root we look at the radicand and find the values that produce nonnegative results.

4. The function defined by $f(x) = \sqrt{x}$.

Example 3

Determine the domain of the function defined by $f(x) = \sqrt{2x + 3}$.

Solution:

Here the radicand is $2x + 3$. This expression must be zero or positive. In other words,

$$2x + 3 \geq 0$$

Solve for *x*.

$$\begin{aligned} 2x + 3 &\geq 0 \\ 2x &\geq -3 \\ x &\geq -\frac{3}{2} \end{aligned}$$

Answer: Domain: $\left[-\frac{3}{2}, \infty\right)$

A **cube root**[5] of a number is a number that when multiplied by itself three times yields the original number. Furthermore, we denote a cube root using the symbol $\sqrt[3]{\ }$, where 3 is called the **index**[6]. For example,

$$\sqrt[3]{64} = 4, \quad \text{because} \quad 4^3 = 64$$

5. A number that when used as a factor with itself three times yields the original number, denoted with the symbol $\sqrt[3]{\ }$.

6. The positive integer *n* in the notation $\sqrt[n]{\ }$ that is used to indicate an *n*th root.

The product of three equal factors will be positive if the factor is positive and negative if the factor is negative. For this reason, any real number will have only one real cube root. Hence the technicalities associated with the principal root do not apply. For example,

$$\sqrt[3]{-64} = -4, \quad \text{because} \quad (-4)^3 = -64$$

In general, given any real number *a*, we have the following property:

$$\sqrt[3]{a^3} = a \quad \text{if} \quad \text{a} \in \mathbb{R}$$

When simplifying cube roots, look for factors that are perfect cubes.

Example 4

Evaluate.

a. $\sqrt[3]{8}$
b. $\sqrt[3]{0}$
c. $\sqrt[3]{\frac{1}{27}}$
d. $\sqrt[3]{-1}$
e. $\sqrt[3]{-125}$

Solution:

a. $\sqrt[3]{8} = \sqrt[3]{2^3} = 2$
b. $\sqrt[3]{0} = \sqrt[3]{0^3} = 0$
c. $\sqrt[3]{\frac{1}{27}} = \sqrt[3]{\left(\frac{1}{3}\right)^3} = \frac{1}{3}$
d. $\sqrt[3]{-1} = \sqrt[3]{(-1)^3} = -1$
e. $\sqrt[3]{-125} = \sqrt[3]{(-5)^3} = -5$

It may be the case that the radicand is not a perfect square or cube. If an integer is not a perfect power of the index, then its root will be irrational. For example, $\sqrt[3]{2}$ is an irrational number that can be approximated on most calculators using the root button $\boxed{\sqrt[x]{\ }}$. Depending on the calculator, we typically type in the index prior to pushing the button and then the radicand as follows:

$$3 \quad \boxed{\sqrt[x]{y}} \quad 2 \quad \boxed{=}$$

Therefore, we have

$$\sqrt[3]{2} \approx 1.260, \quad \text{because} \quad 1.260 \wedge 3 \approx 2$$

Since cube roots can be negative, zero, or positive we do not make use of any absolute values.

Example 5

Simplify: $\sqrt[3]{(y-7)^3}$.

Solution:

The cube root of a quantity cubed is that quantity.

$$\sqrt[3]{(y-7)^3} = y - 7$$

Answer: $y - 7$

Try this! Evaluate: $\sqrt[3]{-1000}$.

Answer: -10

(click to see video)

Next, consider the **cube root function**[7]:

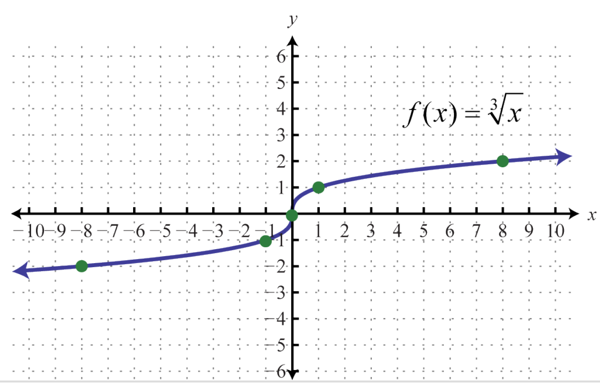
$$f(x) = \sqrt[3]{x} \quad \textit{Cube root function.}$$

7. The function defined by $f(x) = \sqrt[3]{x}$.

Since the cube root could be either negative or positive, we conclude that the domain consists of all real numbers. Sketch the graph by plotting points. Choose some positive and negative values for *x*, as well as zero, and then calculate the corresponding *y*-values.

x	$f(x)=\sqrt[3]{x}$		*Ordered Pairs*
-8	-2	$f(-8)=\sqrt[3]{-8}=-2$	$(-8,-2)$
-1	-1	$f(-1)=\sqrt[3]{-1}=-1$	$(-1,-1)$
0	0	$f(0)=\sqrt[3]{0}=0$	$(0,0)$
1	1	$f(1)=\sqrt[3]{1}=1$	$(1,1)$
8	2	$f(8)=\sqrt[3]{8}=2$	$(8,2)$

Plot the points and sketch the graph of the cube root function.



The graph passes the vertical line test and is indeed a function. In addition, the range consists of all real numbers.

Example 6

Given $g(x) = \sqrt[3]{x+1} + 2$, find $g(-9)$, $g(-2)$, $g(-1)$, and $g(0)$. Sketch the graph of g.

Solution:

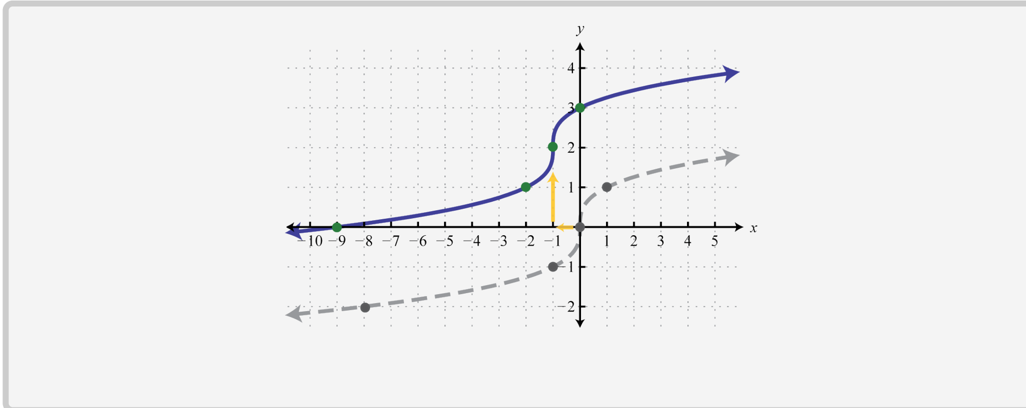
Replace x with the given values.

x	$g(x)$	$g(x)=\sqrt[3]{x+1}+2$	*Ordered Pairs*
-9	0	$g(-9)=\sqrt[3]{-9+1}+2=\sqrt[3]{-8}+2=-2+2=0$	$(-9, 0)$
-2	1	$g(-2)=\sqrt[3]{-2+1}+2=\sqrt[3]{-1}+2=-1+2=1$	$(-2, 1)$
-1	2	$g(-1)=\sqrt[3]{-1+1}+2=\sqrt[3]{0}+2=0+2=2$	$(-1, 2)$
0	3	$g(0)=\sqrt[3]{0+1}+2=\sqrt[3]{1}+2=1+2=3$	$(0, 3)$

We can also sketch the graph using the following translations:

$$y=\sqrt[3]{x} \quad \textit{Basic cube root function}$$
$$y=\sqrt[3]{x+1} \quad \textit{Horizontal shift left 1 unit}$$
$$y=\sqrt[3]{x+1}+2 \quad \textit{Vertical shift up 2 units}$$

Answer:



nth Roots

For any integer $n \geq 2$, we define an **nth root**[8] of a positive real number as a number that when raised to the *n*th power yields the original number. Given any nonnegative real number *a*, we have the following property:

$$\sqrt[n]{d^n} = a, \quad \text{if} \quad a \geq 0$$

Here n is called the index and d^n is called the radicand. Furthermore, we can refer to the entire expression $\sqrt[n]{A}$ as a **radical**[9]. When the index is an integer greater than or equal to 4, we say "fourth root," "fifth root," and so on. The *n*th root of any number is apparent if we can write the radicand with an exponent equal to the index.

8. A number that when raised to the *n*th power $(n \geq 2)$ yields the original number.

9. Used when referring to an expression of the form $\sqrt[n]{A}$.

Example 7

Simplify.

a. $\sqrt[4]{81}$
b. $\sqrt[5]{32}$
c. $\sqrt[7]{1}$
d. $\sqrt[4]{\frac{1}{16}}$

Solution:

a. $\sqrt[4]{81} = \sqrt[4]{3^4} = 3$
b. $\sqrt[5]{32} = \sqrt[5]{2^5} = 2$
c. $\sqrt[7]{1} = \sqrt[7]{1^7} = 1$
d. $\sqrt[4]{\frac{1}{16}} = \sqrt[4]{\left(\frac{1}{2}\right)^4} = \frac{1}{2}$

Note: If the index is $n = 2$, then the radical indicates a square root and it is customary to write the radical without the index; $\sqrt[2]{a} = \sqrt{a}$.

We have already taken care to define the principal square root of a real number. At this point, we extend this idea to *n*th roots when *n* is even. For example, 3 is a fourth root of 81, because $3^4 = 81$. And since $(-3)^4 = 81$, we can say that -3 is a fourth root of 81 as well. Hence we use the radical sign $\sqrt[n]{\ }$ to denote the **principal (nonnegative) *n*th root**[10] when *n* is even. In this case, for any real number *a*, we use the following property:

$$\sqrt[n]{a^n} = |a| \qquad \textit{When n is even}$$

For example,

10. The positive *n*th root when *n* is even.

$$\sqrt[4]{81} = \sqrt[4]{3^4} \quad = |3| \quad = 3$$
$$\sqrt[4]{81} = \sqrt[4]{(-3)^4} = |-3| = 3$$

The negative *n*th root, when *n* is even, will be denoted using a negative sign in front of the radical $-\sqrt[n]{\ }$.

$$-\sqrt[4]{81} = -\sqrt[4]{3^4} = -3$$

We have seen that the square root of a negative number is not real because any real number that is squared will result in a positive number. In fact, a similar problem arises for any even index:

$$\sqrt[4]{-81} = ? \quad or \quad (\ ?\ )^4 = -81$$

We can see that a fourth root of −81 is not a real number because the fourth power of any real number is always positive.

$$\left.\begin{array}{r}\sqrt{-4}\\ \sqrt[4]{-81}\\ \sqrt[6]{-64}\end{array}\right\} \quad \textit{These radicals are not real numbers.}$$

You are encouraged to try all of these on a calculator. What does it say?

Example 8

Simplify.

a. $\sqrt[4]{(-10)^4}$
b. $\sqrt[4]{-10^4}$
c. $\sqrt[6]{(2y+1)^6}$

Solution:

Since the indices are even, use absolute values to ensure nonnegative results.

a. $\sqrt[4]{(-10)^4} = |-10| = 10$
b. $\sqrt[4]{-10^4} = \sqrt[4]{-10{,}000}$ is not a real number.
c. $\sqrt[6]{(2y+1)^6} = |2y+1|$

When the index n is odd, the same problems do not occur. The product of an odd number of positive factors is positive and the product of an odd number of negative factors is negative. Hence when the index n is odd, there is only one real nth root for any real number a. And we have the following property:

$$\sqrt[n]{a^n} = a \qquad \textit{When n is odd}$$

Example 9

Simplify.

a. $\sqrt[5]{(-10)^5}$
b. $\sqrt[5]{-32}$
c. $\sqrt[7]{(2y+1)^7}$

Solution:

Since the indices are odd, the absolute value is not used.

a. $\sqrt[5]{(-10)^5} = -10$
b. $\sqrt[5]{-32} = \sqrt[5]{(-2)^5} = -2$
c. $\sqrt[7]{(2y+1)^7} = 2y + 1$

In summary, for any real number *a* we have,

$$\sqrt[n]{a^n} = |a| \quad \textit{When n is even}$$
$$\sqrt[n]{a^n} = a \quad \textit{When n is odd}$$

When *n is odd*, the *n*th root is *positive or negative* depending on the sign of the radicand.

$$\sqrt[3]{27} = \sqrt[3]{3^3} = 3$$
$$\sqrt[3]{-27} = \sqrt[3]{(-3)^3} = -3$$

When *n is even*, the *n*th root is *positive or not real* depending on the sign of the radicand.

$$\sqrt[4]{16} = \sqrt[4]{2^4} = 2$$
$$\sqrt[4]{16} = \sqrt[4]{(-2)^4} = |-2| = 2$$
$$\sqrt[4]{-16} \quad \textit{Not a real number}$$

Try this! Simplify: $-8\sqrt[5]{-32}$.

Answer: 16

(click to see video)

Simplifying Radicals

It will not always be the case that the radicand is a perfect power of the given index. If it is not, then we use the **product rule for radicals**[11] and the **quotient rule for radicals**[12] to simplify them. Given real numbers $\sqrt[n]{A}$ and $\sqrt[n]{B}$,

Product Rule for Radicals:	$\sqrt[n]{A \cdot B} = \sqrt[n]{A} \cdot \sqrt[n]{B}$
Quotient Rule for Radicals:	$\sqrt[n]{\frac{A}{B}} = \frac{\sqrt[n]{A}}{\sqrt[n]{B}}$

A **radical is simplified**[13] if it does not contain any factors that can be written as perfect powers of the index.

11. Given real numbers $\sqrt[n]{A}$ and $\sqrt[n]{B}$, $\sqrt[n]{A \cdot B} = \sqrt[n]{A} \cdot \sqrt[n]{B}$.

12. Given real numbers $\sqrt[n]{A}$ and $\sqrt[n]{B}$, $\sqrt[n]{\frac{A}{B}} = \frac{\sqrt[n]{A}}{\sqrt[n]{B}}$ where $B \neq 0$.

13. A radical where the radicand does not consist of any factors that can be written as perfect powers of the index.

Example 10

Simplify: $\sqrt{150}$.

Solution:

Here 150 can be written as $2 \cdot 3 \cdot 5^2$.

$$\begin{aligned}\sqrt{150} &= \sqrt{2 \cdot 3 \cdot 5^2} && \textit{Apply the product rule for radicals.}\\ &= \sqrt{2 \cdot 3} \cdot \sqrt{5^2} && \textit{Simplify.}\\ &= \sqrt{6} \cdot 5\\ &= 5\sqrt{6}\end{aligned}$$

We can verify our answer on a calculator:

$$\sqrt{150} \approx 12.25 \quad \text{and} \quad 5\sqrt{6} \approx 12.25$$

Also, it is worth noting that

$$12.25^2 \approx 150$$

Answer: $5\sqrt{6}$

Note: $5\sqrt{6}$ is the exact answer and 12.25 is an approximate answer. We present exact answers unless told otherwise.

Example 11

Simplify: $\sqrt[3]{160}$.

Solution:

Use the prime factorization of 160 to find the largest perfect cube factor:

$$\begin{aligned} 160 &= 2^5 \cdot 5 \\ &= 2^3 \cdot 2^2 \cdot 5 \end{aligned}$$

Replace the radicand with this factorization and then apply the product rule for radicals.

$$\begin{aligned} \sqrt[3]{160} &= \sqrt[3]{2^3 \cdot 2^2 \cdot 5} && \textit{Apply the product rule for radicals.} \\ &= \sqrt[3]{2^3} \cdot \sqrt[3]{2^2 \cdot 5} && \textit{Simplify.} \\ &= 2 \cdot \sqrt[3]{20} \end{aligned}$$

We can verify our answer on a calculator.

$$\sqrt[3]{160} \approx 5.43 \quad \text{and} \quad 2\sqrt[3]{20} \approx 5.43$$

Answer: $2\sqrt[3]{20}$

Example 12

Simplify: $\sqrt[5]{-320}$.

Solution:

Here we note that the index is odd and the radicand is negative; hence the result will be negative. We can factor the radicand as follows:

$$-320 = -1 \cdot 32 \cdot 10 = (-1)^5 \cdot (2)^5 \cdot 10$$

Then simplify:

$$\begin{aligned}\sqrt[5]{-320} &= \sqrt[5]{(-1)^5 \cdot (2)^5 \cdot 10} && \textit{Apply the product rule for radicals.}\\ &= \sqrt[5]{(-1)^5} \cdot \sqrt[5]{(2)^5} \cdot \sqrt[5]{10} && \textit{Simplify.}\\ &= -1 \cdot 2 \cdot \sqrt[5]{10}\\ &= -2 \cdot \sqrt[5]{10}\end{aligned}$$

Answer: $-2\sqrt[5]{10}$

Example 13

Simplify: $\sqrt[3]{-\frac{8}{64}}$.

Solution:

In this case, consider the equivalent fraction with $-8 = (-2)^3$ in the numerator and $64 = 4^3$ in the denominator and then simplify.

$$
\begin{aligned}
\sqrt[3]{-\frac{8}{64}} &= \sqrt[3]{\frac{-8}{64}} && \textit{Apply the quotient rule for radicals.} \\
&= \frac{\sqrt[3]{(-2)^3}}{\sqrt[3]{4^3}} && \textit{Simplify.} \\
&= \frac{-2}{4} \\
&= -\frac{1}{2}
\end{aligned}
$$

Answer: $-\frac{1}{2}$

Try this! Simplify: $\sqrt[4]{\frac{80}{81}}$

Answer: $\frac{2\sqrt[4]{5}}{3}$

(click to see video)

KEY TAKEAWAYS

- To simplify a square root, look for the largest perfect square factor of the radicand and then apply the product or quotient rule for radicals.
- To simplify a cube root, look for the largest perfect cube factor of the radicand and then apply the product or quotient rule for radicals.
- When working with *n*th roots, *n* determines the definition that applies. We use $\sqrt[n]{a^n} = a$ when *n* is odd and $\sqrt[n]{a^n} = |a|$ when *n* is even.
- To simplify *n*th roots, look for the factors that have a power that is equal to the index *n* and then apply the product or quotient rule for radicals. Typically, the process is streamlined if you work with the prime factorization of the radicand.

TOPIC EXERCISES

PART A: SQUARE AND CUBE ROOTS

Simplify.

1. $\sqrt{36}$
2. $\sqrt{100}$
3. $\sqrt{\frac{4}{9}}$
4. $\sqrt{\frac{1}{64}}$
5. $-\sqrt{16}$
6. $-\sqrt{1}$
7. $\sqrt{(-5)^2}$
8. $\sqrt{(-1)^2}$
9. $\sqrt{-4}$
10. $\sqrt{-5^2}$
11. $-\sqrt{(-3)^2}$
12. $-\sqrt{(-4)^2}$
13. $\sqrt{x^2}$
14. $\sqrt{(-x)^2}$
15. $\sqrt{(x-5)^2}$
16. $\sqrt{(2x-1)^2}$
17. $\sqrt[3]{64}$

18. $\sqrt[3]{216}$

19. $\sqrt[3]{-216}$

20. $\sqrt[3]{-64}$

21. $\sqrt[3]{-8}$

22. $\sqrt[3]{1}$

23. $-\sqrt[3]{(-2)^3}$

24. $-\sqrt[3]{(-7)^3}$

25. $\sqrt[3]{\frac{1}{8}}$

26. $\sqrt[3]{\frac{8}{27}}$

27. $\sqrt[3]{(-y)^3}$

28. $-\sqrt[3]{y^3}$

29. $\sqrt[3]{(y-8)^3}$

30. $\sqrt[3]{(2x-3)^3}$

Determine the domain of the given function.

31. $g(x) = \sqrt{x+5}$

32. $g(x) = \sqrt{x-2}$

33. $f(x) = \sqrt{5x+1}$

34. $f(x) = \sqrt{3x+4}$

35. $g(x) = \sqrt{-x+1}$

36. $g(x) = \sqrt{-x-3}$

37. $h(x) = \sqrt{5-x}$

38. $h(x) = \sqrt{2 - 3x}$
39. $g(x) = \sqrt[3]{x + 4}$
40. $g(x) = \sqrt[3]{x - 3}$

Evaluate given the function definition.

41. Given $f(x) = \sqrt{x - 1}$, find $f(1)$, $f(2)$, and $f(5)$
42. Given $f(x) = \sqrt{x + 5}$, find $f(-5)$, $f(-1)$, and $f(20)$
43. Given $f(x) = \sqrt{x} + 3$, find $f(0)$, $f(1)$, and $f(16)$
44. Given $f(x) = \sqrt{x} - 5$, find $f(0)$, $f(1)$, and $f(25)$
45. Given $g(x) = \sqrt[3]{x}$, find $g(-1)$, $g(0)$, and $g(1)$
46. Given $g(x) = \sqrt[3]{x} - 2$, find $g(-1)$, $g(0)$, and $g(8)$
47. Given $g(x) = \sqrt[3]{x + 7}$, find $g(-15)$, $g(-7)$, and $g(20)$
48. Given $g(x) = \sqrt[3]{x - 1} + 2$, find $g(0)$, $g(2)$, and $g(9)$

Sketch the graph of the given function and give its domain and range.

49. $f(x) = \sqrt{x + 9}$
50. $f(x) = \sqrt{x - 3}$
51. $f(x) = \sqrt{x - 1} + 2$
52. $f(x) = \sqrt{x + 1} + 3$
53. $g(x) = \sqrt[3]{x - 1}$
54. $g(x) = \sqrt[3]{x + 1}$
55. $g(x) = \sqrt[3]{x} - 4$
56. $g(x) = \sqrt[3]{x} + 5$
57. $g(x) = \sqrt[3]{x + 2} - 1$

58. $g(x) = \sqrt[3]{x-2} + 3$

59. $f(x) = -\sqrt[3]{x}$

60. $f(x) = -\sqrt[3]{x-1}$

PART B: NTH ROOTS

Simplify.

61. $\sqrt[4]{64}$

62. $\sqrt[4]{16}$

63. $\sqrt[4]{625}$

64. $\sqrt[4]{1}$

65. $\sqrt[4]{256}$

66. $\sqrt[4]{10,000}$

67. $\sqrt[5]{243}$

68. $\sqrt[5]{100,000}$

69. $\sqrt[5]{\frac{1}{32}}$

70. $\sqrt[5]{\frac{1}{243}}$

71. $-\sqrt[4]{16}$

72. $-\sqrt[6]{1}$

73. $\sqrt[5]{-32}$

74. $\sqrt[5]{-1}$

75. $\sqrt{-1}$

76. $\sqrt[4]{-16}$

77. $-6\sqrt[3]{-27}$

78. $-5\sqrt[3]{-8}$

79. $2\sqrt[3]{-1,000}$

80. $7\sqrt[5]{-243}$

81. $6\sqrt[4]{-16}$

82. $12\sqrt[6]{-64}$

83. $3\sqrt{\frac{25}{16}}$

84. $6\sqrt{\frac{16}{9}}$

85. $5\sqrt[3]{\frac{27}{125}}$

86. $7\sqrt[5]{\frac{32}{7^5}}$

87. $-5\sqrt[3]{\frac{8}{27}}$

88. $-8\sqrt[4]{\frac{625}{16}}$

89. $2\sqrt[5]{100,000}$

90. $2\sqrt[7]{128}$

PART C: SIMPLIFYING RADICALS

Simplify.

91. $\sqrt{96}$

92. $\sqrt{500}$

93. $\sqrt{480}$

94. $\sqrt{450}$

95. $\sqrt{320}$

96. $\sqrt{216}$

97. $5\sqrt{112}$

98. $10\sqrt{135}$

99. $-2\sqrt{240}$

100. $-3\sqrt{162}$

101. $\sqrt{\frac{150}{49}}$

102. $\sqrt{\frac{200}{9}}$

103. $\sqrt{\frac{675}{121}}$

104. $\sqrt{\frac{192}{81}}$

105. $\sqrt[3]{54}$

106. $\sqrt[3]{24}$

107. $\sqrt[3]{48}$

108. $\sqrt[3]{81}$

109. $\sqrt[3]{40}$

110. $\sqrt[3]{120}$

111. $\sqrt[3]{162}$

112. $\sqrt[3]{500}$

113. $\sqrt[3]{\frac{54}{125}}$

114. $\sqrt[3]{\frac{40}{343}}$

115. $5\sqrt[3]{-48}$

116. $2\sqrt[3]{-108}$

117. $8\sqrt[4]{96}$

118. $7\sqrt[4]{162}$

119. $\sqrt[5]{160}$

120. $\sqrt[5]{486}$

121. $\sqrt[5]{\frac{224}{243}}$

122. $\sqrt[5]{\frac{5}{32}}$

123. $\sqrt[5]{-\frac{1}{32}}$

124. $\sqrt[6]{-\frac{1}{64}}$

Simplify. Give the exact answer and the approximate answer rounded to the nearest hundredth.

125. $\sqrt{60}$

126. $\sqrt{600}$

127. $\sqrt{\frac{96}{49}}$

128. $\sqrt{\frac{192}{25}}$

129. $\sqrt[3]{240}$

130. $\sqrt[3]{320}$

131. $\sqrt[3]{\frac{288}{125}}$

132. $\sqrt[3]{\frac{625}{8}}$

133. $\sqrt[4]{486}$

134. $\sqrt[5]{288}$

Rewrite the following as a radical expression with coefficient 1.

135. $2\sqrt{15}$

136. $3\sqrt{7}$

137. $5\sqrt{10}$

138. $10\sqrt{3}$

139. $2\sqrt[3]{7}$

140. $3\sqrt[3]{6}$

141. $2\sqrt[4]{5}$

142. $3\sqrt[4]{2}$

143. Each side of a square has a length that is equal to the square root of the square's area. If the area of a square is 72 square units, find the length of each of its sides.

144. Each edge of a cube has a length that is equal to the cube root of the cube's volume. If the volume of a cube is 375 cubic units, find the length of each of its edges.

145. The current I measured in amperes is given by the formula $I = \sqrt{\frac{P}{R}}$ where P is the power usage measured in watts and R is the resistance measured in ohms. If a 100 watt light bulb has 160 ohms of resistance, find the current needed. (Round to the nearest hundredth of an ampere.)

146. The time in seconds an object is in free fall is given by the formula $t = \frac{\sqrt{s}}{4}$ where s represents the distance in feet the object has fallen. How long will it take an object to fall to the ground from the top of an 8-foot stepladder? (Round to the nearest tenth of a second.)

PART D: DISCUSSION BOARD

147. Explain why there are two real square roots for any positive real number and one real cube root for any real number.

148. What is the square root of 1 and what is the cube root of 1? Explain why.

149. Explain why $\sqrt{-1}$ is not a real number and why $\sqrt[3]{-1}$ is a real number.

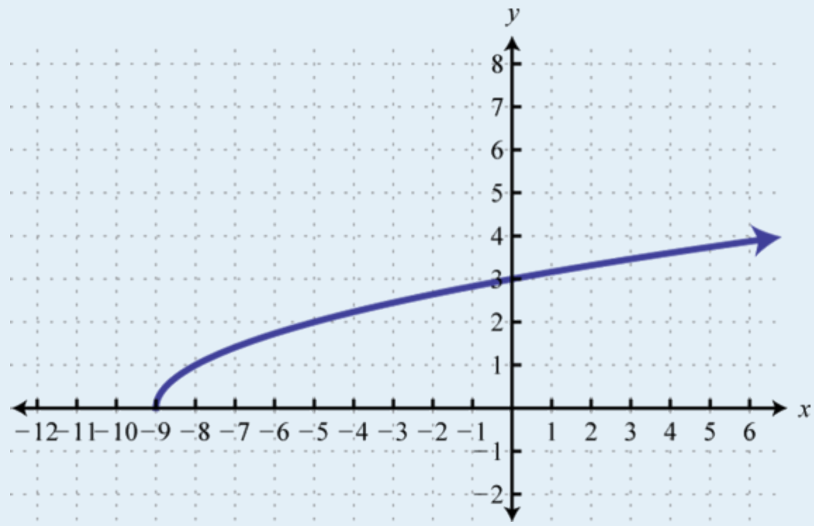
150. Research and discuss the methods used for calculating square roots before the common use of electronic calculators.

ANSWERS

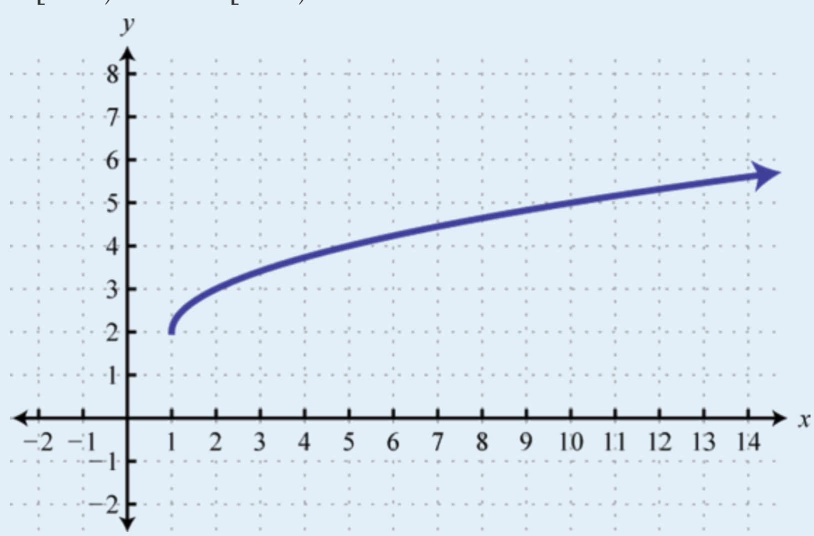
1. 6
3. $\frac{2}{3}$
5. −4
7. 5
9. Not a real number
11. −3
13. $|x|$
15. $|x - 5|$
17. 4
19. −6
21. −2
23. 2
25. $\frac{1}{2}$
27. $-y$
29. $y - 8$
31. $\left[-5, \infty\right)$
33. $\left[-\frac{1}{5}, \infty\right)$
35. $\left(-\infty, 1\right]$
37. $\left(-\infty, 5\right]$
39. $\left(-\infty, \infty\right)$
41. $f(1) = 0; f(2) = 1; f(5) = 2$
43. $f(0) = 3; f(1) = 4; f(16) = 7$
45. $g(-1) = -1; g(0) = 0; g(1) = 1$

47. $g(-15) = -2$; $g(-7) = 0$; $g(20) = 3$

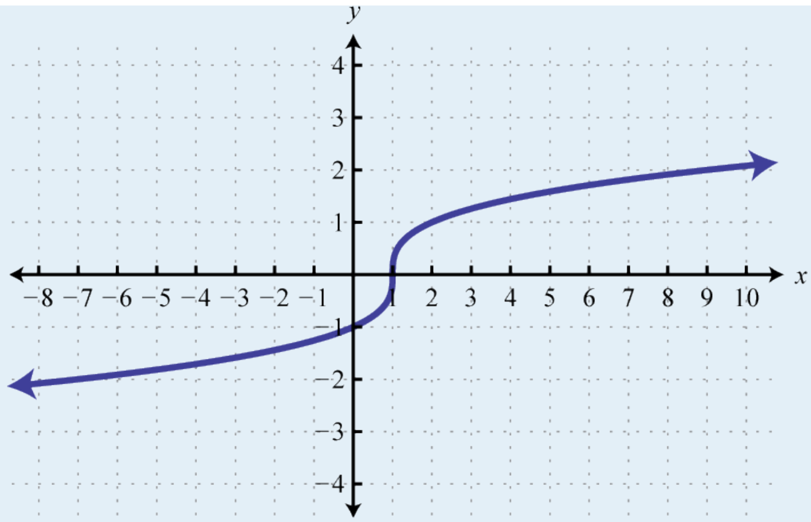
49. Domain: $[-9, \infty)$; range: $[0, \infty)$



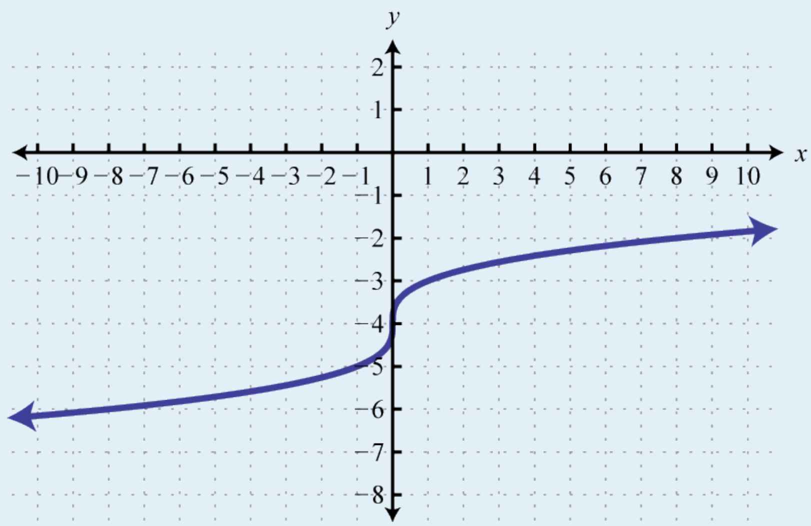
51. Domain: $[1, \infty)$; range: $[2, \infty)$



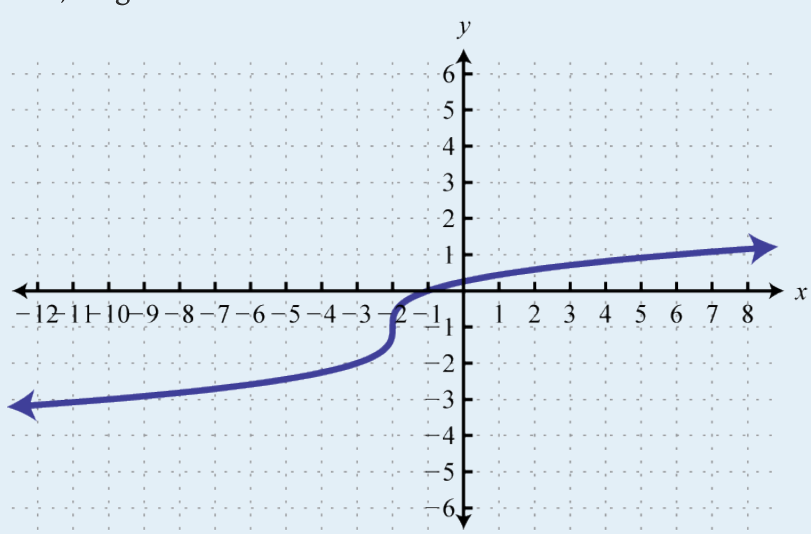
53. Domain: $\mathbb{R}$; range: $\mathbb{R}$



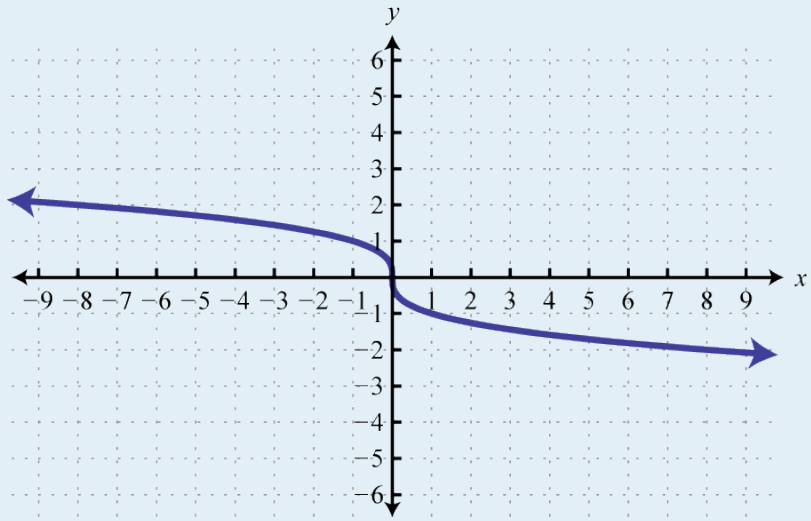
55. Domain: $\mathbb{R}$; range: $\mathbb{R}$



57. Domain: $\mathbb{R}$; range: $\mathbb{R}$



59. Domain: $\mathbb{R}$; range: $\mathbb{R}$



61. 4

63. 5

65. 4

67. 3

69. $\frac{1}{2}$

71. −2

73. −2

75. Not a real number

77. 18

79. −20

81. Not a real number

83. $\frac{15}{4}$

85. 3

87. $-\frac{10}{3}$

89. 20

91. $4\sqrt{6}$

93. $4\sqrt{30}$

95. $8\sqrt{5}$

97. $20\sqrt{7}$

99. $-8\sqrt{15}$

101. $\frac{5\sqrt{6}}{7}$

103. $\frac{15\sqrt{3}}{11}$

105. $3\sqrt[3]{2}$

107. $2\sqrt[3]{6}$

109. $2\sqrt[3]{5}$

111. $3\sqrt[3]{6}$

113. $\frac{3\sqrt[3]{2}}{5}$

115. $-10\sqrt[3]{6}$

117. $16\sqrt[4]{6}$

119. $2\sqrt[5]{5}$

121. $\frac{2\sqrt[5]{7}}{3}$

123. $-\frac{1}{2}$

125. $2\sqrt{15}$; 7.75

127. $\frac{4\sqrt{6}}{7}$; 1.40

129. $2\sqrt[3]{30}$; 6.21

131. $\frac{2\sqrt[3]{36}}{5}$; 1.32

133. $3\sqrt[4]{6}$; 4.70

135. $\sqrt{60}$

137. $\sqrt{250}$

139. $\sqrt[3]{56}$

141. $\sqrt[4]{80}$

143. $6\sqrt{2}$ units

145. Answer: 0.79 ampere

147. Answer may vary

149. Answer may vary

5.2 Simplifying Radical Expressions

LEARNING OBJECTIVES

1. Simplify radical expressions using the product and quotient rule for radicals.
2. Use formulas involving radicals.

Simplifying Radical Expressions

An algebraic expression that contains radicals is called a **radical expression**[14]. We use the product and quotient rules to simplify them.

Example 1

Simplify: $\sqrt[3]{27x^3}$.

Solution:

Use the fact that $\sqrt[n]{a^n} = a$ when n is odd.

$$\begin{aligned} \sqrt[3]{27x^3} &= \sqrt[3]{3^3 \cdot x^3} && \textit{Apply the product rule for radicals.} \\ &= \sqrt[3]{3^3} \cdot \sqrt[3]{x^3} && \textit{Simplify.} \\ &= 3 \cdot x \\ &= 3x \end{aligned}$$

Answer: $3x$

14. An algebraic expression that contains radicals.

Example 2

Simplify: $\sqrt[4]{16y^4}$.

Solution:

Use the fact that $\sqrt[n]{a^n} = |a|$ when n is even.

$$\begin{aligned}\sqrt[4]{16y^4} &= \sqrt[4]{2^4 y^4} && \textit{Apply the product rule for radicals.}\\ &= \sqrt[4]{2^4} \cdot \sqrt[4]{y^4} && \textit{Simplify.}\\ &= 2 \cdot |y|\\ &= 2|y|\end{aligned}$$

Since y is a variable, it may represent a negative number. Thus we need to ensure that the result is positive by including the absolute value.

Answer: $2|y|$

Important Note

Typically, at this point in algebra we note that all variables are assumed to be positive. If this is the case, then y in the previous example is positive and the absolute value operator is not needed. The example can be simplified as follows.

$$\begin{aligned}\sqrt[4]{16y^4} &= \sqrt[4]{2^4 y^4}\\ &= \sqrt[4]{2^4} \cdot \sqrt[4]{y^4}\\ &= 2y\end{aligned}$$

In this section, we will assume that all variables are positive. This allows us to focus on calculating nth roots without the technicalities associated with the principal nth root problem. For this reason, we will use the following property for the rest of the section,

$$\sqrt[n]{a^n} = a, \quad \text{if } \ a \geq 0 \qquad \textit{nth root}$$

When simplifying radical expressions, look for factors with powers that match the index.

Example 3

Simplify: $\sqrt{12x^6y^3}$.

Solution:

Begin by determining the square factors of 12, x^6, and y^3.

$$\left.\begin{aligned} 12 &= 2^2 \cdot 3 \\ x^6 &= \left(x^3\right)^2 \\ y^3 &= y^2 \cdot y \end{aligned}\right\} \textit{Square factors}$$

Make these substitutions, and then apply the product rule for radicals and simplify.

$$\begin{aligned} \sqrt{12x^6y^3} &= \sqrt{2^2 \cdot 3 \cdot \left(x^3\right)^2 \cdot y^2 \cdot y} && \textit{Apply the product rule for radicals.} \\ &= \sqrt{2^2} \cdot \sqrt{\left(x^3\right)^2} \cdot \sqrt{y^2} \cdot \sqrt{3y} && \textit{Simplify.} \\ &= 2 \cdot x^3 \cdot y \cdot \sqrt{3y} \\ &= 2x^3y\sqrt{3y} \end{aligned}$$

Answer: $2x^3y\sqrt{3y}$

Example 4

Simplify: $\sqrt{\frac{18a^5}{b^8}}$.

Solution:

Begin by determining the square factors of 18, a^5, and b^8.

$$\left.\begin{aligned} 18 &= 2 \cdot 3^2 \\ a^5 &= a^2 \cdot a^2 \cdot a = \left(a^2\right)^2 \cdot a \\ b^8 &= b^4 \cdot b^4 \quad\;\; = \left(b^4\right)^2 \end{aligned}\right\} \textit{Square factors}$$

Make these substitutions, apply the product and quotient rules for radicals, and then simplify.

$$\begin{aligned} \sqrt{\frac{18a^5}{b^8}} &= \sqrt{\frac{2 \cdot 3^2 \cdot \left(a^2\right)^2 \cdot a}{\left(b^4\right)^2}} && \textit{Apply the product and quotient rule for radicals.} \\ &= \frac{\sqrt{3^2} \cdot \sqrt{\left(a^2\right)^2} \cdot \sqrt{2a}}{\sqrt{\left(b^4\right)^2}} && \textit{Simplify.} \\ &= \frac{3a^2\sqrt{2a}}{b^4} \end{aligned}$$

Answer: $\frac{3a^2\sqrt{2a}}{b^4}$

Example 5

Simplify: $\sqrt[3]{80x^5y^7}$.

Solution:

Begin by determining the cubic factors of 80, x^5, and y^7.

$$\left.\begin{aligned} 80 &= 2^4 \cdot 5 = 2^3 \cdot 2 \cdot 5 \\ x^5 &= x^3 \cdot x^2 \\ y^7 &= y^6 \cdot y = \left(y^2\right)^3 \cdot y \end{aligned}\right\} \textit{Cubic factors}$$

Make these substitutions, and then apply the product rule for radicals and simplify.

$$\begin{aligned} \sqrt[3]{80x^5y^7} &= \sqrt[3]{2^3 \cdot 2 \cdot 5 \cdot x^3 \cdot x^2 \cdot (y^2)^3 \cdot y} \\ &= \sqrt[3]{2^3} \cdot \sqrt[3]{x^3} \cdot \sqrt[3]{(y^2)^3} \cdot \sqrt[3]{2 \cdot 5 \cdot x^2 \cdot y} \\ &= 2 \cdot xy^2 \cdot \sqrt[3]{10x^2y} \\ &= 2xy^2\sqrt[3]{10x^2y} \end{aligned}$$

Answer: $2xy^2\sqrt[3]{10x^2y}$

Example 6

Simplify $\sqrt[3]{\frac{9x^6}{y^3z^9}}$.

Solution:

The coefficient $9 = 3^2$, and thus does not have any perfect cube factors. It will be left as the only remaining radicand because all of the other factors are cubes, as illustrated below:

$$\left.\begin{aligned} x^6 &= \left(x^2\right)^3 \\ y^3 &= (y)^3 \\ z^9 &= \left(z^3\right)^3 \end{aligned}\right\} \textit{Cubic factors}$$

Replace the variables with these equivalents, apply the product and quotient rules for radicals, and then simplify.

$$\begin{aligned} \sqrt[3]{\frac{9x^6}{y^3z^9}} &= \sqrt[3]{\frac{9 \cdot \left(x^2\right)^3}{y^3 \cdot \left(z^3\right)^3}} \\ &= \frac{\sqrt[3]{9} \cdot \sqrt[3]{\left(x^2\right)^3}}{\sqrt[3]{y^3} \cdot \sqrt[3]{\left(z^3\right)^3}} \\ &= \frac{\sqrt[3]{9} \cdot x^2}{y \cdot z^3} \\ &= \frac{x^2 \sqrt[3]{9}}{yz^3} \end{aligned}$$

Answer: $\frac{x^2 \sqrt[3]{9}}{yz^3}$

Example 7

Simplify: $\sqrt[4]{81a^4b^5}$.

Solution:

Determine all factors that can be written as perfect powers of 4. Here, it is important to see that $b^5 = b^4 \cdot b$. Hence the factor b will be left inside the radical.

$$\begin{aligned}\sqrt[4]{81a^4b^5} &= \sqrt[4]{3^4 \cdot a^4 \cdot b^4 \cdot b}\\ &= \sqrt[4]{3^4} \cdot \sqrt[4]{a^4} \cdot \sqrt[4]{b^4} \cdot \sqrt[4]{b}\\ &= 3 \cdot a \cdot b \cdot \sqrt[4]{b}\\ &= 3ab\sqrt[4]{b}\end{aligned}$$

Answer: $3ab\sqrt[4]{b}$

Example 8

Simplify: $\sqrt[5]{-32x^3y^6z^5}$.

Solution:

Notice that the variable factor x cannot be written as a power of 5 and thus will be left inside the radical. In addition, $y^6 = y^5 \cdot y$; the factor y will be left inside the radical as well.

$$\begin{aligned}\sqrt[5]{-32x^3y^6z^5} &= \sqrt[5]{(-2)^5 \cdot x^3 \cdot y^5 \cdot y \cdot z^5}\\ &= \sqrt[5]{(-2)^5} \cdot \sqrt[5]{y^5} \cdot \sqrt[5]{z^5} \cdot \sqrt[5]{x^3 \cdot y}\\ &= -2 \cdot y \cdot z \cdot \sqrt[5]{x^3 \cdot y}\\ &= -2yz\sqrt[5]{x^3y}\end{aligned}$$

Answer: $-2yz\sqrt[5]{x^3y}$

Tip: To simplify finding an *n*th root, divide the powers by the index.

$$\begin{aligned}\sqrt{a^6} &= a^3, &&\text{which is} \quad a^{6\div 2} = a^3\\ \sqrt[3]{b^6} &= b^2, &&\text{which is} \quad b^{6\div 3} = b^2\\ \sqrt[6]{c^6} &= c, &&\text{which is} \quad c^{6\div 6} = c^1\end{aligned}$$

If the index does not divide into the power evenly, then we can use the quotient and remainder to simplify. For example,

$$\sqrt{a^5} = a^2 \cdot \sqrt{a}, \quad \text{which is} \quad a^{5 \div 2} = a^{2 \ r \ 1}$$
$$\sqrt[3]{b^5} = b \cdot \sqrt[3]{b^2}, \quad \text{which is} \quad b^{5 \div 3} = b^{1 \ r \ 2}$$
$$\sqrt[5]{c^{14}} = c^2 \cdot \sqrt[5]{c^4}, \quad \text{which is} \quad c^{14 \div 5} = c^{2 \ r \ 4}$$

The quotient is the exponent of the factor outside of the radical, and the remainder is the exponent of the factor left inside the radical.

Try this! Simplify: $\sqrt[3]{162a^7 b^5 c^4}$.

Answer: $3a^2 bc\sqrt[3]{6ab^2 c}$

(click to see video)

Formulas Involving Radicals

Formulas often consist of radical expressions. For example, the period of a pendulum, or the time it takes a pendulum to swing from one side to the other and back, depends on its length according to the following formula.

$$T = 2\pi\sqrt{\frac{L}{32}}$$

Here *T* represents the period in seconds and *L* represents the length in feet of the pendulum.

Example 9

If the length of a pendulum measures $1\,\frac{1}{2}$ feet, then calculate the period rounded to the nearest tenth of a second.

Solution:

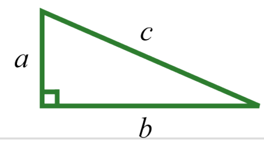
Substitute $1\,\frac{1}{2} = \frac{3}{2}$ for L and then simplify.

$$
\begin{aligned}
T &= 2\pi\sqrt{\frac{L}{32}} \\
&= 2\pi\sqrt{\frac{\frac{3}{2}}{32}} \\
&= 2\pi\sqrt{\frac{3}{2}\cdot\frac{1}{32}} && \textit{Apply the quotient rule for radicals.} \\
&= 2\pi\,\frac{\sqrt{3}}{\sqrt{64}} && \textit{Simplify.} \\
&= \frac{2\pi\sqrt{3}}{8} \\
&= \frac{\pi\sqrt{3}}{4} \approx 1.36
\end{aligned}
$$

Answer: The period is approximately 1.36 seconds.

Frequently you need to calculate the distance between two points in a plane. To do this, form a right triangle using the two points as vertices of the triangle and then apply the Pythagorean theorem. Recall that the Pythagorean theorem states that if given any right triangle with legs measuring a and b units, then the square of the measure of the hypotenuse c is equal to the sum of the squares of the legs:

$a^2 + b^2 = c^2$. In other words, the hypotenuse of any right triangle is equal to the square root of the sum of the squares of its legs.



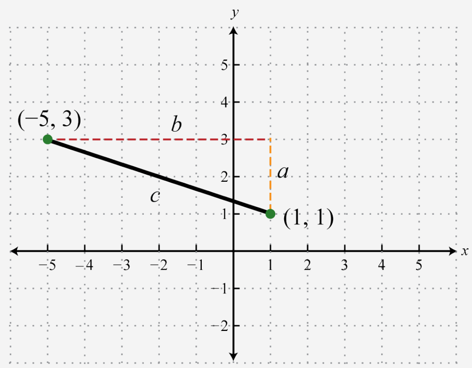
$$a^2 + b^2 = c^2$$
or
$$c = \sqrt{a^2 + b^2}$$

Example 10

Find the distance between (-5, 3) and (1, 1).

Solution:

Form a right triangle by drawing horizontal and vertical lines though the two points. This creates a right triangle as shown below:



The length of leg *b* is calculated by finding the distance between the *x*-values of the given points, and the length of leg *a* is calculated by finding the distance between the given *y*-values.

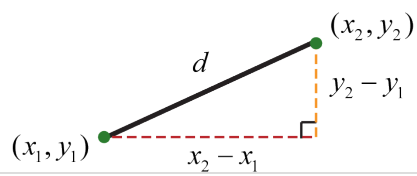
$$a = 3 - 1 = 2\text{ units}$$
$$b = 1 - \left(-5\right) = 1 + 5 = 6\text{ units}$$

Next, use the Pythagorean theorem to find the length of the hypotenuse.

$$\begin{aligned} c &= \sqrt{2^2 + 6^2} \\ &= \sqrt{4 + 36} \\ &= \sqrt{40} \\ &= \sqrt{4 \cdot 10} \\ &= 2\sqrt{10} \text{ units} \end{aligned}$$

Answer: The distance between the two points is $2\sqrt{10}$ units.

Generalize this process to produce a formula that can be used to algebraically calculate the distance between any two given points.



Given two points, (x_1, y_1) and (x_2, y_2), the distance, d, between them is given by the **distance formula**[15], $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

15. Given two points (x_1, y_1) and (x_2, y_2), calculate the distance d between them using the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Example 11

Calculate the distance between (-4, 7) and (2, 1).

Solution:

Use the distance formula with the following points.

$$\begin{array}{cc} (x_1,\ y_1) & (x_2,\ y_2) \\ (-4, 7) & (2,\ 1) \end{array}$$

It is a good practice to include the formula in its general form before substituting values for the variables; this improves readability and reduces the probability of making errors.

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(2 - (-4))^2 + (1 - 7)^2} \\ &= \sqrt{(2 + 4)^2 + (1 - 7)^2} \\ &= \sqrt{(6)^2 + (-6)^2} \\ &= \sqrt{36 + 36} \\ &= \sqrt{72} \\ &= \sqrt{36 \cdot 2} \\ &= 6\sqrt{2} \end{aligned}$$

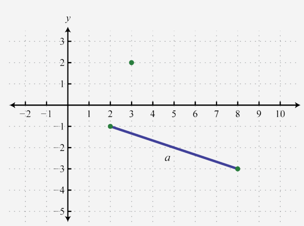
Answer: The distance between the two points is $6\sqrt{2}$ units.

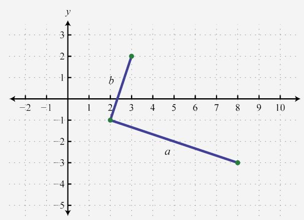
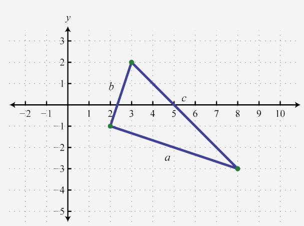
Example 12

Do the three points (2, −1), (3, 2), and (8, −3) form a right triangle?

Solution:

The Pythagorean theorem states that having side lengths that satisfy the property $a^2 + b^2 = c^2$ is a necessary and sufficient condition of right triangles. In other words, if you can show that the sum of the squares of the leg lengths of the triangle is equal to the square of the length of the hypotenuse, then the triangle must be a right triangle. First, calculate the length of each side using the distance formula.

Geometry	*Calculation*
	Points: (2, −1) and (8, −3) $a = \sqrt{(8-2)^2 + [-3-(-1)]^2}$ $= \sqrt{(6)^2 + (-3+1)^2}$ $= \sqrt{36 + (-2)^2}$ $= \sqrt{36+4}$ $= \sqrt{40}$ $= 2\sqrt{10}$

Geometry	*Calculation*
	Points: (2, −1) and (3, 2) $b = \sqrt{(3-2)^2 + [2-(-1)]^2}$ $= \sqrt{(1)^2 + (2+1)^2}$ $= \sqrt{1 + (3)^2}$ $= \sqrt{1+9}$ $= \sqrt{10}$
	Points: (3, 2) and (8, −3) $c = \sqrt{(8-3)^2 + (-3-2)^2}$ $= \sqrt{(5)^2 + (-5)^2}$ $= \sqrt{25+25}$ $= \sqrt{50}$ $= 5\sqrt{2}$

Now we check to see if $a^2 + b^2 = c^2$.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ \left(2\sqrt{10}\right)^2 + \left(\sqrt{10}\right)^2 &= \left(5\sqrt{2}\right)^2 \\ 4\left(\sqrt{10}\right)^2 + \left(\sqrt{10}\right)^2 &= 25\left(\sqrt{2}\right)^2 \\ 4 \cdot 10 + 10 &= 25 \cdot 2 \\ 50 &= 50 \quad \checkmark \end{aligned}$$

Answer: Yes, the three points form a right triangle.

Try this! The speed of a vehicle before the brakes were applied can be estimated by the length of the skid marks left on the road. On wet concrete, the speed v in miles per hour can be estimated by the formula $v = 2\sqrt{3d}$, where d represents the length of the skid marks in feet. Estimate the speed of a vehicle before applying the brakes if the skid marks left behind measure 27 feet. Round to the nearest mile per hour.

Answer: 18 miles per hour

(click to see video)

KEY TAKEAWAYS

- To simplify a radical expression, look for factors of the radicand with powers that match the index. If found, they can be simplified by applying the product and quotient rules for radicals, as well as the property $\sqrt[n]{a^n} = a$, where a is nonnegative.
- A radical expression is simplified if its radicand does not contain any factors that can be written as perfect powers of the index.
- We typically assume that all variable expressions within the radical are nonnegative. This allows us to focus on simplifying radicals without the technical issues associated with the principal nth root. If this assumption is not made, we will ensure a positive result by using absolute values when simplifying radicals with even indices.

TOPIC EXERCISES

PART A: SIMPLIFYING RADICAL EXPRESSIONS

Assume that the variable could represent any real number and then simplify.

1. $\sqrt{9x^2}$
2. $\sqrt{16y^2}$
3. $\sqrt[3]{8y^3}$
4. $\sqrt[3]{125a^3}$
5. $\sqrt[4]{64x^4}$
6. $\sqrt[4]{81y^4}$
7. $\sqrt{36a^4}$
8. $\sqrt{100a^8}$
9. $\sqrt{4a^6}$
10. $\sqrt{a^{10}}$
11. $\sqrt{18a^4b^5}$
12. $\sqrt{48a^5b^3}$
13. $\sqrt[6]{128x^6y^8}$
14. $\sqrt[6]{a^6b^7c^8}$
15. $\sqrt{(5x-4)^2}$
16. $\sqrt{(3x-5)^4}$
17. $\sqrt{x^2-6x+9}$

18. $\sqrt{x^2 - 10x + 25}$
19. $\sqrt{4x^2 + 12x + 9}$
20. $\sqrt{9x^2 + 6x + 1}$

Simplify. (Assume all variable expressions represent positive numbers.)

21. $\sqrt{49a^2}$
22. $\sqrt{64b^2}$
23. $\sqrt{x^2y^2}$
24. $\sqrt{25x^2y^2z^2}$
25. $\sqrt{180x^3}$
26. $\sqrt{150y^3}$
27. $\sqrt{49a^3b^2}$
28. $\sqrt{4a^4b^3c}$
29. $\sqrt{45x^5y^3}$
30. $\sqrt{50x^6y^4}$
31. $\sqrt{64r^2s^6t^5}$
32. $\sqrt{144r^8s^6t^2}$
33. $\sqrt{(x+1)^2}$
34. $\sqrt{(2x+3)^2}$
35. $\sqrt{4(3x-1)^2}$
36. $\sqrt{9(2x+3)^2}$
37. $\sqrt{\frac{9x^3}{25y^2}}$

38. $\sqrt{\dfrac{4x^5}{9y^4}}$

39. $\sqrt{\dfrac{m^7}{36n^4}}$

40. $\sqrt{\dfrac{147m^9}{n^6}}$

41. $\sqrt{\dfrac{2r^2s^5}{25t^4}}$

42. $\sqrt{\dfrac{36r^5}{s^2t^6}}$

43. $\sqrt[3]{27a^3}$

44. $\sqrt[3]{125b^3}$

45. $\sqrt[3]{250x^4y^3}$

46. $\sqrt[3]{162a^3b^5}$

47. $\sqrt[3]{64x^3y^6z^9}$

48. $\sqrt[3]{216x^{12}y^3}$

49. $\sqrt[3]{8x^3y^4}$

50. $\sqrt[3]{27x^5y^3}$

51. $\sqrt[3]{a^4b^5c^6}$

52. $\sqrt[3]{a^7b^5c^3}$

53. $\sqrt[3]{\dfrac{8x^4}{27y^3}}$

54. $\sqrt[3]{\dfrac{x^5}{125y^6}}$

55. $\sqrt[3]{360r^5s^{12}t^{13}}$

56. $\sqrt[3]{540r^3s^2t^9}$

57. $\sqrt[4]{81x^4}$

58. $\sqrt[4]{x^4y^4}$

59. $\sqrt[4]{16x^4y^8}$

60. $\sqrt[4]{81x^{12}y^4}$

61. $\sqrt[4]{a^4b^5c^6}$

62. $\sqrt[4]{5^4a^6c^8}$

63. $\sqrt[4]{128x^6}$

64. $\sqrt[4]{243y^7}$

65. $\sqrt[5]{\dfrac{32m^{10}}{n^5}}$

66. $\sqrt[5]{\dfrac{3^7m^9}{n^{10}}}$

67. $-3\sqrt{4x^2}$

68. $7\sqrt{9y^2}$

69. $-5x\sqrt{4x^2y}$

70. $-3y\sqrt{16x^3y^2}$

71. $12ab\sqrt{a^5b^3}$

72. $6a^2b\sqrt{9a^7b^2}$

73. $2x\sqrt[3]{8x^6}$

74. $-5x^2\sqrt[3]{27x^3}$

75. $2ab\sqrt[3]{-8a^4b^5}$

76. $5a^2b\sqrt[3]{-27a^3b^3}$

Rewrite the following as a radical expression with coefficient 1.

77. $3x\sqrt{6x}$

78. $5y\sqrt{5y}$

79. $ab\sqrt{10a}$

80. $2ab^2\sqrt{a}$

81. $m^2n\sqrt{mn}$

82. $2m^2n^3\sqrt{3n}$

83. $2x\sqrt[3]{3x}$

84. $3y\sqrt[3]{y^2}$

85. $2y^2\sqrt[4]{4y}$

86. $x^2y\sqrt[5]{9xy^2}$

PART B: FORMULAS INVOLVING RADICALS

The period T in seconds of a pendulum is given by the formula

$$T = 2\pi\sqrt{\frac{L}{32}}$$

where L represents the length in feet of the pendulum. Calculate the period, given each of the following lengths. Give the exact value and the approximate value rounded to the nearest tenth of a second.

87. 8 feet

88. 32 feet

89. $\frac{1}{2}$ foot

90. $\frac{1}{8}$ foot

The time t in seconds an object is in free fall is given by the formula

$$t = \frac{\sqrt{s}}{4}$$

where *s* represents the distance in feet the object has fallen. Calculate the time it takes an object to fall, given each of the following distances. Give the exact value and the approximate value rounded to the nearest tenth of a second.

91. 48 feet

92. 80 feet

93. 192 feet

94. 288 feet

95. The speed of a vehicle before the brakes were applied can be estimated by the length of the skid marks left on the road. On dry pavement, the speed *v* in miles per hour can be estimated by the formula $v = 2\sqrt{6d}$, where *d* represents the length of the skid marks in feet. Estimate the speed of a vehicle before applying the brakes on dry pavement if the skid marks left behind measure 27 feet. Round to the nearest mile per hour.

96. The radius *r* of a sphere can be calculated using the formula $r = \frac{\sqrt[3]{6\pi^2 V}}{2\pi}$, where *V* represents the sphere's volume. What is the radius of a sphere if the volume is 36π cubic centimeters?

Given the function find the *y*-intercept

97. $f(x) = \sqrt{x + 12}$

98. $f(x) = \sqrt{x + 8} - 3$

99. $f(x) = \sqrt[3]{x - 8}$

100. $f(x) = \sqrt[3]{x + 27}$

101. $f(x) = \sqrt[3]{x + 16}$

102. $f(x) = \sqrt[3]{x + 3} - 1$

Use the distance formula to calculate the distance between the given two points.

103. (5, −7) and (3, −8)

104. (−9, 7) and (−8, 4)

105. (−3, −4) and (3, −6)

106. (−5, −2) and (1, −6)

107. (−1, 1) and (−4, 10)

108. (8, −3) and (2, −12)

109. (0, −6) and (−3, 0)

110. (0, 0) and (8, −4)

111. $\left(\frac{1}{2}, -\frac{1}{2}\right)$ and $\left(-1, \frac{3}{2}\right)$

112. $\left(-\frac{1}{3}, 2\right)$ and $\left(\frac{5}{3}, -\frac{2}{3}\right)$

Determine whether or not the three points form a right triangle. Use the Pythagorean theorem to justify your answer.

113. (2,−1), (−1,2), and (6,3)

114. (−5,2), (−1, −2), and (−2,5)

115. (−5,0), (0,3), and (6,−1)

116. (−4,−1), (−2,5), and (7,2)

117. (1,−2), (2,3), and (−3,4)

118. (−2,1), (−1,−1), and (1,3)

119. (−4,0), (−2,−10), and (3,−9)

120. (0,0), (2,4), and (−2,6)

PART D: DISCUSSION BOARD

121. Give a value for *x* such that $\sqrt{x^2} \neq x$. Explain why it is important to assume that the variables represent nonnegative numbers.

122. Research and discuss the accomplishments of Christoph Rudolff. What is he credited for?

123. What is a surd, and where does the word come from?

124. Research ways in which police investigators can determine the speed of a vehicle after an accident has occurred. Share your findings on the discussion board.

ANSWERS

1. $3\,|x|$

3. $2y$

5. $2\,|x|$

7. $6a^2$

9. $2\left|a^3\right|$

11. $3a^2b^2\sqrt{2b}$

13. $2\,|xy|\,\sqrt[6]{2y^2}$

15. $|5x-4|$

17. $|x-3|$

19. $|2x+3|$

21. $7a$

23. xy

25. $6x\sqrt{5x}$

27. $7ab\sqrt{a}$

29. $3x^2y\sqrt{5xy}$

31. $8rs^3t^2\sqrt{t}$

33. $x+1$

35. $2\,(3x-1)$

37. $\dfrac{3x\sqrt{x}}{5y}$

39. $\dfrac{m^3\sqrt{m}}{6n^2}$

41. $\dfrac{rs^2\sqrt{2s}}{5t^2}$

43. $3a$

45. $5xy\sqrt[3]{2x}$

47. $4xy^2z^3$

49. $2xy\sqrt[3]{y}$

51. $abc^2\sqrt[3]{ab^2}$

53. $\dfrac{2x\sqrt[3]{x}}{3y}$

55. $2rs^4t^4\sqrt[3]{45r^2t}$

57. $3x$

59. $2xy^2$

61. $abc\sqrt[4]{bc^2}$

63. $2x\sqrt[4]{8x^2}$

65. $\dfrac{2m^2}{n}$

67. $-6x$

69. $-10x^2\sqrt{y}$

71. $12a^3b^2\sqrt{ab}$

73. $4x^3$

75. $-4a^2b^2\sqrt[3]{ab^2}$

77. $\sqrt{54x^3}$

79. $\sqrt{10a^3b^2}$

81. $\sqrt{m^5n^3}$

83. $\sqrt[3]{24x^4}$

85. $\sqrt[4]{64y^9}$

87. π seconds; 3.1 seconds

89. $\frac{\pi}{4}$ seconds; 0.8 seconds

91. $\sqrt{3}$ seconds; 1.7 seconds

93. $2\sqrt{3}$ seconds; 3.5 seconds

95. 25 miles per hour

97. $\left(0, 2\sqrt{3}\right)$

99. $(0, -2)$

101. $\left(0, 2\sqrt[3]{2}\right)$

103. $\sqrt{5}$ units

105. $2\sqrt{10}$ units

107. $3\sqrt{10}$ units

109. $3\sqrt{5}$ units

111. $\frac{5}{2}$ units

113. Right triangle

115. Not a right triangle

117. Right triangle

119. Right triangle

121. Answer may vary

123. Answer may vary

5.3 Adding and Subtracting Radical Expressions

LEARNING OBJECTIVES

1. Add and subtract like radicals.
2. Simplify radical expressions involving like radicals.

Adding and Subtracting Like Radicals

Adding and subtracting radical expressions is similar to adding and subtracting like terms. Radicals are considered to be **like radicals**[16], or **similar radicals**[17], when they share the same index and radicand. For example, the terms $2\sqrt{6}$ and $5\sqrt{6}$ contain like radicals and can be added using the distributive property as follows:

$$\begin{aligned} 2\sqrt{6} + 5\sqrt{6} &= (2+5)\sqrt{6} \\ &= 7\sqrt{6} \end{aligned}$$

Typically, we do not show the step involving the distributive property and simply write,

$$2\sqrt{6} + 5\sqrt{6} = 7\sqrt{6}$$

When adding terms with like radicals, add only the coefficients; the radical part remains the same.

16. Radicals that share the same index and radicand.

17. Term used when referring to like radicals.

Example 1

Add: $7\sqrt[3]{5} + 3\sqrt[3]{5}$.

Solution:

The terms are like radicals; therefore, add the coefficients.

$$7\sqrt[3]{5} + 3\sqrt[3]{5} = 10\sqrt[3]{5}$$

Answer: $10\sqrt[3]{5}$

Subtraction is performed in a similar manner.

Example 2

Subtract: $4\sqrt{10} - 5\sqrt{10}$.

Solution:

$$\begin{aligned} 4\sqrt{10} - 5\sqrt{10} &= (4-5)\sqrt{10} \\ &= -1\sqrt{10} \\ &= -\sqrt{10} \end{aligned}$$

Answer: $-\sqrt{10}$

If the radicand and the index are not exactly the same, then the radicals are not similar and we cannot combine them.

Example 3

Simplify: $10\sqrt{5} + 6\sqrt{2} - 9\sqrt{5} - 7\sqrt{2}$.

Solution:

$$\begin{aligned} 10\sqrt{5} + 6\sqrt{2} - 9\sqrt{5} - 7\sqrt{2} &= 10\sqrt{5} - 9\sqrt{5} + 6\sqrt{2} - 7\sqrt{2} \\ &= \sqrt{5} \quad - \quad \sqrt{2} \end{aligned}$$

We cannot simplify any further because $\sqrt{5}$ and $\sqrt{2}$ are not like radicals; the radicands are not the same.

Answer: $\sqrt{5} - \sqrt{2}$

Caution: It is important to point out that $\sqrt{5} - \sqrt{2} \neq \sqrt{5-2}$. We can verify this by calculating the value of each side with a calculator.

$$\begin{aligned} \sqrt{5} - \sqrt{2} &\approx 0.82 \\ \sqrt{5-2} = \sqrt{3} &\approx 1.73 \end{aligned}$$

In general, note that $\sqrt[n]{a} \pm \sqrt[n]{b} \neq \sqrt[n]{a \pm b}$.

Example 4

Simplify: $5\sqrt[3]{10} + 3\sqrt{10} - \sqrt[3]{10} - 2\sqrt{10}$.

Solution:

$$\begin{aligned} 5\sqrt[3]{10} + 3\sqrt{10} - \sqrt[3]{10} - 2\sqrt{10} &= 5\sqrt[3]{10} - \sqrt[3]{10} + 3\sqrt{10} - 2\sqrt{10} \\ &= \quad 4\sqrt[3]{10} \quad + \quad \sqrt{10} \end{aligned}$$

We cannot simplify any further, because $\sqrt[3]{10}$ and $\sqrt{10}$ are not like radicals; the indices are not the same.

Answer: $4\sqrt[3]{10} + \sqrt{10}$

Adding and Subtracting Radical Expressions

Often, we will have to simplify before we can identify the like radicals within the terms.

Example 5

Subtract: $\sqrt{32} - \sqrt{18} + \sqrt{50}$.

Solution:

At first glance, the radicals do not appear to be similar. However, after simplifying completely, we will see that we can combine them.

$$\begin{aligned}\sqrt{32} - \sqrt{18} + \sqrt{50} &= \sqrt{16 \cdot 2} - \sqrt{9 \cdot 2} + \sqrt{25 \cdot 2} \\ &= 4\sqrt{2} - 3\sqrt{2} + 5\sqrt{2} \\ &= 6\sqrt{2}\end{aligned}$$

Answer: $6\sqrt{2}$

Example 6

Simplify: $\sqrt[3]{108}+\sqrt[3]{24}-\sqrt[3]{32}-\sqrt[3]{81}$.

Solution:

Begin by looking for perfect cube factors of each radicand.

$$\begin{aligned}\sqrt[3]{108}+\sqrt[3]{24}-\sqrt[3]{32}-\sqrt[3]{81}&=\sqrt[3]{27\cdot 4}+\sqrt[3]{8\cdot 3}-\sqrt[3]{8\cdot 4}-\sqrt[3]{27\cdot 3} && \textit{Simplify.}\\ &=3\sqrt[3]{4}+2\sqrt[3]{3}-2\sqrt[3]{4}-3\sqrt[3]{3} && \textit{Combine lik}\\ &=\sqrt[3]{4}-\sqrt[3]{3}\end{aligned}$$

Answer: $\sqrt[3]{4}-\sqrt[3]{3}$

Try this! Simplify: $\sqrt{20}+\sqrt{27}-3\sqrt{5}-2\sqrt{12}$.

Answer: $-\sqrt{5}-\sqrt{3}$

(click to see video)

Next, we work with radical expressions involving variables. In this section, assume all radicands containing variable expressions are nonnegative.

Example 7

Simplify: $-9\sqrt[3]{5x} - \sqrt[3]{2x} + 10\sqrt[3]{5x}$.

Solution:

Combine like radicals.

$$\begin{aligned}-9\sqrt[3]{5x} - \sqrt[3]{2x} + 10\sqrt[3]{5x} &= -9\sqrt[3]{5x} + 10\sqrt[3]{5x} - \sqrt[3]{2x} \\ &= \sqrt[3]{5x} - \sqrt[3]{2x}\end{aligned}$$

We cannot combine any further because the remaining radical expressions do not share the same radicand; they are not like radicals. Note: $\sqrt[3]{5x} - \sqrt[3]{2x} \neq \sqrt[3]{5x - 2x}$.

Answer: $\sqrt[3]{5x} - \sqrt[3]{2x}$

We will often find the need to subtract a radical expression with multiple terms. If this is the case, remember to apply the distributive property before combining like terms.

Example 8

Simplify: $\left(5\sqrt{x} - 4\sqrt{y}\right) - \left(4\sqrt{x} - 7\sqrt{y}\right)$.

Solution:

$$\begin{aligned}\left(5\sqrt{x} - 4\sqrt{y}\right) - \left(4\sqrt{x} - 7\sqrt{y}\right) &= 5\sqrt{x} - 4\sqrt{y} - 4\sqrt{x} + 7\sqrt{y} \quad \textit{Distribute.} \\ &= 5\sqrt{x} - 4\sqrt{x} - 4\sqrt{y} + 7\sqrt{y} \\ &= \sqrt{x} + 3\sqrt{y}\end{aligned}$$

Answer: $\sqrt{x} + 3\sqrt{y}$

Until we simplify, it is often unclear which terms involving radicals are similar. The general steps for simplifying radical expressions are outlined in the following example.

Example 9

Simplify: $5\sqrt[3]{3x^4} + \sqrt[3]{24x^3} - \left(x\sqrt[3]{24x} + 4\sqrt[3]{3x^3}\right)$.

Solution:

Step 1: Simplify the radical expression. In this case, distribute and then simplify each term that involves a radical.

$$\begin{aligned}
&5\sqrt[3]{3x^4} + \sqrt[3]{24x^3} - \left(x\sqrt[3]{24x} + 4\sqrt[3]{3x^3}\right) \\
&= 5\sqrt[3]{3x^4} + \sqrt[3]{24x^3} - x\sqrt[3]{24x} - 4\sqrt[3]{3x^3} \\
&= 5\sqrt[3]{3 \cdot x \cdot x^3} + \sqrt[3]{8 \cdot 3 \cdot x^3} - x\sqrt[3]{8 \cdot 3x} - 4\sqrt[3]{3x^3} \\
&= 5x\sqrt[3]{3x} + 2x\sqrt[3]{3} - 2x\sqrt[3]{3x} - 4x\sqrt[3]{3}
\end{aligned}$$

Step2: Combine all like radicals. Remember to add only the coefficients; the variable parts remain the same.

$$\begin{aligned}
&= 5x\sqrt[3]{3x} + 2x\sqrt[3]{3} - 2x\sqrt[3]{3x} - 4x\sqrt[3]{3} \\
&= 3x\sqrt[3]{3x} - 2x\sqrt[3]{3}
\end{aligned}$$

Answer: $3x\sqrt[3]{3x} - 2x\sqrt[3]{3}$

Example 10

Simplify: $2a\sqrt{125a^2b} - a^2\sqrt{80b} + 4\sqrt{20a^4b}$.

Solution:

$$\begin{aligned}
&2a\sqrt{125a^2b} - a^2\sqrt{80b} + 4\sqrt{20a^4b} \\
&= 2a\sqrt{25 \cdot 5 \cdot a^2 \cdot b} - a^2\sqrt{16 \cdot 5 \cdot b} + 4\sqrt{4 \cdot 5 \cdot \left(a^2\right)^2 b} && \textit{Factor.} \\
&= 2a \cdot 5 \cdot a\sqrt{5b} - a^2 \cdot 4\sqrt{5b} + 4 \cdot 2 \cdot a^2\sqrt{5b} && \textit{Simplify.} \\
&= 10a^2\sqrt{5b} - 4a^2\sqrt{5b} + 8a^2\sqrt{5b} && \textit{Combine like terms.} \\
&= 14a^2\sqrt{5b}
\end{aligned}$$

Answer: $14a^2\sqrt{5b}$

Try this! $\sqrt[3]{2x^6y} + \sqrt[3]{xy^3} - \left(y\sqrt[3]{27x} - 2x\sqrt[3]{2x^3y}\right)$

Answer: $3x^2\sqrt[3]{2y} - 2y\sqrt[3]{x}$

(click to see video)

Tip

Take careful note of the differences between products and sums within a radical. Assume both *x* and *y* are nonnegative.

Products	**Sums**
$\sqrt{x^2y^2}=xy$	$\sqrt{x^2+y^2}\neq x+y$
$\sqrt[3]{x^3y^3}=xy$	$\sqrt[3]{x^3+y^3}\neq x+y$

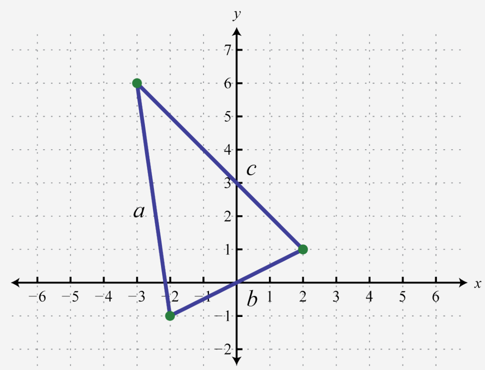
The property $\sqrt[n]{a\cdot b}=\sqrt[n]{a}\cdot\sqrt[n]{b}$ says that we can simplify radicals when the operation in the radicand is multiplication. There is no corresponding property for addition.

Example 11

Calculate the perimeter of the triangle formed by the points $(-2, -1)$, $(-3, 6)$, and $(2, 1)$.

Solution:

The formula for the perimeter of a triangle is $P = a + b + c$ where a, b, and c represent the lengths of each side. Plotting the points we have,



Use the distance formula to calculate the length of each side.

$$
\begin{aligned}
a &= \sqrt{[-3-(-2)]^2 + [6-(-1)]^2} \\
&= \sqrt{(-3+2)^2 + (6+1)^2} \\
&= \sqrt{(-1)^2 + (7)^2} \\
&= \sqrt{1+49} \\
&= \sqrt{50} \\
&= 5\sqrt{2}
\end{aligned}
\qquad
\begin{aligned}
b &= \sqrt{[2-(-2)]^2 + [1-(-1)]^2} \\
&= \sqrt{(2+2)^2 + (1+1)^2} \\
&= \sqrt{(4)^2 + (2)^2} \\
&= \sqrt{16+4} \\
&= \sqrt{20} \\
&= 2\sqrt{5}
\end{aligned}
$$

Similarly we can calculate the distance between (−3, 6) and (2,1) and find that $c = 5\sqrt{2}$ units. Therefore, we can calculate the perimeter as follows:

$$\begin{aligned} P &= a + b + c \\ &= 5\sqrt{2} + 2\sqrt{5} + 5\sqrt{2} \\ &= 10\sqrt{2} + 2\sqrt{5} \end{aligned}$$

Answer: $10\sqrt{2} + 2\sqrt{5}$ units

KEY TAKEAWAYS

- Add and subtract terms that contain like radicals just as you do like terms. If the index and radicand are exactly the same, then the radicals are similar and can be combined. This involves adding or subtracting only the coefficients; the radical part remains the same.
- Simplify each radical completely before combining like terms.

TOPIC EXERCISES

PART A: ADDING AND SUBTRACTING LIKE RADICALS

Simplify

1. $10\sqrt{3} - 5\sqrt{3}$
2. $15\sqrt{6} - 8\sqrt{6}$
3. $9\sqrt{3} + 5\sqrt{3}$
4. $12\sqrt{6} + 3\sqrt{6}$
5. $4\sqrt{5} - 7\sqrt{5} - 2\sqrt{5}$
6. $3\sqrt{10} - 8\sqrt{10} - 2\sqrt{10}$
7. $\sqrt{6} - 4\sqrt{6} + 2\sqrt{6}$
8. $5\sqrt{10} - 15\sqrt{10} - 2\sqrt{10}$
9. $13\sqrt{7} - 6\sqrt{2} - 5\sqrt{7} + 5\sqrt{2}$
10. $10\sqrt{13} - 12\sqrt{15} + 5\sqrt{13} - 18\sqrt{15}$
11. $6\sqrt{5} - \left(4\sqrt{3} - 3\sqrt{5}\right)$
12. $-12\sqrt{2} - \left(6\sqrt{6} + \sqrt{2}\right)$
13. $\left(2\sqrt{5} - 3\sqrt{10}\right) - \left(\sqrt{10} + 3\sqrt{5}\right)$
14. $\left(-8\sqrt{3} + 6\sqrt{15}\right) - \left(\sqrt{3} - \sqrt{15}\right)$
15. $4\sqrt[3]{6} - 3\sqrt[3]{5} + 6\sqrt[3]{6}$
16. $\sqrt[3]{10} + 5\sqrt[3]{10} - 4\sqrt[3]{10}$
17. $\left(7\sqrt[3]{9} - 4\sqrt[3]{3}\right) - \left(\sqrt[3]{9} - 3\sqrt[3]{3}\right)$

18. $\left(-8\sqrt[3]{5}+\sqrt[3]{25}\right)-\left(2\sqrt[3]{5}+6\sqrt[3]{25}\right)$

Simplify. (Assume all radicands containing variable expressions are positive.)

19. $\sqrt{2x}-4\sqrt{2x}$
20. $5\sqrt{3y}-6\sqrt{3y}$
21. $9\sqrt{x}+7\sqrt{x}$
22. $-8\sqrt{y}+4\sqrt{y}$
23. $7x\sqrt{y}-3x\sqrt{y}+x\sqrt{y}$
24. $10y^2\sqrt{x}-12y^2\sqrt{x}-2y^2\sqrt{x}$
25. $2\sqrt{ab}-5\sqrt{a}+6\sqrt{ab}-10\sqrt{a}$
26. $-3x\sqrt{y}+6\sqrt{y}-4x\sqrt{y}-7\sqrt{y}$
27. $5\sqrt{xy}-\left(3\sqrt{xy}-7\sqrt{xy}\right)$
28. $-8a\sqrt{b}-\left(2a\sqrt{b}-4\sqrt{ab}\right)$
29. $\left(3\sqrt{2x}-\sqrt{3x}\right)-\left(\sqrt{2x}-7\sqrt{3x}\right)$
30. $\left(\sqrt{y}-4\sqrt{2y}\right)-\left(\sqrt{y}-5\sqrt{2y}\right)$
31. $5\sqrt[3]{x}-12\sqrt[3]{x}$
32. $-2\sqrt[3]{y}-3\sqrt[3]{y}$
33. $a\sqrt[5]{3b}+4a\sqrt[5]{3b}-a\sqrt[5]{3b}$
34. $-8\sqrt[4]{ab}+3\sqrt[4]{ab}-2\sqrt[4]{ab}$
35. $6\sqrt{2a}-4\sqrt[3]{2a}+7\sqrt{2a}-\sqrt[3]{2a}$
36. $4\sqrt[5]{3a}+\sqrt[3]{3a}-9\sqrt[5]{3a}+\sqrt[3]{3a}$

37. $\left(\sqrt[4]{4xy} - \sqrt[3]{xy}\right) - \left(2\sqrt[4]{4xy} - \sqrt[3]{xy}\right)$

38. $\left(5\sqrt[6]{6y} - 5\sqrt{y}\right) - \left(2\sqrt[6]{6y} + 3\sqrt{y}\right)$

39. $2x^2\sqrt[3]{3x} - \left(x^2\sqrt[3]{3x} - x\sqrt[3]{3x}\right)$

40. $5y^3\sqrt{6y} - \left(\sqrt{6y} - 4y^3\sqrt{6y}\right)$

PART B: ADDING AND SUBTRACTING RADICAL EXPRESSIONS

Simplify.

41. $\sqrt{75} - \sqrt{12}$

42. $\sqrt{24} - \sqrt{54}$

43. $\sqrt{32} + \sqrt{27} - \sqrt{8}$

44. $\sqrt{20} + \sqrt{48} - \sqrt{45}$

45. $\sqrt{28} - \sqrt{27} + \sqrt{63} - \sqrt{12}$

46. $\sqrt{90} + \sqrt{24} - \sqrt{40} - \sqrt{54}$

47. $\sqrt{45} - \sqrt{80} + \sqrt{245} - \sqrt{5}$

48. $\sqrt{108} + \sqrt{48} - \sqrt{75} - \sqrt{3}$

49. $4\sqrt{2} - \left(\sqrt{27} - \sqrt{72}\right)$

50. $-3\sqrt{5} - \left(\sqrt{20} - \sqrt{50}\right)$

51. $\sqrt[3]{16} - \sqrt[3]{54}$

52. $\sqrt[3]{81} - \sqrt[3]{24}$

53. $\sqrt[3]{135} + \sqrt[3]{40} - \sqrt[3]{5}$

54. $\sqrt[3]{108} - \sqrt[3]{32} - \sqrt[3]{4}$
55. $2\sqrt{27} - 2\sqrt{12}$
56. $3\sqrt{50} - 4\sqrt{32}$
57. $3\sqrt{243} - 2\sqrt{18} - \sqrt{48}$
58. $6\sqrt{216} - 2\sqrt{24} - 2\sqrt{96}$
59. $2\sqrt{18} - 3\sqrt{75} - 2\sqrt{98} + 4\sqrt{48}$
60. $2\sqrt{45} - \sqrt{12} + 2\sqrt{20} - \sqrt{108}$
61. $\left(2\sqrt{363} - 3\sqrt{96}\right) - \left(7\sqrt{12} - 2\sqrt{54}\right)$
62. $\left(2\sqrt{288} + 3\sqrt{360}\right) - \left(2\sqrt{72} - 7\sqrt{40}\right)$
63. $3\sqrt[3]{54} + 5\sqrt[3]{250} - 4\sqrt[3]{16}$
64. $4\sqrt[3]{162} - 2\sqrt[3]{384} - 3\sqrt[3]{750}$

Simplify. (Assume all radicands containing variable expressions are positive.)

65. $\sqrt{81b} + \sqrt{4b}$
66. $\sqrt{100a} + \sqrt{a}$
67. $\sqrt{9a^2b} - \sqrt{36a^2b}$
68. $\sqrt{50a^2} - \sqrt{18a^2}$
69. $\sqrt{49x} - \sqrt{9y} + \sqrt{x} - \sqrt{4y}$
70. $\sqrt{9x} + \sqrt{64y} - \sqrt{25x} - \sqrt{y}$
71. $7\sqrt{8x} - \left(3\sqrt{16y} - 2\sqrt{18x}\right)$
72. $2\sqrt{64y} - \left(3\sqrt{32y} - \sqrt{81y}\right)$

73. $2\sqrt{9m^2n} - 5m\sqrt{9n} + \sqrt{m^2n}$

74. $4\sqrt{18n^2m} - 2n\sqrt{8m} + n\sqrt{2m}$

75. $\sqrt{4x^2y} - \sqrt{9xy^2} - \sqrt{16x^2y} + \sqrt{y^2x}$

76. $\sqrt{32x^2y^2} + \sqrt{12x^2y} - \sqrt{18x^2y^2} - \sqrt{27x^2y}$

77. $\left(\sqrt{9x^2y} - \sqrt{16y}\right) - \left(\sqrt{49x^2y} - 4\sqrt{y}\right)$

78. $\left(\sqrt{72x^2y^2} - \sqrt{18x^2y}\right) - \left(\sqrt{50x^2y^2} + x\sqrt{2y}\right)$

79. $\sqrt{12m^4n} - m\sqrt{75m^2n} + 2\sqrt{27m^4n}$

80. $5n\sqrt{27mn^2} + 2\sqrt{12mn^4} - n\sqrt{3mn^2}$

81. $2\sqrt{27a^3b} - a\sqrt{48ab} - a\sqrt{144a^3b}$

82. $2\sqrt{98a^4b} - 2a\sqrt{162a^2b} + a\sqrt{200b}$

83. $\sqrt[3]{125a} - \sqrt[3]{27a}$

84. $\sqrt[3]{1000a^2} - \sqrt[3]{64a^2}$

85. $2x\sqrt[3]{54x} - 2\sqrt[3]{16x^4} + 5\sqrt[3]{2x^4}$

86. $x\sqrt[3]{54x^3} - \sqrt[3]{250x^6} + x^2\sqrt[3]{2}$

87. $\sqrt[4]{16y^2} + \sqrt[4]{81y^2}$

88. $\sqrt[5]{32y^4} - \sqrt[5]{y^4}$

89. $\sqrt[4]{32a^3} - \sqrt[4]{162a^3} + 5\sqrt[4]{2a^3}$

90. $\sqrt[4]{80a^4b} + \sqrt[4]{5a^4b} - a\sqrt[4]{5b}$

91. $\sqrt[3]{27x^3} + \sqrt[3]{8x} - \sqrt[3]{125x^3}$

92. $\sqrt[3]{24x} - \sqrt[3]{128x} - \sqrt[3]{81x}$

93. $\sqrt[3]{27x^4y} - \sqrt[3]{8xy^3} + x\sqrt[3]{64xy} - y\sqrt[3]{x}$

94. $\sqrt[3]{125xy^3} + \sqrt[3]{8x^3y} - \sqrt[3]{216xy^3} + 10x\sqrt[3]{y}$

95. $\left(\sqrt[3]{162x^4y} - \sqrt[3]{250x^4y^2}\right) - \left(\sqrt[3]{2x^4y^2} - \sqrt[3]{384x^4y}\right)$

96. $\left(\sqrt[5]{32x^2y^6} - \sqrt[5]{243x^6y^2}\right) - \left(\sqrt[5]{x^2y^6} - x\sqrt[5]{xy^2}\right)$

Calculate the perimeters of the triangles formed by the following sets of vertices.

97. {(−4, −5), (−4, 3), (2, 3)}

98. {(−1, 1), (3, 1), (3, −2)}

99. {(−3, 1), (−3, 5), (1, 5)}

100. {(−3, −1), (−3, 7), (1, −1)}

101. {(0,0), (2,4), (−2,6)}

102. {(−5,−2), (−3,0), (1,−6)}

103. A square garden that is 10 feet on each side is to be fenced in. In addition, the space is to be partitioned in half using a fence along its diagonal. How much fencing is needed to do this? (Round to the nearest tenth of a foot.)

104. A garden in the shape of a square has an area of 150 square feet. How much fencing is needed to fence it in? (Hint: The length of each side of a square is equal to the square root of the area. Round to the nearest tenth of a foot.)

PART C: DISCUSSION BOARD

105. Choose values for *x* and *y* and use a calculator to show that $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$.

106. Choose values for *x* and *y* and use a calculator to show that $\sqrt{x^2+y^2} \neq x + y$.

ANSWERS

1. $5\sqrt{3}$

3. $14\sqrt{3}$

5. $-5\sqrt{5}$

7. $-\sqrt{6}$

9. $8\sqrt{7} - \sqrt{2}$

11. $9\sqrt{5} - 4\sqrt{3}$

13. $-\sqrt{5} - 4\sqrt{10}$

15. $10\sqrt[3]{6} - 3\sqrt[3]{5}$

17. $6\sqrt[3]{9} - \sqrt[3]{3}$

19. $-3\sqrt{2x}$

21. $16\sqrt{x}$

23. $5x\sqrt{y}$

25. $8\sqrt{ab} - 15\sqrt{a}$

27. $9\sqrt{xy}$

29. $2\sqrt{2x} + 6\sqrt{3x}$

31. $-7\sqrt[3]{x}$

33. $4a\sqrt[5]{3b}$

35. $13\sqrt{2a} - 5\sqrt[3]{2a}$

37. $-\sqrt[4]{4xy}$

39. $x^2\sqrt[3]{3x} + x\sqrt[3]{3x}$

41. $3\sqrt{3}$

43. $2\sqrt{2} + 3\sqrt{3}$

45. $5\sqrt{7} - 5\sqrt{3}$

47. $5\sqrt{5}$

49. $10\sqrt{2} - 3\sqrt{3}$

51. $-\sqrt[3]{2}$

53. $4\sqrt[3]{5}$

55. $2\sqrt{3}$

57. $23\sqrt{3} - 6\sqrt{2}$

59. $-8\sqrt{2} + \sqrt{3}$

61. $8\sqrt{3} - 6\sqrt{6}$

63. $26\sqrt[3]{2}$

65. $11\sqrt{b}$

67. $-3a\sqrt{b}$

69. $8\sqrt{x} - 5\sqrt{y}$

71. $20\sqrt{2x} - 12\sqrt{y}$

73. $-8m\sqrt{n}$

75. $-2x\sqrt{y} - 2y\sqrt{x}$

77. $-4x\sqrt{y}$

79. $3m^2\sqrt{3n}$

81. $2a\sqrt{3ab} - 12a^2\sqrt{ab}$

83. $2\sqrt[3]{a}$

85. $7x\sqrt[3]{2x}$

87. $5\sqrt[4]{y^2}$

89. $4\sqrt[4]{2a^3}$

91. $-2x + 2\sqrt[3]{x}$

93. $7x\sqrt[3]{xy} - 3y\sqrt[3]{x}$

95. $7x\sqrt[3]{6xy} - 6x\sqrt[3]{2xy^2}$

97. 24 units

99. $8 + 4\sqrt{2}$ units

101. $4\sqrt{5} + 2\sqrt{10}$ units

103. 54.1 feet

105. Answer may vary

5.4 Multiplying and Dividing Radical Expressions

LEARNING OBJECTIVES

1. Multiply radical expressions.
2. Divide radical expressions.
3. Rationalize the denominator.

Multiplying Radical Expressions

When multiplying radical expressions with the same index, we use the product rule for radicals. Given real numbers $\sqrt[n]{A}$ and $\sqrt[n]{B}$,

$$\sqrt[n]{A} \cdot \sqrt[n]{B} = \sqrt[n]{A \cdot B}$$

Example 1

Multiply: $\sqrt[3]{12} \cdot \sqrt[3]{6}$.

Solution:

Apply the product rule for radicals, and then simplify.

$$\begin{aligned} \sqrt[3]{12} \cdot \sqrt[3]{6} &= \sqrt[3]{12 \cdot 6} && \textit{Multiply the radicands.} \\ &= \sqrt[3]{72} && \textit{Simplify.} \\ &= \sqrt[3]{2^3 \cdot 3^2} \\ &= 2\sqrt[3]{3^2} \\ &= 2\sqrt[3]{9} \end{aligned}$$

Answer: $2\sqrt[3]{9}$

Often, there will be coefficients in front of the radicals.

Example 2

Multiply: $3\sqrt{6} \cdot 5\sqrt{2}$

Solution:

Using the product rule for radicals and the fact that multiplication is commutative, we can multiply the coefficients and the radicands as follows.

$$
\begin{aligned}
3\sqrt{6} \cdot 5\sqrt{2} &= 3 \cdot 5 \cdot \sqrt{6} \cdot \sqrt{2} && \textit{Multiplication is commutative.} \\
&= 15 \cdot \sqrt{12} && \textit{Multiply the coefficients and} \\
& && \textit{the radicands.} \\
&= 15\sqrt{4 \cdot 3} && \textit{Simplify.} \\
&= 15 \cdot 2 \cdot \sqrt{3} \\
&= 30\sqrt{3}
\end{aligned}
$$

Typically, the first step involving the application of the commutative property is not shown.

Answer: $30\sqrt{3}$

Example 3

Multiply: $-3\sqrt[3]{4y^2}\cdot 5\sqrt[3]{16y}$.

Solution:

$$\begin{aligned}-3\sqrt[3]{4y^2}\cdot 5\sqrt[3]{16y} &= -15\sqrt[3]{64y^3} && \textit{Multiply the coefficients and then multiply the r}\\ &= -15\sqrt[3]{4^3y^3} && \textit{Simplify.}\\ &= -15\cdot 4y\\ &= -60y\end{aligned}$$

Answer: $-60y$

Use the distributive property when multiplying rational expressions with more than one term.

Example 4

Multiply: $5\sqrt{2x}\left(3\sqrt{x} - \sqrt{2x}\right)$.

Solution:

Apply the distributive property and multiply each term by $5\sqrt{2x}$.

$$\begin{aligned} 5\sqrt{2x}\left(3\sqrt{x} - \sqrt{2x}\right) &= 5\sqrt{2x} \cdot 3\sqrt{x} - 5\sqrt{2x} \cdot \sqrt{2x} && \textit{Distribute.} \\ &= 15\sqrt{2x^2} - 5\sqrt{4x^2} && \textit{Simplify.} \\ &= 15x\sqrt{2} - 5 \cdot 2x \\ &= 15x\sqrt{2} - 10x \end{aligned}$$

Answer: $15x\sqrt{2} - 10x$

Example 5

Multiply: $\sqrt[3]{6x^2y}\left(\sqrt[3]{9x^2y^2} - 5 \cdot \sqrt[3]{4xy}\right)$.

Solution:

Apply the distributive property, and then simplify the result.

$$\begin{aligned}
\sqrt[3]{6x^2y}\left(\sqrt[3]{9x^2y^2} - 5 \cdot \sqrt[3]{4xy}\right) &= \sqrt[3]{6x^2y} \cdot \sqrt[3]{9x^2y^2} - \sqrt[3]{6x^2y} \cdot 5\sqrt[3]{4xy} \\
&= \sqrt[3]{54x^4y^3} - 5\sqrt[3]{24x^3y^2} \\
&= \sqrt[3]{27 \cdot 2 \cdot x \cdot x^3 \cdot y^3} - 5\sqrt[3]{8 \cdot 3 \cdot x^3 \cdot y^2} \\
&= 3xy\sqrt[3]{2x} - 5 \cdot 2x\sqrt[3]{3y^2} \\
&= 3xy\sqrt[3]{2x} - 10x\sqrt[3]{3y^2}
\end{aligned}$$

Answer: $3xy\sqrt[3]{2x} - 10x\sqrt[3]{3y^2}$

The process for multiplying radical expressions with multiple terms is the same process used when multiplying polynomials. Apply the distributive property, simplify each radical, and then combine like terms.

Example 6

Multiply: $\left(\sqrt{x} - 5\sqrt{y}\right)^2$.

Solution:

$$\left(\sqrt{x} - 5\sqrt{y}\right)^2 = \left(\sqrt{x} - 5\sqrt{y}\right)\left(\sqrt{x} - 5\sqrt{y}\right)$$

Begin by applying the distributive property.

$$\left(\sqrt{x} - 5\sqrt{y}\right)\left(\sqrt{x} - 5\sqrt{y}\right)$$

$$\begin{aligned} &= \sqrt{x} \cdot \sqrt{x} + \sqrt{x}\left(-5\sqrt{y}\right) + \left(-5\sqrt{y}\right)\sqrt{x} + \left(-5\sqrt{y}\right)\left(-5\sqrt{y}\right) \\ &= \sqrt{x^2} - 5\sqrt{xy} - 5\sqrt{xy} + 25\sqrt{y^2} \\ &= x - 10\sqrt{xy} + 25y \end{aligned}$$

Answer: $x - 10\sqrt{xy} + 25y$

The binomials $(a + b)$ and $(a - b)$ are called **conjugates**[18]. When multiplying conjugate binomials the middle terms are opposites and their sum is zero.

18. The factors $(a + b)$ and $(a - b)$ are conjugates.

Example 7

Multiply: $\left(\sqrt{10}+\sqrt{3}\right)\left(\sqrt{10}-\sqrt{3}\right)$.

Solution:

Apply the distributive property, and then combine like terms.

$$\begin{aligned}\left(\sqrt{10}+\sqrt{3}\right)\left(\sqrt{10}-\sqrt{3}\right) &= \sqrt{10}\cdot\sqrt{10}+\sqrt{10}\left(-\sqrt{3}\right)+\sqrt{3}\cdot \\ &= \sqrt{100}-\sqrt{30}+\sqrt{30}-\sqrt{9} \\ &= 10-\sqrt{30}+\sqrt{30}-3 \\ &= 10 \quad - \quad 3 \\ &= 7\end{aligned}$$

Answer: 7

It is important to note that when multiplying conjugate radical expressions, we obtain a rational expression. This is true in general

$$\begin{aligned}\left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}-\sqrt{y}\right) &= \sqrt{x^2}-\sqrt{xy}+\sqrt{xy}-\sqrt{y^2} \\ &= x-y\end{aligned}$$

Alternatively, using the formula for the difference of squares we have,

$$\left(a+b\right)\left(a-b\right)=a^2-b^2 \qquad \textit{Difference of squares.}$$

$$\begin{aligned}\left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}-\sqrt{y}\right)&=\left(\sqrt{x}\right)^2-\left(\sqrt{y}\right)^2\\&=x-y\end{aligned}$$

Try this! Multiply: $\left(3-2\sqrt{y}\right)\left(3+2\sqrt{y}\right)$. (Assume y is positive.)

Answer: $9-4y$

(click to see video)

Dividing Radical Expressions

To divide radical expressions with the same index, we use the quotient rule for radicals. Given real numbers $\sqrt[n]{A}$ and $\sqrt[n]{B}$,

$$\frac{\sqrt[n]{A}}{\sqrt[n]{B}}=\sqrt[n]{\frac{A}{B}}$$

Example 8

Divide: $\frac{\sqrt[3]{96}}{\sqrt[3]{6}}$.

Solution:

In this case, we can see that 6 and 96 have common factors. If we apply the quotient rule for radicals and write it as a single cube root, we will be able to reduce the fractional radicand.

$$\begin{aligned}\frac{\sqrt[3]{96}}{\sqrt[3]{6}} &= \sqrt[3]{\frac{96}{6}} && \textit{Apply the quotient rule for radicals and reduce the radicand.}\\ &= \sqrt[3]{16} && \textit{Simplify.}\\ &= \sqrt[3]{8 \cdot 2}\\ &= 2\sqrt[3]{2}\end{aligned}$$

Answer: $2\sqrt[3]{2}$

Example 9

Divide: $\dfrac{\sqrt{50x^6y^4}}{\sqrt{8x^3y}}$.

Solution:

Write as a single square root and cancel common factors before simplifying.

$$
\begin{aligned}
\frac{\sqrt{50x^6y^4}}{\sqrt{8x^3y}} &= \sqrt{\frac{50x^6y^4}{8x^3y}} && \textit{Apply the quotient rule for radicals and cancel.}\\
&= \sqrt{\frac{25x^3y^3}{4}} && \textit{Simplify.}\\
&= \frac{\sqrt{25x^3y^3}}{\sqrt{4}}\\
&= \frac{5xy\sqrt{xy}}{2}
\end{aligned}
$$

Answer: $\dfrac{5xy\sqrt{xy}}{2}$

Rationalizing the Denominator

When the denominator (divisor) of a radical expression contains a radical, it is a common practice to find an equivalent expression where the denominator is a rational number. Finding such an equivalent expression is called **rationalizing the denominator**[19].

19. The process of determining an equivalent radical expression with a rational denominator.

$$\begin{array}{ccc} \textit{Radical expression} & & \textit{Rational denominator} \\ \dfrac{1}{\sqrt{2}} & = & \dfrac{\sqrt{2}}{2} \end{array}$$

To do this, multiply the fraction by a special form of 1 so that the radicand in the denominator can be written with a power that matches the index. After doing this, simplify and eliminate the radical in the denominator. For example:

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{4}} = \frac{\sqrt{2}}{2}$$

Remember, to obtain an equivalent expression, you must multiply the numerator and denominator by the exact same nonzero factor.

Example 10

Rationalize the denominator: $\frac{\sqrt{2}}{\sqrt{5x}}$.

Solution:

The goal is to find an equivalent expression without a radical in the denominator. The radicand in the denominator determines the factors that you need to use to rationalize it. In this example, multiply by 1 in the form $\frac{\sqrt{5x}}{\sqrt{5x}}$.

$$\begin{aligned}\frac{\sqrt{2}}{\sqrt{5x}} &= \frac{\sqrt{2}}{\sqrt{5x}} \cdot \frac{\sqrt{5x}}{\sqrt{5x}} && \textit{Multiply by } \frac{\sqrt{5x}}{\sqrt{5x}}. \\ &= \frac{\sqrt{10x}}{\sqrt{25x^2}} && \textit{Simplify.} \\ &= \frac{\sqrt{10x}}{5x}\end{aligned}$$

Answer: $\frac{\sqrt{10x}}{5x}$

Sometimes, we will find the need to reduce, or cancel, after rationalizing the denominator.

Example 11

Rationalize the denominator: $\frac{3a\sqrt{2}}{\sqrt{6ab}}$.

Solution:

In this example, we will multiply by 1 in the form $\frac{\sqrt{6ab}}{\sqrt{6ab}}$.

$$\begin{aligned}
\frac{3a\sqrt{2}}{\sqrt{6ab}} &= \frac{3a\sqrt{2}}{\sqrt{6ab}} \cdot \frac{\sqrt{6ab}}{\sqrt{6ab}} \\
&= \frac{3a\sqrt{12ab}}{\sqrt{36a^2b^2}} && \textit{Simplify.} \\
&= \frac{3a\sqrt{4 \cdot 3ab}}{6ab} \\
&= \frac{6a\sqrt{3ab}}{6ab} && \textit{Cancel.} \\
&= \frac{\sqrt{3ab}}{b}
\end{aligned}$$

Notice that b does not cancel in this example. Do not cancel factors inside a radical with those that are outside.

Answer: $\frac{\sqrt{3ab}}{b}$

Try this! Rationalize the denominator: $\sqrt{\frac{9x}{2y}}$.

Answer: $\frac{3\sqrt{2xy}}{2y}$

(click to see video)

Up to this point, we have seen that multiplying a numerator and a denominator by a square root with the exact same radicand results in a rational denominator. In general, this is true only when the denominator contains a square root. However, this is not the case for a cube root. For example,

$$\frac{1}{\sqrt[3]{x}} \cdot \frac{\sqrt[3]{x}}{\sqrt[3]{x}} = \frac{\sqrt[3]{x}}{\sqrt[3]{x^2}}$$

Note that multiplying by the same factor in the denominator does not rationalize it. In this case, if we multiply by 1 in the form of $\frac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}}$, then we can write the radicand in the denominator as a power of 3. Simplifying the result then yields a rationalized denominator.

$$\frac{1}{\sqrt[3]{x}} = \frac{1}{\sqrt[3]{x}} \cdot \frac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}} = \frac{\sqrt[3]{x^2}}{\sqrt[3]{x^3}} = \frac{\sqrt[3]{x^2}}{x}$$

Therefore, to rationalize the denominator of a radical expression with one radical term in the denominator, begin by factoring the radicand of the denominator. The factors of this radicand and the index determine what we should multiply by. Multiply the numerator and denominator by the *n*th root of factors that produce *n*th powers of all the factors in the radicand of the denominator.

Example 12

Rationalize the denominator: $\frac{\sqrt[3]{2}}{\sqrt[3]{25}}$.

Solution:

The radical in the denominator is equivalent to $\sqrt[3]{5^2}$. To rationalize the denominator, we need: $\sqrt[3]{5^3}$. To obtain this, we need one more factor of 5. Therefore, multiply by 1 in the form of $\frac{\sqrt[3]{5}}{\sqrt[3]{5}}$.

$$\begin{aligned}\frac{\sqrt[3]{2}}{\sqrt[3]{25}} &= \frac{\sqrt[3]{2}}{\sqrt[3]{5^2}} \cdot \frac{\sqrt[3]{5}}{\sqrt[3]{5}} && \textit{Multiply by the cube root of factors that result in powers of 3.}\\ &= \frac{\sqrt[3]{10}}{\sqrt[3]{5^3}} && \textit{Simplify.}\\ &= \frac{\sqrt[3]{10}}{5}\end{aligned}$$

Answer: $\frac{\sqrt[3]{10}}{5}$

Example 13

Rationalize the denominator: $\sqrt[3]{\frac{27a}{2b^2}}$.

Solution:

In this example, we will multiply by 1 in the form $\frac{\sqrt[3]{2^2 b}}{\sqrt[3]{2^2 b}}$.

$$\begin{aligned}
\sqrt[3]{\frac{27a}{2b^2}} &= \frac{\sqrt[3]{3^3 a}}{\sqrt[3]{2b^2}} && \textit{Apply the quotient rule for radicals.} \\
&= \frac{3\sqrt[3]{a}}{\sqrt[3]{2b^2}} \cdot \frac{\sqrt[3]{2^2 b}}{\sqrt[3]{2^2 b}} && \textit{Multiply by the cube root of factors that result in powers o} \\
&= \frac{3\sqrt[3]{2^2 ab}}{\sqrt[3]{2^3 b^3}} && \textit{Simplify.} \\
&= \frac{3\sqrt[3]{4ab}}{2b}
\end{aligned}$$

Answer: $\frac{3\sqrt[3]{4ab}}{2b}$

Example 14

Rationalize the denominator: $\frac{2x\sqrt[5]{5}}{\sqrt[5]{4x^3y}}$.

Solution:

In this example, we will multiply by 1 in the form $\frac{\sqrt[5]{2^3x^2y^4}}{\sqrt[5]{2^3x^2y^4}}$.

$$\begin{aligned}\frac{2x\sqrt[5]{5}}{\sqrt[5]{4x^3y}} &= \frac{2x\sqrt[5]{5}}{\sqrt[5]{2^2x^3y}} \cdot \frac{\sqrt[5]{2^3x^2y^4}}{\sqrt[5]{2^3x^2y^4}} && \textit{Multiply by the fifth root of factors that result in p} \\ &= \frac{2x\sqrt[5]{5 \cdot 2^3x^2y^4}}{\sqrt[5]{2^5x^5y^5}} && \textit{Simplify.} \\ &= \frac{2x\sqrt[5]{40x^2y^4}}{2xy} \\ &= \frac{\sqrt[5]{40x^2y^4}}{y}\end{aligned}$$

Answer: $\frac{\sqrt[5]{40x^2y^4}}{y}$

When two terms involving square roots appear in the denominator, we can rationalize it using a very special technique. This technique involves multiplying the numerator and the denominator of the fraction by the conjugate of the denominator. Recall that multiplying a radical expression by its conjugate produces a rational number.

Example 15

Rationalize the denominator: $\frac{1}{\sqrt{5}-\sqrt{3}}$.

Solution:

In this example, the conjugate of the denominator is $\sqrt{5}+\sqrt{3}$. Therefore, multiply by 1 in the form $\frac{(\sqrt{5}+\sqrt{3})}{(\sqrt{5}+\sqrt{3})}$.

$$
\begin{aligned}
\frac{1}{\sqrt{5}-\sqrt{3}} &= \frac{1}{(\sqrt{5}-\sqrt{3})}\frac{(\sqrt{5}+\sqrt{3})}{(\sqrt{5}+\sqrt{3})} && \textit{Multiply numerator and denominator by the conjugate of the denominator.}\\
&= \frac{\sqrt{5}+\sqrt{3}}{\sqrt{25}+\sqrt{15}-\sqrt{15}-\sqrt{9}} && \textit{Simplify.}\\
&= \frac{\sqrt{5}+\sqrt{3}}{5-3}\\
&= \frac{\sqrt{5}+\sqrt{3}}{2}
\end{aligned}
$$

Answer: $\frac{\sqrt{5}+\sqrt{3}}{2}$

Notice that the terms involving the square root in the denominator are eliminated by multiplying by the conjugate. We can use the property $\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)=a-b$to expedite the process of multiplying the expressions in the denominator.

Example 16

Rationalize the denominator: $\frac{\sqrt{10}}{\sqrt{2}+\sqrt{6}}$.

Solution:

Multiply by 1 in the form $\frac{\sqrt{2}-\sqrt{6}}{\sqrt{2}-\sqrt{6}}$.

$$\frac{\sqrt{10}}{\sqrt{2}+\sqrt{6}} = \frac{\left(\sqrt{10}\right)}{\left(\sqrt{2}+\sqrt{6}\right)} \frac{\left(\sqrt{2}-\sqrt{6}\right)}{\left(\sqrt{2}-\sqrt{6}\right)}$$

Multiply by the conjuga

$$= \frac{\sqrt{20}-\sqrt{60}}{2-6}$$

Simplify.

$$= \frac{\sqrt{4\cdot 5}-\sqrt{4\cdot 15}}{-4}$$

$$= \frac{2\sqrt{5}-2\sqrt{15}}{-4}$$

$$= \frac{2\left(\sqrt{5}-\sqrt{15}\right)}{-4}$$

$$= \frac{\sqrt{5}-\sqrt{15}}{-2} = -\frac{\sqrt{5}-\sqrt{15}}{2} = \frac{-\sqrt{5}+\sqrt{15}}{2}$$

Answer: $\frac{\sqrt{15}-\sqrt{5}}{2}$

Example 17

Rationalize the denominator: $\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$.

Solution:

In this example, we will multiply by 1 in the form $\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}-\sqrt{y}}$.

$$\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}} = \frac{\left(\sqrt{x}-\sqrt{y}\right)}{\left(\sqrt{x}+\sqrt{y}\right)} \frac{\left(\sqrt{x}-\sqrt{y}\right)}{\left(\sqrt{x}-\sqrt{y}\right)} \quad \textit{Multiply by the conjugate of the denomi}$$

$$= \frac{\sqrt{x^2}-\sqrt{xy}-\sqrt{xy}+\sqrt{y^2}}{x-y} \quad \textit{Simplify.}$$

$$= \frac{x-2\sqrt{xy}+y}{x-y}$$

Answer: $\frac{x-2\sqrt{xy}+y}{x-y}$

Try this! Rationalize the denominator: $\frac{2\sqrt{3}}{5-\sqrt{3}}$

Answer: $\frac{5\sqrt{3}+3}{11}$

(click to see video)

KEY TAKEAWAYS

- To multiply two single-term radical expressions, multiply the coefficients and multiply the radicands. If possible, simplify the result.
- Apply the distributive property when multiplying a radical expression with multiple terms. Then simplify and combine all like radicals.
- Multiplying a two-term radical expression involving square roots by its conjugate results in a rational expression.
- It is common practice to write radical expressions without radicals in the denominator. The process of finding such an equivalent expression is called rationalizing the denominator.
- If an expression has one term in the denominator involving a radical, then rationalize it by multiplying the numerator and denominator by the *n*th root of factors of the radicand so that their powers equal the index.
- If a radical expression has two terms in the denominator involving square roots, then rationalize it by multiplying the numerator and denominator by the conjugate of the denominator.

TOPIC EXERCISES

PART A: MULTIPLYING RADICAL EXPRESSIONS

Multiply. (Assume all variables represent non-negative real numbers.)

1. $\sqrt{3} \cdot \sqrt{7}$
2. $\sqrt{2} \cdot \sqrt{5}$
3. $\sqrt{6} \cdot \sqrt{12}$
4. $\sqrt{10} \cdot \sqrt{15}$
5. $\sqrt{2} \cdot \sqrt{6}$
6. $\sqrt{5} \cdot \sqrt{15}$
7. $\sqrt{7} \cdot \sqrt{7}$
8. $\sqrt{12} \cdot \sqrt{12}$
9. $2\sqrt{5} \cdot 7\sqrt{10}$
10. $3\sqrt{15} \cdot 2\sqrt{6}$
11. $\left(2\sqrt{5}\right)^2$
12. $\left(6\sqrt{2}\right)^2$
13. $\sqrt{2x} \cdot \sqrt{2x}$
14. $\sqrt{5y} \cdot \sqrt{5y}$
15. $\sqrt{3a} \cdot \sqrt{12}$
16. $\sqrt{3a} \cdot \sqrt{2a}$
17. $4\sqrt{2x} \cdot 3\sqrt{6x}$

18. $5\sqrt{10y} \cdot 2\sqrt{2y}$
19. $\sqrt[3]{3} \cdot \sqrt[3]{9}$
20. $\sqrt[3]{4} \cdot \sqrt[3]{16}$
21. $\sqrt[3]{15} \cdot \sqrt[3]{25}$
22. $\sqrt[3]{100} \cdot \sqrt[3]{50}$
23. $\sqrt[3]{4} \cdot \sqrt[3]{10}$
24. $\sqrt[3]{18} \cdot \sqrt[3]{6}$
25. $\left(5\sqrt[3]{9}\right)\left(2\sqrt[3]{6}\right)$
26. $\left(2\sqrt[3]{4}\right)\left(3\sqrt[3]{4}\right)$
27. $\left(2\sqrt[3]{2}\right)^3$
28. $\left(3\sqrt[3]{4}\right)^3$
29. $\sqrt[3]{3a^2} \cdot \sqrt[3]{9a}$
30. $\sqrt[3]{7b} \cdot \sqrt[3]{49b^2}$
31. $\sqrt[3]{6x^2} \cdot \sqrt[3]{4x^2}$
32. $\sqrt[3]{12y} \cdot \sqrt[3]{9y^2}$
33. $\sqrt[3]{20x^2y} \cdot \sqrt[3]{10x^2y^2}$
34. $\sqrt[3]{63xy} \cdot \sqrt[3]{12x^4y^2}$
35. $\sqrt{5}\left(3 - \sqrt{5}\right)$
36. $\sqrt{2}\left(\sqrt{3} - \sqrt{2}\right)$

37. $3\sqrt{7}\left(2\sqrt{7}-\sqrt{3}\right)$

38. $2\sqrt{5}\left(6-3\sqrt{10}\right)$

39. $\sqrt{6}\left(\sqrt{3}-\sqrt{2}\right)$

40. $\sqrt{15}\left(\sqrt{5}+\sqrt{3}\right)$

41. $\sqrt{x}\left(\sqrt{x}+\sqrt{xy}\right)$

42. $\sqrt{y}\left(\sqrt{xy}+\sqrt{y}\right)$

43. $\sqrt{2ab}\left(\sqrt{14a}-2\sqrt{10b}\right)$

44. $\sqrt{6ab}\left(5\sqrt{2a}-\sqrt{3b}\right)$

45. $\sqrt[3]{6}\left(\sqrt[3]{9}-\sqrt[3]{20}\right)$

46. $\sqrt[3]{12}\left(\sqrt[3]{36}+\sqrt[3]{14}\right)$

47. $\left(\sqrt{2}-\sqrt{5}\right)\left(\sqrt{3}+\sqrt{7}\right)$

48. $\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{5}-\sqrt{7}\right)$

49. $\left(2\sqrt{3}-4\right)\left(3\sqrt{6}+1\right)$

50. $\left(5-2\sqrt{6}\right)\left(7-2\sqrt{3}\right)$

51. $\left(\sqrt{5}-\sqrt{3}\right)^2$

52. $\left(\sqrt{7}-\sqrt{2}\right)^2$

53. $\left(2\sqrt{3}+\sqrt{2}\right)\left(2\sqrt{3}-\sqrt{2}\right)$
54. $\left(\sqrt{2}+3\sqrt{7}\right)\left(\sqrt{2}-3\sqrt{7}\right)$
55. $\left(\sqrt{a}-\sqrt{2b}\right)^2$
56. $\left(\sqrt{ab}+1\right)^2$
57. What is the perimeter and area of a rectangle with length measuring $5\sqrt{3}$ centimeters and width measuring $3\sqrt{2}$ centimeters?
58. What is the perimeter and area of a rectangle with length measuring $2\sqrt{6}$ centimeters and width measuring $\sqrt{3}$ centimeters?
59. If the base of a triangle measures $6\sqrt{2}$ meters and the height measures $3\sqrt{2}$ meters, then calculate the area.
60. If the base of a triangle measures $6\sqrt{3}$ meters and the height measures $3\sqrt{6}$ meters, then calculate the area.

PART B: DIVIDING RADICAL EXPRESSIONS

Divide. (Assume all variables represent positive real numbers.)

61. $\frac{\sqrt{75}}{\sqrt{3}}$
62. $\frac{\sqrt{360}}{\sqrt{10}}$
63. $\frac{\sqrt{72}}{\sqrt{75}}$
64. $\frac{\sqrt{90}}{\sqrt{98}}$

65. $\dfrac{\sqrt{90x^5}}{\sqrt{2x}}$

66. $\dfrac{\sqrt{96y^3}}{\sqrt{3y}}$

67. $\dfrac{\sqrt{162x^7y^5}}{\sqrt{2xy}}$

68. $\dfrac{\sqrt{363x^4y^9}}{\sqrt{3xy}}$

69. $\dfrac{\sqrt[3]{16a^5b^2}}{\sqrt[3]{2a^2b^2}}$

70. $\dfrac{\sqrt[3]{192a^2b^7}}{\sqrt[3]{2a^2b^2}}$

PART C: RATIONALIZING THE DENOMINATOR

Rationalize the denominator. (Assume all variables represent positive real numbers.)

71. $\dfrac{1}{\sqrt{5}}$

72. $\dfrac{1}{\sqrt{6}}$

73. $\dfrac{\sqrt{2}}{\sqrt{3}}$

74. $\dfrac{\sqrt{3}}{\sqrt{7}}$

75. $\dfrac{5}{2\sqrt{10}}$

76. $\frac{3}{5\sqrt{6}}$

77. $\frac{\sqrt{3}-\sqrt{5}}{\sqrt{3}}$

78. $\frac{\sqrt{6}-\sqrt{2}}{\sqrt{2}}$

79. $\frac{1}{\sqrt{7x}}$

80. $\frac{1}{\sqrt{3y}}$

81. $\frac{a}{5\sqrt{ab}}$

82. $\frac{3b^2}{2\sqrt{3ab}}$

83. $\frac{2}{\sqrt[3]{36}}$

84. $\frac{14}{\sqrt[3]{7}}$

85. $\frac{1}{\sqrt[3]{4x}}$

86. $\frac{1}{\sqrt[3]{3y^2}}$

87. $\frac{9x\sqrt[3]{2}}{\sqrt[3]{9xy^2}}$

88. $\frac{5y^2\sqrt[3]{x}}{\sqrt[3]{5x^2y}}$

89. $\frac{3a}{2\sqrt[3]{3a^2b^2}}$

90. $\dfrac{25n}{3\sqrt[3]{25m^2n}}$

91. $\dfrac{3}{\sqrt[5]{27x^2y}}$

92. $\dfrac{2}{\sqrt[5]{16xy^2}}$

93. $\dfrac{ab}{\sqrt[5]{9a^3b}}$

94. $\dfrac{abc}{\sqrt[5]{ab^2c^3}}$

95. $\sqrt[5]{\dfrac{3x}{8y^2z}}$

96. $\sqrt[5]{\dfrac{4xy^2}{9x^3yz^4}}$

97. $\dfrac{3}{\sqrt{10}-3}$

98. $\dfrac{2}{\sqrt{6}-2}$

99. $\dfrac{1}{\sqrt{5}+\sqrt{3}}$

100. $\dfrac{1}{\sqrt{7}-\sqrt{2}}$

101. $\dfrac{\sqrt{3}}{\sqrt{3}+\sqrt{6}}$

102. $\dfrac{\sqrt{5}}{\sqrt{5}+\sqrt{15}}$

103. $\dfrac{10}{5-3\sqrt{5}}$

104. $\dfrac{-2\sqrt{2}}{4-3\sqrt{2}}$

105. $\dfrac{\sqrt{3}+\sqrt{5}}{\sqrt{3}-\sqrt{5}}$

106. $\dfrac{\sqrt{10}-\sqrt{2}}{\sqrt{10}+\sqrt{2}}$

107. $\dfrac{2\sqrt{3}-3\sqrt{2}}{4\sqrt{3}+\sqrt{2}}$

108. $\dfrac{6\sqrt{5}+2}{2\sqrt{5}-\sqrt{2}}$

109. $\dfrac{x-y}{\sqrt{x}+\sqrt{y}}$

110. $\dfrac{x-y}{\sqrt{x}-\sqrt{y}}$

111. $\dfrac{x+\sqrt{y}}{x-\sqrt{y}}$

112. $\dfrac{x-\sqrt{y}}{x+\sqrt{y}}$

113. $\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$

114. $\dfrac{\sqrt{ab}+\sqrt{2}}{\sqrt{ab}-\sqrt{2}}$

115. $\dfrac{\sqrt{x}}{5-2\sqrt{x}}$

116. $\dfrac{1}{\sqrt{x}-y}$

117. $\dfrac{\sqrt{x}+\sqrt{2y}}{\sqrt{2x}-\sqrt{y}}$

118. $\dfrac{\sqrt{3x}-\sqrt{y}}{\sqrt{x}+\sqrt{3y}}$

119. $\dfrac{\sqrt{2x+1}}{\sqrt{2x+1}-1}$

120. $\dfrac{\sqrt{x+1}}{1-\sqrt{x+1}}$

121. $\dfrac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}$

122. $\dfrac{\sqrt{2x+3}-\sqrt{2x-3}}{\sqrt{2x+3}+\sqrt{2x-3}}$

123. The radius of the base of a right circular cone is given by $r=\sqrt{\frac{3V}{\pi h}}$ where V represents the volume of the cone and h represents its height. Find the radius of a right circular cone with volume 50 cubic centimeters and height 4 centimeters. Give the exact answer and the approximate answer rounded to the nearest hundredth.

124. The radius of a sphere is given by $r=\sqrt[3]{\frac{3V}{4\pi}}$ where V represents the volume of the sphere. Find the radius of a sphere with volume 135 square centimeters. Give the exact answer and the approximate answer rounded to the nearest hundredth.

PART D: DISCUSSION

125. Research and discuss some of the reasons why it is a common practice to rationalize the denominator.

126. Explain in your own words how to rationalize the denominator.

ANSWERS

1. $\sqrt{21}$
3. $6\sqrt{2}$
5. $2\sqrt{3}$
7. 7
9. $70\sqrt{2}$
11. 20
13. $2x$
15. $6\sqrt{a}$
17. $24x\sqrt{3}$
19. 3
21. $5\sqrt[3]{3}$
23. $2\sqrt[3]{5}$
25. $30\sqrt[3]{2}$
27. 16
29. $3a$
31. $2x\sqrt[3]{3x}$
33. $2xy\sqrt[3]{25x}$
35. $3\sqrt{5} - 5$
37. $42 - 3\sqrt{21}$
39. $3\sqrt{2} - 2\sqrt{3}$
41. $x + x\sqrt{y}$
43. $2a\sqrt{7b} - 4b\sqrt{5a}$

45. $3\sqrt[3]{2} - 2\sqrt[3]{15}$

47. $\sqrt{6} + \sqrt{14} - \sqrt{15} - \sqrt{35}$

49. $18\sqrt{2} + 2\sqrt{3} - 12\sqrt{6} - 4$

51. $8 - 2\sqrt{15}$

53. 10

55. $a - 2\sqrt{2ab} + 2b$

57. Perimeter: $\left(10\sqrt{3} + 6\sqrt{2}\right)$ centimeters; area: $15\sqrt{6}$ square centimeters

59. 18 square meters

61. 5

63. $\dfrac{2\sqrt{6}}{5}$

65. $3x^2\sqrt{5}$

67. $9x^3y^2$

69. $2a$

71. $\dfrac{\sqrt{5}}{5}$

73. $\dfrac{\sqrt{6}}{3}$

75. $\dfrac{\sqrt{10}}{4}$

77. $\dfrac{3 - \sqrt{15}}{3}$

79. $\dfrac{\sqrt{7x}}{7x}$

81. $\dfrac{\sqrt{ab}}{5b}$

83. $\dfrac{\sqrt[3]{6}}{3}$

85. $\dfrac{\sqrt[3]{2x^2}}{2x}$

87. $\dfrac{3\sqrt[3]{6x^2y}}{y}$

89. $\dfrac{\sqrt[3]{9ab}}{2b}$

91. $\dfrac{\sqrt[5]{9x^3y^4}}{xy}$

93. $\dfrac{\sqrt[5]{27a^2b^4}}{3}$

95. $\dfrac{\sqrt[5]{12xy^3z^4}}{2yz}$

97. $3\sqrt{10} + 9$

99. $\dfrac{\sqrt{5} - \sqrt{3}}{2}$

101. $-1 + \sqrt{2}$

103. $\dfrac{-5 - 3\sqrt{5}}{2}$

105. $-4 - \sqrt{15}$

107. $\dfrac{15 - 7\sqrt{6}}{23}$

109. $\sqrt{x} - \sqrt{y}$

111. $\dfrac{x^2 + 2x\sqrt{y} + y}{x^2 - y}$

113. $\dfrac{a - 2\sqrt{ab} + b}{a - b}$

115. $\dfrac{5\sqrt{x} + 2x}{25 - 4x}$

117. $\dfrac{x\sqrt{2} + 3\sqrt{xy} + y\sqrt{2}}{2x - y}$

119. $\dfrac{2x + 1 + \sqrt{2x + 1}}{2x}$

121. $x + \sqrt{x^2 - 1}$

123. $\frac{5\sqrt{6\pi}}{2\pi}$ centimeters; 3.45 centimeters

125. Answer may vary

5.5 Rational Exponents

LEARNING OBJECTIVES

1. Write expressions with rational exponents in radical form.
2. Write radical expressions with rational exponents.
3. Perform operations and simplify expressions with rational exponents.
4. Perform operations on radicals with different indices.

Rational Exponents

So far, exponents have been limited to integers. In this section, we will define what rational (or fractional) exponents mean and how to work with them. All of the rules for exponents developed up to this point apply. In particular, recall the product rule for exponents. Given any rational numbers *m* and *n*, we have

$$x^m \cdot x^n = x^{m+n}$$

For example, if we have an exponent of 1/2, then the product rule for exponents implies the following:

$$5^{1/2} \cdot 5^{1/2} = 5^{1/2+1/2} = 5^1 = 5$$

Here $5^{1/2}$ is one of two equal factors of 5; hence it is a square root of 5, and we can write

$$5^{1/2} = \sqrt{5}$$

Furthermore, we can see that $2^{1/3}$ is one of three equal factors of 2.

$$2^{1/3} \cdot 2^{1/3} \cdot 2^{1/3} = 2^{1/3+1/3+1/3} = 2^{3/3} = 2^1 = 2$$

Therefore, $2^{1/3}$ is a cube root of 2, and we can write

$$2^{1/3} = \sqrt[3]{2}$$

This is true in general, given any nonzero real number *a* and integer $n \geq 2$,

$$a^{1/n} = \sqrt[n]{a}$$

In other words, the denominator of a fractional exponent determines the index of an *n*th root.

Example 1

Rewrite as a radical.

a. $6^{1/2}$
b. $6^{1/3}$

Solution:

a. $6^{1/2} = \sqrt[2]{6} = \sqrt{6}$
b. $6^{1/3} = \sqrt[3]{6}$

Example 2

Rewrite as a radical and then simplify.

a. $16^{1/2}$
b. $16^{1/4}$

Solution:

a. $16^{1/2} = \sqrt{16} = \sqrt{4^2} = 4$
b. $16^{1/4} = \sqrt[4]{16} = \sqrt[4]{2^4} = 2$

Example 3

Rewrite as a radical and then simplify.

a. $\left(64x^3\right)^{1/3}$
b. $\left(-32x^5y^{10}\right)^{1/5}$

Solution:

a.

$$\begin{aligned}\left(64x^3\right)^{1/3} &= \sqrt[3]{64x^3}\\ &= \sqrt[3]{4^3x^3}\\ &= 4x\end{aligned}$$

b.

$$\begin{aligned}\left(-32x^5y^{10}\right)^{1/5} &= \sqrt[5]{-32x^5y^{10}}\\ &= \sqrt[5]{(-2)^5x^5\left(y^2\right)^5}\\ &= -2xy^2\end{aligned}$$

Next, consider fractional exponents where the numerator is an integer other than 1. For example, consider the following:

$$5^{2/3}\cdot 5^{2/3}\cdot 5^{2/3} = 5^{2/3+2/3+2/3} = 5^{6/3} = 5^2$$

This shows that $5^{2/3}$ is one of three equal factors of 5^2. In other words, $5^{2/3}$ is a cube root of 5^2 and we can write:

$$5^{2/3} = \sqrt[3]{5^2}$$

In general, given any nonzero real number a where m and n are positive integers $(n \geq 2)$,

$$a^{m/n} = \sqrt[n]{a^m}$$

An expression with a **rational exponent**[20] is equivalent to a radical where the denominator is the index and the numerator is the exponent. Any radical expression can be written with a rational exponent, which we call **exponential form**[21].

$$\begin{array}{ccc} \textit{Radical form} & & \textit{Exponential form} \\ \sqrt[5]{x^2} & = & x^{2/5} \end{array}$$

Example 4

Rewrite as a radical.

a. $6^{2/5}$
b. $3^{3/4}$

Solution:

a. $6^{2/5} = \sqrt[5]{6^2} = \sqrt[5]{36}$
b. $3^{3/4} = \sqrt[4]{3^3} = \sqrt[4]{27}$

20. The fractional exponent m/n that indicates a radical with index n and exponent m: $a^{m/n} = \sqrt[n]{a^m}$.

21. An equivalent expression written using a rational exponent.

Example 5

Rewrite as a radical and then simplify.

a. $27^{2/3}$
b. $(12)^{5/3}$

Solution:

We can often avoid very large integers by working with their prime factorization.

a.

$$\begin{aligned} 27^{2/3} &= \sqrt[3]{27^2} \\ &= \sqrt[3]{\left(3^3\right)^2} && \textit{Replace 27 with } 3^3. \\ &= \sqrt[3]{3^6} && \textit{Simplify.} \\ &= 3^2 \\ &= 9 \end{aligned}$$

b.

$$\begin{aligned} (12)^{5/3} &= \sqrt[3]{(12)^5} && \textit{Replace 12 with } 2^2 \cdot 3. \\ &= \sqrt[3]{\left(2^2 \cdot 3\right)^5} && \textit{Apply the rules for exponents.} \\ &= \sqrt[3]{2^{10} \cdot 3^5} && \textit{Simplify.} \\ &= \sqrt[3]{2^9 \cdot 2 \cdot 3^3 \cdot 3^2} \\ &= 2^3 \cdot 3 \cdot \sqrt[3]{2 \cdot 3^2} \\ &= 24\sqrt[3]{18} \end{aligned}$$

Given a radical expression, we might want to find the equivalent in exponential form. Assume all variables are positive.

Example 6

Rewrite using rational exponents: $\sqrt[5]{x^3}$.

Solution:

Here the index is 5 and the power is 3. We can write

$$\sqrt[5]{x^3} = x^{3/5}$$

Answer: $x^{3/5}$

Example 7

Rewrite using rational exponents: $\sqrt[6]{y^3}$.

Solution:

Here the index is 6 and the power is 3. We can write

$$\begin{aligned}\sqrt[6]{y^3} &= y^{3/6} \\ &= y^{1/2}\end{aligned}$$

Answer: $y^{1/2}$

It is important to note that the following are equivalent.

$$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

In other words, it does not matter if we apply the power first or the root first. For example, we can apply the power before the *n*th root:

$$27^{2/3} = \sqrt[3]{27^2} = \sqrt[3]{\left(3^3\right)^2} = \sqrt[3]{3^6} = 3^2 = 9$$

Or we can apply the *n*th root before the power:

$$27^{2/3} = \left(\sqrt[3]{27}\right)^2 = \left(\sqrt[3]{3^3}\right)^2 = (3)^2 = 9$$

The results are the same.

Example 8

Rewrite as a radical and then simplify: $(-8)^{2/3}$.

Solution:

Here the index is 3 and the power is 2. We can write

$$(-8)^{2/3} = \left(\sqrt[3]{-8}\right)^2 = (-2)^2 = 4$$

Answer: 4

Try this! Rewrite as a radical and then simplify: $100^{3/2}$.

Answer: 1,000

(click to see video)

Some calculators have a caret button $\boxed{\wedge}$ which is used for entering exponents. If so, we can calculate approximations for radicals using it and rational exponents.

For example, to calculate $\sqrt{2} = 2^{1/2} = 2 \wedge (1/2) \approx 1.414$, we make use of the parenthesis buttons and type

$$2 \ \boxed{\wedge} \ \boxed{(} \ 1 \ \boxed{\div} \ 2 \ \boxed{)} \ \boxed{=}$$

To calculate $\sqrt[3]{2^2} = 2^{2/3} = 2 \wedge (2/3) \approx 1.587$, we would type

$$2 \ \boxed{\wedge} \ \boxed{(} \ 2 \ \boxed{\div} \ 3 \ \boxed{)} \ \boxed{=}$$

Operations Using the Rules of Exponents

In this section, we review all of the rules of exponents, which extend to include rational exponents. If given any rational numbers *m* and *n*, then we have

Product rule for exponents:	$x^m \cdot x^n = x^{m+n}$
Quotient rule for exponents:	$\frac{x^m}{x^n} = x^{m-n}, \ x \neq 0$
Power rule for exponents:	$(x^m)^n = x^{m \cdot n}$
Power rule for a product:	$(xy)^n = x^n y^n$

Power rule for a quotient:	$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}\ , y \neq 0$
Negative exponents:	$x^{-n} = \frac{1}{x^n}$
Zero exponent:	$x^0 = 1,\ x \neq 0$

These rules allow us to perform operations with rational exponents.

Example 9

Simplify: $7^{1/3} \cdot 7^{4/9}$.

Solution:

$$\begin{aligned} 7^{1/3} \cdot 7^{4/9} &= 7^{1/3+4/9} && \textit{Apply the product rule } x^m \cdot x^n = x^{m+n}. \\ &= 7^{3/9+4/9} \\ &= 7^{7/9} \end{aligned}$$

Answer: $7^{7/9}$

Example 10

Simplify: $\frac{x^{3/2}}{x^{2/3}}$.

Solution:

$$\begin{aligned} \frac{x^{3/2}}{x^{2/3}} &= x^{3/2-2/3} \quad \textit{Apply the quotient rule } \frac{x^m}{x^n} = x^{m-n}. \\ &= x^{9/6-4/6} \\ &= x^{5/6} \end{aligned}$$

Answer: $x^{5/6}$

Example 11

Simplify: $\left(y^{3/4}\right)^{2/3}$.

Solution:

$$\begin{aligned} \left(y^{3/4}\right)^{2/3} &= y^{(3/4)(2/3)} \quad \textit{Apply the power rule } \left(x^m\right)^n = x^{m \cdot n}. \\ &= y^{6/12} \quad \textit{Multiply the exponents and reduce.} \\ &= y^{1/2} \end{aligned}$$

Answer: $y^{1/2}$

Example 12

Simplify: $(81a^8b^{12})^{3/4}$.

Solution:

$$\begin{aligned}
(81a^8b^{12})^{3/4} &= (3^4a^8b^{12})^{3/4} && \color{blue}{\textit{Rewrite 81 as } 3^4.} \\
&= (3^4)^{3/4}(a^8)^{3/4}(b^{12})^{3/4} && \color{blue}{\textit{Apply the power rule for a product } (xy)^n = x} \\
&= 3^{4(3/4)}a^{8(3/4)}b^{12(3/4)} && \color{blue}{\textit{Apply the power rule to each factor.}} \\
&= 3^3a^6b^9 && \color{blue}{\textit{Simplify.}} \\
&= 27a^6b^9
\end{aligned}$$

Answer: $27a^6b^9$

Example 13

Simplify: $\left(9x^4\right)^{-3/2}$.

Solution:

$$\begin{aligned}
\left(9x^4\right)^{-3/2} &= \frac{1}{\left(9x^4\right)^{3/2}} && \textit{Apply the definition of negative exponents } x^{-n} = \frac{1}{x^n}. \\
&= \frac{1}{\left(3^2x^4\right)^{3/2}} && \textit{Write 9 as } 3^2 \textit{ and apply the rules of exponents.} \\
&= \frac{1}{3^{2(3/2)}x^{4(3/2)}} \\
&= \frac{1}{3^3 \cdot x^6} \\
&= \frac{1}{27x^6}
\end{aligned}$$

Answer: $\frac{1}{27x^6}$

Try this! Simplify: $\frac{\left(125a^{1/4}b^6\right)^{2/3}}{a^{1/6}}$.

Answer: $25b^4$

(click to see video)

Radical Expressions with Different Indices

To apply the product or quotient rule for radicals, the indices of the radicals involved must be the same. If the indices are different, then first rewrite the radicals in exponential form and then apply the rules for exponents.

Example 14

Multiply: $\sqrt{2} \cdot \sqrt[3]{2}$.

Solution:

In this example, the index of each radical factor is different. Hence the product rule for radicals does not apply. Begin by converting the radicals into an equivalent form using rational exponents. Then apply the product rule for exponents.

$$\begin{aligned} \sqrt{2} \cdot \sqrt[3]{2} &= 2^{1/2} \cdot 2^{1/3} && \textit{Equivalents using rational exponents} \\ &= 2^{1/2+1/3} && \textit{Apply the product rule for exponents.} \\ &= 2^{5/6} \\ &= \sqrt[6]{2^5} \end{aligned}$$

Answer: $\sqrt[6]{2^5}$

Example 15

Divide: $\frac{\sqrt[3]{4}}{\sqrt[5]{2}}$.

Solution:

In this example, the index of the radical in the numerator is different from the index of the radical in the denominator. Hence the quotient rule for radicals does not apply. Begin by converting the radicals into an equivalent form using rational exponents and then apply the quotient rule for exponents.

$$\begin{aligned}\frac{\sqrt[3]{4}}{\sqrt[5]{2}} &= \frac{\sqrt[3]{2^2}}{\sqrt[5]{2}} \\ &= \frac{2^{2/3}}{2^{1/5}} && \textit{Equivalents using rational exponents} \\ &= 2^{2/3-1/5} && \textit{Apply the quotient rule for exponents.} \\ &= 2^{7/15} \\ &= \sqrt[15]{2^7}\end{aligned}$$

Answer: $\sqrt[15]{2^7}$

Example 16

Simplify: $\sqrt{\sqrt[3]{4}}$.

Solution:

Here the radicand of the square root is a cube root. After rewriting this expression using rational exponents, we will see that the power rule for exponents applies.

$$\begin{aligned}\sqrt{\sqrt[3]{4}} &= \sqrt{\sqrt[3]{2^2}} \\ &= \left(2^{2/3}\right)^{1/2} && \textit{Equivalents using rational exponents} \\ &= 2^{(2/3)(1/2)} && \textit{Apply the power rule for exponents.} \\ &= 2^{1/3} \\ &= \sqrt[3]{2}\end{aligned}$$

Answer: $\sqrt[3]{2}$

KEY TAKEAWAYS

- Any radical expression can be written in exponential form: $\sqrt[n]{a^m} = a^{m/n}$.
- Fractional exponents indicate radicals. Use the numerator as the power and the denominator as the index of the radical.
- All the rules of exponents apply to expressions with rational exponents.
- If operations are to be applied to radicals with different indices, first rewrite the radicals in exponential form and then apply the rules for exponents.

TOPIC EXERCISES

PART A: RATIONAL EXPONENTS

Express using rational exponents.

1. $\sqrt{10}$
2. $\sqrt{6}$
3. $\sqrt[3]{3}$
4. $\sqrt[4]{5}$
5. $\sqrt[3]{5^2}$
6. $\sqrt[4]{2^3}$
7. $\sqrt[3]{49}$
8. $\sqrt[3]{9}$
9. $\sqrt[5]{x}$
10. $\sqrt[6]{x}$
11. $\sqrt[6]{x^7}$
12. $\sqrt[5]{x^4}$
13. $\dfrac{1}{\sqrt{x}}$
14. $\dfrac{1}{\sqrt[3]{x^2}}$

Express in radical form.

15. $10^{1/2}$
16. $11^{1/3}$

17. $7^{2/3}$

18. $2^{3/5}$

19. $x^{3/4}$

20. $x^{5/6}$

21. $x^{-1/2}$

22. $x^{-3/4}$

23. $\left(\frac{1}{x}\right)^{-1/3}$

24. $\left(\frac{1}{x}\right)^{-3/5}$

25. $(2x+1)^{2/3}$

26. $(5x-1)^{1/2}$

Write as a radical and then simplify.

27. $64^{1/2}$

28. $49^{1/2}$

29. $\left(\frac{1}{4}\right)^{1/2}$

30. $\left(\frac{4}{9}\right)^{1/2}$

31. $4^{-1/2}$

32. $9^{-1/2}$

33. $\left(\frac{1}{4}\right)^{-1/2}$

34. $\left(\frac{1}{16}\right)^{-1/2}$

35. $8^{1/3}$

36. $125^{1/3}$

37. $\left(\frac{1}{27}\right)^{1/3}$

38. $\left(\frac{8}{125}\right)^{1/3}$

39. $(-27)^{1/3}$

40. $(-64)^{1/3}$

41. $16^{1/4}$

42. $625^{1/4}$

43. $81^{-1/4}$

44. $16^{-1/4}$

45. $100{,}000^{1/5}$

46. $(-32)^{1/5}$

47. $\left(\frac{1}{32}\right)^{1/5}$

48. $\left(\frac{1}{243}\right)^{1/5}$

49. $9^{3/2}$

50. $4^{3/2}$

51. $8^{5/3}$

52. $27^{2/3}$

53. $16^{3/2}$

54. $32^{2/5}$

55. $\left(\frac{1}{16}\right)^{3/4}$

56. $\left(\frac{1}{81}\right)^{3/4}$

57. $(-27)^{2/3}$

58. $(-27)^{4/3}$

59. $(-32)^{3/5}$

60. $(-32)^{4/5}$

Use a calculator to approximate an answer rounded to the nearest hundredth.

61. $2^{1/2}$

62. $2^{1/3}$

63. $2^{3/4}$

64. $3^{2/3}$

65. $5^{1/5}$

66. $7^{1/7}$

67. $(-9)^{3/2}$

68. $-9^{3/2}$

69. Explain why (−4)^(3/2) gives an error on a calculator and −4^(3/2) gives an answer of −8.

70. Marcy received a text message from Mark asking her age. In response, Marcy texted back "125^(2/3) years old." Help Mark determine Marcy's age.

PART B: OPERATIONS USING THE RULES OF EXPONENTS

Perform the operations and simplify. Leave answers in exponential form.

71. $5^{3/2} \cdot 5^{1/2}$

72. $3^{2/3} \cdot 3^{7/3}$

73. $5^{1/2} \cdot 5^{1/3}$

74. $2^{1/6} \cdot 2^{3/4}$

75. $y^{1/4} \cdot y^{2/5}$

76. $x^{1/2} \cdot x^{1/4}$

77. $\dfrac{5^{11/3}}{5^{2/3}}$

78. $\dfrac{2^{9/2}}{2^{1/2}}$

79. $\dfrac{2a^{2/3}}{a^{1/6}}$

80. $\dfrac{3b^{1/2}}{b^{1/3}}$

81. $\left(8^{1/2}\right)^{2/3}$

82. $\left(3^{6}\right)^{2/3}$

83. $\left(x^{2/3}\right)^{1/2}$

84. $\left(y^{3/4}\right)^{4/5}$

85. $\left(y^{8}\right)^{-1/2}$

86. $\left(y^{6}\right)^{-2/3}$

87. $\left(4x^{2}y^{4}\right)^{1/2}$

88. $\left(9x^{6}y^{2}\right)^{1/2}$

89. $\left(2x^{1/3}y^{2/3}\right)^{3}$

90. $\left(8x^{3/2}y^{1/2}\right)^{2}$

91. $\left(36x^{4}y^{2}\right)^{-1/2}$

92. $\left(8x^{3}y^{6}z^{-3}\right)^{-1/3}$

93. $\left(\dfrac{a^{3/4}}{a^{1/2}}\right)^{4/3}$

94. $\left(\dfrac{b^{4/5}}{b^{1/10}}\right)^{10/3}$

95. $\left(\frac{4x^{2/3}}{y^4}\right)^{1/2}$

96. $\left(\frac{27x^{3/4}}{y^9}\right)^{1/3}$

97. $\frac{y^{1/2}y^{2/3}}{y^{1/6}}$

98. $\frac{x^{2/5}x^{1/2}}{x^{1/10}}$

99. $\frac{xy}{x^{1/2}y^{1/3}}$

100. $\frac{x^{5/4}y}{xy^{2/5}}$

101. $\frac{49a^{5/7}b^{3/2}}{7a^{3/7}b^{1/4}}$

102. $\frac{16a^{5/6}b^{5/4}}{8a^{1/2}b^{2/3}}$

103. $\frac{\left(9x^{2/3}y^6\right)^{3/2}}{x^{1/2}y}$

104. $\frac{\left(125x^3y^{3/5}\right)^{2/3}}{xy^{1/3}}$

105. $\frac{\left(27a^{1/4}b^{3/2}\right)^{2/3}}{a^{1/6}b^{1/2}}$

106. $\frac{\left(25a^{2/3}b^{4/3}\right)^{3/2}}{a^{1/6}b^{1/3}}$

107. $\left(16x^2y^{-1/3}z^{2/3}\right)^{-3/2}$

108. $\left(81x^8y^{-4/3}z^{-4}\right)^{-3/4}$

109. $\left(100a^{-2/3}b^4c^{-3/2}\right)^{-1/2}$

110. $\left(125a^9b^{-3/4}c^{-1}\right)^{-1/3}$

PART C: RADICAL EXPRESSIONS WITH DIFFERENT INDICES

Perform the operations.

111. $\sqrt[3]{9} \cdot \sqrt[5]{3}$
112. $\sqrt{5} \cdot \sqrt[5]{25}$
113. $\sqrt{x} \cdot \sqrt[3]{x}$
114. $\sqrt{y} \cdot \sqrt[4]{y}$
115. $\sqrt[3]{x^2} \cdot \sqrt[4]{x}$
116. $\sqrt[5]{x^3} \cdot \sqrt[3]{x}$
117. $\dfrac{\sqrt[3]{100}}{\sqrt{10}}$
118. $\dfrac{\sqrt[5]{16}}{\sqrt[3]{4}}$
119. $\dfrac{\sqrt[3]{a^2}}{\sqrt{a}}$
120. $\dfrac{\sqrt[5]{b^4}}{\sqrt[3]{b}}$
121. $\dfrac{\sqrt[3]{x^2}}{\sqrt[5]{x^3}}$
122. $\dfrac{\sqrt[4]{x^3}}{\sqrt[3]{x^2}}$
123. $\sqrt{\sqrt[5]{16}}$
124. $\sqrt{\sqrt[3]{9}}$

125. $\sqrt[3]{\sqrt[5]{2}}$

126. $\sqrt[3]{\sqrt[5]{5}}$

127. $\sqrt[3]{\sqrt{7}}$

128. $\sqrt[3]{\sqrt{3}}$

PART D: DISCUSSION BOARD

129. Who is credited for devising the notation that allows for rational exponents? What are some of his other accomplishments?

130. When using text, it is best to communicate *n*th roots using rational exponents. Give an example.

ANSWERS

1. $10^{1/2}$
3. $3^{1/3}$
5. $5^{2/3}$
7. $7^{2/3}$
9. $x^{1/5}$
11. $x^{7/6}$
13. $x^{-1/2}$
15. $\sqrt{10}$
17. $\sqrt[3]{49}$
19. $\sqrt[4]{x^3}$
21. $\dfrac{1}{\sqrt{x}}$
23. $\sqrt[3]{x}$
25. $\sqrt[3]{(2x+1)^2}$
27. 8
29. $\frac{1}{2}$
31. $\frac{1}{2}$
33. 2
35. 2
37. $\frac{1}{3}$
39. −3
41. 2

43. $\frac{1}{3}$

45. 10

47. $\frac{1}{2}$

49. 27

51. 32

53. 64

55. $\frac{1}{8}$

57. 9

59. −8

61. 1.41

63. 1.68

65. 1.38

67. Not a real number

69. Answer may vary

71. 25

73. $5^{5/6}$

75. $y^{13/20}$

77. 125

79. $2a^{1/2}$

81. 2

83. $x^{1/3}$

85. $\frac{1}{y^4}$

87. $2xy^2$

89. $8xy^2$

91. $\dfrac{1}{6x^{2}y}$

93. $a^{1/3}$

95. $\dfrac{2x^{1/3}}{y^{2}}$

97. y

99. $x^{1/2}y^{2/3}$

101. $7a^{2/7}b^{5/4}$

103. $27x^{1/2}y^{8}$

105. $9b^{1/2}$

107. $\dfrac{y^{1/2}}{64x^{3}z}$

109. $\dfrac{a^{1/3}b^{3/4}}{10b^{2}}$

111. $\sqrt[15]{3^{13}}$

113. $\sqrt[6]{x^{5}}$

115. $\sqrt[12]{x^{11}}$

117. $\sqrt[6]{10}$

119. $\sqrt[6]{a}$

121. $\sqrt[15]{x}$

123. $\sqrt[5]{4}$

125. $\sqrt[15]{2}$

127. $\sqrt[6]{7}$

129. Answer may vary

5.6 Solving Radical Equations

LEARNING OBJECTIVES

1. Solve equations involving square roots.
2. Solve equations involving cube roots.

Radical Equations

A **radical equation**[22] is any equation that contains one or more radicals with a variable in the radicand. Following are some examples of radical equations, all of which will be solved in this section:

$\sqrt{2x-1}=3$	$\sqrt[3]{4x^2+7}-2=0$	$\sqrt{x+2}-\sqrt{x}=1$

We begin with the **squaring property of equality**[23]; given real numbers *a* and *b*, we have the following:

$$\text{If } a=b, \text{ then } a^2=b^2.$$

In other words, equality is retained if we square both sides of an equation.

$$\begin{aligned} -3=-3 \Rightarrow (-3)^2 &= (-3)^2 \\ 9 &= 9 \quad ✓ \end{aligned}$$

The converse, on the other hand, is not necessarily true,

22. Any equation that contains one or more radicals with a variable in the radicand.

23. Given real numbers *a* and *b*, where $a=b$, then $a^2=b^2$.

$$9 = 9$$
$$(-3)^2 = (3)^2 \Rightarrow -3 \neq 3 \quad ✗$$

This is important because we will use this property to solve radical equations. Consider a very simple radical equation that can be solved by inspection,

$$\sqrt{x} = 5$$

Here we can see that $x = 25$ is a solution. To solve this equation algebraically, make use of the squaring property of equality and the fact that $\left(\sqrt{a}\right)^2 = \sqrt{a^2} = a$ when *a* is nonnegative. Eliminate the square root by squaring both sides of the equation as follows:

$$\left(\sqrt{x}\right)^2 = (5)^2$$
$$x = 25$$

As a check, we can see that $\sqrt{25} = 5$ as expected. Because the converse of the squaring property of equality is not necessarily true, solutions to the squared equation may not be solutions to the original. Hence squaring both sides of an equation introduces the possibility of **extraneous solutions**[24], which are solutions that do not solve the original equation. For example,

$$\sqrt{x} = -5$$

This equation clearly does not have a real number solution. However, squaring both sides gives us a solution:

24. A properly found solution that does not solve the original equation.

$$\left(\sqrt{x}\right)^2 = (-5)^2$$
$$x = 25$$

As a check, we can see that $\sqrt{25} \neq -5$. For this reason, we must check the answers that result from squaring both sides of an equation.

Example 1

Solve: $\sqrt{3x+1} = 4$.

Solution:

We can eliminate the square root by applying the squaring property of equality.

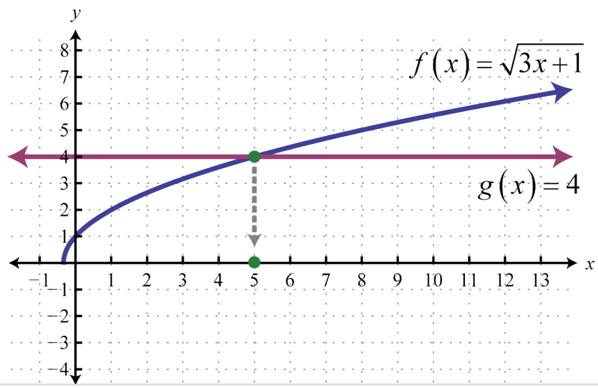
$$
\begin{aligned}
\sqrt{3x+1} &= 4 \\
\left(\sqrt{3x+1}\right)^2 &= (4)^2 \quad \textit{Square both sides.} \\
3x+1 &= 16 \quad \textit{Solve.} \\
3x &= 15 \\
x &= 5
\end{aligned}
$$

Next, we must check.

$$
\begin{aligned}
\sqrt{3\left(5\right)+1} &= 4 \\
\sqrt{15+1} &= 4 \\
\sqrt{16} &= 4 \\
4 &= 4 \quad \checkmark
\end{aligned}
$$

Answer: The solution is 5.

There is a geometric interpretation to the previous example. Graph the function defined by $f(x) = \sqrt{3x+1}$ and determine where it intersects the graph defined by $g(x) = 4$.



As illustrated, $f(x) = g(x)$ where $x = 5$.

Example 2

Solve: $\sqrt{x-3} = x - 5$.

Solution:

Begin by squaring both sides of the equation.

$$\begin{aligned} \sqrt{x-3} &= x - 5 \\ \left(\sqrt{x-3}\right)^2 &= (x-5)^2 \qquad \textit{Square both sides.} \\ x - 3 &= x^2 - 10x + 25 \end{aligned}$$

The resulting quadratic equation can be solved by factoring.

$$\begin{aligned} x - 3 &= x^2 - 10x + 25 \\ 0 &= x^2 - 11x + 28 \\ 0 &= (x-4)\ (x-7) \end{aligned}$$

$$\begin{aligned} x - 4 &= 0 \qquad \text{or} \qquad x - 7 = 0 \\ x &= 4 \qquad\qquad\qquad\quad x = 7 \end{aligned}$$

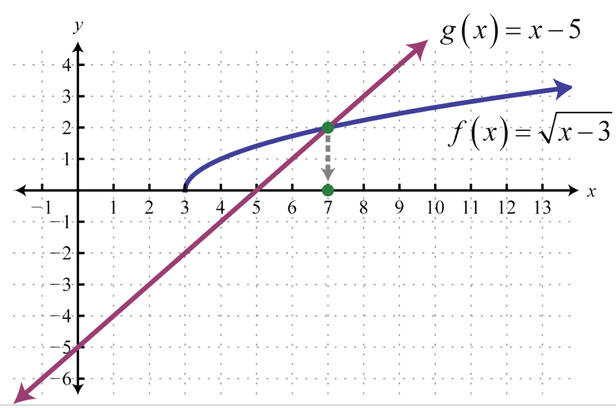
Checking the solutions after squaring both sides of an equation is not optional. Use the original equation when performing the check.

Check $x = 4$	*Check* $x = 7$
$\begin{aligned} \sqrt{x-3} &= x-5 \\ \sqrt{4-3} &= 4-5 \\ \sqrt{1} &= -1 \\ 1 &= -1 \ ✗ \end{aligned}$	$\begin{aligned} \sqrt{x-3} &= x-5 \\ \sqrt{7-3} &= 7-5 \\ \sqrt{4} &= 2 \\ 2 &= 2 \ ✓ \end{aligned}$

After checking, you can see that $x = 4$ is an extraneous solution; it does not solve the original radical equation. Disregard that answer. This leaves $x = 7$ as the only solution.

Answer: The solution is 7.

Geometrically we can see that $f(x) = \sqrt{x+3}$ is equal to $g(x) = x - 5$ where $x = 7$.



In the previous two examples, notice that the radical is isolated on one side of the equation. Typically, this is not the case. The steps for solving radical equations involving square roots are outlined in the following example.

Example 3

Solve: $\sqrt{2x-1}+2=x$.

Solution:

Step 1: Isolate the square root. Begin by subtracting 2 from both sides of the equation.

$$\begin{aligned}\sqrt{2x-1}+2&=x\\ \sqrt{2x-1}&=x-2\end{aligned}$$

Step 2: Square both sides. Squaring both sides eliminates the square root.

$$\begin{aligned}\left(\sqrt{2x-1}\right)^2&=(x-2)^2\\ 2x-1&=x^2-4x+4\end{aligned}$$

Step 3: Solve the resulting equation. Here we are left with a quadratic equation that can be solved by factoring.

$$
\begin{aligned}
2x - 1 &= x^2 - 4x + 4 \\
0 &= x^2 - 6x + 5 \\
0 &= (x - 1)(x - 5)
\end{aligned}
$$

$$
\begin{aligned}
x - 1 &= 0 \quad & \text{or} \quad x - 5 &= 0 \\
x &= 1 & x &= 5
\end{aligned}
$$

Step 4: Check the solutions in the original equation. Squaring both sides introduces the possibility of extraneous solutions; hence the check is required.

Check $x = 1$	*Check* $x = 5$
$\begin{aligned} \sqrt{2x-1}+2 &= x \\ \sqrt{2(1)-1}+2 &= 1 \\ \sqrt{1}+2 &= 1 \\ 1+2 &= 1 \\ 3 &= 1 \; ✗ \end{aligned}$	$\begin{aligned} \sqrt{2x-1}+2 &= x \\ \sqrt{2(5)-1}+2 &= 5 \\ \sqrt{9}+2 &= 5 \\ 3+2 &= 5 \\ 5 &= 5 \; ✓ \end{aligned}$

After checking, we can see that $x = 1$ is an extraneous solution; it does not solve the original radical equation. This leaves $x = 5$ as the only solution.

Answer: The solution is 5.

Sometimes there is more than one solution to a radical equation.

Example 4

Solve: $2\sqrt{2x+5} - x = 4$.

Solution:

Begin by isolating the term with the radical.

$$\begin{aligned} 2\sqrt{2x+5} - x &= 4 && \textit{Add } x \textit{ to both sides.} \\ 2\sqrt{2x+5} &= x + 4 \end{aligned}$$

Despite the fact that the term on the left side has a coefficient, we still consider it to be isolated. Recall that terms are separated by addition or subtraction operators.

$$\begin{aligned} 2\sqrt{2x+5} &= x + 4 \\ \left(2\sqrt{2x+5}\right)^2 &= (x+4)^2 && \textit{Square both sides.} \\ 4\left(2x+5\right) &= x^2 + 8x + 16 \end{aligned}$$

Solve the resulting quadratic equation.

$$
\begin{aligned}
4\left(2x+5\right) &= x^2+8x+16\\
8x+20 &= x^2+8x+16\\
0 &= x^2-4\\
0 &= (x+2)\ (x-2)
\end{aligned}
$$

$$
\begin{aligned}
x+2 &= 0 & \text{or}\quad x-2 &= 0\\
x &= -2 & x &= 2
\end{aligned}
$$

Since we squared both sides, we must check our solutions.

Check $x=-2$	*Check* $x=2$
$\begin{aligned} 2\sqrt{2x+5}-x &= 4\\ 2\sqrt{2(-2)+5}-(-2) &= 4\\ 2\sqrt{-4+5}+2 &= 4\\ 2\sqrt{1}+2 &= 4\\ 2+2 &= 4\\ 4 &= 4\ \checkmark \end{aligned}$	$\begin{aligned} 2\sqrt{2x+5}-x &= 4\\ 2\sqrt{2(2)+5}-(2) &= 4\\ 2\sqrt{4+5}-2 &= 4\\ 2\sqrt{9}-2 &= 4\\ 6-2 &= 4\\ 4 &= 4\ \checkmark \end{aligned}$

After checking, we can see that both are solutions to the original equation.

Answer: The solutions are ±2.

Sometimes both of the possible solutions are extraneous.

Example 5

Solve: $\sqrt{4 - 11x} - x + 2 = 0$.

Solution:

Begin by isolating the radical.

$$\begin{aligned} \sqrt{4 - 11x} - x + 2 &= 0 && \color{blue}{\textit{Isolate the radical.}} \\ \sqrt{4 - 11x} &= x - 2 \\ \left(\sqrt{4 - 11x}\right)^2 &= (x - 2)^2 && \color{blue}{\textit{Square both sides.}} \\ 4 - 11x &= x^2 - 4x + 4 && \color{blue}{\textit{Solve.}} \\ 0 &= x^2 + 7x \\ 0 &= x\,(x + 7) \end{aligned}$$

$$\begin{aligned} x = 0 \text{ or } x + 7 &= 0 \\ x &= -7 \end{aligned}$$

Since we squared both sides, we must check our solutions.

Check $x = 0$	*Check* $x = -7$
$\begin{aligned} \sqrt{4-11x}-x+2 &= 0 \\ \sqrt{4-11(0)}-0+2 &= 0 \\ \sqrt{4}+2 &= 0 \\ 2+2 &= 0 \\ 4 &= 0 \end{aligned}$ ✗	$\begin{aligned} \sqrt{4-11x}-x+2 &= 0 \\ \sqrt{4-11(-7)}-(-7)+2 &= 0 \\ \sqrt{4+77}+7+2 &= 0 \\ \sqrt{81}+9 &= 0 \\ 9+9 &= 0 \\ 18 &= 0 \end{aligned}$

Since both possible solutions are extraneous, the equation has no solution.

Answer: No solution, Ø

The squaring property of equality extends to any positive integer power n. Given real numbers a and b, we have the following:

$$\text{If } a = b, \text{ then } a^n = b^n.$$

This is often referred to as the **power property of equality**[25]. Use this property, along with the fact that $\left(\sqrt[n]{a}\right)^n = \sqrt[n]{a^n} = a$, when a is nonnegative, to solve radical equations with indices greater than 2.

25. Given any positive integer n and real numbers a and b where $a = b$, then $a^n = b^n$.

Example 6

Solve: $\sqrt[3]{4x^2+7}-2=0$.

Solution:

Isolate the radical, and then cube both sides of the equation.

$$\begin{aligned} \sqrt[3]{4x^2+7}-2&=0 && \textit{Isolate the radical.} \\ \sqrt[3]{4x^2+7}&=2 \\ \left(\sqrt[3]{4x^2+7}\right)^3&=(2)^3 && \textit{Cube both sides.} \\ 4x^2+7&=8 && \textit{Solve.} \\ 4x^2-1&=0 \\ (2x+1)\,(2x-1)&=0 \end{aligned}$$

$$\begin{aligned} 2x+1&=0 \quad \text{or} \quad & 2x-1&=0 \\ 2x&=-1 & 2x&=1 \\ x&=-\frac{1}{2} & x&=\frac{1}{2} \end{aligned}$$

Check.

Check $x = -\frac{1}{2}$	*Check* $x = \frac{1}{2}$
$$\begin{aligned} \sqrt[3]{4x^2+7}-2 &= 0 \\ \sqrt[3]{4\left(-\frac{1}{2}\right)^2+7}-2 &= 0 \\ \sqrt[3]{4\cdot\frac{1}{4}+7}-2 &= 0 \\ \sqrt[3]{1+7}-2 &= 0 \\ \sqrt[3]{8}-2 &= 0 \\ 2-2 &= 0 \\ 0 &= 0\ \checkmark \end{aligned}$$	$$\begin{aligned} \sqrt[3]{4x^2+7}-2 &= 0 \\ \sqrt[3]{4\left(\frac{1}{2}\right)^2+7}-2 &= 0 \\ \sqrt[3]{4\cdot\frac{1}{4}+7}-2 &= 0 \\ \sqrt[3]{1+7}-2 &= 0 \\ \sqrt[3]{8}-2 &= 0 \\ 2-2 &= 0 \\ 0 &= 0\ \checkmark \end{aligned}$$

Answer: The solutions are $\pm\frac{1}{2}$.

Try this! $x - 3\sqrt{3x+1} = 3$

Answer: The solution is 33.

(click to see video)

It may be the case that the equation has more than one term that consists of radical expressions.

Example 7

Solve: $\sqrt{5x-3} = \sqrt{4x-1}$.

Solution:

Both radicals are considered isolated on separate sides of the equation.

$$\begin{aligned}
\sqrt{5x-3} &= \sqrt{4x-1} \\
\left(\sqrt{5x-3}\right)^2 &= \left(\sqrt{4x-1}\right)^2 && \textit{Square both sides.} \\
5x-3 &= 4x-1 && \textit{Solve.} \\
x &= 2
\end{aligned}$$

Check $x = 2$.

$$\begin{aligned}
\sqrt{5x-3} &= \sqrt{4x-1} \\
\sqrt{5(2)-3} &= \sqrt{4(2)-1} \\
\sqrt{10-3} &= \sqrt{8-1} \\
\sqrt{7} &= \sqrt{7} \quad ✓
\end{aligned}$$

Answer: The solution is 2.

Example 8

Solve: $\sqrt[3]{x^2 + x - 14} = \sqrt[3]{x + 50}$.

Solution:

Eliminate the radicals by cubing both sides.

$$
\begin{aligned}
\sqrt[3]{x^2 + x - 14} &= \sqrt[3]{x + 50} \\
\left(\sqrt[3]{x^2 + x - 14}\right)^3 &= \left(\sqrt[3]{x + 50}\right)^3 && \textit{Cube both sides.} \\
x^2 + x - 14 &= x + 50 && \textit{Solve.} \\
x^2 - 64 &= 0 \\
(x + 8)\ (x - 8) &= 0
\end{aligned}
$$

$$
\begin{aligned}
x + 8 &= 0 \qquad \text{or} \qquad & x - 8 &= 0 \\
x &= -8 & x &= 8
\end{aligned}
$$

Check.

Check $x = -8$	*Check* $x = 8$
$\sqrt[3]{x^2+x-14} = \sqrt[3]{x+50}$ $\sqrt[3]{(-8)^2+(-8)-14} = \sqrt[3]{(-8)+50}$ $\sqrt[3]{64-8-14} = \sqrt[3]{42}$ $\sqrt[3]{42} = \sqrt[3]{42}$ ✓	$\sqrt[3]{x^2+x-14} =$ $\sqrt[3]{(8)^2+(8)-14} =$ $\sqrt[3]{64+8-14} =$ $\sqrt[3]{58} =$

Answer: The solutions are ± 8.

It may not be possible to isolate a radical on both sides of the equation. When this is the case, isolate the radicals, one at a time, and apply the squaring property of equality multiple times until only a polynomial remains.

Example 9

Solve: $\sqrt{x+2} - \sqrt{x} = 1$

Solution:

Begin by isolating one of the radicals. In this case, add $\sqrt{x}$ to both sides of the equation.

$$\begin{aligned} \sqrt{x+2} - \sqrt{x} &= 1 \\ \sqrt{x+2} &= \sqrt{x} + 1 \end{aligned}$$

Next, square both sides. Take care to apply the distributive property to the right side.

$$\begin{aligned} \left(\sqrt{x+2}\right)^2 &= \left(\sqrt{x}+1\right)^2 \\ x+2 &= \left(\sqrt{x}+1\right)\left(\sqrt{x}+1\right) \\ x+2 &= \sqrt{x^2} + \sqrt{x} + \sqrt{x} + 1 \\ x+2 &= x + 2\sqrt{x} + 1 \end{aligned}$$

At this point we have one term that contains a radical. Isolate it and square both sides again.

$$
\begin{aligned}
x + 2 &= x + 2\sqrt{x} + 1 \\
1 &= 2\sqrt{x} \\
(1)^2 &= \left(2\sqrt{x}\right)^2 \\
1 &= 4x \\
\frac{1}{4} &= x
\end{aligned}
$$

Check to see if $x = \frac{1}{4}$ satisfies the original equation $\sqrt{x+2} - \sqrt{x} = 1$.

$$
\begin{aligned}
\sqrt{\frac{1}{4} + 2} - \sqrt{\frac{1}{4}} &= 1 \\
\sqrt{\frac{9}{4}} - \frac{1}{2} &= 1 \\
\frac{3}{2} - \frac{1}{2} &= 1 \\
\frac{2}{2} &= 1 \\
1 &= 1 \quad \checkmark
\end{aligned}
$$

Answer: The solution is $\frac{1}{4}$.

Note: Because $(A + B)^2 \neq A^2 + B^2$, we cannot simply square each term. For example, it is incorrect to square each term as follows.

$$
\left(\sqrt{x+2}\right)^2 - \left(\sqrt{x}\right)^2 = (1)^2
$$

Incorrect!

This is a common mistake and leads to an incorrect result. When squaring both sides of an equation with multiple terms, we must take care to apply the distributive property.

Example 10

Solve: $\sqrt{2x+10}-\sqrt{x+6}=1$

Solution:

Begin by isolating one of the radicals. In this case, add $\sqrt{x+6}$ to both sides of the equation.

$$\begin{aligned}\sqrt{2x+10}-\sqrt{x+6}&=1\\ \sqrt{2x+10}&=\sqrt{x+6}+1\end{aligned}$$

Next, square both sides. Take care to apply the distributive property to the right side.

$$\begin{aligned}\left(\sqrt{2x+10}\right)^2&=\left(\sqrt{x+6}+1\right)^2\\ 2x+10&=x+6+2\sqrt{x+6}+1\\ 2x+10&=x+7+2\sqrt{x+6}\end{aligned}$$

At this point we have one term that contains a radical. Isolate it and square both sides again.

$$\begin{aligned}
2x+10 &= x+7+2\sqrt{x+6}\\
x+3 &= 2\sqrt{x+6}\\
(x+3)^2 &= \left(2\sqrt{x+6}\right)^2\\
x^2+6x+9 &= 4\left(x+6\right)\\
x^2+6x+9 &= 4x+24\\
x^2+2x-15 &= 0\\
(x-3)\ \left(x+5\right) &= 0
\end{aligned}$$

$$\begin{aligned}
x-3&=0 \qquad\qquad \text{or} \quad & x+5&=0\\
x&=3 & x&=-5
\end{aligned}$$

Check.

Check $x=3$	*Check* $x=-5$
$\sqrt{2x+10}-\sqrt{x+6} = 1$	$\sqrt{2x+10}-\sqrt{x+6} =$
$\sqrt{2(3)+10}-\sqrt{3+6} = 1$	$\sqrt{2\left(-5\right)+10}-\sqrt{-5+6} =$
$\sqrt{16}-\sqrt{9} = 1$	$\sqrt{0}-\sqrt{1} =$
$4-3 = 1$	$0-1 =$
$1 = 1$ ✓	$-1 =$

Answer: The solution is 3.

Try this! Solve: $\sqrt{4x+21}-\sqrt{2x+22}=1$

Answer: The solution is 7.

(click to see video)

KEY TAKEAWAYS

- Solve equations involving square roots by first isolating the radical and then squaring both sides. Squaring a square root eliminates the radical, leaving us with an equation that can be solved using the techniques learned earlier in our study of algebra.
- Squaring both sides of an equation introduces the possibility of extraneous solutions. For this reason, you must check your solutions in the original equation.
- Solve equations involving *n*th roots by first isolating the radical and then raise both sides to the *n*th power. This eliminates the radical and results in an equation that may be solved with techniques you have already mastered.
- When more than one radical term is present in an equation, isolate them one at a time, and apply the power property of equality multiple times until only a polynomial remains.

TOPIC EXERCISES

PART A: SOLVING RADICAL EQUATIONS

Solve

1. $\sqrt{x} = 7$
2. $\sqrt{x} = 4$
3. $\sqrt{x} + 8 = 9$
4. $\sqrt{x} - 4 = 5$
5. $\sqrt{x} + 7 = 4$
6. $\sqrt{x} + 3 = 1$
7. $5\sqrt{x} - 1 = 0$
8. $3\sqrt{x} - 2 = 0$
9. $\sqrt{3x + 1} = 2$
10. $\sqrt{5x - 4} = 4$
11. $\sqrt{7x + 4} + 6 = 11$
12. $\sqrt{3x - 5} + 9 = 14$
13. $2\sqrt{x - 1} - 3 = 0$
14. $3\sqrt{x + 1} - 2 = 0$
15. $\sqrt{x + 1} = \sqrt{x} + 1$
16. $\sqrt{2x - 1} = \sqrt{2x} - 1$
17. $\sqrt{4x - 1} = 2\sqrt{x} - 1$
18. $\sqrt{4x - 11} = 2\sqrt{x} - 1$

19. $\sqrt{x+8} = \sqrt{x} - 4$
20. $\sqrt{25x-1} = 5\sqrt{x} + 1$
21. $\sqrt[3]{x} = 3$
22. $\sqrt[3]{x} = -4$
23. $\sqrt[3]{2x+9} = 3$
24. $\sqrt[3]{4x-11} = 1$
25. $\sqrt[3]{5x+7} + 3 = 1$
26. $\sqrt[3]{3x-6} + 5 = 2$
27. $4 - 2\sqrt[3]{x+2} = 0$
28. $6 - 3\sqrt[3]{2x-3} = 0$
29. $\sqrt[5]{3(x+10)} = 2$
30. $\sqrt[5]{4x+3} + 5 = 4$
31. $\sqrt{8x+11} = 3\sqrt{x+1}$
32. $2\sqrt{3x-4} = \sqrt{2(3x+1)}$
33. $\sqrt{2(x+10)} = \sqrt{7x-15}$
34. $\sqrt{5(x-4)} = \sqrt{x+4}$
35. $\sqrt[3]{5x-2} = \sqrt[3]{4x}$
36. $\sqrt[3]{9(x-1)} = \sqrt[3]{3(x+7)}$
37. $\sqrt[3]{3x+1} = \sqrt[3]{2(x-1)}$
38. $\sqrt[3]{9x} = \sqrt[3]{3(x-6)}$
39. $\sqrt[5]{3x-5} = \sqrt[5]{2x+8}$
40. $\sqrt[5]{x+3} = \sqrt[5]{2x+5}$

41. $\sqrt{4x+21} = x$

42. $\sqrt{8x+9} = x$

43. $\sqrt{4(2x-3)} = x$

44. $\sqrt{3(4x-9)} = x$

45. $2\sqrt{x-1} = x$

46. $3\sqrt{2x-9} = x$

47. $\sqrt{9x+9} = x+1$

48. $\sqrt{3x+10} = x+4$

49. $\sqrt{x-1} = x-3$

50. $\sqrt{2x-5} = x-4$

51. $\sqrt{16-3x} = x-6$

52. $\sqrt{7-3x} = x-3$

53. $3\sqrt{2x+10} = x+9$

54. $2\sqrt{2x+5} = x+4$

55. $3\sqrt{x-1} - 1 = x$

56. $2\sqrt{2x+2} - 1 = x$

57. $\sqrt{10x+41} - 5 = x$

58. $\sqrt{6(x+3)} - 3 = x$

59. $\sqrt{8x^2-4x+1} = 2x$

60. $\sqrt{18x^2-6x+1} = 3x$

61. $5\sqrt{x+2} = x+8$

62. $4\sqrt{2(x+1)} = x+7$

63. $\sqrt{x^2 - 25} = x$

64. $\sqrt{x^2 + 9} = x$

65. $3 + \sqrt{6x - 11} = x$

66. $2 + \sqrt{9x - 8} = x$

67. $\sqrt{4x + 25} - x = 7$

68. $\sqrt{8x + 73} - x = 10$

69. $2\sqrt{4x + 3} - 3 = 2x$

70. $2\sqrt{6x + 3} - 3 = 3x$

71. $2x - 4 = \sqrt{14 - 10x}$

72. $3x - 6 = \sqrt{33 - 24x}$

73. $\sqrt[3]{x^2 - 24} = 1$

74. $\sqrt[3]{x^2 - 54} = 3$

75. $\sqrt[3]{x^2 + 6x} + 1 = 4$

76. $\sqrt[3]{x^2 + 2x} + 5 = 7$

77. $\sqrt[3]{25x^2 - 10x - 7} = -2$

78. $\sqrt[3]{9x^2 - 12x - 23} = -3$

79. $\sqrt[3]{4x^2 - 1} - 2 = 0$

80. $4\sqrt[3]{x^2} - 1 = 0$

81. $\sqrt[5]{x(2x + 1)} - 1 = 0$

82. $\sqrt[5]{3x^2 - 20x} - 2 = 0$

83. $\sqrt{2x^2 - 15x + 25} = \sqrt{(x + 5)(x - 5)}$

84. $\sqrt{x^2 - 4x + 4} = \sqrt{x(5 - x)}$

85. $\sqrt[3]{2(x^2 + 3x - 20)} = \sqrt[3]{(x + 3)^2}$

86. $\sqrt[3]{3x^2 + 3x + 40} = \sqrt[3]{(x - 5)^2}$

87. $\sqrt{2x - 5} + \sqrt{2x} = 5$

88. $\sqrt{4x + 13} - 2\sqrt{x} = 3$

89. $\sqrt{8x + 17} - 2\sqrt{2 - x} = 3$

90. $\sqrt{3x - 6} - \sqrt{2x - 3} = 1$

91. $\sqrt{2(x - 2)} - \sqrt{x - 1} = 1$

92. $\sqrt{2x + 5} - \sqrt{x + 3} = 2$

93. $\sqrt{2(x + 1)} - \sqrt{3x + 4} - 1 = 0$

94. $\sqrt{6 - 5x} + \sqrt{3 - 3x} - 1 = 0$

95. $\sqrt{x - 2} - 1 = \sqrt{2(x - 3)}$

96. $\sqrt{14 - 11x} + \sqrt{7 - 9x} = 1$

97. $\sqrt{x + 1} = \sqrt{3} - \sqrt{2 - x}$

98. $\sqrt{2x + 9} - \sqrt{x + 1} = 2$

99. $x^{1/2} - 10 = 0$

100. $x^{1/2} - 6 = 0$

101. $x^{1/3} + 2 = 0$

102. $x^{1/3} + 4 = 0$

103. $(x - 1)^{1/2} - 3 = 0$

104. $(x + 2)^{1/2} - 6 = 0$

105. $(2x-1)^{1/3}+3=0$

106. $(3x-1)^{1/3}-2=0$

107. $(4x+15)^{1/2}-2x=0$

108. $(3x+2)^{1/2}-3x=0$

109. $(2x+12)^{1/2}-x=6$

110. $(4x+36)^{1/2}-x=9$

111. $2(5x+26)^{1/2}=x+10$

112. $3(x-1)^{1/2}=x+1$

113. $x^{1/2}+(3x-2)^{1/2}=2$

114. $(6x+1)^{1/2}-(3x)^{1/2}=1$

115. $(3x+7)^{1/2}+(x+3)^{1/2}-2=0$

116. $(3x)^{1/2}+(x+1)^{1/2}-5=0$

Determine the roots of the given functions. Recall that a root is a value in the domain that results in zero. In other words, find x where $f(x)=0$.

117. $f(x)=\sqrt{x+5}-2$

118. $f(x)=\sqrt{2x-3}-1$

119. $f(x)=2\sqrt{x+2}-8$

120. $f(x)=3\sqrt{x-7}-6$

121. $f(x)=\sqrt[3]{x+1}+2$

122. $f(x)=2\sqrt[3]{x-1}+6$

Solve for the indicated variable.

123. Solve for P: $r=\sqrt{P}-1$

124. Solve for x: $y = \sqrt{x - h} + k$

125. Solve for s: $t = \sqrt{\frac{2s}{g}}$

126. Solve for L: $T = 2\pi\sqrt{\frac{L}{32}}$

127. Solve for R: $I = \sqrt{\frac{P}{R}}$

128. Solve for h: $r = \sqrt{\frac{3V}{\pi h}}$

129. Solve for V: $r = \sqrt[3]{\frac{3V}{4\pi}}$

130. Solve for c: $a = \sqrt[3]{\frac{b^2\pi}{2c}}$

131. The square root of 1 less than twice a number is equal to 2 less than the number. Find the number.

132. The square root of 4 less than twice a number is equal to 6 less than the number. Find the number.

133. The square root of twice a number is equal to one-half of that number. Find the number.

134. The square root of twice a number is equal to one-third of that number. Find the number.

135. The distance d in miles a person can see an object on the horizon is given by the formula

$$d = \frac{\sqrt{6h}}{2}$$

where h represents the height in feet of the person's eyes above sea level. How high must a person's eyes be to see an object 5 miles away?

136. The current I measured in amperes is given by the formula

$$I = \sqrt{\frac{P}{R}}$$

where P is the power usage measured in watts and R is the resistance measured in ohms. If a light bulb requires 1/2 amperes of current and uses 60 watts of power, then what is the resistance through the bulb?

The period of a pendulum T in seconds is given by the formula

$$T = 2\pi\sqrt{\frac{L}{32}}$$

where L represents the length in feet. Calculate the length of a pendulum given the period. Give the exact value and the approximate value rounded to the nearest tenth of a foot.

137. 1 second
138. 2 seconds
139. $\frac{1}{2}$ second
140. $\frac{1}{3}$ second

The time t in seconds, an object is in free fall is given by the formula

$$t = \frac{\sqrt{s}}{4}$$

where s represents the distance it has fallen, in feet. Calculate the distance an object will fall given the amount of time.

141. 1 second
142. 2 seconds
143. $\frac{1}{2}$ second
144. $\frac{1}{4}$ second

PART B: DISCUSSION BOARD

145. Discuss reasons why we sometimes obtain extraneous solutions when solving radical equations. Are there ever any conditions where we do not need to check for extraneous solutions? Why or why not?

146. If an equation has multiple terms, explain why squaring all of them is incorrect. Provide an example.

ANSWERS

1. 49

3. 1

5. Ø

7. $\frac{1}{25}$

9. 1

11. 3

13. $\frac{13}{4}$

15. 0

17. $\frac{1}{4}$

19. Ø

21. 27

23. 9

25. −3

27. 6

29. $\frac{2}{3}$

31. 2

33. 7

35. 2

37. −3

39. 13

41. 7

43. 2, 6

45. 2

47. −1, 8

49. 5

51. Ø

53. −3, 3

55. 2, 5

57. −4, 4

59. $\frac{1}{2}$

61. 2, 7

63. Ø

65. 10

67. −6, −4

69. $-\frac{1}{2}, \frac{3}{2}$

71. Ø

73. −5, 5

75. −9, 3

77. $\frac{1}{5}$

79. $-\frac{3}{2}, \frac{3}{2}$

81. −1, 1/2

83. 5, 10

85. −7, 7

87. $\frac{9}{2}$

89. 1

91. 10

93. Ø

95. 3

97. −1, 2

99. 100

101. −8

103. 10

105. −13

107. $\frac{5}{2}$

109. −6, −4

111. −2, 2

113. 1

115. −2

117. −1

119. 14

121. −9

123. $P = (r + 1)^2$

125. $s = \frac{gt^2}{2}$

127. $R = \frac{P}{I^2}$

129. $V = \frac{4\pi r^3}{3}$

131. 5

133. 0, 8

135. $16\frac{2}{3}$ feet

137. $\frac{8}{\pi^2}$ feet; 0.8 feet

139. $\frac{2}{\pi^2}$ feet; 0.2 feet

141. 16 feet

143. 4 feet

145. Answer may vary

5.7 Complex Numbers and Their Operations

LEARNING OBJECTIVES

1. Define the imaginary unit and complex numbers.
2. Add and subtract complex numbers.
3. Multiply and divide complex numbers.

Introduction to Complex Numbers

Up to this point the square root of a negative number has been left undefined. For example, we know that $\sqrt{-9}$ is not a real number.

$$\sqrt{-9} = ? \qquad \text{or} \qquad (\ ?\ )^2 = -9$$

There is no real number that when squared results in a negative number. We begin to resolve this issue by defining the **imaginary unit**[26], i, as the square root of −1.

$$i = \sqrt{-1} \quad \text{and} \quad i^2 = -1$$

To express a square root of a negative number in terms of the imaginary unit i, we use the following property where a represents any non-negative real number:

$$\sqrt{-a} = \sqrt{-1 \cdot a} = \sqrt{-1} \cdot \sqrt{a} = i\sqrt{a}$$

With this we can write

26. Defined as $i = \sqrt{-1}$ where $i^2 = -1$.

$$\sqrt{-9} = \sqrt{-1 \cdot 9} = \sqrt{-1} \cdot \sqrt{9} = i \cdot 3 = 3i$$

If $\sqrt{-9} = 3i$, then we would expect that $3i$ squared will equal -9:

$$(3i)^2 = 9i^2 = 9\,(-1) = -9 \quad ✓$$

In this way any square root of a negative real number can be written in terms of the imaginary unit. Such a number is often called an **imaginary number**[27].

Example 1

Rewrite in terms of the imaginary unit *i*.

a. $\sqrt{-7}$
b. $\sqrt{-25}$
c. $\sqrt{-72}$

Solution:

a. $\sqrt{-7} = \sqrt{-1 \cdot 7} = \sqrt{-1} \cdot \sqrt{7} = i\sqrt{7}$
b. $\sqrt{-25} = \sqrt{-1 \cdot 25} = \sqrt{-1} \cdot \sqrt{25} = i \cdot 5 = 5i$
c. $\sqrt{-72} = \sqrt{-1 \cdot 36 \cdot 2} = \sqrt{-1} \cdot \sqrt{36} \cdot \sqrt{2} = i \cdot 6 \cdot \sqrt{2} = 6i\sqrt{2}$

Notation Note: When an imaginary number involves a radical, we place *i* in front of the radical. Consider the following:

27. A square root of any negative real number.

$$6i\sqrt{2} = 6\sqrt{2}i$$

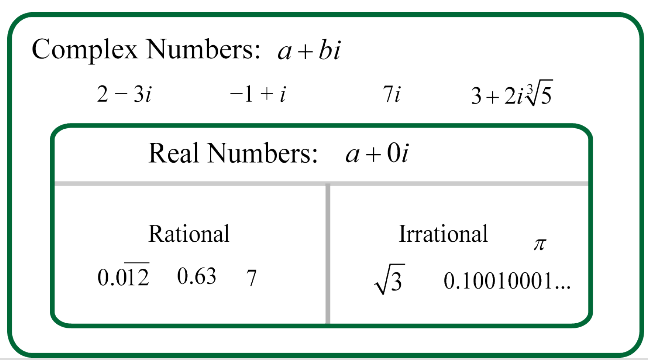
Since multiplication is commutative, these numbers are equivalent. However, in the form $6\sqrt{2}i$, the imaginary unit i is often misinterpreted to be part of the radicand. To avoid this confusion, it is a best practice to place i in front of the radical and use $6i\sqrt{2}$.

A **complex number**[28] is any number of the form,

$$a + bi$$

where a and b are real numbers. Here, a is called the **real part**[29] and b is called the **imaginary part**[30]. For example, $3 - 4i$ is a complex number with a real part of 3 and an imaginary part of −4. It is important to note that any real number is also a complex number. For example, 5 is a real number; it can be written as $5 + 0i$ with a real part of 5 and an imaginary part of 0. Hence, the set of real numbers, denoted $\mathbb{R}$, is a subset of the set of complex numbers, denoted $\mathbb{C}$.

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$



Complex numbers are used in many fields including electronics, engineering, physics, and mathematics. In this textbook we will use them to better understand solutions to equations such as $x^2 + 4 = 0$. For this reason, we next explore algebraic operations with them.

28. A number of the form $a + bi$, where a and b are real numbers.

29. The real number a of a complex number $a + bi$.

30. The real number b of a complex number $a + bi$.

Adding and Subtracting Complex Numbers

Adding or subtracting complex numbers is similar to adding and subtracting polynomials with like terms. We add or subtract the real parts and then the imaginary parts.

Example 2

Add: $(5 - 2i) + (7 + 3i)$.

Solution:

Add the real parts and then add the imaginary parts.

$$\begin{aligned}(5 - 2i) + (7 + 3i) &= 5 - 2i + 7 + 3i \\ &= 5 + 7 - 2i + 3i \\ &= 12 + i\end{aligned}$$

Answer: $12 + i$

To subtract complex numbers, we subtract the real parts and subtract the imaginary parts. This is consistent with the use of the distributive property.

Example 3

Subtract: $(10 - 7i) - (9 + 5i)$.

Solution:

Distribute the negative sign and then combine like terms.

$$\begin{aligned}(10 - 7i) - (9 + 5i) &= 10 - 7i - 9 - 5i \\ &= 10 - 9 - 7i - 5i \\ &= 1 - 12i\end{aligned}$$

Answer: $1 - 12i$

In general, given real numbers *a*, *b*, *c* and *d*:

$$\begin{aligned}(a + bi) + (c + di) &= (a + c) + (b + d)\,i \\ (a + bi) - (c + di) &= (a - c) + (b - d)\,i\end{aligned}$$

Example 4

Simplify: $\left(5 + i\right) + \ (2 - 3i) - \ (4 - 7i)$.

Solution:

$$\begin{aligned}\left(5 + i\right) + \ (2 - 3i) - \ (4 - 7i) &= 5 + i + 2 - 3i - 4 + 7i \\ &= 3 + 5i\end{aligned}$$

Answer: $3 + 5i$

In summary, adding and subtracting complex numbers results in a complex number.

Multiplying and Dividing Complex Numbers

Multiplying complex numbers is similar to multiplying polynomials. The distributive property applies. In addition, we make use of the fact that $i^2 = -1$ to simplify the result into standard form $a + bi$.

Example 5

Multiply: $-6i\,(2-3i)$.

Solution:

We begin by applying the distributive property.

$$
\begin{aligned}
-6i\,(2-3i) &= (-6i)\cdot 2 - (-6i)\cdot 3i && \textit{Distribute.}\\
&= -12i + 18i^2 && \textit{Substitute } i^2 = -1.\\
&= -12i + 18\,(-1) && \textit{Simplify.}\\
&= -12i - 18\\
&= -18 - 12i
\end{aligned}
$$

Answer: $-18 - 12i$

Example 6

Multiply: $(3-4i)\left(4+5i\right)$.

Solution:

$$\begin{aligned}(3-4i)\left(4+5i\right) &= 3\cdot 4+3\cdot 5i-4i\cdot 4-4i\cdot 5i && \textit{Distribute.}\\ &= 12+15i-16i-20i^2 && \textit{Substitute } i^2=-1.\\ &= 12+15i-16i-20\left(-1\right)\\ &= 12-i+20\\ &= 32-i\end{aligned}$$

Answer: $32-i$

In general, given real numbers a, b, c and d:

$$\begin{aligned}\left(a+bi\right)\left(c+di\right) &= ac+adi+bci+bdi^2\\ &= ac+adi+bci+bd\left(-1\right)\\ &= ac+\left(ad+bc\right)i-bd\\ &= \left(ac-bd\right)+\left(ad+bc\right)i\end{aligned}$$

Try this! Simplify: $(3-2i)^2$.

Answer: $5-12i$

(click to see video)

Given a complex number $a + bi$, its **complex conjugate**[31] is $a - bi$. We next explore the product of complex conjugates.

Example 7

Multiply: $(5 + 2i)\ (5 - 2i)$.

Solution:

$$\begin{aligned}(5 + 2i)(5 - 2i) &= 5 \cdot 5 - 5 \cdot 2i + 2i \cdot 5 - 2i \cdot 2i \\ &= 25 - 10i + 10i - 4i^2 \\ &= 25 - 4(-1) \\ &= 25 + 4 \\ &= 29\end{aligned}$$

Answer: 29

In general, the **product of complex conjugates**[32] follows:

$$\begin{aligned}(a + bi)\ (a - bi) &= a^2 - a \cdot bi + bi \cdot a - b^2 i^2 \\ &= a^2 - abi + abi - b^2\ (-1) \\ &= a^2 + b^2\end{aligned}$$

Note that the result does not involve the imaginary unit; hence, it is real. This leads us to the very useful property

31. Two complex numbers whose real parts are the same and imaginary parts are opposite. If given $a + bi$, then its complex conjugate is $a - bi$.

32. The real number that results from multiplying complex conjugates: $(a + bi)\ (a - bi) = a^2 + b^2$.

$$(a + bi)\ (a - bi) = a^2 + b^2$$

To divide complex numbers, we apply the technique used to rationalize the denominator. Multiply the numerator and denominator by the conjugate of the denominator. The result can then be simplified into standard form $a + bi$.

Example 8

Divide: $\frac{1}{2-3i}$.

Solution:

In this example, the conjugate of the denominator is $2 + 3i$. Therefore, we will multiply by 1 in the form $\frac{(2+3i)}{(2+3i)}$.

$$\begin{aligned}\frac{1}{2-3i} &= \frac{1}{(2-3i)} \cdot \frac{(2+3i)}{(2+3i)} \\ &= \frac{(2+3i)}{2^2+3^2} \\ &= \frac{2+3i}{4+9} \\ &= \frac{2+3i}{13}\end{aligned}$$

To write this complex number in standard form, we make use of the fact that 13 is a common denominator.

$$\begin{aligned}\frac{2+3i}{13} &= \frac{2}{13} + \frac{3i}{13} \\ &= \frac{2}{13} + \frac{3}{13}i\end{aligned}$$

Answer: $\frac{2}{13} + \frac{3}{13}i$

Example 9

Divide: $\frac{1-5i}{4+i}$.

Solution:

$$\begin{aligned}\frac{1-5i}{4+i} &= \frac{(1-5i)}{(4+i)} \cdot \frac{(4-i)}{(4-i)} \\ &= \frac{4-i-20i+5i^2}{4^2+1^2} \\ &= \frac{4-21i+5(-1)}{16+1} \\ &= \frac{4-21i-5}{17} \\ &= \frac{-1-21i}{17} \\ &= -\frac{1}{17}-\frac{21}{17}i\end{aligned}$$

Answer: $-\frac{1}{17}-\frac{21}{17}i$

In general, given real numbers *a*, *b*, *c* and *d* where *c* and *d* are not both 0:

$$\begin{aligned}\frac{(a+bi)}{(c+di)} &= \frac{(a+bi)}{(c+di)} \cdot \frac{(c-di)}{(c-di)} \\ &= \frac{ac - adi + bci - bdi^2}{c^2+d^2} \\ &= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2} \\ &= \left(\frac{ac+bd}{c^2+d^2}\right) + \left(\frac{bc-ad}{c^2+d^2}\right)i\end{aligned}$$

Example 10

Divide: $\frac{8-3i}{2i}$.

Solution:

Here we can think of $2i = 0 + 2i$ and thus we can see that its conjugate is $-2i = 0 - 2i$.

$$\begin{aligned}
\frac{8-3i}{2i} &= \frac{(8-3i)}{(2i)} \cdot \frac{(-2i)}{(-2i)} \\
&= \frac{-16i + 6i^2}{-4i^2} \\
&= \frac{-16i + 6(-1)}{-4(-1)} \\
&= \frac{-16i - 6}{4} \\
&= \frac{-6 - 16i}{4} \\
&= \frac{-6}{4} - \frac{16i}{4} \\
&= -\frac{3}{2} - 4i
\end{aligned}$$

Because the denominator is a monomial, we could multiply numerator and denominator by 1 in the form of $\frac{i}{i}$ and save some steps reducing in the end.

$$\begin{aligned}\frac{8-3i}{2i} &= \frac{(8-3i)}{(2i)} \cdot \frac{i}{i} \\ &= \frac{8i-3i^2}{2i^2} \\ &= \frac{8i-3(-1)}{2(-1)} \\ &= \frac{8i+3}{-2} \\ &= \frac{8i}{-2} + \frac{3}{-2} \\ &= -4i - \frac{3}{2}\end{aligned}$$

Answer: $-\frac{3}{2} - 4i$

Try this! Divide: $\frac{3+2i}{1-i}$.

Answer: $\frac{1}{2} + \frac{5}{2}i$

(click to see video)

When multiplying and dividing complex numbers we must take care to understand that the product and quotient rules for radicals require that both *a* and *b* are positive. In other words, if $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are both real numbers then we have the following rules.

$$\textbf{Product rule for radicals}: \sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$$

$$\textbf{Quotient rule for radicals}: \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

For example, we can demonstrate that the product rule is true when *a* and *b* are both positive as follows:

$$\begin{aligned} \sqrt{4} \cdot \sqrt{9} &= \sqrt{36} \\ 2 \cdot 3 &= 6 \\ 6 &= 6 \quad \checkmark \end{aligned}$$

However, when *a* and *b* are both negative the property is not true.

$$\begin{aligned} \sqrt{-4} \cdot \sqrt{-9} &\stackrel{?}{=} \sqrt{36} \\ 2i \cdot 3i &= 6 \\ 6i^2 &= 6 \\ -6 &= 6 \quad \text{✗} \end{aligned}$$

Here $\sqrt{-4}$ and $\sqrt{-9}$ both are not real numbers and the product rule for radicals fails to produce a true statement. Therefore, to avoid some common errors associated with this technicality, ensure that any complex number is written in terms of the imaginary unit *i* before performing any operations.

Example 11

Multiply: $\sqrt{-6} \cdot \sqrt{-15}$.

Solution:

Begin by writing the radicals in terms of the imaginary unit *i*.

$$\sqrt{-6} \cdot \sqrt{-15} = i\sqrt{6} \cdot i\sqrt{15}$$

Now the radicands are both positive and the product rule for radicals applies.

$$\begin{aligned}\sqrt{-6} \cdot \sqrt{-15} &= i\sqrt{6} \cdot i\sqrt{15} \\ &= i^2\sqrt{6 \cdot 15} \\ &= (-1)\sqrt{90} \\ &= (-1)\sqrt{9 \cdot 10} \\ &= (-1) \cdot 3 \cdot \sqrt{10} \\ &= -3\sqrt{10}\end{aligned}$$

Answer: $-3\sqrt{10}$

Example 12

Multiply: $\sqrt{-10}\left(\sqrt{-6}-\sqrt{10}\right)$.

Solution:

Begin by writing the radicals in terms of the imaginary unit i and then distribute.

$$\begin{aligned}\sqrt{-10}\left(\sqrt{-6}-\sqrt{10}\right) &= i\sqrt{10}\left(i\sqrt{6}-\sqrt{10}\right)\\ &= i^2\sqrt{60}-i\sqrt{100}\\ &= (-1)\sqrt{4\cdot 15}-i\sqrt{100}\\ &= (-1)\cdot 2\cdot\sqrt{15}-i\cdot 10\\ &= -2\sqrt{15}-10i\end{aligned}$$

Answer: $-2\sqrt{15}-10i$

In summary, multiplying and dividing complex numbers results in a complex number.

Try this! Simplify: $\left(2i\sqrt{2}\right)^2-\left(3-i\sqrt{5}\right)^2$.

Answer: $-12+6i\sqrt{5}$

(click to see video)

KEY TAKEAWAYS

- The imaginary unit i is defined to be the square root of negative one. In other words, $i = \sqrt{-1}$ and $i^2 = -1$.
- Complex numbers have the form $a + bi$ where a and b are real numbers.
- The set of real numbers is a subset of the complex numbers.
- The result of adding, subtracting, multiplying, and dividing complex numbers is a complex number.
- The product of complex conjugates, $a + bi$ and $a - bi$, is a real number. Use this fact to divide complex numbers. Multiply the numerator and denominator of a fraction by the complex conjugate of the denominator and then simplify.
- Ensure that any complex number is written in terms of the imaginary unit i before performing any operations.

TOPIC EXERCISES

PART A: INTRODUCTION TO COMPLEX NUMBERS

Rewrite in terms of imaginary unit *i*.

1. $\sqrt{-81}$
2. $\sqrt{-64}$
3. $-\sqrt{-4}$
4. $-\sqrt{-36}$
5. $\sqrt{-20}$
6. $\sqrt{-18}$
7. $\sqrt{-50}$
8. $\sqrt{-48}$
9. $-\sqrt{-45}$
10. $-\sqrt{-8}$
11. $\sqrt{-\frac{1}{16}}$
12. $\sqrt{-\frac{2}{9}}$
13. $\sqrt{-0.25}$
14. $\sqrt{-1.44}$

Write the complex number in standard form $a + bi$.

15. $5 - 2\sqrt{-4}$
16. $3 - 5\sqrt{-9}$
17. $-2 + 3\sqrt{-8}$

18. $4-2\sqrt{-18}$

19. $\frac{3-\sqrt{-24}}{6}$

20. $\frac{2+\sqrt{-75}}{10}$

21. $\frac{\sqrt{-63}-\sqrt{5}}{-12}$

22. $\frac{-\sqrt{-72}+\sqrt{8}}{-24}$

Given that $i^2 = -1$ compute the following powers of i.

23. i^3

24. i^4

25. i^5

26. i^6

27. i^{15}

28. i^{24}

PART B: ADDING AND SUBTRACTING COMPLEX NUMBERS

Perform the operations.

29. $(3+5i)+(7-4i)$

30. $(6-7i)+(-5-2i)$

31. $(-8-3i)+(5+2i)$

32. $(-10+15i)+(15-20i)$

33. $\left(\frac{1}{2}+\frac{3}{4}i\right)+\left(\frac{1}{6}-\frac{1}{8}i\right)$

34. $\left(\frac{2}{5}-\frac{1}{6}i\right)+\left(\frac{1}{10}-\frac{3}{2}i\right)$

35. $(5+2i)-(8-3i)$

36. $(7-i)-(-6-9i)$

37. $(-9-5i)-(8+12i)$

38. $(-11+2i)-(13-7i)$

39. $\left(\frac{1}{14}+\frac{3}{2}i\right)-\left(\frac{4}{7}-\frac{3}{4}i\right)$

40. $\left(\frac{3}{8}-\frac{1}{3}i\right)-\left(\frac{1}{2}-\frac{1}{2}i\right)$

41. $(2-i)+(3+4i)-(6-5i)$

42. $(7+2i)-(6-i)-(3-4i)$

43. $\left(\frac{1}{3}-i\right)-\left(1-\frac{1}{2}i\right)-\left(\frac{1}{6}+\frac{1}{6}i\right)$

44. $\left(1-\frac{3}{4}i\right)+\left(\frac{5}{2}+i\right)-\left(\frac{1}{4}-\frac{5}{8}i\right)$

45. $(5-3i)-(2+7i)-(1-10i)$

46. $(6-11i)+(2+3i)-(8-4i)$

47. $\sqrt{-16}-\left(3-\sqrt{-1}\right)$

48. $\sqrt{-100}+\left(\sqrt{-9}+7\right)$

49. $\left(1+\sqrt{-1}\right)-\left(1-\sqrt{-1}\right)$

50. $\left(3-\sqrt{-81}\right)-\left(5-3\sqrt{-9}\right)$

51. $\left(5-2\sqrt{-25}\right)-\left(-3+4\sqrt{-1}\right)$

52. $\left(-12-\sqrt{-1}\right)-\left(3-\sqrt{-49}\right)$

PART C: MULTIPLYING AND DIVIDING COMPLEX NUMBERS

Perform the operations.

53. $i(1 - i)$
54. $i(1 + i)$
55. $2i(7 - 4i)$
56. $6i(1 - 2i)$
57. $-2i(3 - 4i)$
58. $-5i(2 - i)$
59. $(2 + i)(2 - 3i)$
60. $(3 - 5i)(1 - 2i)$
61. $(1 - i)(8 - 9i)$
62. $(1 + 5i)(5 + 2i)$
63. $(4 + 3i)^2$
64. $(-1 + 2i)^2$
65. $(2 - 5i)^2$
66. $(5 - i)^2$
67. $(1 + i)(1 - i)$
68. $(2 - i)(2 + i)$
69. $(4 - 2i)(4 + 2i)$
70. $(6 + 5i)(6 - 5i)$
71. $\left(\frac{1}{2} + \frac{2}{3}i\right)\left(\frac{1}{3} - \frac{1}{2}i\right)$
72. $\left(\frac{2}{3} - \frac{1}{3}i\right)\left(\frac{1}{2} - \frac{3}{2}i\right)$
73. $(2 - i)^3$

74. $(1 - 3i)^3$

75. $\sqrt{-2}\left(\sqrt{-2} - \sqrt{6}\right)$

76. $\sqrt{-1}\left(\sqrt{-1} + \sqrt{8}\right)$

77. $\sqrt{-6}\left(\sqrt{10} - \sqrt{-6}\right)$

78. $\sqrt{-15}\left(\sqrt{3} - \sqrt{-10}\right)$

79. $\left(2 - 3\sqrt{-2}\right)\left(2 + 3\sqrt{-2}\right)$

80. $\left(1 + \sqrt{-5}\right)\left(1 - \sqrt{-5}\right)$

81. $\left(1 - 3\sqrt{-4}\right)\left(2 + \sqrt{-9}\right)$

82. $\left(2 - 3\sqrt{-1}\right)\left(1 + 2\sqrt{-16}\right)$

83. $\left(2 - 3i\sqrt{2}\right)\left(3 + i\sqrt{2}\right)$

84. $\left(-1 + i\sqrt{3}\right)\left(2 - 2i\sqrt{3}\right)$

85. $\dfrac{-3}{i}$

86. $\dfrac{5}{i}$

87. $\dfrac{1}{5 + 4i}$

88. $\dfrac{1}{3 - 4i}$

89. $\dfrac{15}{1 - 2i}$

90. $\dfrac{29}{5 + 2i}$

91. $\dfrac{20i}{1 - 3i}$

92. $\dfrac{10i}{1+2i}$

93. $\dfrac{10-5i}{3-i}$

94. $\dfrac{5-2i}{1-2i}$

95. $\dfrac{5+10i}{3+4i}$

96. $\dfrac{2-4i}{5+3i}$

97. $\dfrac{26+13i}{2-3i}$

98. $\dfrac{4+2i}{1+i}$

99. $\dfrac{3-i}{2i}$

100. $\dfrac{-5+2i}{4i}$

101. $\dfrac{1}{a-bi}$

102. $\dfrac{i}{a+bi}$

103. $\dfrac{1-\sqrt{-1}}{1+\sqrt{-1}}$

104. $\dfrac{1+\sqrt{-9}}{1-\sqrt{-9}}$

105. $\dfrac{-\sqrt{-6}}{\sqrt{18}+\sqrt{-4}}$

106. $\dfrac{\sqrt{-12}}{\sqrt{2}-\sqrt{-27}}$

Given that $i^{-n} = \frac{1}{i^n}$ compute the following powers of i.

107. i^{-1}

108. i^{-2}

109. i^{-3}

110. i^{-4}

Perform the operations and simplify.

111. $2i\,(2 - i) - i\,(3 - 4i)$

112. $i\left(5 - i\right) - 3i\left(1 - 6i\right)$

113. $5 - 3(1 - i)^2$

114. $2(1 - 2i)^2 + 3i$

115. $(1 - i)^2 - 2\,(1 - i) + 2$

116. $(1 + i)^2 - 2\,(1 + i) + 2$

117. $\left(2i\sqrt{2}\right)^2 + 5$

118. $\left(3i\sqrt{5}\right)^2 - \left(i\sqrt{3}\right)^2$

119. $\left(\sqrt{2} - i\right)^2 - \left(\sqrt{2} + i\right)^2$

120. $\left(i\sqrt{3} + 1\right)^2 - \left(4i\sqrt{2}\right)^2$

121. $\left(\dfrac{1}{1 + i}\right)^2$

122. $\left(\dfrac{1}{1 + i}\right)^3$

123. $\left(a - bi\right)^2 - \left(a + bi\right)^2$

124. $\left(a^2 + ai + 1\right)\left(a^2 - ai + 1\right)$

125. Show that both $-2i$ and $2i$ satisfy $x^2 + 4 = 0$.

126. Show that both $-i$ and i satisfy $x^2 + 1 = 0$.

127. Show that both $3 - 2i$ and $3 + 2i$ satisfy $x^2 - 6x + 13 = 0$.

128. Show that both $5 - i$ and $5 + i$ satisfy $x^2 - 10x + 26 = 0$.

129. Show that 3, $-2i$, and $2i$ are all solutions to $x^3 - 3x^2 + 4x - 12 = 0$.

130. Show that -2, $1 - i$, and $1 + i$ are all solutions to $x^3 - 2x + 4 = 0$.

PART D: DISCUSSION BOARD.

131. Research and discuss the history of the imaginary unit and complex numbers.

132. How would you define i^0 and why?

133. Research what it means to calculate the absolute value of a complex number $|a + bi|$ .Illustrate your finding with an example.

134. Explore the powers of *i*. Look for a pattern and share your findings.

ANSWERS

1. $9i$
3. $-2i$
5. $2i\sqrt{5}$
7. $5i\sqrt{2}$
9. $-3i\sqrt{5}$
11. $\frac{i}{4}$
13. $0.5i$
15. $5 - 4i$
17. $-2 + 6i\sqrt{2}$
19. $$\frac{1}{2} - \frac{\sqrt{6}}{3}i$$
21. $$\frac{\sqrt{5}}{12} - \frac{\sqrt{7}}{4}i$$
23. $-i$
25. i
27. $-i$
29. $10 + i$
31. $-3 - i$
33. $\frac{2}{3} + \frac{5}{8}i$
35. $-3 + 5i$
37. $-17 - 17i$
39. $$-\frac{1}{2} + \frac{9}{4}i$$
41. $-1 + 8i$
43. $$-\frac{5}{6} - \frac{2}{3}i$$

45. 2

47. $-3 + 5i$

49. $2i$

51. $8 - 14i$

53. $1 + i$

55. $8 + 14i$

57. $-8 - 6i$

59. $7 - 4i$

61. $-1 - 17i$

63. $7 + 24i$

65. $-21 - 20i$

67. 2

69. 20

71. $\frac{1}{2} - \frac{1}{36}i$

73. $2 - 11i$

75. $-2 - 2i\sqrt{3}$

77. $6 + 2i\sqrt{15}$

79. 22

81. $20 - 9i$

83. $12 - 7i\sqrt{2}$

85. $3i$

87. $\frac{5}{41} - \frac{4}{41}i$

89. $3 + 6i$

91. $-6 + 2i$

93. $\frac{7}{2} - \frac{1}{2}i$

95. $\frac{11}{5} - \frac{2}{5}i$

97. $1 + 8i$

99. $-\frac{1}{2} - \frac{3}{2}i$

101. $\frac{a}{a^2 + b^2} + \frac{b}{a^2 + b^2}i$

103. $-i$

105. $-\frac{\sqrt{6}}{11} - \frac{3\sqrt{3}}{11}i$

107. $-i$

109. i

111. $-2 + i$

113. $5 + 6i$

115. 0

117. −3

119. $-4i\sqrt{2}$

121. $-\frac{i}{2}$

123. $-4abi$

125. Proof

127. Proof

129. Proof

131. Answer may vary

133. Answer may vary

5.8 Review Exercises and Sample Exam

REVIEW EXERCISES

ROOTS AND RADICALS

Simplify.

1. $-\sqrt{121}$
2. $\sqrt{(-7)^2}$
3. $\sqrt{(xy)^2}$
4. $\sqrt{(6x-7)^2}$
5. $\sqrt[3]{125}$
6. $\sqrt[3]{-27}$
7. $\sqrt[3]{(xy)^3}$
8. $\sqrt[3]{(6x+1)^3}$
9. Given $f(x) = \sqrt{x+10}$, find $f(-1)$ and $f(6)$.
10. Given $g(x) = \sqrt[3]{x-5}$, find $g(4)$ and $g(13)$.
11. Determine the domain of the function defined by $g(x) = \sqrt{5x+2}$.
12. Determine the domain of the function defined by $g(x) = \sqrt[3]{3x-1}$.

Simplify.

13. $\sqrt[3]{250}$
14. $4\sqrt[3]{120}$
15. $-3\sqrt[3]{108}$
16. $10\sqrt[5]{\frac{1}{32}}$

17. $-6\sqrt[4]{\frac{81}{16}}$

18. $\sqrt[6]{128}$

19. $\sqrt[5]{-192}$

20. $-3\sqrt{420}$

SIMPLIFYING RADICAL EXPRESSIONS

Simplify.

21. $\sqrt{20x^4y^3}$

22. $-4\sqrt{54x^6y^3}$

23. $\sqrt{x^2 - 14x + 49}$

24. $\sqrt{(x-8)^4}$

Simplify. (Assume all variable expressions are nonzero.)

25. $\sqrt{100x^2y^4}$

26. $\sqrt{36a^6b^2}$

27. $\sqrt{\frac{8a^2}{b^4}}$

28. $\sqrt{\frac{72x^4y}{z^6}}$

29. $10x\sqrt{150x^7y^4}$

30. $-5n^2\sqrt{25m^{10}n^6}$

31. $\sqrt[3]{48x^6y^3z^2}$

32. $\sqrt[3]{270a^{10}b^8c^3}$

33. $\sqrt[3]{\frac{a^3 b^5}{64c^6}}$

34. $\sqrt[5]{\frac{a^{26}}{32b^5 c^{10}}}$

35. The period T in seconds of a pendulum is given by the formula $T = 2\pi\sqrt{\frac{L}{32}}$ where L represents the length in feet of the pendulum. Calculate the period of a pendulum that is $2\frac{1}{2}$ feet long. Give the exact answer and the approximate answer to the nearest hundredth of a second.

36. The time in seconds an object is in free fall is given by the formula $t = \frac{\sqrt{s}}{4}$ where s represents the distance in feet the object has fallen. How long does it take an object to fall 28 feet? Give the exact answer and the approximate answer to the nearest tenth of a second.

37. Find the distance between (−5, 6) and (−3,−4).

38. Find the distance between $\left(\frac{2}{3}, -\frac{1}{2}\right)$ and $\left(1, -\frac{3}{4}\right)$.

Determine whether or not the three points form a right triangle. Use the Pythagorean theorem to justify your answer.

39. (−4,5), (−3,−1), and (3,0)

40. (−1,−1), (1,3), and (−6,1)

ADDING AND SUBTRACTING RADICAL EXPRESSIONS

Simplify. Assume all radicands containing variables are nonnegative.

41. $7\sqrt{2} + 5\sqrt{2}$

42. $8\sqrt{15} - 2\sqrt{15}$

43. $14\sqrt{3} + 5\sqrt{2} - 5\sqrt{3} - 6\sqrt{2}$

44. $22\sqrt{ab} - 5a\sqrt{b} + 7\sqrt{ab} - 2a\sqrt{b}$

45. $7\sqrt{x} - \left(3\sqrt{x} + 2\sqrt{y}\right)$

46. $\left(8y\sqrt{x} - 7x\sqrt{y}\right) - \left(5x\sqrt{y} - 12y\sqrt{x}\right)$

47. $\left(3\sqrt{5} + 2\sqrt{6}\right) + \left(8\sqrt{5} - 3\sqrt{6}\right)$

48. $\left(4\sqrt[3]{3} - \sqrt[3]{12}\right) - \left(5\sqrt[3]{3} - 2\sqrt[3]{12}\right)$

49. $\left(2 - \sqrt{10x} + 3\sqrt{y}\right) - \left(1 + 2\sqrt{10x} - 6\sqrt{y}\right)$

50. $\left(3a\sqrt[3]{ab^2} + 6\sqrt[3]{a^2b}\right) + \left(9a\sqrt[3]{ab^2} - 12\sqrt[3]{a^2b}\right)$

51. $\sqrt{45} + \sqrt{12} - \sqrt{20} - \sqrt{75}$

52. $\sqrt{24} - \sqrt{32} + \sqrt{54} - 2\sqrt{32}$

53. $2\sqrt{3x^2} + \sqrt{45x} - x\sqrt{27} + \sqrt{20x}$

54. $5\sqrt{6a^2b} + \sqrt{8a^2b^2} - 2\sqrt{24a^2b} - a\sqrt{18b^2}$

55. $5y\sqrt{4x^2y} - \left(x\sqrt{16y^3} - 2\sqrt{9x^2y^3}\right)$

56. $\left(2b\sqrt{9a^2c} - 3a\sqrt{16b^2c}\right) - \left(\sqrt{64a^2b^2c} - 9b\sqrt{a^2c}\right)$

57. $\sqrt[3]{216x} - \sqrt[3]{125xy} - \sqrt[3]{8x}$

58. $\sqrt[3]{128x^3} - 2x\sqrt[3]{54} + 3\sqrt[3]{2x^3}$

59. $\sqrt[3]{8x^3y} - 2x\sqrt[3]{8y} + \sqrt[3]{27x^3y} + x\sqrt[3]{y}$

60. $\sqrt[3]{27a^3b} - 3\sqrt[3]{8ab^3} + a\sqrt[3]{64b} - b\sqrt[3]{a}$

61. Calculate the perimeter of the triangle formed by the following set of vertices: $\{(-3, -2), (-1, 1), (1, -2)\}$.

62. Calculate the perimeter of the triangle formed by the following set of vertices: $\{(0, -4), (2, 0), (-3, 0)\}$.

MULTIPLYING AND DIVIDING RADICAL EXPRESSIONS

Multiply.

63. $\sqrt{6} \cdot \sqrt{15}$

64. $\left(4\sqrt{2}\right)^2$

65. $\sqrt{2}\left(\sqrt{2} - \sqrt{10}\right)$

66. $\left(\sqrt{5} - \sqrt{6}\right)^2$

67. $\left(5 - \sqrt{3}\right)\left(5 + \sqrt{3}\right)$

68. $\left(2\sqrt{6} + \sqrt{3}\right)\left(\sqrt{2} - 5\sqrt{3}\right)$

69. $\left(\sqrt{a} - 5\sqrt{b}\right)^2$

70. $3\sqrt{xy}\left(\sqrt{x} - 2\sqrt{y}\right)$

71. $\sqrt[3]{3a^2} \cdot \sqrt[3]{18a}$

72. $\sqrt[3]{49a^2b} \cdot \sqrt[3]{7a^2b^2}$

Divide. Assume all variables represent nonzero numbers and rationalize the denominator where appropriate.

73. $\dfrac{\sqrt{72}}{\sqrt{9}}$

74. $\dfrac{10\sqrt{48}}{\sqrt{64}}$

75. $\dfrac{5}{\sqrt{5}}$

76. $\dfrac{\sqrt{15}}{\sqrt{2}}$

77. $\dfrac{3}{2\sqrt{6}}$

78. $\dfrac{2+\sqrt{5}}{\sqrt{10}}$

79. $\dfrac{18}{\sqrt{3x}}$

80. $\dfrac{2\sqrt{3x}}{\sqrt{6xy}}$

81. $\dfrac{1}{\sqrt[3]{3x^2}}$

82. $\dfrac{5ab^2}{\sqrt[3]{5a^2b}}$

83. $\sqrt[3]{\dfrac{5xz^2}{49x^2y^2z}}$

84. $\dfrac{1}{\sqrt[5]{8x^4y^2z}}$

85. $\dfrac{9x^2y}{\sqrt[5]{81xy^2z^3}}$

86. $\sqrt[5]{\dfrac{27ab^3}{15a^4bc^2}}$

87. $\dfrac{1}{\sqrt{5}-\sqrt{3}}$

88. $\dfrac{\sqrt{3}}{\sqrt{2}+1}$

89. $\dfrac{-3\sqrt{6}}{2-\sqrt{10}}$

90. $$\frac{\sqrt{xy}}{\sqrt{x}-\sqrt{y}}$$

91. $$\frac{\sqrt{2}-\sqrt{6}}{\sqrt{2}+\sqrt{6}}$$

92. $$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$$

93. The base of a triangle measures $2\sqrt{6}$ units and the height measures $3\sqrt{15}$ units. Find the area of the triangle.

94. If each side of a square measures $5+2\sqrt{10}$ units, find the area of the square.

RATIONAL EXPONENTS

Express in radical form.

95. $11^{1/2}$

96. $2^{2/3}$

97. $x^{3/5}$

98. $a^{-4/5}$

Write as a radical and then simplify.

99. $16^{1/2}$

100. $72^{1/2}$

101. $8^{2/3}$

102. $32^{1/3}$

103. $$\left(\frac{1}{9}\right)^{3/2}$$

104. $\left(\frac{1}{216}\right)^{-1/3}$

Perform the operations and simplify. Leave answers in exponential form.

105. $6^{1/2} \cdot 6^{3/2}$

106. $3^{1/3} \cdot 3^{1/2}$

107. $\frac{6^{5/2}}{6^{3/2}}$

108. $\frac{4^{3/4}}{4^{1/4}}$

109. $\left(64x^{6}y^{2}\right)^{1/2}$

110. $\left(27x^{12}y^{6}\right)^{1/3}$

111. $\left(\frac{a^{4/3}}{a^{1/2}}\right)^{2/5}$

112. $\left(\frac{16x^{4/3}}{y^{2}}\right)^{1/2}$

113. $\frac{56x^{3/4}y^{3/2}}{14x^{1/2}y^{2/3}}$

114. $\frac{\left(4a^{4}b^{2/3}c^{4/3}\right)^{1/2}}{2a^{2}b^{1/6}c^{2/3}}$

115. $\left(9x^{-4/3}y^{1/3}\right)^{-3/2}$

116. $\left(16x^{-4/5}y^{1/2}z^{-2/3}\right)^{-3/4}$

Perform the operations with mixed indices.

117. $\sqrt{y} \cdot \sqrt[5]{y^{2}}$

118. $\sqrt[3]{y} \cdot \sqrt[5]{y^{3}}$

119. $\dfrac{\sqrt[3]{y^2}}{\sqrt[5]{y}}$

120. $\sqrt{\sqrt[3]{y^2}}$

SOLVING RADICAL EQUATIONS

Solve.

121. $2\sqrt{x} + 3 = 13$

122. $\sqrt{3x - 2} = 4$

123. $\sqrt{x - 5} + 4 = 8$

124. $5\sqrt{x + 3} + 7 = 2$

125. $\sqrt{4x - 3} = \sqrt{2x + 15}$

126. $\sqrt{8x - 15} = x$

127. $x - 1 = \sqrt{13 - x}$

128. $\sqrt{4x - 3} = 2x - 3$

129. $\sqrt{x + 5} = 5 - \sqrt{x}$

130. $\sqrt{x + 3} = 3\sqrt{x} - 1$

131. $\sqrt{2(x + 1)} - \sqrt{x + 2} = 1$

132. $\sqrt{6 - x} + \sqrt{x - 2} = 2$

133. $\sqrt{3x - 2} + \sqrt{x - 1} = 1$

134. $\sqrt{9 - x} = \sqrt{x + 16} - 1$

135. $\sqrt[3]{4x - 3} = 2$

136. $\sqrt[3]{x - 8} = -1$

137. $\sqrt[3]{x(3x+10)} = 2$

138. $\sqrt[3]{2x^2 - x} + 4 = 5$

139. $\sqrt[3]{3(x+4)(x+1)} = \sqrt[3]{5x+37}$

140. $\sqrt[3]{3x^2 - 9x + 24} = \sqrt[3]{(x+2)^2}$

141. $y^{1/2} - 3 = 0$

142. $y^{1/3} + 3 = 0$

143. $(x-5)^{1/2} - 2 = 0$

144. $(2x-1)^{1/3} - 5 = 0$

145. $(x-1)^{1/2} = x^{1/2} - 1$

146. $(x-2)^{1/2} - (x-6)^{1/2} = 2$

147. $(x+4)^{1/2} - (3x)^{1/2} = -2$

148. $(5x+6)^{1/2} = 3 - (x+3)^{1/2}$

149. Solve for *g*: $t = \sqrt{\frac{2s}{g}}$.

150. Solve for *x*: $y = \sqrt[3]{x+4} - 2$.

151. The period in seconds of a pendulum is given by the formula $T = 2\pi\sqrt{\frac{L}{32}}$ where *L* represents the length in feet of the pendulum. Find the length of a pendulum that has a period of $1\frac{1}{2}$ seconds. Find the exact answer and the approximate answer rounded off to the nearest tenth of a foot.

152. The outer radius of a spherical shell is given by the formula $r = \sqrt[3]{\frac{3V}{4\pi}} + 2$ where *V* represents the inner volume in cubic centimeters. If the outer radius measures 8 centimeters, find the inner volume of the sphere.

153. The speed of a vehicle before the brakes are applied can be estimated by the length of the skid marks left on the road. On dry pavement, the speed *v* in miles per hour can be estimated by the formula $v = 2\sqrt{6d}$, where *d*

represents the length of the skid marks in feet. Estimate the length of a skid mark if the vehicle is traveling 30 miles per hour before the brakes are applied.

154. Find the real root of the function defined by $f(x) = \sqrt[3]{x-3} + 2$.

COMPLEX NUMBERS AND THEIR OPERATIONS

Write the complex number in standard form $a + bi$.

155. $5 - \sqrt{-16}$

156. $-\sqrt{-25} - 6$

157. $\dfrac{3 + \sqrt{-8}}{10}$

158. $\dfrac{\sqrt{-12} - 4}{6}$

Perform the operations.

159. $(6 - 12i) + (4 + 7i)$

160. $(-3 + 2i) - (6 - 4i)$

161. $\left(\dfrac{1}{2} - i\right) - \left(\dfrac{3}{4} - \dfrac{3}{2}i\right)$

162. $\left(\dfrac{5}{8} - \dfrac{1}{5}i\right) + \left(\dfrac{3}{2} - \dfrac{2}{3}i\right)$

163. $(5 - 2i) - (6 - 7i) + (4 - 4i)$

164. $(10 - 3i) + (20 + 5i) - (30 - 15i)$

165. $4i(2 - 3i)$

166. $(2 + 3i)(5 - 2i)$

167. $(4 + i)^2$

168. $(8 - 3i)^2$

169. $(3 + 2i)(3 - 2i)$

170. $\left(-1+5i\right)\left(-1-5i\right)$

171. $\frac{2+9i}{2i}$

172. $\frac{i}{1-2i}$

173. $\frac{4+5i}{2-i}$

174. $\frac{3-2i}{3+2i}$

175. $10-5(2-3i)^2$

176. $(2-3i)^2-(2-3i)+4$

177. $\left(\frac{1}{1-i}\right)^2$

178. $\left(\frac{1+2i}{3i}\right)^2$

179. $\sqrt{-8}\left(\sqrt{3}-\sqrt{-4}\right)$

180. $\left(1-\sqrt{-18}\right)\left(3-\sqrt{-2}\right)$

181. $\left(\sqrt{-5}-\sqrt{-10}\right)^2$

182. $\left(1-\sqrt{-2}\right)^2-\left(1+\sqrt{-2}\right)^2$

183. Show that both $-5i$ and $5i$ satisfy $x^2+25=0$.

184. Show that both $1-2i$ and $1+2i$ satisfy $x^2-2x+5=0$.

ANSWERS

1. -11
3. $|xy|$
5. 5
7. xy
9. $f(-1) = 3; f(6) = 4$
11. $\left[-\frac{2}{5}, \infty\right)$
13. $5\sqrt[3]{2}$
15. $-9\sqrt[3]{4}$
17. -9
19. $-2\sqrt[5]{6}$
21. $2x^2|y|\sqrt{5y}$
23. $|x-7|$
25. $10xy^2$
27. $\frac{2a\sqrt{2}}{b^2}$
29. $50x^4y^2\sqrt{6x}$
31. $2x^2y\sqrt[3]{6z^2}$
33. $\frac{ab\sqrt[3]{b^2}}{4c^2}$
35. $\frac{\pi\sqrt{5}}{4}$ seconds; 1.76 seconds
37. $2\sqrt{26}$ units
39. Right triangle
41. $12\sqrt{2}$

43. $9\sqrt{3} - \sqrt{2}$

45. $4\sqrt{x} - 2\sqrt{y}$

47. $11\sqrt{5} - \sqrt{6}$

49. $1 - 3\sqrt{10x} + 9\sqrt{y}$

51. $\sqrt{5} - 3\sqrt{3}$

53. $-x\sqrt{3} + 5\sqrt{5x}$

55. $12xy\sqrt{y}$

57. $4\sqrt[3]{x} - 5\sqrt[3]{xy}$

59. $2x\sqrt[3]{y}$

61. $4 + 2\sqrt{13}$ units

63. $3\sqrt{10}$

65. $2 - 2\sqrt{5}$

67. 22

69. $a - 10\sqrt{ab} + 25b$

71. $3a\sqrt[3]{2}$

73. $2\sqrt{2}$

75. $\sqrt{5}$

77. $\frac{\sqrt{6}}{4}$

79. $\frac{6\sqrt{3x}}{x}$

81. $\frac{\sqrt[3]{9x}}{3x}$

83. $\frac{\sqrt[3]{35x^2yz}}{7xy}$

85. $\frac{3xy\sqrt[5]{3x^4y^3z^2}}{z}$

87. $\frac{\sqrt{5}+\sqrt{3}}{2}$

89. $\sqrt{6}+\sqrt{15}$

91. $-2+\sqrt{3}$

93. $9\sqrt{10}$ square units

95. $\sqrt{11}$

97. $\sqrt[5]{x^3}$

99. 4

101. 4

103. 1/27

105. 36

107. 6

109. $8x^3y$

111. $a^{1/3}$

113. $4x^{1/4}y^{5/6}$

115. $\frac{x^2}{27y^{1/2}}$

117. $\sqrt[10]{y^9}$

119. $\sqrt[15]{y^7}$

121. 25

123. 21

125. 9

127. 4

129. 4

131. 7

133. 1

135. $\frac{11}{4}$

137. $-4, \frac{2}{3}$

139. $-5, \frac{5}{3}$

141. 9

143. 9

145. 1

147. 12

149. $g = \frac{2s}{t^2}$

151. $\frac{18}{\pi^2}$ feet; 1.8 feet

153. 37.5 feet

155. $5 - 4i$

157. $\frac{3}{10} + \frac{\sqrt{2}}{5} i$

159. $10 - 5i$

161. $-\frac{1}{4} + \frac{1}{2} i$

163. $3 + i$

165. $12 + 8i$

167. $15 + 8i$

169. 13

171. $\frac{9}{2} - i$

173. $\frac{3}{5} + \frac{14}{5} i$

175. $35 + 60i$

177. $\frac{1}{2} i$

179. $4\sqrt{2} + 2i\sqrt{6}$

181. $-15 + 10\sqrt{2}$

183. Answer may vary

SAMPLE EXAM

Simplify. (Assume all variables are positive.)

1. $5x\sqrt{121x^2y^4}$
2. $2xy^2\sqrt[3]{-64x^6y^9}$
3. Calculate the distance between $(-5, -3)$ and $(-2, 6)$.
4. The time in seconds an object is in free fall is given by the formula $t = \frac{\sqrt{s}}{4}$ where s represents the distance in feet that the object has fallen. If a stone is dropped into a 36-foot pit, how long will it take to hit the bottom of the pit?

Perform the operations and simplify. (Assume all variables are positive and rationalize the denominator where appropriate.)

5. $\sqrt{150xy^2} - 2\sqrt{18x^3} + y\sqrt{24x} + x\sqrt{128x}$
6. $3\sqrt[3]{16x^3y^2} - \left(2x\sqrt[3]{250y^2} - \sqrt[3]{54x^3y^2}\right)$
7. $2\sqrt{2}\left(\sqrt{2} - 3\sqrt{6}\right)$
8. $\left(\sqrt{10} - \sqrt{5}\right)^2$
9. $\dfrac{\sqrt{6}}{\sqrt{2} + \sqrt{3}}$
10. $\dfrac{2x}{\sqrt{2xy}}$
11. $\dfrac{1}{\sqrt[5]{8xy^2z^4}}$
12. Simplify: $81^{3/4}$.
13. Express in radical form: $x^{-3/5}$.

Simplify. Assume all variables are nonzero and leave answers in exponential form.

14. $\left(81x^4y^2\right)^{-1/2}$

15. $\dfrac{\left(25a^{4/3}b^8\right)^{3/2}}{a^{1/2}b}$

Solve.

16. $\sqrt{x} - 5 = 1$

17. $\sqrt[3]{5x-2} + 6 = 4$

18. $5\sqrt{2x+5} - 2x = 11$

19. $\sqrt{4-3x} + 2 = x$

20. $\sqrt{2x+5} - \sqrt{x+3} = 2$

21. The time in seconds an object is in free fall is given by the formula $t = \frac{\sqrt{s}}{4}$ where s represents the distance in feet that the object has fallen. If a stone is dropped into a pit and it takes 4 seconds to reach the bottom, how deep is the pit?

22. The width in inches of a container is given by the formula $w = \frac{\sqrt[3]{4V}}{2} + 1$ where V represents the inside volume in cubic inches of the container. What is the inside volume of the container if the width is 6 inches?

Perform the operations and write the answer in standard form.

23. $\sqrt{-3}\left(\sqrt{6} - \sqrt{-3}\right)$

24. $\frac{4+3i}{2-i}$

25. $6 - 3(2-3i)^2$

ANSWERS

1. $55x^2y^2$
3. $3\sqrt{10}$ units
5. $7y\sqrt{6x} + 2x\sqrt{2x}$
7. $4 - 12\sqrt{3}$
9. $-2\sqrt{3} + 3\sqrt{2}$
11. $\dfrac{\sqrt[5]{4x^4y^3z}}{2xyz}$
13. $\dfrac{1}{\sqrt[5]{x^3}}$
15. $125a^{3/2}b^{11}$
17. $-\frac{6}{5}$
19. Ø
21. 256 feet
23. $3 + 3i\sqrt{2}$
25. $21 + 36i$

Chapter 6

Solving Equations and Inequalities

6.1 Extracting Square Roots and Completing the Square

LEARNING OBJECTIVES

1. Solve certain quadratic equations by extracting square roots.
2. Solve any quadratic equation by completing the square.

Extracting Square Roots

Recall that a quadratic equation is in **standard form**[1] if it is equal to 0:

$$ax^2 + bx + c = 0$$

where *a*, *b*, and *c* are real numbers and $a \neq 0$. A solution to such an equation is a root of the quadratic function defined by $f(x) = ax^2 + bx + c$. Quadratic equations can have two real solutions, one real solution, or no real solution—in which case there will be two complex solutions. If the quadratic expression factors, then we can solve the equation by factoring. For example, we can solve $4x^2 - 9 = 0$ by factoring as follows:

$$\begin{aligned} 4x^2 - 9 &= 0 \\ (2x + 3)(2x - 3) &= 0 \end{aligned}$$

$$\begin{aligned} 2x + 3 &= 0 \quad &\text{or}\quad 2x - 3 &= 0 \\ 2x &= -3 & 2x &= 3 \\ x &= -\frac{3}{2} & x &= \frac{3}{2} \end{aligned}$$

1. Any quadratic equation in the form $ax^2 + bx + c = 0$, where *a*, *b*, and *c* are real numbers and $a \neq 0$.

The two solutions are $\pm\frac{3}{2}$ .Here we use ± to write the two solutions in a more compact form. The goal in this section is to develop an alternative method that can be used to easily solve equations where $b = 0$, giving the form

$$ax^2 + c = 0$$

The equation $4x^2 - 9 = 0$ is in this form and can be solved by first isolating x^2.

$$\begin{aligned} 4x^2 - 9 &= 0 \\ 4x^2 &= 9 \\ x^2 &= \frac{9}{4} \end{aligned}$$

If we take the square root of both sides of this equation, we obtain the following:

$$\begin{aligned} \sqrt{x^2} &= \sqrt{\frac{9}{4}} \\ |x| &= \frac{3}{2} \end{aligned}$$

Here we see that $x = \pm\frac{3}{2}$ are solutions to the resulting equation. In general, this describes the **square root property**[2]; for any real number k,

$$\text{if} \quad x^2 = k, \quad \text{then} \quad x = \pm\sqrt{k}$$

2. For any real number k, if $x^2 = k$, then $x = \pm\sqrt{k}$.

Applying the square root property as a means of solving a quadratic equation is called **extracting the root**[3]. This method allows us to solve equations that do not factor.

3. Applying the square root property as a means of solving a quadratic equation.

Example 1

Solve: $9x^2 - 8 = 0$.

Solution:

Notice that the quadratic expression on the left does not factor. However, it is in the form $ax^2 + c = 0$ and so we can solve it by extracting the roots. Begin by isolating x^2.

$$\begin{aligned} 9x^2 - 8 &= 0 \\ 9x^2 &= 8 \\ x^2 &= \frac{8}{9} \end{aligned}$$

Next, apply the square root property. Remember to include the ± and simplify.

$$\begin{aligned} x &= \pm\sqrt{\frac{8}{9}} \\ &= \pm\frac{2\sqrt{2}}{3} \end{aligned}$$

For completeness, check that these two real solutions solve the original quadratic equation.

$Check\, x = -\frac{2\sqrt{2}}{3}$	$Check\, x = \frac{2\sqrt{2}}{3}$
$\begin{aligned} 9x^2 - 8 &= 0 \\ 9\left(-\frac{2\sqrt{2}}{3}\right)^2 - 8 &= 0 \\ 9\left(\frac{4 \cdot 2}{9}\right) - 8 &= 0 \\ 8 - 8 &= 0 \\ 0 &= 0 \ \checkmark \end{aligned}$	$\begin{aligned} 9x^2 - 8 &= 0 \\ 9\left(\frac{2\sqrt{2}}{3}\right)^2 - 8 &= 0 \\ 9\left(\frac{4 \cdot 2}{9}\right) - 8 &= 0 \\ 8 - 8 &= 0 \\ 0 &= 0 \ \checkmark \end{aligned}$

Answer: Two real solutions, $\pm \frac{2\sqrt{2}}{3}$

Sometimes quadratic equations have no real solution. In this case, the solutions will be complex numbers.

Example 2

Solve: $x^2 + 25 = 0$.

Solution:

Begin by isolating x^2 and then apply the square root property.

$$\begin{aligned} x^2 + 25 &= 0 \\ x^2 &= -25 \\ x &= \pm\sqrt{-25} \end{aligned}$$

After applying the square root property, we are left with the square root of a negative number. Therefore, there is no real solution to this equation; the solutions are complex. We can write these solutions in terms of the imaginary unit $i = \sqrt{-1}$.

$$\begin{aligned} x &= \pm\sqrt{-25} \\ &= \pm\sqrt{-1 \cdot 25} \\ &= \pm i \cdot 5 \\ &= \pm 5i \end{aligned}$$

$Check\, x = -5i$	$Check\, x = 5i$
$\begin{aligned} x^2 + 25 &= 0 \\ (-5i)^2 + 25 &= 0 \\ 25i^2 + 25 &= 0 \\ 25(-1) + 25 &= 0 \\ -25 + 25 &= 0 \\ 0 &= 0 \checkmark \end{aligned}$	$\begin{aligned} x^2 + 25 &= 0 \\ (5i)^2 + 25 &= 0 \\ 25i^2 + 25 &= 0 \\ 25(-1) + 25 &= 0 \\ -25 + 25 &= 0 \\ 0 &= 0 \checkmark \end{aligned}$

Answer: Two complex solutions, $\pm 5i$.

Try this! Solve: $2x^2 - 3 = 0$.

Answer: The solutions are $\pm \frac{\sqrt{6}}{2}$.

(click to see video)

Consider solving the following equation:

$$(x + 5)^2 = 9$$

To solve this equation by factoring, first square $x + 5$ and then put the equation in standard form, equal to zero, by subtracting 9 from both sides.

$$
\begin{aligned}
(x+5)^2 &= 9 \\
x^2+10x+25 &= 9 \\
x^2+10x+16 &= 0
\end{aligned}
$$

Factor and then apply the zero-product property.

$$
\begin{aligned}
x^2+10x+16 &= 0 \\
(x+8)(x+2) &= 0
\end{aligned}
$$

$$
\begin{aligned}
x+8 &= 0 \quad \text{or} \quad x+2=0 \\
x &= -8 \qquad\qquad x=-2
\end{aligned}
$$

The two solutions are −8 and −2. When an equation is in this form, we can obtain the solutions in fewer steps by extracting the roots.

Example 3

Solve by extracting roots: $(x+5)^2 = 9$.

Solution:

The term with the square factor is isolated so we begin by applying the square root property.

$$\begin{aligned} (x+5)^2 &= 9 && \textit{Apply the square root property.} \\ x+5 &= \pm\sqrt{9} && \textit{Simplify.} \\ x+5 &= \pm 3 \\ x &= -5 \pm 3 \end{aligned}$$

At this point, separate the "plus or minus" into two equations and solve each individually.

$$\begin{aligned} x &= -5+3 \quad \text{or} \quad x = -5-3 \\ x &= -2 \qquad\qquad\quad\;\; x = -8 \end{aligned}$$

Answer: The solutions are -2 and -8.

In addition to fewer steps, this method allows us to solve equations that do not factor.

Example 4

Solve: $2(x-2)^2-5=0$.

Solution:

Begin by isolating the term with the square factor.

$$\begin{aligned} 2(x-2)^2-5 &= 0 \\ 2(x-2)^2 &= 5 \\ (x-2)^2 &= \frac{5}{2} \end{aligned}$$

Next, extract the roots, solve for x, and then simplify.

$$\begin{aligned} x-2 &= \pm\sqrt{\frac{5}{2}} && \textit{Rationalize the denominator.} \\ x &= 2 \pm \frac{\sqrt{5}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\ x &= 2 \pm \frac{\sqrt{10}}{2} \\ x &= \frac{4 \pm \sqrt{10}}{2} \end{aligned}$$

Answer: The solutions are $\frac{4-\sqrt{10}}{2}$ and $\frac{4+\sqrt{10}}{2}$.

Try this! Solve: $2(3x-1)^2+9=0$.

Answer: The solutions are $\frac{1}{3} \pm \frac{\sqrt{2}}{2} i$.

(click to see video)

Completing the Square

In this section, we will devise a method for rewriting any quadratic equation of the form

$$ax^2+bx+c=0$$

as an equation of the form

$$(x-p)^2=q$$

This process is called **completing the square**[4]. As we have seen, quadratic equations in this form can be easily solved by extracting roots. We begin by examining perfect square trinomials:

$$(x+3)^2 = x^2 + \underbrace{6x}_{\downarrow\ \left(\frac{6}{2}\right)^2=(3)^2=9} + 9$$

4. The process of rewriting a quadratic equation to be in the form $(x-p)^2=q$.

The last term, 9, is the square of one-half of the coefficient of x. In general, this is true for any perfect square trinomial of the form $x^2 + bx + c$.

$$\begin{aligned}\left(x + \frac{b}{2}\right)^2 &= x^2 + 2 \cdot \frac{b}{2} x + \left(\frac{b}{2}\right)^2 \\ &= x^2 + bx + \left(\frac{b}{2}\right)^2\end{aligned}$$

In other words, any trinomial of the form $x^2 + bx + c$ will be a perfect square trinomial if

$$c = \left(\frac{b}{2}\right)^2$$

Note: It is important to point out that the leading coefficient must be equal to 1 for this to be true.

Example 5

Complete the square: $x^2 - 6x + \ ? \ = (x + \ ? \ )^2$.

Solution:

In this example, the coefficient *b* of the middle term is -6. Find the value that completes the square as follows:

$$\left(\frac{b}{2}\right)^2 = \left(\frac{-6}{2}\right)^2 = (-3)^2 = 9$$

The value that completes the square is 9.

$$\begin{aligned} x^2 - 6x + 9 &= (x-3)\ (x-3) \\ &= (x-3)^2 \end{aligned}$$

Answer: $x^2 - 6x + 9 = (x - 3)^2$

Example 6

Complete the square: $x^2 + x + \ ? \ = (x + \ ?)^2$.

Solution:

Here $b = 1$. Find the value that will complete the square as follows:

$$\left(\frac{b}{2}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

The value $\frac{1}{4}$ completes the square:

$$\begin{aligned} x^2 + x + \frac{1}{4} &= \left(x + \frac{1}{2}\right)\left(x + \frac{1}{2}\right) \\ &= \left(x + \frac{1}{2}\right)^2 \end{aligned}$$

Answer: $x^2 + x + \frac{1}{4} = \left(x + \frac{1}{2}\right)^2$

We can use this technique to solve quadratic equations. The idea is to take any quadratic equation in standard form and complete the square so that we can solve it by extracting roots. The following are general steps for solving a quadratic equation with leading coefficient 1 in standard form by completing the square.

Example 7

Solve by completing the square: $x^2 - 8x - 2 = 0$.

Solution:

It is important to notice that the leading coefficient is 1.

Step 1: Add or subtract the constant term to obtain an equation of the form $x^2 + bx = c$. Here we add 2 to both sides of the equation.

$$\begin{aligned} x^2 - 8x - 2 &= 0 \\ x^2 - 8x &= 2 \end{aligned}$$

Step 2: Use $\left(\frac{b}{2}\right)^2$ to determine the value that completes the square. In this case, $b = -8$:

$$\left(\frac{b}{2}\right)^2 = \left(\frac{-8}{2}\right)^2 = (-4)^2 = 16$$

Step 3: Add $\left(\frac{b}{2}\right)^2$ to both sides of the equation and complete the square.

$$\begin{aligned} x^2 - 8x &= 2 \\ x^2 - 8x + 16 &= 2 + 16 \\ (x - 4)(x - 4) &= 18 \\ (x - 4)^2 &= 18 \end{aligned}$$

Step 4: Solve by extracting roots.

$$\begin{aligned}(x-4)^2 &= 18\\ x-4 &= \pm\sqrt{18}\\ x &= 4 \pm \sqrt{9 \cdot 2}\\ x &= 4 \pm 3\sqrt{2}\end{aligned}$$

Answer: The solutions are $4 - 3\sqrt{2}$ and $4 + 3\sqrt{2}$. The check is left to the reader.

Example 8

Solve by completing the square: $x^2 + 2x - 48 = 0$.

Solution:

Begin by adding 48 to both sides.

$$\begin{aligned} x^2 + 2x - 48 &= 0 \\ x^2 + 2x &= 48 \end{aligned}$$

Next, find the value that completes the square using $b = 2$.

$$\left(\frac{b}{2}\right)^2 = \left(\frac{2}{2}\right)^2 = (1)^2 = 1$$

To complete the square, add 1 to both sides, complete the square, and then solve by extracting the roots.

$$\begin{aligned} x^2 + 2x &= 48 && \textit{Complete the square.} \\ x^2 + 2x + 1 &= 48 + 1 \\ (x+1)(x+1) &= 49 \\ (x+1)^2 &= 49 && \textit{Extract the roots.} \\ x + 1 &= \pm\sqrt{49} \\ x + 1 &= \pm 7 \\ x &= -1 \pm 7 \end{aligned}$$

At this point, separate the "plus or minus" into two equations and solve each individually.

$$\begin{aligned} x &= -1 - 7 \quad \text{or} \quad x = -1 + 7 \\ x &= -8 \qquad\qquad\quad\; x = 6 \end{aligned}$$

Answer: The solutions are −8 and 6.

Note: In the previous example the solutions are integers. If this is the case, then the original equation will factor.

$$\begin{aligned} x^2 + 2x - 48 &= 0 \\ (x - 6)(x + 8) &= 0 \end{aligned}$$

If an equation factors, we can solve it by factoring. However, not all quadratic equations will factor. Furthermore, equations often have complex solutions.

Example 9

Solve by completing the square: $x^2 - 10x + 26 = 0$.

Solution:

Begin by subtracting 26 from both sides of the equation.

$$\begin{aligned} x^2 - 10x + 26 &= 0 \\ x^2 - 10x &= -26 \end{aligned}$$

Here $b = -10$, and we determine the value that completes the square as follows:

$$\left(\frac{b}{2}\right)^2 = \left(\frac{-10}{2}\right)^2 = (-5)^2 = 25$$

To complete the square, add 25 to both sides of the equation.

$$\begin{aligned} x^2 - 10x &= -26 \\ x^2 - 10x + 25 &= -26 + 25 \\ x^2 - 10x + 25 &= -1 \end{aligned}$$

Factor and then solve by extracting roots.

$$\begin{aligned} x^2 - 10x + 25 &= -1 \\ (x-5)\ (x-5) &= -1 \\ (x-5)^2 &= -1 \\ x - 5 &= \pm\sqrt{-1} \\ x - 5 &= \pm i \\ x &= 5 \pm i \end{aligned}$$

Answer: The solutions are $5 \pm i$.

Try this! Solve by completing the square: $x^2 - 2x - 17 = 0$.

Answer: The solutions are $x = 1 \pm 3\sqrt{2}$.

(click to see video)

The coefficient of x is not always divisible by 2.

Example 10

Solve by completing the square: $x^2 + 3x + 4 = 0$.

Solution:

Begin by subtracting 4 from both sides.

$$\begin{aligned} x^2 + 3x + 4 &= 0 \\ x^2 + 3x &= -4 \end{aligned}$$

Use $b = 3$ to find the value that completes the square:

$$\left(\frac{b}{2}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

To complete the square, add $\frac{9}{4}$ to both sides of the equation.

$$\begin{aligned} x^2 + 3x &= -4 \\ x^2 + 3x + \frac{9}{4} &= -4 + \frac{9}{4} \\ \left(x + \frac{3}{2}\right)\left(x + \frac{3}{2}\right) &= \frac{-16}{4} + \frac{9}{4} \\ \left(x + \frac{3}{2}\right)^2 &= \frac{-7}{4} \end{aligned}$$

Solve by extracting roots.

$$\begin{aligned}\left(x+\frac{3}{2}\right)^2 &= -\frac{7}{4}\\ x+\frac{3}{2} &= \pm\sqrt{\frac{-1\cdot 7}{4}}\\ x+\frac{3}{2} &= \pm\frac{i\sqrt{7}}{2}\\ x &= -\frac{3}{2}\pm\frac{\sqrt{7}}{2}i\end{aligned}$$

Answer: The solutions are $-\frac{3}{2}\pm\frac{\sqrt{7}}{2}i$.

So far, all of the examples have had a leading coefficient of 1. The formula $\left(\frac{b}{2}\right)^2$ determines the value that completes the square only if the leading coefficient is 1. If this is not the case, then simply divide both sides by the leading coefficient before beginning the steps outlined for completing the square.

Example 11

Solve by completing the square: $2x^2 + 5x - 1 = 0$.

Solution:

Notice that the leading coefficient is 2. Therefore, divide both sides by 2 before beginning the steps required to solve by completing the square.

$$\begin{aligned} \frac{2x^2 + 5x - 1}{2} &= \frac{0}{2} \\ \frac{2x^2}{2} + \frac{5x}{2} - \frac{1}{2} &= 0 \\ x^2 + \frac{5}{2}x - \frac{1}{2} &= 0 \end{aligned}$$

Add $\frac{1}{2}$ to both sides of the equation.

$$\begin{aligned} x^2 + \frac{5}{2}x - \frac{1}{2} &= 0 \\ x^2 + \frac{5}{2}x &= \frac{1}{2} \end{aligned}$$

Here $b = \frac{5}{2}$, and we can find the value that completes the square as follows:

$$\left(\frac{b}{2}\right)^2 = \left(\frac{5/2}{2}\right)^2 = \left(\frac{5}{2} \cdot \frac{1}{2}\right)^2 = \left(\frac{5}{4}\right)^2 = \frac{25}{16}$$

To complete the square, add $\frac{25}{16}$ to both sides of the equation.

$$\begin{aligned} x^2+\frac{5}{2}x &= \frac{1}{2} \\ x^2+\frac{5}{2}x + \frac{25}{16} &= \frac{1}{2} + \frac{25}{16} \\ \left(x+\frac{5}{4}\right)\left(x+\frac{5}{4}\right) &= \frac{8}{16}+\frac{25}{16} \\ \left(x+\frac{5}{4}\right)^2 &= \frac{33}{16} \end{aligned}$$

Next, solve by extracting roots.

$$\begin{aligned} \left(x+\frac{5}{4}\right)^2 &= \frac{33}{16} \\ x+\frac{5}{4} &= \pm\sqrt{\frac{33}{16}} \\ x+\frac{5}{4} &= \pm\frac{\sqrt{33}}{4} \\ x &= -\frac{5}{4}\pm\frac{\sqrt{33}}{4} \\ x &= \frac{-5\pm\sqrt{33}}{4} \end{aligned}$$

Answer: The solutions are $\frac{-5\pm\sqrt{33}}{4}$.

Try this! Solve by completing the square: $3x^2 - 2x + 1 = 0$.

Answer: The solutions are $x = \frac{1}{3} \pm \frac{\sqrt{2}}{3} i$.

(click to see video)

KEY TAKEAWAYS

- Solve equations of the form $ax^2 + c = 0$by extracting the roots.
- Extracting roots involves isolating the square and then applying the square root property. Remember to include "±" when taking the square root of both sides.
- After applying the square root property, solve each of the resulting equations. Be sure to simplify all radical expressions and rationalize the denominator if necessary.
- Solve any quadratic equation by completing the square.
- You can apply the square root property to solve an equation if you can first convert the equation to the form $\left(x - p\right)^2 = q$.
- To complete the square, first make sure the equation is in the form $x^2 + bx = c$. The leading coefficient must be 1. Then add the value $\left(\frac{b}{2}\right)^2$ to both sides and factor.
- The process for completing the square always works, but it may lead to some tedious calculations with fractions. This is the case when the middle term, *b*, is not divisible by 2.

TOPIC EXERCISES

PART A: EXTRACTING SQUARE ROOTS

Solve by factoring and then solve by extracting roots. Check answers.

1. $x^2 - 16 = 0$
2. $x^2 - 36 = 0$
3. $9y^2 - 1 = 0$
4. $4y^2 - 25 = 0$
5. $(x - 2)^2 - 1 = 0$
6. $(x + 1)^2 - 4 = 0$
7. $4\left(y - 2\right)^2 - 9 = 0$
8. $9\left(y + 1\right)^2 - 4 = 0$
9. $\left(u - 5\right)^2 - 25 = 0$
10. $(u + 2)^2 - 4 = 0$

Solve by extracting the roots.

11. $x^2 = 81$
12. $x^2 = 1$
13. $y^2 = \frac{1}{9}$
14. $y^2 = \frac{1}{16}$
15. $x^2 = 12$
16. $x^2 = 18$
17. $16x^2 = 9$
18. $4x^2 = 25$

19. $2t^2 = 1$
20. $3t^2 = 2$
21. $x^2 - 40 = 0$
22. $x^2 - 24 = 0$
23. $x^2 + 1 = 0$
24. $x^2 + 100 = 0$
25. $5x^2 - 1 = 0$
26. $6x^2 - 5 = 0$
27. $8x^2 + 1 = 0$
28. $12x^2 + 5 = 0$
29. $y^2 + 4 = 0$
30. $y^2 + 1 = 0$
31. $x^2 - \frac{4}{9} = 0$
32. $x^2 - \frac{9}{25} = 0$
33. $x^2 - 8 = 0$
34. $t^2 - 18 = 0$
35. $x^2 + 8 = 0$
36. $x^2 + 125 = 0$
37. $5y^2 - 2 = 0$
38. $3x^2 - 1 = 0$
39. $(x + 7)^2 - 4 = 0$
40. $(x + 9)^2 - 36 = 0$
41. $\left(x - 5\right)^2 - 20 = 0$

42. $(x+1)^2 - 28 = 0$
43. $(3t+2)^2 + 6 = 0$
44. $(3t-5)^2 + 10 = 0$
45. $4(3x+1)^2 - 27 = 0$
46. $9(2x-3)^2 - 8 = 0$
47. $2(3x-1)^2 + 3 = 0$
48. $5(2x-1)^2 + 2 = 0$
49. $3\left(y - \frac{2}{3}\right)^2 - \frac{3}{2} = 0$
50. $2\left(3y - \frac{1}{3}\right)^2 - \frac{5}{2} = 0$
51. $-3(t-1)^2 + 12 = 0$
52. $-2(t+1)^2 + 8 = 0$
53. Solve for *x*: $px^2 - q = 0, p, q > 0$
54. Solve for *x*: $(x-p)^2 - q = 0, p, q > 0$
55. The diagonal of a square measures 3 centimeters. Find the length of each side.
56. The length of a rectangle is twice its width. If the diagonal of the rectangle measures 10 meters, then find the dimensions of the rectangle.
57. If a circle has an area of 50π square centimeters, then find its radius.
58. If a square has an area of 27 square centimeters, then find the length of each side.
59. The height in feet of an object dropped from an 18-foot stepladder is given by $h(t) = -16t^2 + 18$, where *t* represents the time in seconds after the object is dropped. How long does it take the object to hit the ground? (Hint: The height is 0 when the object hits the ground. Round to the nearest hundredth of a second.)
60. The height in feet of an object dropped from a 50-foot platform is given by $h(t) = -16t^2 + 50$, where *t* represents the time in seconds after the object

is dropped. How long does it take the object to hit the ground? (Round to the nearest hundredth of a second.)

61. How high does a 22-foot ladder reach if its base is 6 feet from the building on which it leans? Round to the nearest tenth of a foot.

62. The height of a triangle is $\frac{1}{2}$ the length of its base. If the area of the triangle is 72 square meters, find the exact length of the triangle's base.

PART B: COMPLETING THE SQUARE

Complete the square.

63. $x^2 - 2x + \ ? \ = (x - \ ?)^2$

64. $x^2 - 4x + \ ? \ = (x - \ ?)^2$

65. $x^2 + 10x + \ ? \ = (x + \ ?)^2$

66. $x^2 + 12x + \ ? \ = (x + \ ?)^2$

67. $x^2 + 7x + \ ? \ = (x + \ ?)^2$

68. $x^2 + 5x + \ ? \ = (x + \ ?)^2$

69. $x^2 - x + \ ? \ = (x - \ ?)^2$

70. $x^2 - \frac{1}{2}x + \ ? \ = (x - \ ?)^2$

71. $x^2 + \frac{2}{3}x + \ ? \ = (x + \ ?)^2$

72. $x^2 + \frac{4}{5}x + \ ? \ = (x + \ ?)^2$

Solve by factoring and then solve by completing the square. Check answers.

73. $x^2 + 2x - 8 = 0$

74. $x^2 - 8x + 15 = 0$

75. $y^2 + 2y - 24 = 0$

76. $y^2 - 12y + 11 = 0$

77. $t^2 + 3t - 28 = 0$

78. $t^2 - 7t + 10 = 0$

79. $2x^2 + 3x - 2 = 0$

80. $3x^2 - x - 2 = 0$

81. $2y^2 - y - 1 = 0$

82. $2y^2 + 7y - 4 = 0$

Solve by completing the square.

83. $x^2 + 6x - 1 = 0$

84. $x^2 + 8x + 10 = 0$

85. $x^2 - 2x - 7 = 0$

86. $x^2 - 6x - 3 = 0$

87. $y^2 - 2y + 4 = 0$

88. $y^2 - 4y + 9 = 0$

89. $t^2 + 10t - 75 = 0$

90. $t^2 + 12t - 108 = 0$

91. $u^2 - \frac{2}{3}u - \frac{1}{3} = 0$

92. $u^2 - \frac{4}{5}u - \frac{1}{5} = 0$

93. $x^2 + x - 1 = 0$

94. $x^2 + x - 3 = 0$

95. $y^2 + 3y - 2 = 0$

96. $y^2 + 5y - 3 = 0$

97. $x^2 + 3x + 5 = 0$

98. $x^2 + x + 1 = 0$

99. $x^2 - 7x + \frac{11}{2} = 0$

100. $x^2 - 9x + \frac{3}{2} = 0$

101. $t^2 - \frac{1}{2}t - 1 = 0$

102. $t^2 - \frac{1}{3}t - 2 = 0$

103. $4x^2 - 8x - 1 = 0$

104. $2x^2 - 4x - 3 = 0$

105. $3x^2 + 6x + 1 = 0$

106. $5x^2 + 10x + 2 = 0$

107. $3x^2 + 2x - 3 = 0$

108. $5x^2 + 2x - 5 = 0$

109. $4x^2 - 12x - 15 = 0$

110. $2x^2 + 4x - 43 = 0$

111. $2x^2 - 4x + 10 = 0$

112. $6x^2 - 24x + 42 = 0$

113. $2x^2 - x - 2 = 0$

114. $2x^2 + 3x - 1 = 0$

115. $3u^2 + 2u - 2 = 0$

116. $3u^2 - u - 1 = 0$

117. $x^2 - 4x - 1 = 15$

118. $x^2 - 12x + 8 = -10$

119. $x(x + 1) - 11(x - 2) = 0$

120. $(x + 1)(x + 7) - 4(3x + 2) = 0$

121. $y^2 = (2y + 3)(y - 1) - 2(y - 1)$

122. $(2y+5)(y-5)-y(y-8)=-24$

123. $(t+2)^2=3(3t+1)$

124. $(3t+2)(t-4)-(t-8)=1-10t$

Solve by completing the square and round the solutions to the nearest hundredth.

125. $(2x-1)^2=2x$

126. $(3x-2)^2=5-15x$

127. $(2x+1)(3x+1)=9x+4$

128. $(3x+1)(4x-1)=17x-4$

129. $9x(x-1)-2(2x-1)=-4x$

130. $(6x+1)^2-6(6x+1)=0$

PART C: DISCUSSION BOARD

131. Create an equation of your own that can be solved by extracting the roots. Share it, along with the solution, on the discussion board.

132. Explain why the technique of extracting roots greatly expands our ability to solve quadratic equations.

133. Explain why the technique for completing the square described in this section requires that the leading coefficient be equal to 1.

134. Derive a formula for the diagonal of a square in terms of its sides.

ANSWERS

1. −4, 4

3. $-\frac{1}{3}, \frac{1}{3}$

5. 1, 3

7. $\frac{1}{2}, \frac{7}{2}$

9. 0, 10

11. ±9

13. $\pm \frac{1}{3}$

15. $\pm 2\sqrt{3}$

17. $\pm \frac{3}{4}$

19. $\pm \frac{\sqrt{2}}{2}$

21. $\pm 2\sqrt{10}$

23. $\pm i$

25. $\pm \frac{\sqrt{5}}{5}$

27. $\pm \frac{\sqrt{2}}{4} i$

29. $\pm 2i$

31. $\pm \frac{2}{3}$

33. $\pm 2\sqrt{2}$

35. $\pm 2i\sqrt{2}$

37. $\pm \frac{\sqrt{10}}{5}$

39. −9, −5

41. $5 \pm 2\sqrt{5}$

43. $-\frac{2}{3} \pm \frac{\sqrt{6}}{3}i$

45. $\frac{-2 \pm 3\sqrt{3}}{6}$

47. $\frac{1}{3} \pm \frac{\sqrt{6}}{6}i$

49. $\frac{4 \pm 3\sqrt{2}}{6}$

51. −1, 3

53. $x = \pm \frac{\sqrt{pq}}{p}$

55. $\frac{3\sqrt{2}}{2}$ centimeters

57. $5\sqrt{2}$ centimeters

59. 1.06 seconds

61. 21.2 feet

63. $x^2 - 2x + 1 = (x - 1)^2$

65. $x^2 + 10x + 25 = (x + 5)^2$

67. $x^2 + 7x + \frac{49}{4} = \left(x + \frac{7}{2}\right)^2$

69. $x^2 - x + \frac{1}{4} = \left(x - \frac{1}{2}\right)^2$

71. $x^2 + \frac{2}{3}x + \frac{1}{9} = \left(x + \frac{1}{3}\right)^2$

73. −4, 2

75. −6, 4

77. −7, 4

79. $-2, \frac{1}{2}$

81. $-\frac{1}{2}, 1$

83. $-3 \pm \sqrt{10}$

85. $1 \pm 2\sqrt{2}$

87. $1 \pm i\sqrt{3}$

89. −15, 5

91. $-\frac{1}{3}, 1$

93. $\frac{-1 \pm \sqrt{5}}{2}$

95. $\frac{-3 \pm \sqrt{17}}{2}$

97. $-\frac{3}{2} \pm \frac{\sqrt{11}}{2} i$

99. $\frac{7 \pm 3\sqrt{3}}{2}$

101. $\frac{1 \pm \sqrt{17}}{4}$

103. $\frac{2 \pm \sqrt{5}}{2}$

105. $\frac{-3 \pm \sqrt{6}}{3}$

107. $\frac{-1 \pm \sqrt{10}}{3}$

109. $\frac{3 \pm 2\sqrt{6}}{2}$

111. $1 \pm 2i$

113. $\frac{1 \pm \sqrt{17}}{4}$

115. $\frac{-1 \pm \sqrt{7}}{3}$

117. $2 \pm 2\sqrt{5}$

119. $5 \pm \sqrt{3}$

121. $\dfrac{1 \pm \sqrt{5}}{2}$

123. $\dfrac{5 \pm \sqrt{21}}{2}$

125. 0.19, 1.31

127. −0.45, 1.12

129. 0.33, 0.67

131. Answer may vary

133. Answer may vary

6.2 Quadratic Formula

LEARNING OBJECTIVES

1. Solve quadratic equations using the quadratic formula.
2. Use the determinant to determine the number and type of solutions to a quadratic equation.

The Quadratic Formula

In this section, we will develop a formula that gives the solutions to any quadratic equation in standard form. To do this, we begin with a general quadratic equation in standard form and solve for x by completing the square. Here a, b, and c are real numbers and $a \neq 0$:

$$\begin{aligned} ax^2 + bx + c &= 0 && \textit{Standard form of a quadratic equation.} \\ \frac{ax^2 + bx + c}{a} &= \frac{0}{a} && \textit{Divide both sides by a.} \\ x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 && \textit{Subtract } \frac{c}{a} \textit{ from both sides.} \\ x^2 + \frac{b}{a}x &= -\frac{c}{a} \end{aligned}$$

Determine the constant that completes the square: take the coefficient of x, divide it by 2, and then square it.

$$\left(\frac{b/a}{2}\right)^2 = \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

Add this to both sides of the equation to complete the square and then factor.

$$\begin{aligned}
x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} &= -\frac{c}{a} + \frac{b^2}{4a^2}\\
\left(x + \frac{b}{2a}\right)\left(x + \frac{b}{2a}\right) &= -\frac{c}{a} + \frac{b^2}{4a^2}\\
\left(x + \frac{b}{2a}\right)^2 &= -\frac{4ac}{4a^2} + \frac{b^2}{4a^2}\\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2}
\end{aligned}$$

Solve by extracting roots.

$$\begin{aligned}
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2}\\
x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}\\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a}\\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}\\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

This derivation gives us a formula that solves any quadratic equation in standard form. Given $ax^2 + bx + c = 0$, where *a*, *b*, and *c* are real numbers and $a \neq 0$, the solutions can be calculated using the **quadratic formula**[5]:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

5. The formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, which gives the solutions to any quadratic equation in the standard form $ax^2 + bx + c = 0$, where *a*, *b*, and *c* are real numbers and $a \neq 0$.

Example 1

Solve using the quadratic formula: $2x^2 - 7x - 15 = 0$.

Solution:

Begin by identifying the coefficients of each term: *a*, *b*, and *c*.

$$a = 2 \qquad b = -7 \qquad c = -15$$

Substitute these values into the quadratic formula and then simplify.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-7) \pm \sqrt{(-7)^2 - 4(2)\left(-15\right)}}{2(2)} \\ &= \frac{7 \pm \sqrt{49 + 120}}{4} \\ &= \frac{7 \pm \sqrt{169}}{4} \\ &= \frac{7 \pm 13}{4} \end{aligned}$$

Separate the "plus or minus" into two equations and simplify further.

$$\begin{aligned} x &= \frac{7-13}{4} \quad \text{or} \quad & x &= \frac{7+13}{4} \\ x &= \frac{-6}{4} & x &= \frac{20}{4} \\ x &= -\frac{3}{2} & x &= 5 \end{aligned}$$

Answer: The solutions are $-\frac{3}{2}$ and 5.

The previous example can be solved by factoring as follows:

$$\begin{aligned} 2x^2 - 7x - 15 &= 0 \\ (2x+3)\left(x-5\right) &= 0 \end{aligned}$$

$$\begin{aligned} 2x + 3 &= 0 \quad \text{or} & x - 5 &= 0 \\ 2x &= -3 & x &= 5 \\ x &= -\frac{3}{2} \end{aligned}$$

Of course, if the quadratic expression factors, then it is a best practice to solve the equation by factoring. However, not all quadratic polynomials factor so easily. The quadratic formula provides us with a means to solve all quadratic equations.

Example 2

Solve using the quadratic formula: $3x^2 + 6x - 2 = 0$.

Solution:

Begin by identifying *a*, *b*, and *c*.

$$a = 3 \qquad b = 6 \qquad c = -2$$

Substitute these values into the quadratic formula.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(6) \pm \sqrt{(6)^2 - 4(3)(-2)}}{2(3)} \\ &= \frac{-6 \pm \sqrt{36 + 24}}{6} \\ &= \frac{-6 \pm \sqrt{60}}{6} \end{aligned}$$

At this point we see that $60 = 4 \times 15$ and thus the fraction can be simplified further.

$$
\begin{aligned}
&= \frac{-6 \pm \sqrt{60}}{6} \\
&= \frac{-6 \pm \sqrt{4 \times 15}}{6} \\
&= \frac{-6 \pm 2\sqrt{15}}{6} \\
&= \frac{\cancel{2}\left(-3 \pm \sqrt{15}\right)}{\underset{3}{\cancel{6}}} \\
&= \frac{-3 \pm \sqrt{15}}{3}
\end{aligned}
$$

It is important to point out that there are two solutions here:

$$x = \frac{-3 - \sqrt{15}}{3} \quad \text{or} \quad x = \frac{-3 + \sqrt{15}}{3}$$

We may use ± to write the two solutions in a more compact form.

Answer: The solutions are $\frac{-3 \pm \sqrt{15}}{3}$.

Sometimes terms are missing. When this is the case, use 0 as the coefficient.

Example 3

Solve using the quadratic formula: $x^2 - 45 = 0$.

Solution:

This equation is equivalent to

$$1x^2 + 0x - 45 = 0$$

And we can use the following coefficients:

$$a = 1 \qquad b = 0 \qquad c = -45$$

Substitute these values into the quadratic formula.

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(0) \pm \sqrt{(0)^2 - 4(1)\left(-45\right)}}{2(1)} \\
&= \frac{0 \pm \sqrt{0 + 180}}{2} \\
&= \frac{\pm\sqrt{180}}{2} \\
&= \frac{\pm\sqrt{36 \times 5}}{2} \\
&= \frac{\pm 6\sqrt{5}}{2} \\
&= \pm 3\sqrt{5}
\end{aligned}
$$

Since the coefficient of x was 0, we could have solved this equation by extracting the roots. As an exercise, solve it using this method and verify that the results are the same.

Answer: The solutions are $\pm 3\sqrt{5}$.

Often solutions to quadratic equations are not real.

Example 4

Solve using the quadratic formula: $x^2 - 4x + 29 = 0$.

Solution:

Begin by identifying a, b, and c. Here

$$a = 1 \qquad b = -4 \qquad c = 29$$

Substitute these values into the quadratic formula and then simplify.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(29)}}{2(1)} \\ &= \frac{4 \pm \sqrt{16 - 116}}{2} \\ &= \frac{4 \pm \sqrt{-100}}{2} && \textit{Negative radicand} \\ &= \frac{4 \pm 10i}{2} && \textit{Two complex solutions} \\ &= \frac{4}{2} \pm \frac{10i}{2} \\ &= 2 \pm 5i \end{aligned}$$

Check these solutions by substituting them into the original equation.

Check $x = 2 - 5i$	***Check*** $x = 2$
$\begin{aligned} x^2 - 4x + 29 &= 0 \\ (2 - 5i)^2 - 4(2 - 5i) + 29 &= 0 \\ 4 - 20i + 25i^2 - 8 + 20i + 29 &= 0 \\ 25i^2 + 25 &= 0 \\ 25(-1) + 25 &= 0 \\ -25 + 25 &= 0 \ ✓ \end{aligned}$	$\begin{aligned} &x^2 - 4x \\ &(2 + 5i)^2 - 4(2 + 5i) \\ &4 + 20i + 25i^2 - 8 - 20i \\ &25i^2 \\ &25(-1) \\ &-25 \end{aligned}$

Answer: The solutions are $2 \pm 5i$.

The equation may not be given in standard form. The general steps for using the quadratic formula are outlined in the following example.

Example 5

Solve: $(5x+1)(x-1) = x(x+1)$.

Solution:

Step 1: Write the quadratic equation in standard form, with zero on one side of the equal sign.

$$\begin{aligned}(5x+1)(x-1) &= x(x+1)\\ 5x^2-5x+x-1 &= x^2+x\\ 5x^2-4x-1 &= x^2+x\\ 4x^2-5x-1 &= 0\end{aligned}$$

Step 2: Identify a, b, and c for use in the quadratic formula. Here

$$a=4 \qquad b=-5 \qquad c=-1$$

Step 3: Substitute the appropriate values into the quadratic formula and then simplify.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(4)(-1)}}{2(4)} \\ &= \frac{5 \pm \sqrt{25 + 16}}{8} \\ &= \frac{5 \pm \sqrt{41}}{8} \end{aligned}$$

Answer: The solution is $\frac{5 \pm \sqrt{41}}{8}$.

Try this! Solve: $(x + 3)(x - 5) = -19$

Answer: $1 \pm i\sqrt{3}$

(click to see video)

The Discriminant

If given a quadratic equation in standard form, $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$, then the solutions can be calculated using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

As we have seen, the solutions can be rational, irrational, or complex. We can determine the number and type of solutions by studying the **discriminant**[6], the

6. The expression inside the radical of the quadratic formula, $b^2 - 4ac$.

expression inside the radical, $b^2 - 4ac$. If the value of this expression is negative, then the equation has two complex solutions. If the discriminant is positive, then the equation has two real solutions. And if the discriminant is 0, then the equation has one real solution, a double root.

Example 6

Determine the type and number of solutions: $2x^2 + x + 3 = 0$.

Solution:

We begin by identifying a, b, and c. Here

$$a = 2 \qquad b = 1 \qquad c = 3$$

Substitute these values into the discriminant and simplify.

$$\begin{aligned} b^2 - 4ac &= (1)^2 - 4\,(2)\,(3) \\ &= 1 - 24 \\ &= -23 \end{aligned}$$

Since the discriminant is negative, we conclude that there are no real solutions. They are complex.

Answer: Two complex solutions.

If we use the quadratic formula in the previous example, we find that a negative radicand introduces the imaginary unit and we are left with two complex solutions.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(1) \pm \sqrt{-23}}{2(2)} \\ &= \frac{-1 \pm i\sqrt{23}}{4} \\ &= -\frac{1}{4} \pm \frac{\sqrt{23}}{4}i \qquad \textit{Two complex solutions} \end{aligned}$$

Note: Irrational and complex solutions of quadratic equations always appear in conjugate pairs.

Example 7

Determine the type and number of solutions: $6x^2 - 5x - 1 = 0$.

Solution:

In this example,

$$a = 6 \qquad b = -5 \qquad c = -1$$

Substitute these values into the discriminant and simplify.

$$\begin{aligned} b^2 - 4ac &= (-5)^2 - 4\,(6)\,(-1) \\ &= 25 + 24 \\ &= 49 \end{aligned}$$

Since the discriminant is positive, we conclude that the equation has two real solutions. Furthermore, since the discriminant is a perfect square, we obtain two rational solutions.

Answer: Two rational solutions

Because the discriminant is a perfect square, we could solve the previous quadratic equation by factoring or by using the quadratic formula.

Solve by factoring:	Solve using the quadratic formula:
$6x^2 - 5x - 1 = 0$ $(6x+1)(x-1) = 0$ $6x + 1 = 0 \quad \text{or } x - 1 = 0$ $6x = -1 \qquad x = 1$ $x = -\frac{1}{6}$	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $= \frac{-(-5) \pm \sqrt{49}}{2(6)}$ $= \frac{5 \pm 7}{12}$ $x = \frac{5-7}{12} \qquad \text{or } x = \frac{5+7}{12}$ $x = \frac{-2}{12} \qquad x = \frac{12}{12}$ $x = -\frac{1}{6} \qquad x = 1$

Given the special condition where the discriminant is 0, we obtain only one solution, a double root.

Example 8

Determine the type and number of solutions: $25x^2 - 20x + 4 = 0$.

Solution:

Here $a = 25$, $b = -20$, and $c = 4$, and we have

$$\begin{aligned} b^2 - 4ac &= (-20)^2 - 4\left(25\right)(4) \\ &= 400 - 400 \\ &= 0 \end{aligned}$$

Since the discriminant is 0, we conclude that the equation has only one real solution, a double root.

Answer: One rational solution

Since 0 is a perfect square, we can solve the equation above by factoring.

$$\begin{aligned} 25x^2 - 20x + 4 &= 0 \\ (5x - 2)(5x - 2) &= 0 \end{aligned}$$

$$\begin{aligned} 5x - 2 &= 0 \quad \text{or} & 5x - 2 &= 0 \\ 5x &= 2 & 5x &= 2 \\ x &= \frac{2}{5} & x &= \frac{2}{5} \end{aligned}$$

Here $\frac{2}{5}$ is a solution that occurs twice; it is a double root.

Example 9

Determine the type and number of solutions: $x^2 - 2x - 4 = 0$.

Solution:

Here $a = 1$, $b = -2$, and $c = -4$, and we have

$$\begin{aligned} b^2 - 4ac &= (-2)^2 - 4(1)(-4) \\ &= 4 + 16 \\ &= 20 \end{aligned}$$

Since the discriminant is positive, we can conclude that the equation has two real solutions. Furthermore, since 20 is not a perfect square, both solutions are irrational.

Answer: Two irrational solutions.

If we use the quadratic formula in the previous example, we find that a positive radicand in the quadratic formula leads to two real solutions.

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-2) \pm \sqrt{20}}{2(1)} && \textit{Positive discriminant} \\
&= \frac{2 \pm \sqrt{4 \times 5}}{2} \\
&= \frac{2 \pm 2\sqrt{5}}{2} \\
&= \frac{\cancel{2}\left(1 \pm \sqrt{5}\right)}{\underset{1}{\cancel{2}}} \\
&= 1 \pm \sqrt{5} && \textit{Two irrational solutions}
\end{aligned}
$$

The two real solutions are $1 - \sqrt{5}$ and $1 + \sqrt{5}$. Note that these solutions are irrational; we can approximate the values on a calculator.

$$1 - \sqrt{5} \approx -1.24 \quad \text{and} \quad 1 + \sqrt{5} \approx 3.24$$

In summary, if given any quadratic equation in standard form, $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$, then we have the following:

$$
\begin{aligned}
&\textbf{\textit{Positive discriminant}}: && b^2 - 4ac > 0 \quad \text{Two real solutions} \\
&\textbf{\textit{Zero discriminant}}: && b^2 - 4ac = 0 \quad \text{One real solution} \\
&\textbf{\textit{Negative discriminant}}: && b^2 - 4ac < 0 \quad \text{Two complex solutions}
\end{aligned}
$$

Furthermore, if the discriminant is nonnegative and a perfect square, then the solutions to the equation are rational; otherwise they are irrational. As we will see, knowing the number and type of solutions ahead of time helps us determine which method is best for solving a quadratic equation.

Try this! Determine the number and type of solutions: $2x^2 = x - 2$.

Answer: Two complex solutions.

(click to see video)

KEY TAKEAWAYS

- We can use the quadratic formula to solve any quadratic equation in standard form.
- To solve any quadratic equation, we first rewrite it in standard form $ax^2 + bx + c = 0$, substitute the appropriate coefficients into the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, and then simplify.
- We can determine the number and type of solutions to any quadratic equation in standard form using the discriminant, $b^2 - 4ac$. If the value of this expression is negative, then the equation has two complex solutions. If the discriminant is positive, then the equation has two real solutions. And if the discriminant is 0, then the equation has one real solution, a double root.
- We can further classify real solutions into rational or irrational numbers. If the discriminant is a perfect square, the roots are rational and the equation will factor. If the discriminant is not a perfect square, the roots are irrational.

TOPIC EXERCISES

PART A: THE QUADRATIC FORMULA

Identify the coefficients, *a*, *b* and *c*, used in the quadratic formula. Do not solve.

1. $x^2 - x + 3 = 0$
2. $5x^2 - 2x - 8 = 0$
3. $4x^2 - 9 = 0$
4. $x^2 + 3x = 0$
5. $-x^2 + 2x - 7 = 0$
6. $-2x^2 - 5x + 2 = 0$
7. $px^2 - qx - 1 = 0$
8. $p^2x^2 - x + 2q = 0$
9. $\left(x - 5\right)^2 = 49$
10. $(2x + 1)^2 = 2x - 1$

Solve by factoring and then solve using the quadratic formula. Check answers.

11. $x^2 - 6x - 16 = 0$
12. $x^2 - 3x - 18 = 0$
13. $2x^2 + 7x - 4 = 0$
14. $3x^2 + 5x - 2 = 0$
15. $4y^2 - 9 = 0$
16. $9y^2 - 25 = 0$
17. $5t^2 - 6t = 0$
18. $t^2 + 6t = 0$

19. $-x^2 + 9x - 20 = 0$

20. $-2x^2 - 3x + 5 = 0$

21. $16y^2 - 24y + 9 = 0$

22. $4y^2 - 20y + 25 = 0$

Solve by extracting the roots and then solve using the quadratic formula. Check answers.

23. $x^2 - 18 = 0$

24. $x^2 - 12 = 0$

25. $x^2 + 12 = 0$

26. $x^2 + 20 = 0$

27. $3x^2 + 2 = 0$

28. $5x^2 + 3 = 0$

29. $(x + 2)^2 + 9 = 0$

30. $(x - 4)^2 + 1 = 0$

31. $(2x + 1)^2 - 2 = 0$

32. $(3x + 1)^2 - 5 = 0$

Solve using the quadratic formula.

33. $x^2 - 5x + 1 = 0$

34. $x^2 - 7x + 2 = 0$

35. $x^2 + 8x + 5 = 0$

36. $x^2 - 4x + 2 = 0$

37. $y^2 - 2y + 10 = 0$

38. $y^2 - 4y + 13 = 0$

39. $2x^2 - 10x - 1 = 0$

40. $2x^2 - 4x - 3 = 0$
41. $3x^2 - x + 2 = 0$
42. $4x^2 - 3x + 1 = 0$
43. $5u^2 - 2u + 1 = 0$
44. $8u^2 - 20u + 13 = 0$
45. $-y^2 + 16y - 62 = 0$
46. $-y^2 + 14y - 46 = 0$
47. $-2t^2 + 4t + 3 = 0$
48. $-4t^2 + 8t + 1 = 0$
49. $\frac{1}{2}y^2 + 5y + \frac{3}{2} = 0$
50. $3y^2 + \frac{1}{2}y - \frac{1}{3} = 0$
51. $2x^2 - \frac{1}{2}x + \frac{1}{4} = 0$
52. $3x^2 - \frac{2}{3}x + \frac{1}{3} = 0$
53. $1.2x^2 - 0.5x - 3.2 = 0$
54. $0.4x^2 + 2.3x + 1.1 = 0$
55. $2.5x^2 - x + 3.6 = 0$
56. $-0.8x^2 + 2.2x - 6.1 = 0$
57. $-2y^2 = 3(y - 1)$
58. $3y^2 = 5(2y - 1)$
59. $(t + 1)^2 = 2t + 7$
60. $(2t - 1)^2 = 73 - 4t$
61. $(x + 5)(x - 1) = 2x + 1$
62. $(x + 7)(x - 2) = 3(x + 1)$

63. $2x(x-1) = -1$

64. $x(2x+5) = 3x-5$

65. $3t(t-2) + 4 = 0$

66. $5t(t-1) = t-4$

67. $(2x+3)^2 = 16x+4$

68. $(2y+5)^2 - 12(y+1) = 0$

Assume *p* and *q* are nonzero integers and use the quadratic formula to solve for *x*.

69. $px^2 + x + 1 = 0$

70. $x^2 + px + 1 = 0$

71. $x^2 + x - p = 0$

72. $x^2 + px + q = 0$

73. $p^2x^2 + 2px + 1 = 0$

74. $x^2 - 2qx + q^2 = 0$

Solve using algebra.

75. The height in feet reached by a baseball tossed upward at a speed of 48 feet per second from the ground is given by $h(t) = -16t^2 + 48t$, where t represents time in seconds after the ball is tossed. At what time does the baseball reach 24 feet? (Round to the nearest tenth of a second.)

76. The height in feet of a projectile launched upward at a speed of 32 feet per second from a height of 64 feet is given by $h(t) = -16t^2 + 32t + 64$. At what time after launch does the projectile hit the ground? (Round to the nearest tenth of a second.)

77. The profit in dollars of running an assembly line that produces custom uniforms each day is given by $P(t) = -40t^2 + 960t - 4{,}000$ where t represents the number of hours the line is in operation. Determine the number of hours the assembly line should run in order to make a profit of $1,760 per day.

78. A manufacturing company has determined that the daily revenue R in thousands of dollars is given by $R(n) = 12n - 0.6n^2$ where n represents the number of pallets of product sold. Determine the number of pallets that must be sold in order to maintain revenues at 60 thousand dollars per day.

79. The area of a rectangle is 10 square inches. If the length is 3 inches more than twice the width, then find the dimensions of the rectangle. (Round to the nearest hundredth of an inch.)

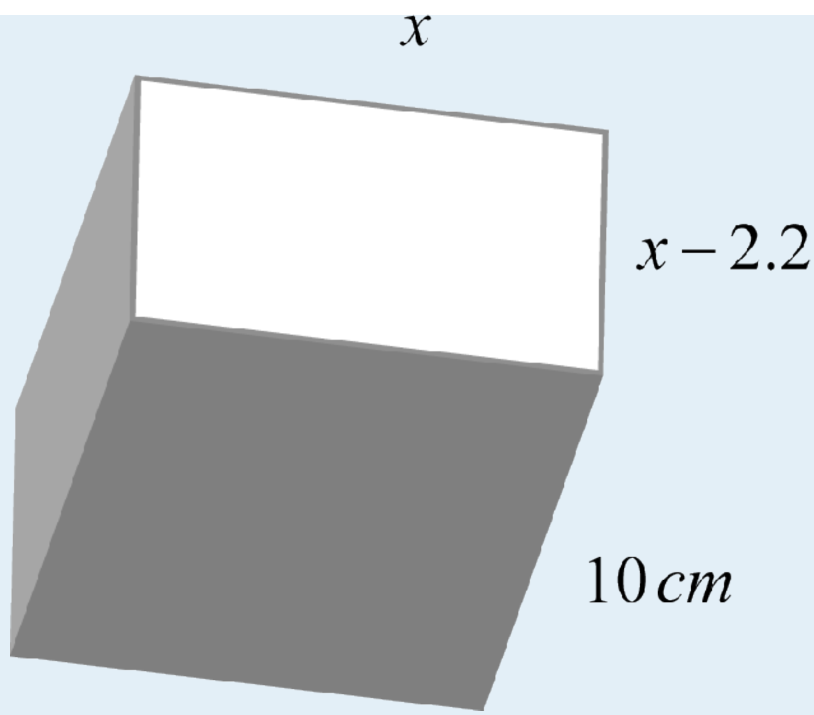
80. The area of a triangle is 2 square meters. If the base is 2 meters less than the height, then find the base and the height. (Round to the nearest hundredth of a meter.)

81. To safely use a ladder, the base should be placed about $\frac{1}{4}$ of the ladder's length away from the wall. If a 32-foot ladder is used safely, then how high against a building does the top of the ladder reach? (Round to the nearest tenth of a foot.)

82. The length of a rectangle is twice its width. If the diagonal of the rectangle measures 10 centimeters, then find the dimensions of the rectangle. (Round to the nearest tenth of a centimeter.)

83. Assuming dry road conditions and average reaction times, the safe stopping distance in feet of a certain car is given by $d(x) = \frac{1}{20}x^2 + x$ where x represents the speed of the car in miles per hour. Determine the safe speed of the car if you expect to stop in 50 feet. (Round to the nearest mile per hour.)

84. The width of a rectangular solid is 2.2 centimeters less than its length and the depth measures 10 centimeters.



Determine the length and width if the total volume of the solid is 268.8 cubic centimeters.

85. An executive traveled 25 miles in a car and then another 30 miles on a helicopter. If the helicopter was 10 miles per hour less than twice as fast as the car and the total trip took 1 hour, then what was the average speed of the car? (Round to the nearest mile per hour.)

86. Joe can paint a typical room in 1.5 hours less time than James. If Joe and James can paint 2 rooms working together in an 8-hour shift, then how long does it take James to paint a single room? (Round to the nearest tenth of an hour.)

PART B: THE DISCRIMINANT

Calculate the discriminant and use it to determine the number and type of solutions. Do not solve.

87. $x^2 - x + 1 = 0$

88. $x^2 + 2x + 3 = 0$

89. $x^2 - 2x - 3 = 0$

90. $x^2 - 5x - 5 = 0$

91. $3x^2 - 1x - 2 = 0$

92. $3x^2 - 1x + 2 = 0$

93. $9y^2 + 2 = 0$

94. $9y^2 - 2 = 0$

95. $2x^2 + 3x = 0$

96. $4x^2 - 5x = 0$

97. $\frac{1}{2}x^2 - 2x + \frac{5}{2} = 0$

98. $\frac{1}{2}x^2 - x - \frac{1}{2} = 0$

99. $-x^2 - 3x + 4 = 0$

100. $-x^2 - 5x + 3 = 0$

101. $25t^2 + 30t + 9 = 0$

102. $9t^2 - 12t + 4 = 0$

Find a nonzero integer *p* so that the following equations have one real solution. (Hint: If the discriminant is zero, then there will be one real solution.)

103. $px^2 - 4x - 1 = 0$

104. $x^2 - 8x + p = 0$

105. $x^2 + px + 25 = 0$

106. $x^2 - 2x + p^2 = 0$

PART C: DISCUSSION BOARD

107. When talking about a quadratic equation in standard form $ax^2 + bx + c = 0$, why is it necessary to state that $a \neq 0$? What would happen if *a* is equal to zero?

108. Research and discuss the history of the quadratic formula and solutions to quadratic equations.

109. Solve $mx^2 + nx + p = 0$ for *x* by completing the square.

ANSWERS

1. $a = 1; b = -1; c = 3$
3. $a = 4; b = 0; c = -9$
5. $a = -1; b = 2; c = -7$
7. $a = p; b = -q; c = -1$
9. $a = 1; b = -10; c = -24$
11. −2, 8
13. $-4, \frac{1}{2}$
15. $\pm \frac{3}{2}$
17. $0, \frac{6}{5}$
19. 4, 5
21. $\frac{3}{4}$
23. $\pm 3\sqrt{2}$
25. $\pm 2i\sqrt{3}$
27. $\pm \frac{i\sqrt{6}}{3}$
29. $-2 \pm 3i$
31. $\frac{-1 \pm \sqrt{2}}{2}$
33. $\frac{5 \pm \sqrt{21}}{2}$
35. $-4 \pm \sqrt{11}$
37. $1 \pm 3i$
39. $\frac{5 \pm 3\sqrt{3}}{2}$
41. $\frac{1}{6} \pm \frac{\sqrt{23}}{6} i$

43. $\frac{1}{5} \pm \frac{2}{5} i$

45. $8 \pm \sqrt{2}$

47. $\frac{2 \pm \sqrt{10}}{2}$

49. $-5 \pm \sqrt{22}$

51. $\frac{1}{8} \pm \frac{\sqrt{7}}{8} i$

53. $x \approx -1.4$ or $x \approx 1.9$

55. $x \approx 0.2 \pm 1.2i$

57. $\frac{-3 \pm \sqrt{33}}{4}$

59. $\pm\sqrt{6}$

61. $-1 \pm \sqrt{7}$

63. $\frac{1}{2} \pm \frac{1}{2} i$

65. $1 \pm \frac{\sqrt{3}}{3} i$

67. $\frac{1}{2} \pm i$

69. $x = \frac{-1 \pm \sqrt{1 - 4p}}{2p}$

71. $x = \frac{-1 \pm \sqrt{1 + 4p}}{2}$

73. $x = -\frac{1}{p}$

75. 0.6 seconds and 2.4 seconds

77. 12 hours

79. Length: 6.22 inches; width: 1.61 inches

81. 31.0 feet

83. 23 miles per hour

85. 42 miles per hour

87. −3; two complex solutions

89. 16; two rational solutions

91. 25; two rational solutions

93. −72; two complex solutions

95. 9; two rational solutions

97. −1; two complex solutions

99. 25; two rational solutions

101. 0; one rational solution

103. $p = -4$

105. $p = \pm 10$

107. Answer may vary

109. Answer may vary

6.3 Solving Equations Quadratic in Form

LEARNING OBJECTIVES

1. Develop a general strategy for solving quadratic equations.
2. Solve equations that are quadratic in form.

General Guidelines for Solving Quadratic Equations

Use the coefficients of a quadratic equation to help decide which method is most appropriate for solving it. While the quadratic formula always works, it is sometimes not the most efficient method. If given any quadratic equation in standard form,

$$ax^2 + bx + c = 0$$

where $c = 0$, then it is best to factor out the GCF and solve by factoring.

Example 1

Solve: $12x^2 - 3x = 0$.

Solution:

In this case, $c = 0$ and we can solve by factoring out the GCF $3x$.

$$\begin{aligned} 12x^2 - 3x &= 0 \\ 3x(4x-1) &= 0 \end{aligned}$$

Then apply the zero-product property and set each factor equal to zero.

$$\begin{array}{rcr} 3x = 0 & \text{or} & 4x - 1 = 0 \\ x = 0 & & 4x = 1 \\ & & x = \dfrac{1}{4} \end{array}$$

Answer: The solutions are 0 and $\frac{1}{4}$.

If $b = 0$, then we can solve by extracting the roots.

Example 2

Solve: $5x^2 + 8 = 0$.

Solution:

In this case, $b = 0$ and we can solve by extracting the roots. Begin by isolating the square.

$$\begin{aligned} 5x^2 + 8 &= 0 \\ 5x^2 &= -8 \\ x^2 &= -\frac{8}{5} \end{aligned}$$

Next, apply the square root property. Remember to include the ±.

$$\begin{aligned} x &= \pm\sqrt{-\frac{8}{5}} && \textit{Rationalize the denominator.} \\ &= \pm\frac{\sqrt{-4 \cdot 2}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} && \textit{Simplify.} \\ &= \pm\frac{2i\sqrt{2} \cdot \sqrt{5}}{5} \\ &= \pm\frac{2i\sqrt{10}}{5} \end{aligned}$$

Answer: The solutions are $\pm\frac{2i\sqrt{10}}{5}$.

When given a quadratic equation in standard form where a, b, and c are all nonzero, determine the value for the discriminant using the formula $b^2 - 4ac$.

a. If the discriminant is a perfect square, then solve by factoring.
b. If the discriminant is not a perfect square, then solve using the quadratic formula.

Recall that if the discriminant is not a perfect square and positive, the quadratic equation will have two irrational solutions. And if the discriminant is negative, the quadratic equation will have two complex conjugate solutions.

Example 3

Solve: $\left(3x+5\right)\ (3x+7)=6x+10$.

Solution:

Begin by rewriting the quadratic equation in standard form.

$$\begin{aligned}\left(3x+5\right)\ (3x+7)&=6x+10\\ 9x^2+21x+15x+35&=6x+10\\ 9x^2+36x+35&=6x+10\\ 9x^2+30x+25&=0\end{aligned}$$

Substitute a = 9, b = 30, and c = 25 into the discriminant formula.

$$\begin{aligned}b^2-4ac&=(30)^2-4\,(9)\ \left(25\right)\\ &=900-900\\ &=0\end{aligned}$$

Since the discriminant is 0, solve by factoring and expect one real solution, a double root.

$$\begin{aligned} 9x^2 + 30x + 25 &= 0 \\ (3x+5)\,(3x+5) &= 0 \end{aligned}$$

$$\begin{aligned} 3x + 5 &= 0 \quad &\text{or}\quad 3x + 5 &= 0 \\ 3x &= -5 & 3x &= -5 \\ x &= -\frac{5}{3} & x &= -\frac{5}{3} \end{aligned}$$

Answer: The solution is $-\frac{5}{3}$.

It is good to know that the quadratic formula will work to find the solutions to all of the examples in this section. However, it is not always the best solution. If the equation can be solved by factoring or by extracting the roots, you should use that method.

Solving Equations Quadratic in Form

In this section we outline an algebraic technique that is used extensively in mathematics to transform equations into familiar forms. We begin by defining **quadratic form**[7],

$$au^2 + bu + c = 0$$

Here u represents an algebraic expression. Some examples follow:

7. An equation of the form $au^2 + bu + c = 0$ where a, b and c are real numbers and u represents an algebraic expression.

$$\left(\frac{t+2}{t}\right)^2+8\left(\frac{t+2}{t}\right)+7=0 \xRightarrow{u=\frac{t+2}{t}} u^2+8u+7=0$$

$$x^{2/3}-3x^{1/3}-10=0 \xRightarrow{u=x^{1/3}} u^2-3u-10=0$$

$$3y^{-2}+7y^{-1}-6=0 \xRightarrow{u=y^{-1}} 3u^2+7u-6=0$$

If we can express an equation in quadratic form, then we can use any of the techniques used to solve quadratic equations. For example, consider the following fourth-degree polynomial equation,

$$x^4-4x^2-32=0$$

If we let $u=x^2$ then $u^2=\left(x^2\right)^2=x^4$ and we can write

$$x^4-4x^2-32=0 \Rightarrow \underbrace{\left(x^2\right)^2}_{u^2}-4\underbrace{\left(x^2\right)}_{u}-32=0$$

$$u^2-4u-32=0$$

This substitution transforms the equation into a familiar quadratic equation in terms of u which, in this case, can be solved by factoring.

$$\begin{aligned} u^2-4u-32&=0\\ (u-8)(u+4)&=0\\ u=8 \quad \mathit{or} \quad u&=-4 \end{aligned}$$

Since $u=x^2$ we can back substitute and then solve for x.

$$\begin{array}{lll} u=8 & \text{or} & u=-4 \\ \downarrow & & \downarrow \\ x^2=8 & & x^2=-4 \\ x=\pm\sqrt{8} & & x=\pm\sqrt{-4} \\ x=\pm2\sqrt{2} & & x=\pm2i \end{array}$$

Therefore, the equation $x^4 - 4x^2 - 32 = 0$ has four solutions $\left\{\pm2\sqrt{2}, \pm2i\right\}$, two real and two complex. This technique, often called a ***u*-substitution**[8], can also be used to solve some non-polynomial equations.

8. A technique in algebra using substitution to transform equations into familiar forms.

Example 4

Solve: $x - 2\sqrt{x} - 8 = 0$.

Solution:

This is a radical equation that can be written in quadratic form. If we let $u = \sqrt{x}$ then $u^2 = \left(\sqrt{x}\right)^2 = x$ and we can write

$$\begin{array}{c} x - 2\sqrt{x} - 8 = 0 \\ \downarrow \quad\quad \downarrow \\ u^2 - 2u - 8 = 0 \end{array}$$

Solve for u.

$$\begin{aligned} u^2 - 2u - 8 &= 0 \\ (u - 4)\ (u + 2) &= 0 \\ u = 4 \quad or \quad u &= -2 \end{aligned}$$

Back substitute $u = \sqrt{x}$ and solve for x.

$$\begin{array}{ccc} \sqrt{x} = 4 & \text{or} & \sqrt{x} = -2 \\ \left(\sqrt{x}\right)^2 = (4)^2 & & \left(\sqrt{x}\right)^2 = (-2)^2 \\ x = 16 & & x = 4 \end{array}$$

Recall that squaring both sides of an equation introduces the possibility of extraneous solutions. Therefore we must check our potential solutions.

$$\begin{array}{r|r} \textit{Check}\ x = 16 & \textit{Check}\ x = 4 \\ x - 2\sqrt{x} - 8 = 0 & x - 2\sqrt{x} - 8 = 0 \\ 16 - 2\sqrt{16} - 8 = 0 & 4 - 2\sqrt{4} - 8 = 0 \\ 16 - 2 \cdot 4 - 8 = 0 & 4 - 2 \cdot 2 - 8 = 0 \\ 16 - 8 - 8 = 0 & 4 - 4 - 8 = 0 \\ 0 = 0\ ✓ & -8 = 0\ ✗ \end{array}$$

Because $x = 4$ is extraneous, there is only one solution, $x = 16$.

Answer: The solution is 16.

Example 5

Solve: $x^{2/3} - 3x^{1/3} - 10 = 0$.

Solution:

If we let $u = x^{1/3}$, then $u^2 = \left(x^{1/3}\right)^2 = x^{2/3}$ and we can write

$$\begin{array}{ccccc} x^{2/3} & - & 3x^{1/3} & - & 10 = 0 \\ \downarrow & & \downarrow & & \\ u^2 & - & 3u & - & 10 = 0 \end{array}$$

Solve for u.

$$\begin{aligned} u^2 - 3u - 10 &= 0 \\ (u-5)\ (u+2) &= 0 \\ u = 5 \quad or \quad u &= -2 \end{aligned}$$

Back substitute $u = x^{1/3}$ and solve for x.

$$\begin{array}{rcr} x^{1/3} = 5 & \text{or} & x^{1/3} = -2 \\ \left(x^{1/3}\right)^3 = (5)^3 & & \left(x^{1/3}\right)^3 = (-2)^3 \\ x = 125 & & x = -8 \end{array}$$

Check.

$$\begin{aligned}
&\textit{Check } x = 125\\
x^{2/3} - 3x^{1/3} - 10 &= 0\\
(125)^{2/3} - 3(125)^{1/3} - 10 &= 0\\
\left(5^3\right)^{2/3} - 3\left(5^3\right)^{1/3} - 10 &= 0\\
5^2 - 3 \cdot 5 - 10 &= 0\\
25 - 15 - 10 &= 0\\
0 &= 0 \ \checkmark
\end{aligned}$$

$$\begin{aligned}
&\textit{Check } x = -8\\
x^{2/3} - 3x^{1/3} - 10 &= 0\\
(-8)^{2/3} - 3(-8)^{1/3} - 10 &= 0\\
\left[(-2)^3\right]^{2/3} - 3\left[(-2)^3\right]^{1/3} - 10 &= 0\\
(-2)^2 - 3 \cdot (-2) - 10 &= 0\\
4 + 6 - 10 &= 0\\
0 &= 0 \ \checkmark
\end{aligned}$$

Answer: The solutions are −8, 125.

Example 6

Solve: $3y^{-2} + 7y^{-1} - 6 = 0$.

Solution:

If we let $u = y^{-1}$, then $u^2 = \left(y^{-1}\right)^2 = y^{-2}$ and we can write

$$\begin{array}{c} 3y^{-2} + 7y^{-1} - 6 = 0 \\ \downarrow \qquad \downarrow \\ 3u^2 \ + \ 7u \ - \ 6 = 0 \end{array}$$

Solve for u.

$$\begin{aligned} 3u^2 + 7u - 6 &= 0 \\ (3u - 2)\ (u + 3) &= 0 \\ u = \frac{2}{3} \quad or \quad u &= -3 \end{aligned}$$

Back substitute $u = y^{-1}$ and solve for y.

$$\begin{array}{ll} y^{-1} = \frac{2}{3} & \text{or } y^{-1} = -3 \\ \frac{1}{y} = \frac{2}{3} & \frac{1}{y} = -3 \\ y = \frac{3}{2} & y = -\frac{1}{3} \end{array}$$

The original equation is actually a rational equation where $y \neq 0$. In this case, the solutions are not restrictions; they solve the original equation.

Answer: The solutions are $-\frac{1}{3}$, $\frac{3}{2}$.

Example 7

Solve: $\left(\frac{t+2}{t}\right)^2 + 8\left(\frac{t+2}{t}\right) + 7 = 0.$

Solution:

If we let $u = \frac{t+2}{t}$, then $u^2 = \left(\frac{t+2}{t}\right)^2$ and we can write

$$\begin{array}{ccccc} \left(\frac{t+2}{t}\right)^2 & +8 & \left(\frac{t+2}{t}\right) & +7=0 \\ \downarrow & & \downarrow & \\ u^2 & + & 8u & +7=0 \end{array}$$

Solve for u.

$$\begin{aligned} u^2 + 8u + 7 &= 0 \\ (u+1)\ (u+7) &= 0 \\ u = -1 \quad or \quad u &= -7 \end{aligned}$$

Back substitute $u = \frac{t+2}{t}$ and solve for t.

$$\begin{aligned} \frac{t+2}{t} &= -1 \quad \text{or} \quad & \frac{t+2}{t} &= -7 \\ t+2 &= -t & t+2 &= -7t \\ 2t &= -2 & 8t &= -2 \\ t &= -1 & t &= -\frac{1}{4} \end{aligned}$$

Answer: The solutions are -1, $-\frac{1}{4}$ .The check is left to the reader.

Try this! Solve: $12x^{-2} - 16x^{-1} + 5 = 0$

Answer: The solutions are $\frac{6}{5}$, 2.

(click to see video)

So far all of the examples were of equations that factor. As we know, not all quadratic equations factor. If this is the case, we use the quadratic formula.

Example 8

Solve: $x^4 - 10x^2 + 23 = 0$. Approximate to the nearest hundredth.

Solution:

If we let $u = x^2$, then $u^2 = \left(x^2\right)^2 = x^4$ and we can write

$$\begin{array}{l} x^4 - 10x^2 + 23 = 0 \\ \downarrow \qquad \downarrow \\ u^2 - 10u + 23 = 0 \end{array}$$

This equation does not factor; therefore, use the quadratic formula to find the solutions for u. Here $a = 1$, $b = -10$, and $c = 23$.

$$\begin{aligned} u &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(23)}}{2(1)} \\ &= \frac{10 \pm \sqrt{8}}{2} \\ &= \frac{10 \pm 2\sqrt{2}}{2} \\ &= 5 \pm \sqrt{2} \end{aligned}$$

Therefore, $u = 5 \pm \sqrt{2}$. Now back substitute $u = x^2$ and solve for x.

$$\begin{array}{lll} u=5-\sqrt{2} & \text{or} & u=5+\sqrt{2} \\ \downarrow & & \downarrow \\ x^2=5-\sqrt{2} & & x^2=5+\sqrt{2} \\ x=\pm\sqrt{5-\sqrt{2}} & & x=\pm\sqrt{5+\sqrt{2}} \end{array}$$

Round the four solutions as follows.

$$\begin{array}{ll} x=-\sqrt{5-\sqrt{2}}\approx -1.89 & x=-\sqrt{5+\sqrt{2}}\approx -2.53 \\ x=\sqrt{5-\sqrt{2}}\approx 1.89 & x=\sqrt{5+\sqrt{2}}\approx 2.53 \end{array}$$

Answer: The solutions are approximately ±1.89, ±2.53.

If multiple roots and complex roots are counted, then the **fundamental theorem of algebra**[9] implies that every polynomial with one variable will have as many roots as its degree. For example, we expect $f(x) = x^3 - 8$ to have three roots. In other words, the equation

$$x^3 - 8 = 0$$

should have three solutions. To find them one might first think of trying to extract the cube roots just as we did with square roots,

9. If multiple roots and complex roots are counted, then every polynomial with one variable will have as many roots as its degree.

$$\begin{aligned} x^3 - 8 &= 0 \\ x^3 &= 8 \\ x &= \sqrt[3]{8} \\ x &= 2 \end{aligned}$$

As you can see, this leads to one solution, the real cube root. There should be two others; let’s try to find them.

Example 9

Find the set of all roots: $f\,(x) = x^3 - 8$.

Solution:

Notice that the expression $x^3 - 8$ is a difference of cubes and recall that $a^3 - b^3 = \left(a - b\right)\left(a^2 + ab + b^2\right)$ Here $a = x$ and $b = 2$ and we can write

$$\begin{aligned} x^3 - 8 &= 0 \\ (x - 2)\left(x^2 + 2x + 4\right) &= 0 \end{aligned}$$

Next apply the zero-product property and set each factor equal to zero. After setting the factors equal to zero we can then solve the resulting equation using the appropriate methods.

$$x - 2 = 0 \text{ or } x^2 + 2x + 4 = 0$$

$$\begin{aligned} x &= 2 & x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ & & &= \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(4)}}{2(1)} \\ & & &= \frac{-2 \pm \sqrt{-12}}{2} \\ & & &= \frac{-2 \pm 2i\sqrt{3}}{2} \\ & & &= -1 \pm i\sqrt{3} \end{aligned}$$

Using this method we were able to obtain the set of all three roots $\left\{2, -1 \pm i\sqrt{3}\right\}$, one real and two complex.

Answer: $\left\{2, -1 \pm i\sqrt{3}\right\}$

Sometimes the roots of a function will occur multiple times. For example, $g\left(x\right) = (x-2)^3$ has degree three where the roots can be found as follows:

$$\begin{aligned}(x-2)^3 &= 0\\ (x-2)(x-2)(x-2) &= 0\end{aligned}$$

$$\begin{array}{ccccc} x-2=0 & \text{or} & x-2=0 & \text{or} & x-2=0 \\ x=2 & & x=2 & & x=2 \end{array}$$

Even though g is of degree 3 there is only one real root {2}; it occurs 3 times.

KEY TAKEAWAYS

- The quadratic formula can solve any quadratic equation. However, it is sometimes not the most efficient method.
- If a quadratic equation can be solved by factoring or by extracting square roots you should use that method.
- We can sometimes transform equations into equations that are quadratic in form by making an appropriate u-substitution. After solving the equivalent equation, back substitute and solve for the original variable.
- Counting multiple and complex roots, the fundamental theorem of algebra guarantees as many roots as the degree of a polynomial equation with one variable.

TOPIC EXERCISES

PART A: SOLVING QUADRATIC EQUATIONS

Solve.

1. $x^2 - 9x = 0$
2. $x^2 + 10x = 0$
3. $15x^2 + 6x = 0$
4. $36x^2 - 18x = 0$
5. $x^2 - 90 = 0$
6. $x^2 + 48 = 0$
7. $2x^2 + 1 = 0$
8. $7x^2 - 1 = 0$
9. $6x^2 - 11x + 4 = 0$
10. $9x^2 + 12x - 5 = 0$
11. $x^2 + x + 6 = 0$
12. $x^2 + 2x + 8 = 0$
13. $4t^2 + 28t + 49 = 0$
14. $25t^2 - 20t + 4 = 0$
15. $u^2 - 4u - 1 = 0$
16. $u^2 - 2u - 11 = 0$
17. $2(x + 2)^2 = 11 + 4x - 2x^2$
18. $(2x + 1)(x - 3) + 2x^2 = 3(x - 1)$
19. $(3x + 2)^2 = 6(2x + 1)$
20. $(2x - 3)^2 + 5x^2 = 4(2 - 3x)$

21. $4(3x-1)^2-5=0$

22. $9(2x+3)^2-2=0$

PART B: SOLVING EQUATIONS QUADRATIC IN FORM

Find all solutions.

23. $x^4+x^2-72=0$

24. $x^4-17x^2-18=0$

25. $x^4-13x^2+36=0$

26. $4x^4-17x^2+4=0$

27. $x+2\sqrt{x}-3=0$

28. $x-\sqrt{x}-2=0$

29. $x-5\sqrt{x}+6=0$

30. $x-6\sqrt{x}+5=0$

31. $x^{2/3}+5x^{1/3}+6=0$

32. $x^{2/3}-2x^{1/3}-35=0$

33. $4x^{2/3}-4x^{1/3}+1=0$

34. $3x^{2/3}-2x^{1/3}-1=0$

35. $5x^{-2}+9x^{-1}-2=0$

36. $3x^{-2}+8x^{-1}-3=0$

37. $8x^{-2}+14x^{-1}-15=0$

38. $9x^{-2}-24x^{-1}+16=0$

39. $\left(\frac{x-3}{x}\right)^2-2\left(\frac{x-3}{x}\right)-24=0$

40. $\left(\frac{2x+1}{x}\right)^2+9\left(\frac{2x+1}{x}\right)-36=0$

41. $2\left(\frac{x}{x+1}\right)^2 - 5\left(\frac{x}{x+1}\right) - 3 = 0$
42. $3\left(\frac{x}{3x-1}\right)^2 + 13\left(\frac{x}{3x-1}\right) - 10 = 0$
43. $4y^{-2} - 9 = 0$
44. $16y^{-2} + 4y^{-1} = 0$
45. $30y^{2/3} - 15y^{1/3} = 0$
46. $y^{2/3} - 9 = 0$
47. $81y^4 - 1 = 0$
48. $5\left(\frac{1}{x+2}\right)^2 - 3\left(\frac{1}{x+2}\right) - 2 = 0$
49. $12\left(\frac{x}{2x-3}\right)^2 - 11\left(\frac{x}{2x-3}\right) + 2 = 0$
50. $10x^{-2} - 19x^{-1} - 2 = 0$
51. $x^{1/2} - 3x^{1/4} + 2 = 0$
52. $x + 5\sqrt{x} - 50 = 0$
53. $8x^{2/3} + 7x^{1/3} - 1 = 0$
54. $x^{4/3} - 13x^{2/3} + 36 = 0$
55. $y^4 - 14y^2 + 46 = 0$
56. $x^{4/3} - 2x^{2/3} + 1 = 0$
57. $2y^{-2} - y^{-1} - 1 = 0$
58. $2x^{-2/3} - 3x^{-1/3} - 2 = 0$
59. $4x^{-1} - 17x^{-1/2} + 4 = 0$
60. $3x^{-1} - 8x^{-1/2} + 4 = 0$
61. $2x^{1/3} - 3x^{1/6} + 1 = 0$
62. $x^{1/3} - x^{1/6} - 2 = 0$

Find all solutions. Round your answers to the nearest hundredth.

63. $x^4 - 6x^2 + 7 = 0$

64. $x^4 - 6x^2 + 6 = 0$

65. $x^4 - 8x^2 + 14 = 0$

66. $x^4 - 12x^2 + 31 = 0$

67. $4x^4 - 16x^2 + 13 = 0$

68. $9x^4 - 30x^2 + 1 = 0$

Find the set of all roots.

69. $f(x) = x^3 - 1$

70. $g(x) = x^3 + 1$

71. $f(x) = x^3 - 27$

72. $g(x) = x^4 - 16$

73. $h(x) = x^4 - 1$

74. $h(x) = x^6 - 1$

75. $f(x) = (2x - 1)^3$

76. $g(x) = x^2(x - 4)^2$

77. $f(x) = x^3 - q^3, q > 0$

78. $f(x) = x^3 + q^3, q > 0$

Find all solutions.

79. $x^6 + 7x^3 - 8 = 0$

80. $x^6 - 7x^3 - 8 = 0$

81. $x^6 + 28x^3 + 27 = 0$

82. $x^6 + 16x^3 + 64 = 0$

83. $\left|x^2 + 2x - 5\right| = 1$

84. $\left|x^2 - 2x - 3\right| = 3$
85. $\left|2x^2 - 5\right| = 4$
86. $\left|3x^2 - 9x\right| = 6$

Find a quadratic function with integer coefficients and the given set of roots. (Hint: If r_1 and r_2 are roots, then $(x - r_1)(x - r_2) = 0$.)

87. $\{\pm 3i\}$
88. $\left\{\pm i\sqrt{5}\right\}$
89. $\left\{\pm\sqrt{3}\right\}$
90. $\left\{\pm 2\sqrt{6}\right\}$
91. $\left\{1 \pm \sqrt{3}\right\}$
92. $\left\{2 \pm 3\sqrt{2}\right\}$
93. $\{1 \pm 6i\}$
94. $\{2 \pm 3i\}$

PART C: DISCUSSION BOARD

95. On a note card, write out your strategy for solving a quadratic equation. Share your strategy on the discussion board.
96. Make up your own equation that is quadratic in form. Share it and the solution on the discussion board.

ANSWERS

1. 0, 9

3. $-\frac{2}{5}, 0$

5. $\pm 3\sqrt{10}$

7. $\pm \frac{\sqrt{2}}{2} i$

9. $\frac{1}{2}, \frac{4}{3}$

11. $-\frac{1}{2} \pm \frac{\sqrt{23}}{2} i$

13. $-\frac{7}{2}$

15. $2 \pm \sqrt{5}$

17. $-\frac{3}{2}, \frac{1}{2}$

19. $\pm \frac{\sqrt{2}}{3}$

21. $\frac{2 \pm \sqrt{5}}{6}$

23. $\pm 2\sqrt{2}, \pm 3i$

25. ±2, ±3

27. 1

29. 4, 9

31. −27, −8

33. $\frac{1}{8}$

35. $-\frac{1}{2}, 5$

37. $-\frac{2}{5}, \frac{4}{3}$

39. $\pm \frac{3}{5}$

41. $-\frac{3}{2}, -\frac{1}{3}$

43. $\pm\frac{2}{3}$

45. $0, \frac{1}{8}$

47. $\pm\frac{1}{3}, \pm\frac{i}{3}$

49. $-\frac{3}{2}, 6$

51. 1, 16

53. $-1, \frac{1}{512}$

55. $\pm\sqrt{7-\sqrt{3}}, \pm\sqrt{7+\sqrt{3}}$

57. −2, 1

59. $\frac{1}{16}, 16$

61. $\frac{1}{64}, 1$

63. ±1.26, ±2.10

65. ±1.61, ±2.33

67. ±1.06, ±1.69

69. $\left\{1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right\}$

71. $\left\{3, -\frac{3}{2} \pm \frac{3\sqrt{3}}{2}i\right\}$

73. $\{\pm 1, \pm i\}$

75. $\left\{\frac{1}{2}\right\}$

77. $\left\{q, -\frac{q}{2} \pm \frac{q\sqrt{3}}{2}i\right\}$

79. $-2, 1, 1 \pm i\sqrt{3}, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

81. $-3, -1, \frac{3}{2} \pm \frac{3\sqrt{3}}{2} i, \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

83. $-1 \pm \sqrt{7}, -1 \pm \sqrt{5}$

85. $\pm \frac{\sqrt{2}}{2}, \pm \frac{3\sqrt{2}}{2}$

87. $f(x) = x^2 + 9$

89. $f(x) = x^2 - 3$

91. $f(x) = x^2 - 2x - 2$

93. $f(x) = x^2 - 2x + 37$

95. Answer may vary

6.4 Quadratic Functions and Their Graphs

LEARNING OBJECTIVES

1. Graph a parabola.
2. Find the intercepts and vertex of a parabola.
3. Find the maximum and minimum y-value.
4. Find the vertex of a parabola by completing the square.

The Graph of a Quadratic Function

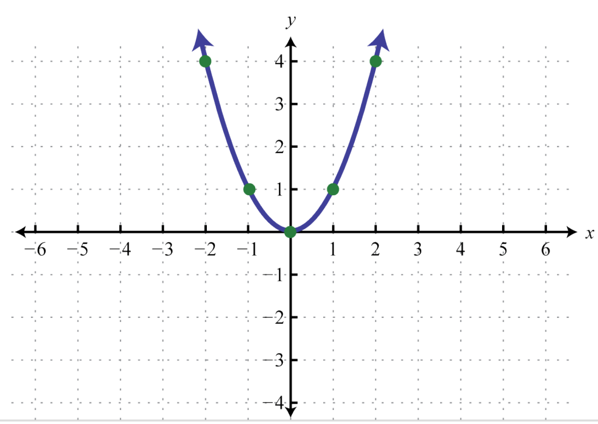
A **quadratic function** is a polynomial function of degree 2 which can be written in the general form,

$$f(x) = ax^2 + bx + c$$

Here a, b and c represent real numbers where $a \neq 0$. The squaring function $f(x) = x^2$ is a quadratic function whose graph follows.

$f(x) = x^2$

x	$f(x)$
−2	4
−1	1
0	0
1	1
2	4

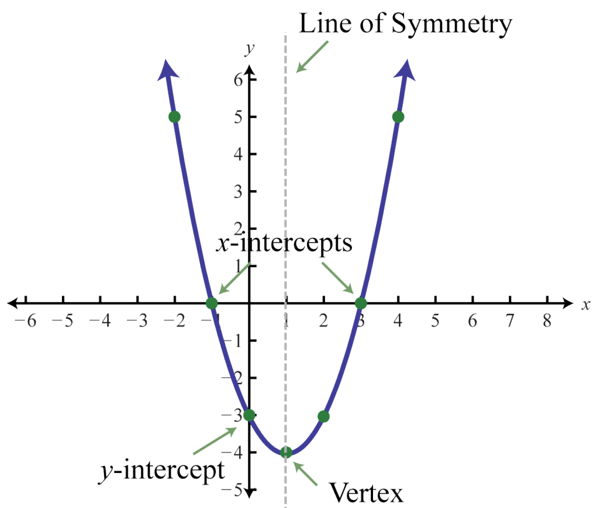


This general curved shape is called a **parabola**[10] and is shared by the graphs of all quadratic functions. Note that the graph is indeed a function as it passes the vertical line test. Furthermore, the domain of this function consists of the set of all

10. The U-shaped graph of any quadratic function defined by $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$.

real numbers $(-\infty, \infty)$ and the range consists of the set of nonnegative numbers $[0, \infty)$.

When graphing parabolas, we want to include certain special points in the graph. The *y*-intercept is the point where the graph intersects the *y*-axis. The *x*-intercepts are the points where the graph intersects the *x*-axis. The **vertex**[11] is the point that defines the minimum or maximum of the graph. Lastly, the **line of symmetry**[12] (also called the **axis of symmetry**[13]) is the vertical line through the vertex, about which the parabola is symmetric.



For any parabola, we will find the vertex and *y*-intercept. In addition, if the *x*-intercepts exist, then we will want to determine those as well. Guessing at the *x*-values of these special points is not practical; therefore, we will develop techniques that will facilitate finding them. Many of these techniques will be used extensively as we progress in our study of algebra.

Given a quadratic function $f(x) = ax^2 + bx + c$, find the *y*-intercept by evaluating the function where $x = 0$. In general, $f(0) = a(0)^2 + b(0) + c = c$, and we have

y-intercept

$$(0, c)$$

11. The point that defines the minimum or maximum of a parabola.

12. The vertical line through the vertex, $x = -\frac{b}{2a}$, about which the parabola is symmetric.

13. A term used when referencing the line of symmetry.

Next, recall that the x-intercepts, if they exist, can be found by setting $f\left(x\right) = 0$. Doing this, we have $a^2 + bx + c = 0$, which has general solutions given by the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Therefore, the x-intercepts have this general form:

$$\begin{gathered} x - intercepts \\ \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}, 0\right) \text{ and } \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}, 0\right) \end{gathered}$$

Using the fact that a parabola is symmetric, we can determine the vertical line of symmetry using the x-intercepts. To do this, we find the x-value midway between the x-intercepts by taking an average as follows:

$$\begin{aligned} x &= \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a} + \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) \div 2 \\ &= \left(\frac{-b - \cancel{\sqrt{b^2 - 4ac}} - b + \cancel{\sqrt{b^2 - 4ac}}}{2a}\right) \div \left(\frac{2}{1}\right) \\ &= \frac{-2b}{2a} \cdot \frac{1}{2} \\ &= -\frac{b}{2a} \end{aligned}$$

Therefore, the line of symmetry is the vertical line $x = -\frac{b}{2a}$. We can use the line of symmetry to find the the vertex.

$$\begin{array}{cc} \text{Line of symmetry} & \text{Vertex} \\ x = -\dfrac{b}{2a} & \left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right) \end{array}$$

Generally three points determine a parabola. However, in this section we will find five points so that we can get a better approximation of the general shape. The steps for graphing a parabola are outlined in the following example.

Example 1

Graph: $f(x) = -x^2 - 2x + 3$.

Solution:

Step 1: Determine the y-intercept. To do this, set $x = 0$ and find $f(0)$.

$$\begin{aligned} f(x) &= -x^2 - 2x + 3 \\ f(0) &= -(0)^2 - 2(0) + 3 \\ &= 3 \end{aligned}$$

The y-intercept is $(0, 3)$.

Step 2: Determine the x-intercepts if any. To do this, set $f(x) = 0$ and solve for x.

$$\begin{aligned} f(x) &= -x^2 - 2x + 3 && \textit{Set } f(x) = 0. \\ 0 &= -x^2 - 2x + 3 && \textit{Multiply both sides by } -1. \\ 0 &= x^2 + 2x - 3 && \textit{Factor.} \\ 0 &= (x+3)(x-1) && \textit{Set each factor equal to zero.} \end{aligned}$$

$$\begin{aligned} x + 3 &= 0 \quad & \text{or} \quad x - 1 &= 0 \\ x &= -3 & x &= 1 \end{aligned}$$

Here where $f(x) = 0$, we obtain two solutions. Hence, there are two x-intercepts, $(-3, 0)$ and $(1, 0)$.

Step 3: Determine the vertex. One way to do this is to first use $x = -\frac{b}{2a}$ to find the x-value of the vertex and then substitute this value in the function to find the corresponding y-value. In this example, $a = -1$ and $b = -2$.

$$\begin{aligned} x &= \frac{-b}{2a} \\ &= \frac{-(-2)}{2(-1)} \\ &= \frac{2}{-2} \\ &= -1 \end{aligned}$$

Substitute -1 into the original function to find the corresponding y-value.

$$\begin{aligned} f(x) &= -x^2 - 2x + 3 \\ f(-1) &= -(-1)^2 - 2(-1) + 3 \\ &= -1 + 2 + 3 \\ &= 4 \end{aligned}$$

The vertex is $(-1, 4)$.

Step 4: Determine extra points so that we have at least five points to plot. Ensure a good sampling on either side of the line of symmetry. In this example, one other point will suffice. Choose $x = -2$ and find the corresponding y-value.

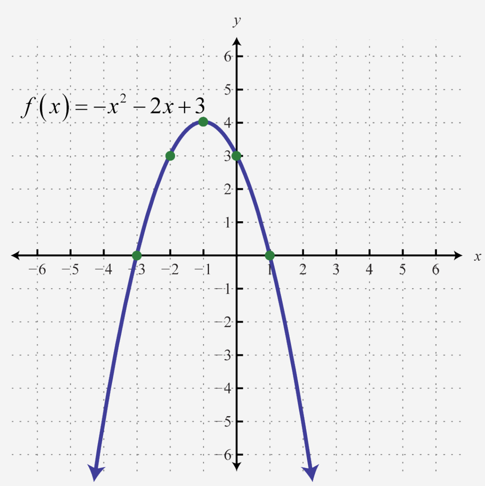
x	y		Point
-2	3	$f(-2) = -(-2)^2 - 2(-2) + 3 = -4 + 4 + 3 = 3$	$(-2, 3)$

Our fifth point is $(-2, 3)$.

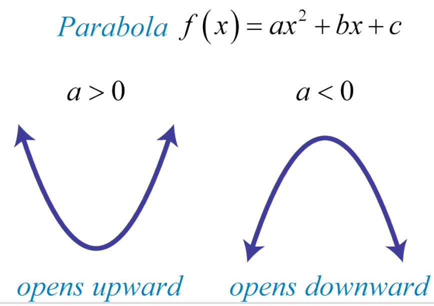
Step 5: Plot the points and sketch the graph. To recap, the points that we have found are

$$\begin{aligned} y-intercept &: (0, 3) \\ x-intercepts &: (-3, 0) \text{ and } (1, 0) \\ Vertex &: (-1, 4) \\ Extra\,point &: (-2, 3) \end{aligned}$$

Answer:



The parabola opens downward. In general, use the leading coefficient to determine if the parabola opens upward or downward. If the leading coefficient is negative, as in the previous example, then the parabola opens downward. If the leading coefficient is positive, then the parabola opens upward.



All quadratic functions of the form $f(x) = ax^2 + bx + c$ have parabolic graphs with y-intercept $(0, c)$. However, not all parabolas have x-intercepts.

Example 2

Graph: $f(x) = 2x^2 + 4x + 5$.

Solution:

Because the leading coefficient 2 is positive, we note that the parabola opens upward. Here $c = 5$ and the y-intercept is $(0, 5)$. To find the x-intercepts, set $f(x) = 0$.

$$\begin{aligned} f(x) &= 2x^2 + 4x + 5 \\ 0 &= 2x^2 + 4x + 5 \end{aligned}$$

In this case, $a = 2$, $b = 4$, and $c = 5$. Use the discriminant to determine the number and type of solutions.

$$\begin{aligned} b^2 - 4ac &= (4)^2 - 4(2)(5) \\ &= 16 - 40 \\ &= -24 \end{aligned}$$

Since the discriminant is negative, we conclude that there are no real solutions. Because there are no real solutions, there are no x-intercepts. Next, we determine the x-value of the vertex.

$$
\begin{aligned}
x &= \frac{-b}{2a} \\
&= \frac{-(4)}{2(2)} \\
&= \frac{-4}{4} \\
&= -1
\end{aligned}
$$

Given that the x-value of the vertex is −1, substitute −1 into the original equation to find the corresponding y-value.

$$
\begin{aligned}
f(x) &= 2x^2 + 4x + 5 \\
f(-1) &= 2(-1)^2 + 4(-1) + 5 \\
&= 2 - 4 + 5 \\
&= 3
\end{aligned}
$$

The vertex is (−1, 3). So far, we have only two points. To determine three more, choose some x-values on either side of the line of symmetry, $x = -1$. Here we choose x-values −3, −2, and 1.

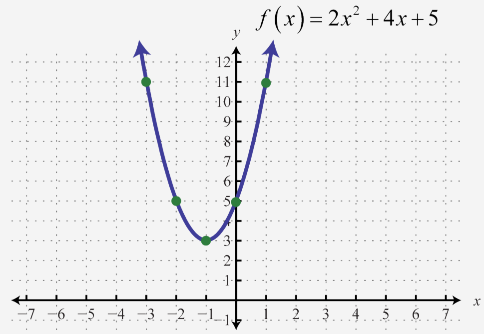
x	y		Points
-3	11	$f(-3) = 2(-3)^2 + 4(-3) + 5 = 18 - 12 + 5 = 11$	$(-3, 11)$
-2	5	$f(-2) = 2(-2)^2 + 4(-2) + 5 = 8 - 8 + 5 = 5$	$(-2, 5)$
1	11	$f(1) = 2(1)^2 + 4(1) + 5 = 2 + 4 + 5 = 11$	$(1, 11)$

To summarize, we have

$$\begin{aligned} y-intercept &: (0,5) \\ x-intercepts &: \text{None} \\ Vertex &: (-1,3) \\ Extra\,points &: (-3,11),\ (-2,5),\ (1,11) \end{aligned}$$

Plot the points and sketch the graph.

Answer:



Example 3

Graph: $f(x) = x^2 - 2x - 1$.

Solution:

Since $a = 1$, the parabola opens upward. Furthermore, $c = -1$, so the y-intercept is $(0, -1)$. To find the x-intercepts, set $f(x) = 0$.

$$\begin{aligned} f(x) &= x^2 - 2x - 1 \\ 0 &= x^2 - 2x - 1 \end{aligned}$$

In this case, solve using the quadratic formula with $a = 1$, $b = -2$, and $c = -1$.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)} \\ &= \frac{2 \pm \sqrt{8}}{2} \\ &= \frac{2 \pm 2\sqrt{2}}{2} \\ &= \frac{2(1 \pm \sqrt{2})}{2} \\ &= 1 \pm \sqrt{2} \end{aligned}$$

Here we obtain two real solutions for x, and thus there are two x-intercepts:

$$\left(1-\sqrt{2},0\right) \text{ and } \left(1+\sqrt{2},0\right) \quad \textit{Exact values}$$
$$(-0.41,0) \qquad (2.41,0) \qquad \textit{Approximate values}$$

Approximating the *x*-intercepts using a calculator will help us plot the points. However, we will present the exact *x*-intercepts on the graph. Next, find the vertex.

$$\begin{aligned} x &= \frac{-b}{2a} \\ &= \frac{-(-2)}{2(1)} \\ &= \frac{2}{2} \\ &= 1 \end{aligned}$$

Given that the *x*-value of the vertex is 1, substitute into the original equation to find the corresponding *y*-value.

$$\begin{aligned} y &= x^2 - 2x - 1 \\ &= (1)^2 - 2(1) - 1 \\ &= 1 - 2 - 1 \\ &= -2 \end{aligned}$$

The vertex is (1, -2). We need one more point.

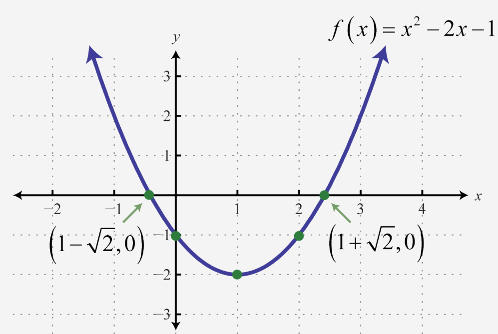
x	y		Point
2	-1	$f(2) = (2)^2 - 2(2) - 1 = 4 - 4 - 1 = -1$	$(2, -1)$

To summarize, we have

$$\begin{aligned} y - intercept &: (0, -1) \\ x - intercepts &: (1 - \sqrt{2}, 0) \quad \text{and} \quad (1 + \sqrt{2}, 0) \\ Vertex &: (1, -2) \\ Extra\,point &: (2, -1) \end{aligned}$$

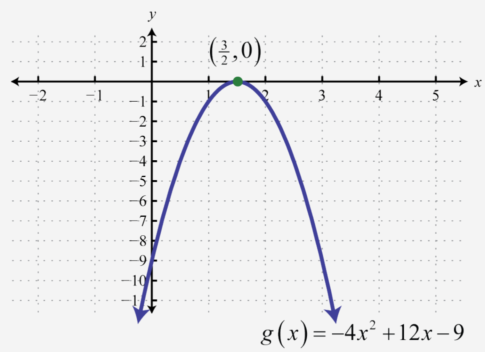
Plot the points and sketch the graph.

Answer:



Try this! Graph: $g(x) = -4x^2 + 12x - 9$.

Answer:



(click to see video)

Finding the Maximum or Minimum

It is often useful to find the maximum and/or minimum values of functions that model real-life applications. To find these important values given a quadratic function, we use the vertex. If the leading coefficient *a* is positive, then the parabola opens upward and there will be a minimum *y*-value. If the leading coefficient *a* is negative, then the parabola opens downward and there will be a maximum *y*-value.

Example 4

Determine the maximum or minimum: $y = -4x^2 + 24x - 35$.

Solution:

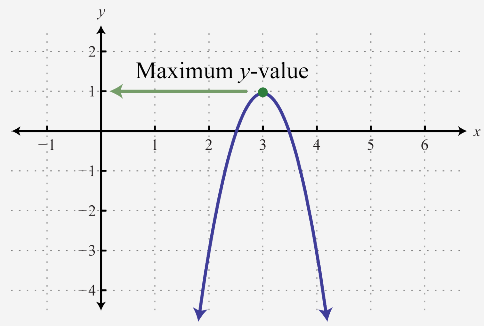
Since $a = -4$, we know that the parabola opens downward and there will be a maximum y-value. To find it, first find the x-value of the vertex.

$$\begin{aligned} x &= -\frac{b}{2a} && \textit{x-value of the vertex.} \\ &= -\frac{24}{2(-4)} && \textit{Substitute } a = -4 \textit{ and } b = 24. \\ &= -\frac{24}{-8} && \textit{Simplify.} \\ &= 3 \end{aligned}$$

The x-value of the vertex is 3. Substitute this value into the original equation to find the corresponding y-value.

$$\begin{aligned} y &= -4x^2 + 24x - 35 && \textit{Substitute } x = 3. \\ &= -4(3)^2 + 24(3) - 35 && \textit{Simplify.} \\ &= -36 + 72 - 35 \\ &= 1 \end{aligned}$$

The vertex is (3, 1). Therefore, the maximum y-value is 1, which occurs where $x = 3$, as illustrated below:



Note: The graph is not required to answer this question.

Answer: The maximum is 1.

Example 5

Determine the maximum or minimum: $y = 4x^2 - 32x + 62$.

Solution:

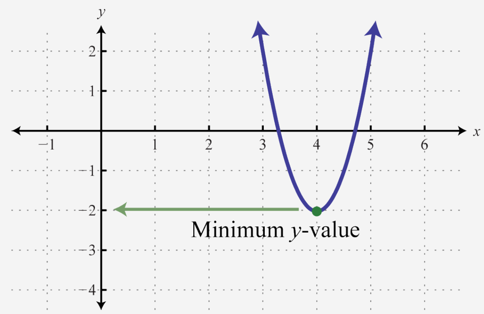
Since a = 4, the parabola opens upward and there is a minimum y-value. Begin by finding the x-value of the vertex.

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\frac{-32}{2(4)} && \textit{Substitute } a = 4 \textit{ and } b = -32. \\ &= -\frac{-32}{8} && \textit{Simplify.} \\ &= 4 \end{aligned}$$

Substitute x = 4 into the original equation to find the corresponding y-value.

$$\begin{aligned} y &= 4x^2 - 32x + 62 \\ &= 4(4)^2 - 32(4) + 62 \\ &= 64 - 128 + 62 \\ &= -2 \end{aligned}$$

The vertex is (4, −2). Therefore, the minimum y-value of −2 occurs where x = 4, as illustrated below:



Answer: The minimum is −2.

Example 6

The height in feet of a projectile is given by the function $h(t) = -16t^2 + 72t$, where t represents the time in seconds after launch. What is the maximum height reached by the projectile?

Solution:

Here $a = -16$, and the parabola opens downward. Therefore, the y-value of the vertex determines the maximum height. Begin by finding the time at which the vertex occurs.

$$t = -\frac{b}{2a} = -\frac{72}{2\left(-16\right)} = \frac{72}{32} = \frac{9}{4}$$

The maximum height will occur in $\frac{9}{4}$ seconds (or $2\,\frac{1}{4}$ seconds). Substitute this time into the function to determine the maximum height attained.

$$\begin{aligned} h\left(\frac{9}{4}\right) &= -16\left(\frac{9}{4}\right)^2 + 72\left(\frac{9}{4}\right) \\ &= -16\left(\frac{81}{16}\right) + 72\left(\frac{9}{4}\right) \\ &= -81 + 162 \\ &= 81 \end{aligned}$$

Answer: The maximum height of the projectile is 81 feet.

Finding the Vertex by Completing the Square

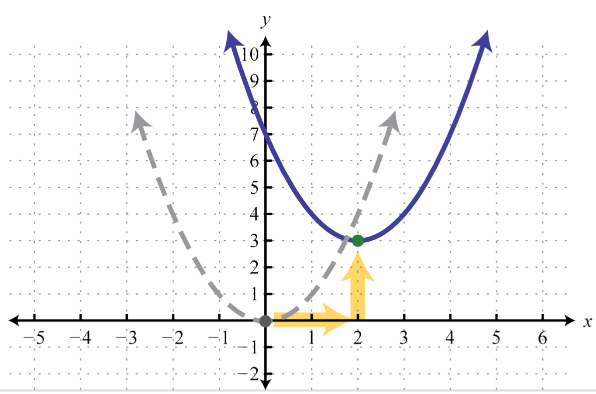
In this section, we demonstrate an alternate approach for finding the vertex. Any quadratic function $f(x) = ax^2 + bx + c$ can be rewritten in **vertex form**[14],

$$f(x) = a(x - h)^2 + k$$

In this form, the vertex is (h, k). To see that this is the case, consider graphing $f(x) = (x - 2)^2 + 3$ using the transformations.

$$\begin{aligned} y &= x^2 && \textit{Basic squaring function} \\ y &= (x-2)^2 && \textit{Horizontal shift right 2 units} \\ y &= (x-2)^2 + 3 && \textit{Vertical shift up 3 units} \end{aligned}$$

Use these translations to sketch the graph,



Here we can see that the vertex is (2, 3).

14. A quadratic function written in the form $f(x) = a(x - h)^2 + k$.

$$\begin{aligned} f(x) &= a(x-h)^2+k \\ &\qquad\ \downarrow \qquad \downarrow \\ f(x) &= \ \ (x-2)^2+3 \end{aligned}$$

When the equation is in this form, we can read the vertex directly from it.

Example 7

Determine the vertex: $f(x) = 2(x+3)^2 - 2$.

Solution:

Rewrite the equation as follows before determining *h* and *k*.

$$\begin{aligned} f(x) &= a(x - \ \ h \ \ )^2 + \ \ k \\ &\qquad\quad \downarrow \qquad\quad \downarrow \\ f(x) &= 2\left[x-(-3)\right]^2+(-2) \end{aligned}$$

Here $h = -3$ and $k = -2$.

Answer: The vertex is $(-3, -2)$.

Often the equation is not given in vertex form. To obtain this form, complete the square.

Example 8

Rewrite in vertex form and determine the vertex: $f(x) = x^2 + 4x + 9$.

Solution:

Begin by making room for the constant term that completes the square.

$$\begin{aligned} f(x) &= x^2 + 4x + 9 \\ &= x^2 + 4x + \_\_\_ + 9 - \_\_\_ \end{aligned}$$

The idea is to add and subtract the value that completes the square, $\left(\frac{b}{2}\right)^2$, and then factor. In this case, add and subtract $\left(\frac{4}{2}\right)^2 = (2)^2 = 4$.

$$\begin{aligned} f(x) &= x^2 + 4x + 9 && \textit{Add and subtract } 4. \\ &= x^2 + 4x + 4 + 9 - 4 && \textit{Factor.} \\ &= (x^2 + 4x + 4) + 5. \\ &= (x+3)(x+2) + 5 \\ &= (x+2)^2 + 5 \end{aligned}$$

Adding and subtracting the same value within an expression does not change it. Doing so is equivalent to adding 0. Once the equation is in this form, we can easily determine the vertex.

$$\begin{aligned} f(x) &= a(x - \;\; h \;\;)^2 + k \\ & \qquad\quad\;\, \downarrow \qquad\; \downarrow \\ f(x) &= \;\; (x - (-2))^2 + 5 \end{aligned}$$

Here $h = -2$ and $k = 5$.

Answer: The vertex is $(-2, 5)$.

If there is a leading coefficient other than 1, then we must first factor out the leading coefficient from the first two terms of the trinomial.

Example 9

Rewrite in vertex form and determine the vertex: $f\left(x\right) = 2x^2 - 4x + 8$.

Solution:

Since $a = 2$, factor this out of the first two terms in order to complete the square. Leave room inside the parentheses to add and subtract the value that completes the square.

$$\begin{aligned} f\left(x\right) &= 2x^2 - 4x + 8 \\ &= 2\left(x^2 - 2x \quad\right) + 8 \end{aligned}$$

Now use −2 to determine the value that completes the square. In this case, $\left(\frac{-2}{2}\right)^2 = (-1)^2 = 1$. Add and subtract 1 and factor as follows:

$$\begin{aligned} f\left(x\right) &= 2x^2 - 4x + 8 \\ &= 2\left(x^2 - 2x + \_\_ - \_\,\_\right) + 8 && \textit{Add and subtract 1.} \\ &= 2(x^2 - 2x + 1 - 1) + 8 && \textit{Factor.} \\ &= 2\left[(x-1)\ (x-1) - 1\right] + 8 \\ &= 2\left[(x-1)^2 - 1\right] + 8 && \textit{Distribute the 2.} \\ &= 2(x-1)^2 - 2 + 8 \\ &= 2(x-1)^2 + 6 \end{aligned}$$

In this form, we can easily determine the vertex.

$$f(x) = a(x-h)^2 + k$$
$$\downarrow \qquad \downarrow$$
$$f(x) = 2(x-1)^2 + 6$$

Here $h = 1$ and $k = 6$.

Answer: The vertex is (1, 6).

Try this! Rewrite in vertex form and determine the vertex: $f(x) = -2x^2 - 12x + 3$.

Answer: $f(x) = -2(x+3)^2 + 21$; vertex: $(-3, 21)$

(click to see video)

KEY TAKEAWAYS

- The graph of any quadratic function $f(x) = ax^2 + bx + c$, where *a*, *b*, and *c* are real numbers and $a \neq 0$, is called a parabola.
- When graphing a parabola always find the vertex and the *y*-intercept. If the *x*-intercepts exist, find those as well. Also, be sure to find ordered pair solutions on either side of the line of symmetry, $x = -\frac{b}{2a}$.
- Use the leading coefficient, *a*, to determine if a parabola opens upward or downward. If *a* is positive, then it opens upward. If *a* is negative, then it opens downward.
- The vertex of any parabola has an *x*-value equal to $-\frac{b}{2a}$. After finding the *x*-value of the vertex, substitute it into the original equation to find the corresponding *y*-value. This *y*-value is a maximum if the parabola opens downward, and it is a minimum if the parabola opens upward.
- The domain of a parabola opening upward or downward consists of all real numbers. The range is bounded by the *y*-value of the vertex.
- An alternate approach to finding the vertex is to rewrite the quadratic function in the form $f(x) = a(x - h)^2 + k$. When in this form, the vertex is (h, k) and can be read directly from the equation. To obtain this form, take $f(x) = ax^2 + bx + c$ and complete the square.

TOPIC EXERCISES

PART A: THE GRAPH OF QUADRATIC FUNCTIONS

Does the parabola open upward or downward? Explain.

1. $y = x^2 - 9x + 20$
2. $y = x^2 - 12x + 32$
3. $y = -2x^2 + 5x + 12$
4. $y = -6x^2 + 13x - 6$
5. $y = 64 - x^2$
6. $y = -3x + 9x^2$

Determine the *x*- and *y*-intercepts.

7. $y = x^2 + 4x - 12$
8. $y = x^2 - 13x + 12$
9. $y = 2x^2 + 5x - 3$
10. $y = 3x^2 - 4x - 4$
11. $y = -5x^2 - 3x + 2$
12. $y = -6x^2 + 11x - 4$
13. $y = 4x^2 - 27$
14. $y = 9x^2 - 50$
15. $y = x^2 - x + 1$
16. $y = x^2 - 6x + 4$

Find the vertex and the line of symmetry.

17. $y = -x^2 + 10x - 34$
18. $y = -x^2 - 6x + 1$

19. $y = -4x^2 + 12x - 7$

20. $y = -9x^2 + 6x + 2$

21. $y = 4x^2 - 1$

22. $y = x^2 - 16$

Graph. Find the vertex and the *y*-intercept. In addition, find the *x*-intercepts if they exist.

23. $f(x) = x^2 - 2x - 8$

24. $f(x) = x^2 - 4x - 5$

25. $f(x) = -x^2 + 4x + 12$

26. $f(x) = -x^2 - 2x + 15$

27. $f(x) = x^2 - 10x$

28. $f(x) = x^2 + 8x$

29. $f(x) = x^2 - 9$

30. $f(x) = x^2 - 25$

31. $f(x) = 1 - x^2$

32. $f(x) = 4 - x^2$

33. $f(x) = x^2 - 2x + 1$

34. $f(x) = x^2 + 4x + 4$

35. $f(x) = -4x^2 + 12x - 9$

36. $f(x) = -4x^2 - 4x + 3$

37. $f(x) = x^2 - 2$

38. $f(x) = x^2 - 3$

39. $f(x) = -4x^2 + 4x - 3$

40. $f(x) = 4x^2 + 4x + 3$

41. $f(x) = x^2 - 2x - 2$
42. $f(x) = x^2 - 6x + 6$
43. $f(x) = -2x^2 + 6x - 3$
44. $f(x) = -4x^2 + 4x + 1$
45. $f(x) = x^2 + 3x + 4$
46. $f(x) = -x^2 + 3x - 4$
47. $f(x) = -2x^2 + 3$
48. $f(x) = -2x^2 - 1$
49. $f(x) = 2x^2 + 4x - 3$
50. $f(x) = 3x^2 + 2x - 2$

PART B: FINDING THE MAXIMUM OR MINIMUM

Determine the maximum or minimum *y*-value.

51. $y = -x^2 - 6x + 1$
52. $y = -x^2 - 4x + 8$
53. $y = 25x^2 - 10x + 5$
54. $y = 16x^2 - 24x + 7$
55. $y = -x^2$
56. $y = 1 - 9x^2$
57. $y = 20x - 10x^2$
58. $y = 12x + 4x^2$
59. $y = 3x^2 - 4x - 2$
60. $y = 6x^2 - 8x + 5$
61. $y = x^2 - 5x + 1$

62. $y = 1 - x - x^2$

Given the following quadratic functions, determine the domain and range.

63. $f(x) = 3x^2 + 30x + 50$

64. $f(x) = 5x^2 - 10x + 1$

65. $g(x) = -2x^2 + 4x + 1$

66. $g(x) = -7x^2 - 14x - 9$

67. $f(x) = x^2 + x - 1$

68. $f(x) = -x^2 + 3x - 2$

69. The height in feet reached by a baseball tossed upward at a speed of 48 feet per second from the ground is given by the function $h(t) = -16t^2 + 48t$, where *t* represents the time in seconds after the ball is thrown. What is the baseball's maximum height and how long does it take to attain that height?

70. The height in feet of a projectile launched straight up from a mound is given by the function $h(t) = -16t^2 + 96t + 4$, where *t* represents seconds after launch. What is the maximum height?

71. The profit in dollars generated by producing and selling *x* custom lamps is given by the function $P(x) = -10x^2 + 800x - 12{,}000$. What is the maximum profit?

72. The profit in dollars generated from producing and selling a particular item is modeled by the formula $P(x) = 100x - 0.0025x^2$, where *x* represents the number of units produced and sold. What number of units must be produced and sold to maximize revenue?

73. The average number of hits to a radio station Web site is modeled by the formula $f(x) = 450t^2 - 3{,}600t + 8{,}000$, where *t* represents the number of hours since 8:00 a.m. At what hour of the day is the number of hits to the Web site at a minimum?

74. The value in dollars of a new car is modeled by the formula $V(t) = 125t^2 - 3{,}000t + 22{,}000$, where *t* represents the number of years since it was purchased. Determine the minimum value of the car.

75. The daily production cost in dollars of a textile manufacturing company producing custom uniforms is modeled by the formula

$C(x) = 0.02x^2 - 20x + 10{,}000$, where x represents the number of uniforms produced.

a. How many uniforms should be produced to minimize the daily production costs?
b. What is the minimum daily production cost?

76. The area in square feet of a certain rectangular pen is given by the formula $A = 14w - w^2$, where w represents the width in feet. Determine the width that produces the maximum area.

PART C: FINDING THE VERTEX BY COMPLETING THE SQUARE

Determine the vertex.

77. $y = -(x - 5)^2 + 3$
78. $y = -2(x - 1)^2 + 7$
79. $y = 5(x + 1)^2 + 6$
80. $y = 3(x + 4)^2 + 10$
81. $y = -5(x + 8)^2 - 1$
82. $y = (x + 2)^2 - 5$

Rewrite in vertex form $y = a(x - h)^2 + k$ and determine the vertex.

83. $y = x^2 - 14x + 24$
84. $y = x^2 - 12x + 40$
85. $y = x^2 + 4x - 12$
86. $y = x^2 + 6x - 1$
87. $y = 2x^2 - 12x - 3$
88. $y = 3x^2 - 6x + 5$
89. $y = -x^2 + 16x + 17$

90. $y = -x^2 + 10x$

Graph. Find the vertex and the *y*-intercept. In addition, find the *x*-intercepts if they exist.

91. $f(x) = x^2 - 1$
92. $f(x) = x^2 + 1$
93. $f(x) = (x - 1)^2$
94. $f(x) = (x + 1)^2$
95. $f(x) = (x - 4)^2 - 9$
96. $f(x) = (x - 1)^2 - 4$
97. $f(x) = -2(x + 1)^2 + 8$
98. $f(x) = -3(x + 2)^2 + 12$
99. $f(x) = -5(x - 1)^2$
100. $f(x) = -(x + 2)^2$
101. $f(x) = -4(x - 1)^2 - 2$
102. $f(x) = 9(x + 1)^2 + 2$
103. $f(x) = \left(x + 5\right)^2 - 15$
104. $f(x) = 2\left(x - 5\right)^2 - 3$
105. $f(x) = -2(x - 4)^2 + 22$
106. $f(x) = 2(x + 3)^2 - 13$

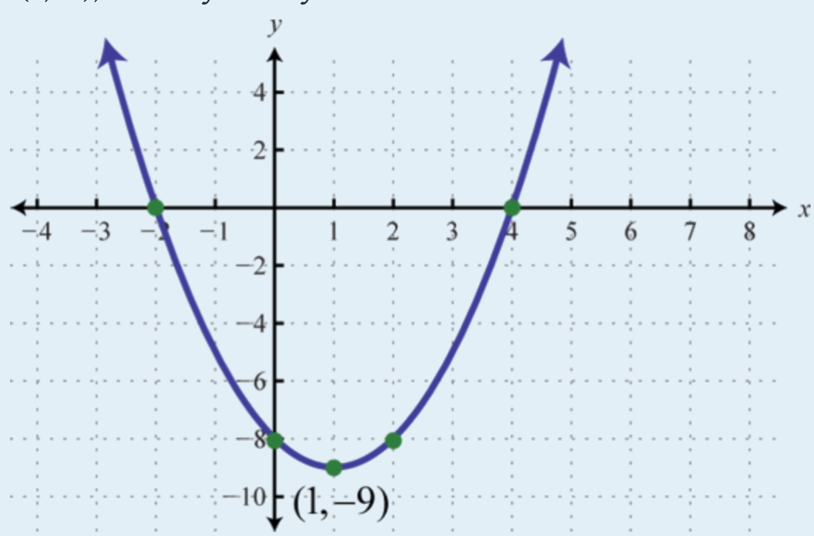
PART D: DISCUSSION BOARD

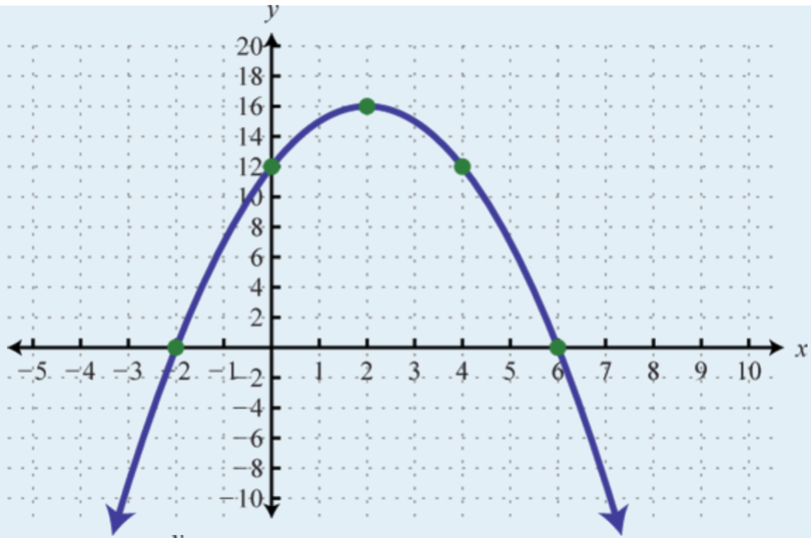
107. Write down your plan for graphing a parabola on an exam. What will you be looking for and how will you present your answer? Share your plan on the discussion board.

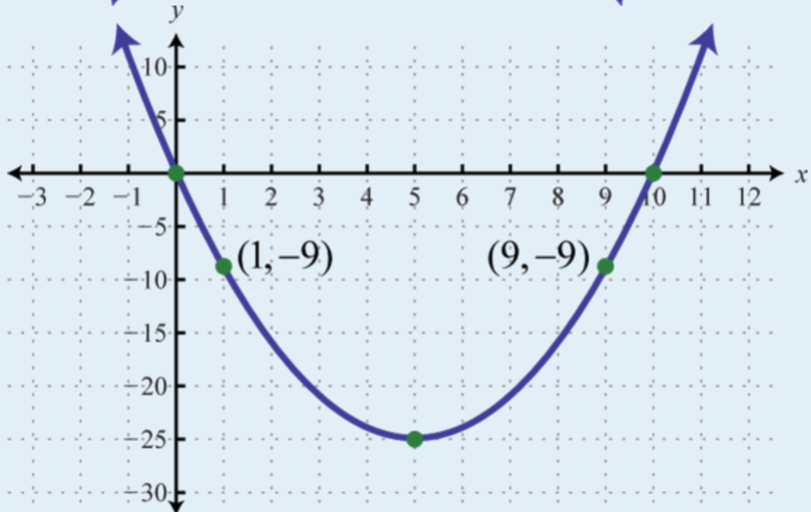
108. Why is any parabola that opens upward or downward a function? Explain to a classmate how to determine the domain and range.

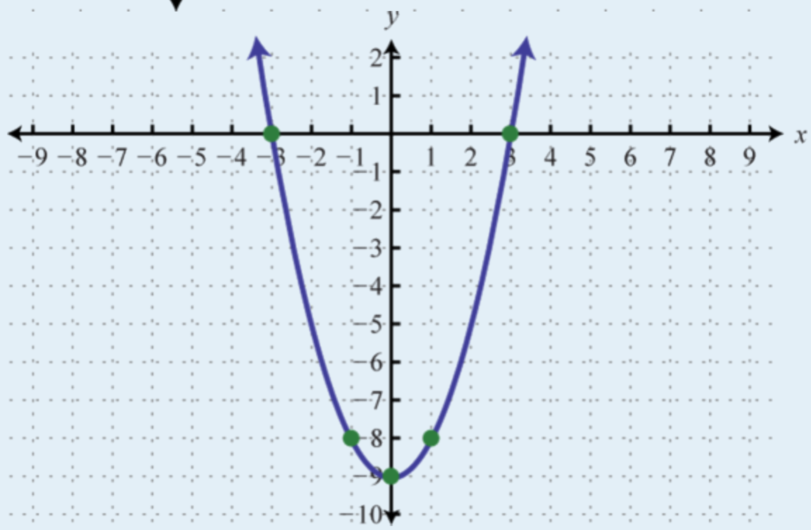
109. Research and discuss ways of finding a quadratic function that has a graph passing through any three given points. Share a list of steps as well as an example of how to do this.

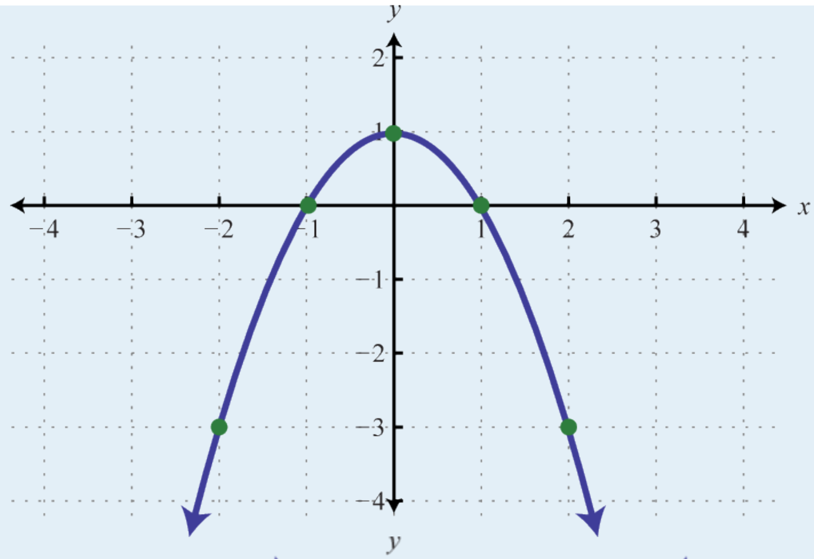
ANSWERS

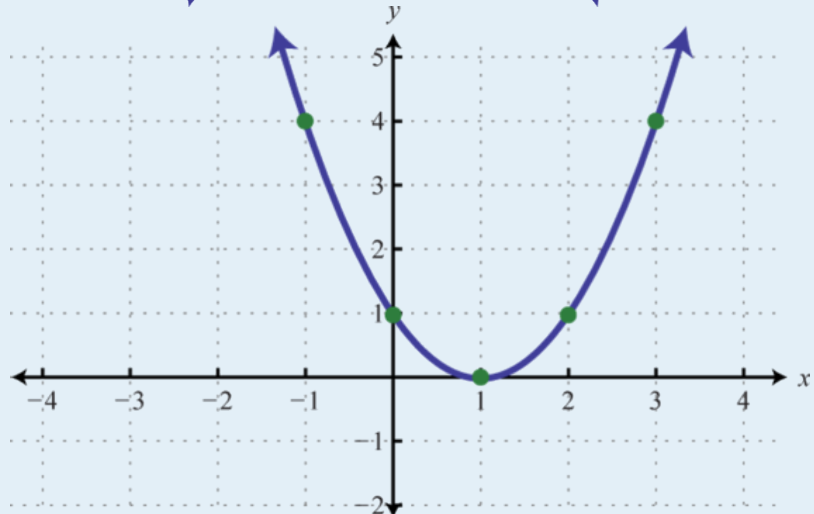
1. Upward
3. Downward
5. Downward
7. x-intercepts: (−6, 0), (2, 0);y-intercept: (0, −12)
9. x-intercepts: (−3, 0), $\left(\frac{1}{2}, 0\right)$; y-intercept: (0, −3)
11. x-intercepts: (−1, 0), $\left(\frac{2}{5}, 0\right)$; y-intercept: (0, 2)
13. x-intercepts: $\left(-\frac{3\sqrt{3}}{2}, 0\right), \left(\frac{3\sqrt{3}}{2}, 0\right)$; y-intercept: (0, −27)
15. x-intercepts: none; y-intercept: (0, 1)
17. Vertex: (5, −9); line of symmetry: $x = 5$
19. Vertex: $\left(\frac{3}{2}, 2\right)$; line of symmetry: $x = \frac{3}{2}$
21. Vertex: (0, −1); line of symmetry: $x = 0$
23. 

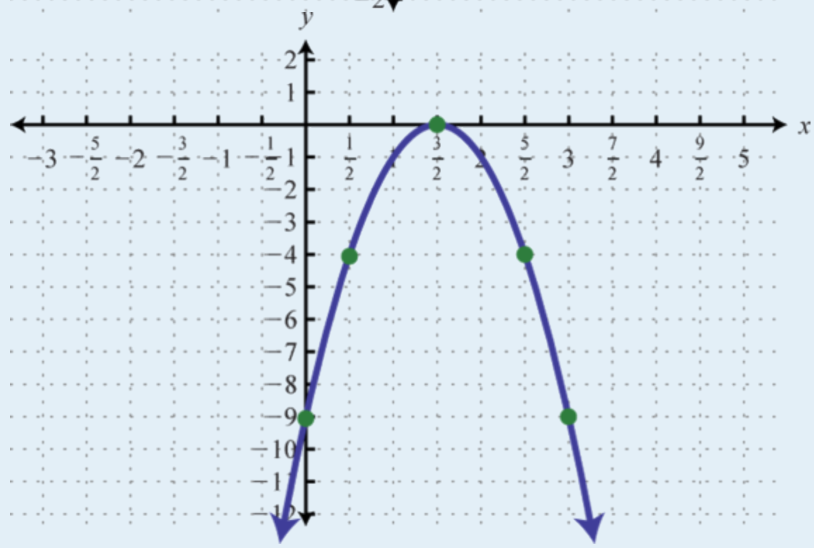
25. 

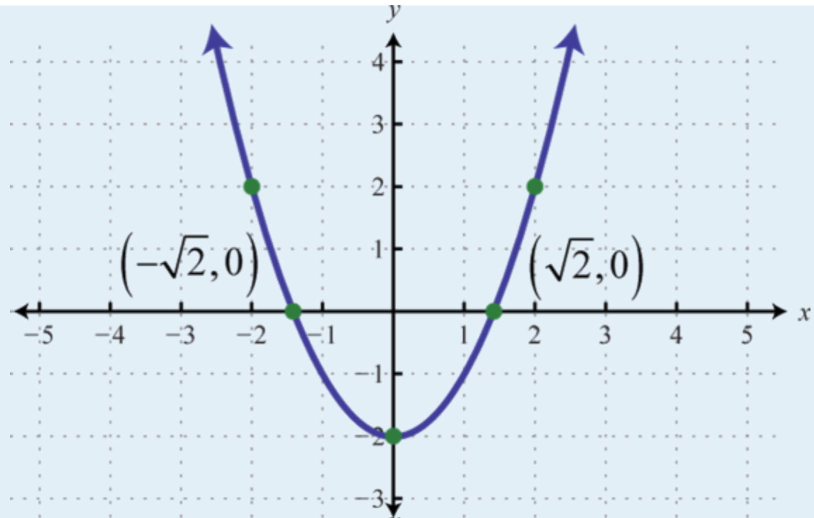
27. 

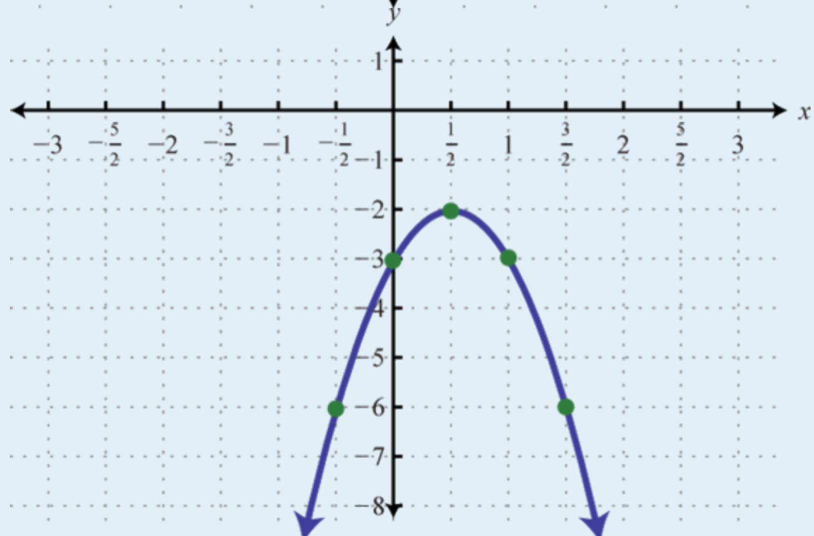
29. 

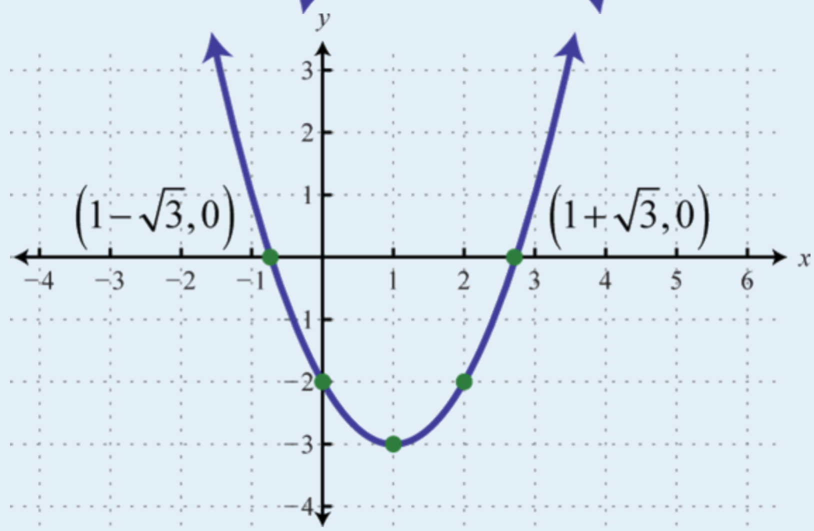
31. 

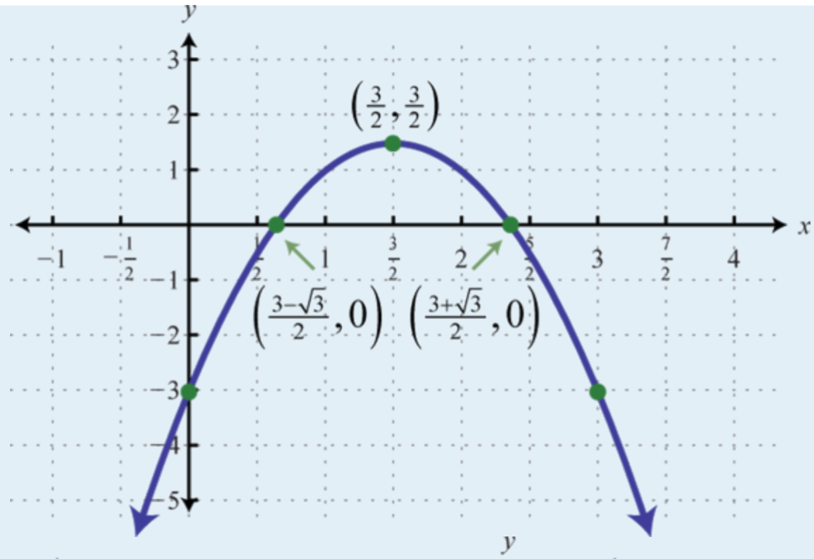
33. 

35. 

37. 

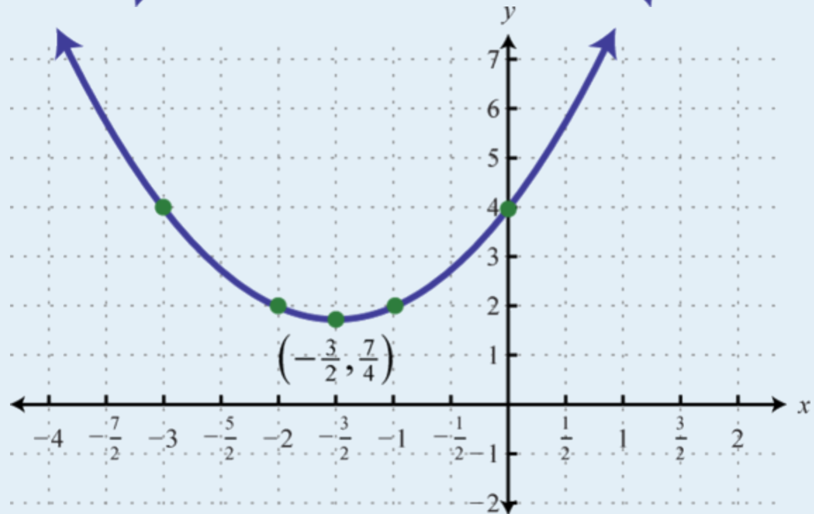
39. 

41. 



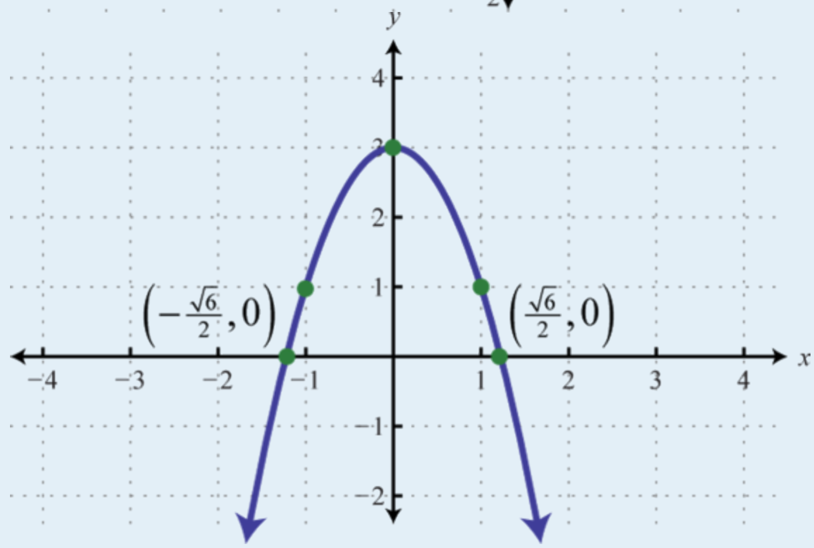
$\left(\frac{3}{2}, \frac{3}{2}\right)$
$\left(\frac{3-\sqrt{3}}{2}, 0\right)$
$\left(\frac{3+\sqrt{3}}{2}, 0\right)$

43.



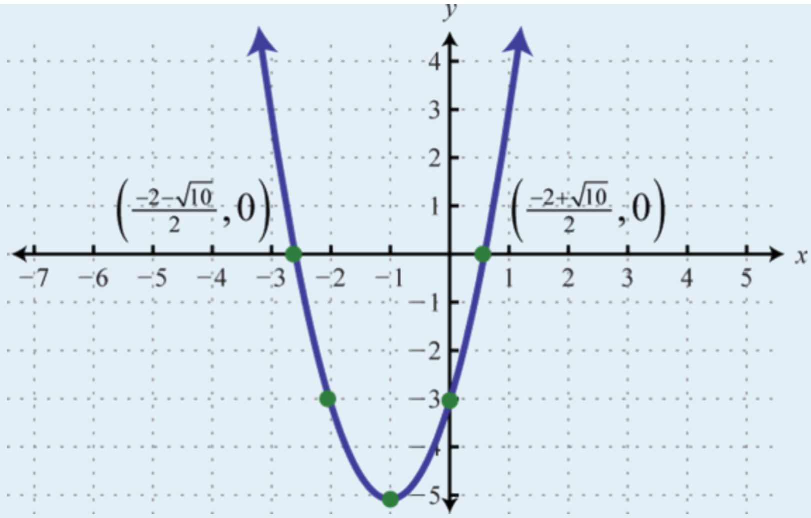
$\left(-\frac{3}{2}, \frac{7}{4}\right)$

45.



$\left(-\frac{\sqrt{6}}{2}, 0\right)$
$\left(\frac{\sqrt{6}}{2}, 0\right)$

47.

49. 

51. Maximum: $y = 10$

53. Minimum: $y = 4$

55. Maximum: $y = 0$

57. Maximum: $y = 10$

59. Minimum: $y = -\frac{10}{3}$

61. Minimum: $y = -\frac{21}{4}$

63. Domain: $(-\infty, \infty)$; range: $[-25, \infty)$

65. Domain: $(-\infty, \infty)$; range: $(-\infty, 3]$

67. Domain: $(-\infty, \infty)$; range: $\left[-\frac{5}{4}, \infty\right)$

69. The maximum height of 36 feet occurs after 1.5 seconds.

71. $4,000

73. 12:00 p.m.

75.
   a. 500 uniforms
   b. $5,000

77. (5, 3)

79. (-1, 6)

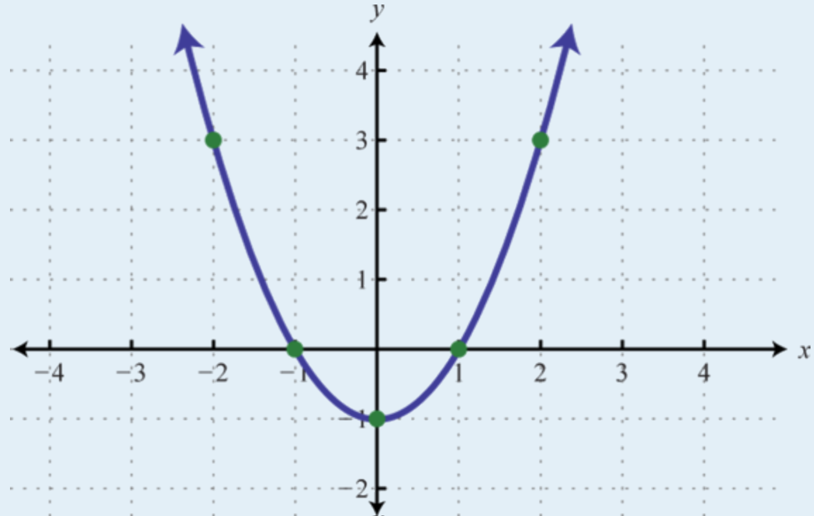
81. (-8, -1)

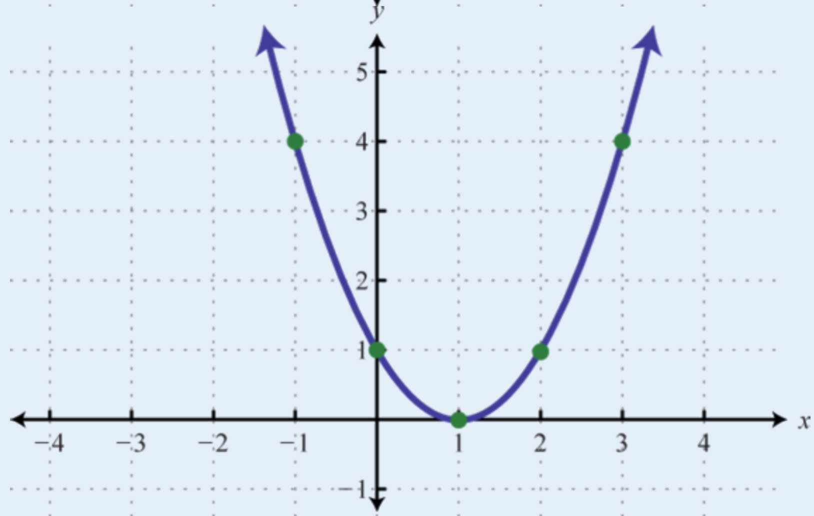
83. $y = (x - 7)^2 - 25$; vertex: (7, -25)

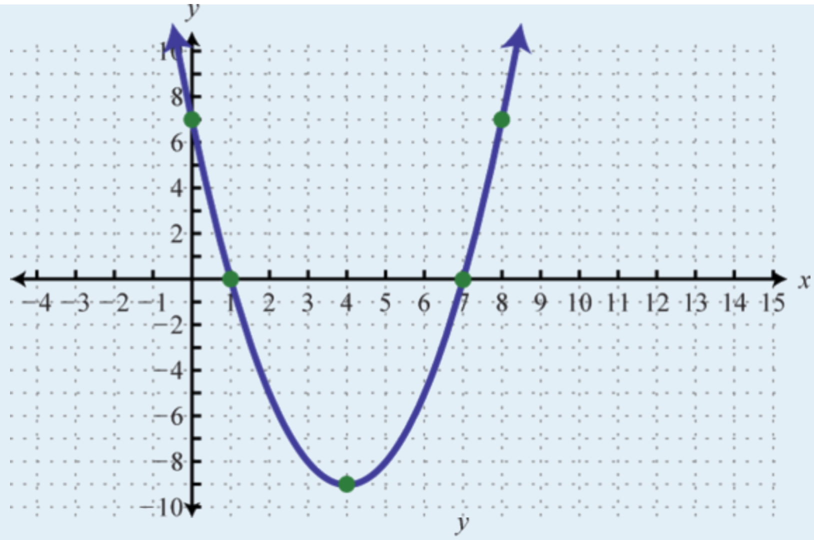
85. $y = (x+2)^2 - 16$; vertex: (−2, −16)

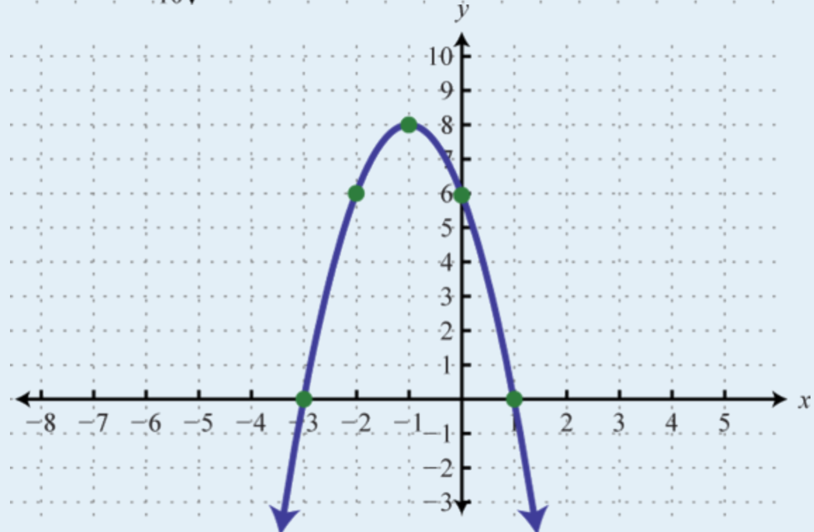
87. $y = 2(x-3)^2 - 21$; vertex: (3, −21)

89. $y = -(x-8)^2 + 81$; vertex: (8, 81)

91. 

93. 

95.

y
x
10
8
6
4
2
−2
−4
−6
−8
−10
−4 −3 −2 −1 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

97.

y
x
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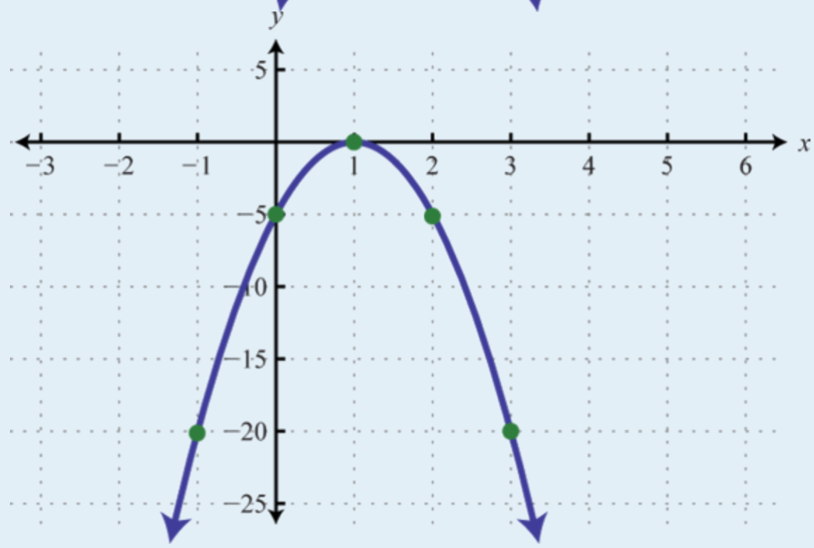
−8 −7 −6 −5 −4 −3 −2 −1 1 2 3 4 5

99.

y
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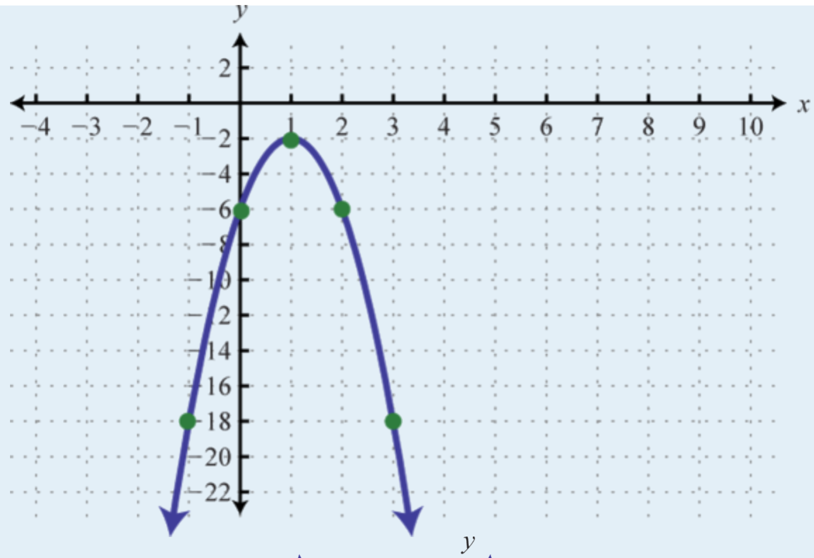
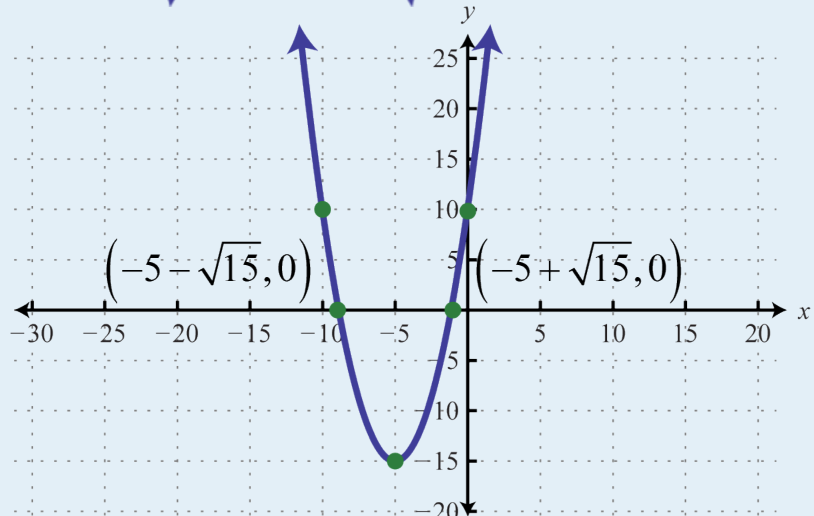
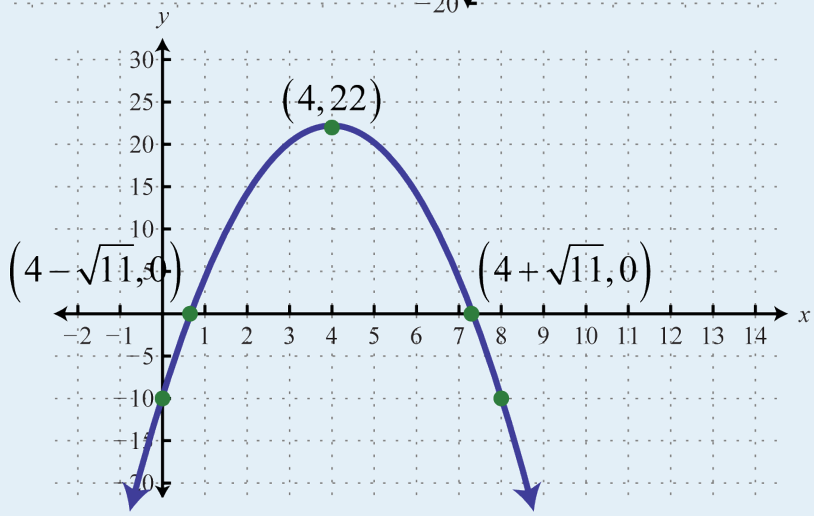
−5
−10
−15
−20
−25
−3 −2 −1 1 2 3 4 5 6

101. 

103. 

105. 

107. Answer may vary

109. Answer may vary

6.5 Solving Quadratic Inequalities

LEARNING OBJECTIVES

1. Check solutions to quadratic inequalities with one variable.
2. Understand the geometric relationship between solutions to quadratic inequalities and their graphs.
3. Solve quadratic inequalities.

Solutions to Quadratic Inequalities

A **quadratic inequality**[15] is a mathematical statement that relates a quadratic expression as either less than or greater than another. Some examples of quadratic inequalities solved in this section follow.

$x^2 - 2x - 11 \leq 0$	$2x^2 - 7x + 3 > 0$	$9 - x^2 > 0$

A solution to a quadratic inequality is a real number that will produce a true statement when substituted for the variable.

15. A mathematical statement that relates a quadratic expression as either less than or greater than another.

Example 1

Are -3, -2, and -1 solutions to $x^2 - x - 6 \leq 0$?

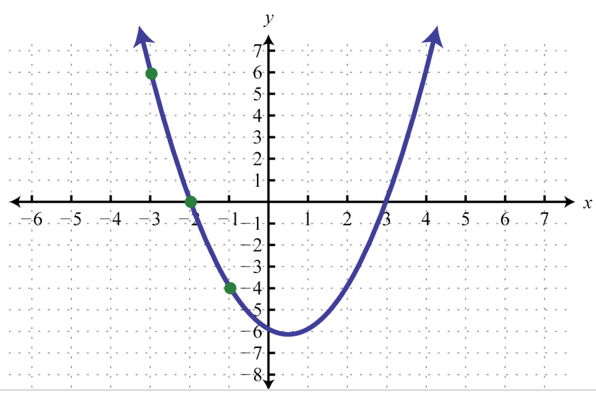
Solution:

Substitute the given value in for *x* and simplify.

$x^2 - x - 6 \leq 0$	$x^2 - x - 6 \leq 0$	$x^2 - x - 6 \leq 0$
$(-3)^2 - (-3) - 6 \leq 0$	$(-2)^2 - (-2) - 6 \leq 0$	$(-1)^2 - (-1) - 6 \leq 0$
$9 + 3 - 6 \leq 0$	$4 + 2 - 6 \leq 0$	$1 + 1 - 6 \leq 0$
$6 \leq 0$ ✗	$0 \leq 0$ ✓	$-4 \leq 0$ ✓

Answer: -2 and -1 are solutions and -3 is not.

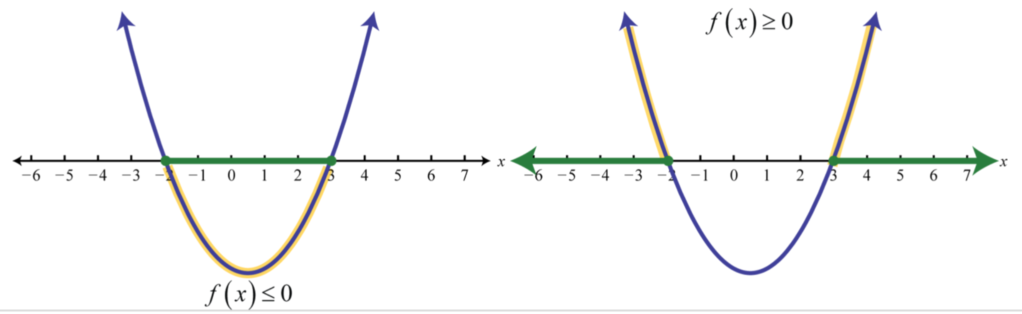
Quadratic inequalities can have infinitely many solutions, one solution, or no solution. If there are infinitely many solutions, graph the solution set on a number line and/or express the solution using interval notation. Graphing the function defined by $f(x) = x^2 - x - 6$ found in the previous example we have



The result of evaluating for any *x*-value will be negative, zero, or positive.

$$\begin{array}{lll} f(-3)=6 & \textit{Positive} & f(x)>0 \\ \mathrm{f}(-2)=0 & \textit{Zero} & f(x)=0 \\ \mathrm{f}(-1)=-4 & \textit{Negative} & f(x)<0 \end{array}$$

The values in the domain of a function that separate regions that produce positive or negative results are called **critical numbers**[16]. In the case of a quadratic function, the critical numbers are the roots, sometimes called the zeros. For example, $f(x) = x^2 - x - 6 = (x+2)(x-3)$ has roots -2 and 3. These values bound the regions where the function is positive (above the *x*-axis) or negative (below the *x*-axis).

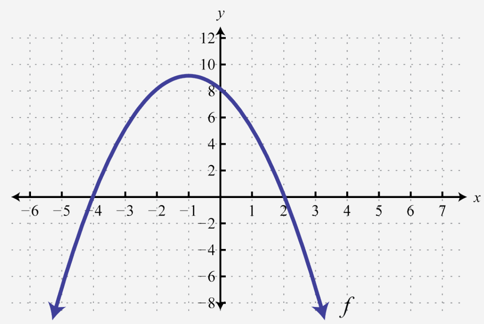


Therefore $x^2 - x - 6 \leq 0$ has solutions where $-2 \leq x \leq 3$, using interval notation $[-2, 3]$. Furthermore, $x^2 - x - 6 \geq 0$ has solutions where $x \leq -2$ or $x \geq 3$, using interval notation $(-\infty, -2] \cup [-3, \infty)$.

16. The values in the domain of a function that separate regions that produce positive or negative results.

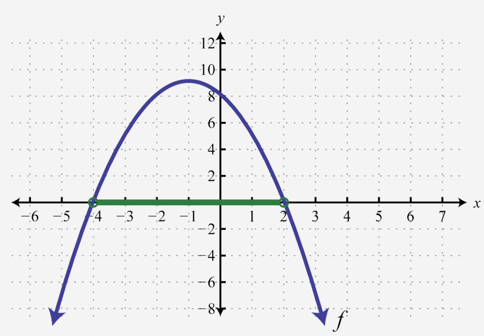
Example 2

Given the graph of f determine the solutions to $f(x) > 0$:



Solution:

From the graph we can see that the roots are -4 and 2. The graph of the function lies above the *x*-axis ($f(x) > 0$) in between these roots.



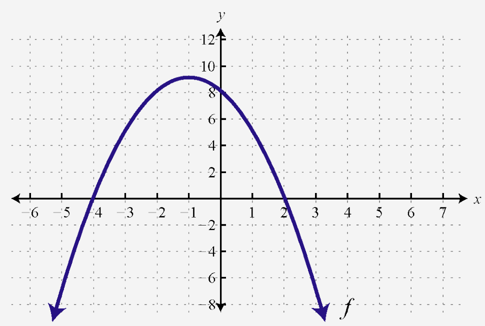
Because of the strict inequality, the solution set is shaded with an open dot on each of the boundaries. This indicates that these critical numbers are not actually included in the solution set. This solution set can be expressed two ways,

$$\{x \mid -4 < x < 2\} \quad \textit{Set Notation}$$
$$(-4, 2) \quad \textit{Interval Notation}$$

In this textbook, we will continue to present answers in interval notation.

Answer: $(-4, 2)$

Try this! Given the graph of f determine the solutions to $f(x) < 0$:



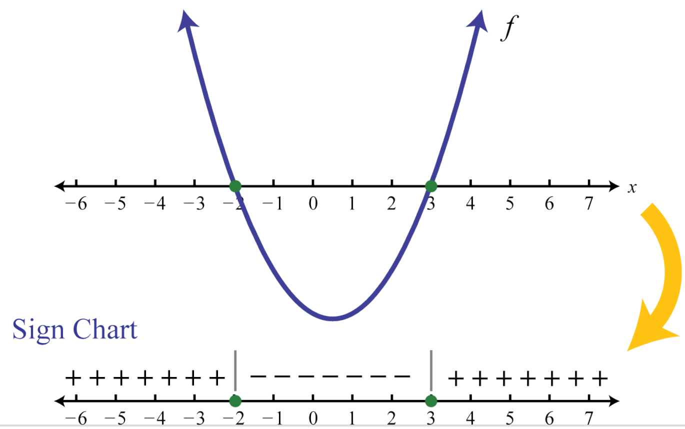
Answer: $(-\infty, -4) \cup (2, \infty)$

(click to see video)

Solving Quadratic Inequalities

Next we outline a technique used to solve quadratic inequalities without graphing the parabola. To do this we make use of a **sign chart**[17] which models a function using a number line that represents the *x*-axis and signs (+ or –) to indicate where the function is positive or negative. For example,

17. A model of a function using a number line and signs (+ or –) to indicate regions in the domain where the function is positive or negative.



The plus signs indicate that the function is positive on the region. The negative signs indicate that the function is negative on the region. The boundaries are the critical numbers, −2 and 3 in this case. Sign charts are useful when a detailed picture of the graph is not needed and are used extensively in higher level mathematics. The steps for solving a quadratic inequality with one variable are outlined in the following example.

Example 3

Solve: $-x^2 + 6x + 7 \geq 0$.

Solution:

It is important to note that this quadratic inequality is in standard form, with zero on one side of the inequality.

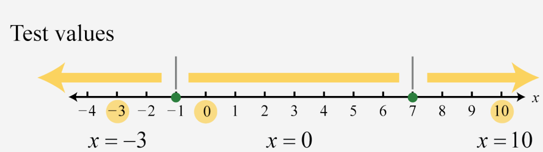
Step 1: Determine the critical numbers. For a quadratic inequality in standard form, the critical numbers are the roots. Therefore, set the function equal to zero and solve.

$$\begin{aligned} -x^2 + 6x + 7 &= 0 \\ -\left(x^2 - 6x - 7\right) &= 0 \\ -(x+1)\,(x-7) &= 0 \end{aligned}$$

$$\begin{aligned} x + 1 &= 0 \quad \text{or} \quad x - 7 = 0 \\ x &= -1 \qquad\qquad x = 7 \end{aligned}$$

The critical numbers are -1 and 7.

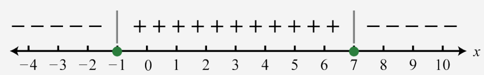
Step 2: Create a sign chart. Since the critical numbers bound the regions where the function is positive or negative, we need only test a single value in each region. In this case the critical numbers partition the number line into three regions and we choose test values $x = -3$, $x = 0$, and $x = 10$.



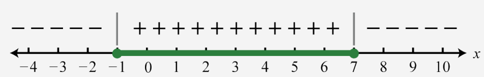
Test values may vary. In fact, we need only determine the sign (+ or –) of the result when evaluating $f(x) = -x^2 + 6x + 7 = -(x+1)(x-7)$. Here we evaluate using the factored form.

$$\begin{aligned} f(-3) &= -(-3+1)(-3-7) = -(-2)(-10) = - \quad \textit{Negative} \\ f(0) &= -(0+1)(0-7) \quad = -(1)(-7) \quad = + \quad \textit{Positive} \\ f(10) &= -(10+1)(10-7) = -(11)(3) \quad = - \quad \textit{Negative} \end{aligned}$$

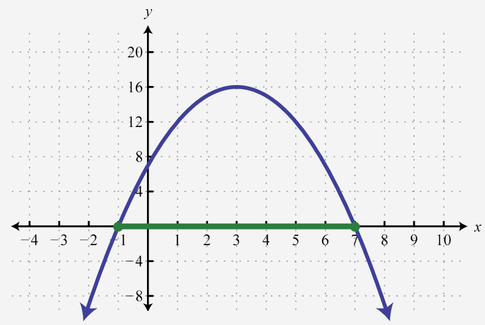
Since the result of evaluating for -3 was negative, we place negative signs above the first region. The result of evaluating for 0 was positive, so we place positive signs above the middle region. Finally, the result of evaluating for 10 was negative, so we place negative signs above the last region, and the sign chart is complete.



Step 3: Use the sign chart to answer the question. In this case, we are asked to determine where $f(x) \geq 0$, or where the function is positive or zero. From the sign chart we see this occurs when *x*-values are inclusively between -1 and 7.



Using interval notation, the shaded region is expressed as $[-1, 7]$. The graph is not required; however, for the sake of completeness it is provided below.



Indeed the function is greater than or equal to zero, above or on the *x*-axis, for *x*-values in the specified interval.

Answer: $\left[-1, 7\right]$

Example 4

Solve: $2x^2 - 7x + 3 > 0$.

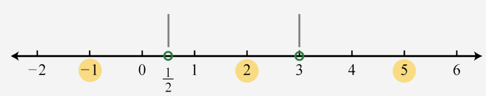
Solution:

Begin by finding the critical numbers, in this case, the roots of $f(x) = 2x^2 - 7x + 3$.

$$\begin{aligned} 2x^2 - 7x + 3 &= 0 \\ (2x - 1)(x - 3) &= 0 \end{aligned}$$

$$\begin{aligned} 2x - 1 &= 0 \quad \text{or} \quad x - 3 = 0 \\ 2x &= 1 \qquad\qquad x = 3 \\ x &= \frac{1}{2} \end{aligned}$$

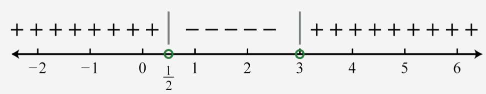
The critical numbers are $\frac{1}{2}$ and 3. Because of the strict inequality > we will use open dots.



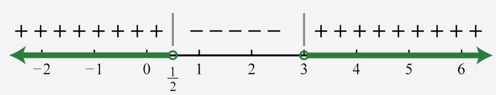
Next choose a test value in each region and determine the sign after evaluating $f(x) = 2x^2 - 7x + 3 = (2x - 1)(x - 3)$. Here we choose test values −1, 2, and 5.

$$\begin{aligned} f(-1) &= [2(-1)-1]\ (-1-3) = (-)\ (-) = + \\ f(2) &= [2(2)-1]\ (2-3) \quad = (+)\ (-) = - \\ f(5) &= [2(5)-1]\ (5-3) \quad = (+)\ (+) = + \end{aligned}$$

And we can complete the sign chart.



The question asks us to find the *x*-values that produce positive results (greater than zero). Therefore, shade in the regions with a + over them. This is the solution set.



Answer: $\left(-\infty, \frac{1}{2}\right) \cup (3, \infty)$

Sometimes the quadratic function does not factor. In this case we can make use of the quadratic formula.

Example 5

Solve: $x^2 - 2x - 11 \leq 0$.

Solution:

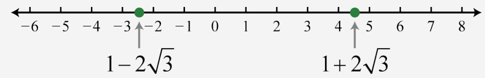
Find the critical numbers.

$$x^2 - 2x - 11 = 0$$

Identify *a*, *b*, and *c* for use in the quadratic formula. Here $a = 1$, $b = -2$, and $c = -11$. Substitute the appropriate values into the quadratic formula and then simplify.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-11)}}{2(1)} \\ &= \frac{2 \pm \sqrt{48}}{2} \\ &= \frac{2 \pm 4\sqrt{3}}{2} \\ &= 1 \pm 2\sqrt{3} \end{aligned}$$

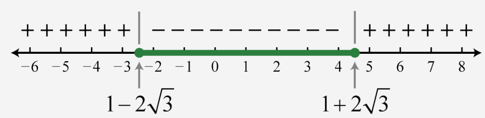
Therefore the critical numbers are $1 - 2\sqrt{3} \approx -2.5$ and $1 + 2\sqrt{3} \approx 4.5$. Use a closed dot on the number to indicate that these values will be included in the solution set.



Here we will use test values -5, 0, and 7.

$$\begin{aligned} f\left(-5\right) &= \left(-5\right)^2 - 2\left(-5\right) - 11 = 25 + 10 - 11 = + \\ f\left(0\right) &= \left(0\right)^2 - 2\left(0\right) - 11 \qquad = 0 + 0 - 11 \quad = - \\ f\left(7\right) &= \left(7\right)^2 - 2\left(7\right) - 11 \qquad = 49 - 14 - 11 = + \end{aligned}$$

After completing the sign chart shade in the values where the function is negative as indicated by the question ($f\left(x\right) \leq 0$).



Answer: $\left[1 - 2\sqrt{3}, 1 + 2\sqrt{3}\right]$

Try this! Solve: $9 - x^2 > 0$.

Answer: $(-3, 3)$

(click to see video)

It may be the case that there are no critical numbers.

Example 6

Solve: $x^2 - 2x + 3 > 0$.

Solution:

To find the critical numbers solve,

$$x^2 - 2x + 3 = 0$$

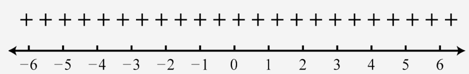
Substitute $a = 1$, $b = -2$, and $c = 3$ into the quadratic formula and then simplify.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(3)}}{2(1)} \\ &= \frac{2 \pm \sqrt{-8}}{2} \\ &= \frac{2 \pm 2i\sqrt{2}}{2} \\ &= 1 + i\sqrt{2} \end{aligned}$$

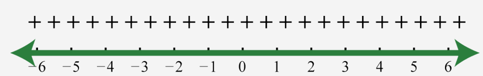
Because the solutions are not real, we conclude there are no real roots; hence there are no critical numbers. When this is the case, the graph has no *x*-intercepts and is completely above or below the *x*-axis. We can test any value to create a sign chart. Here we choose $x = 0$.

$$f(0) = (0)^2 - 2(0) + 3 = +$$

Because the test value produced a positive result the sign chart looks as follows:

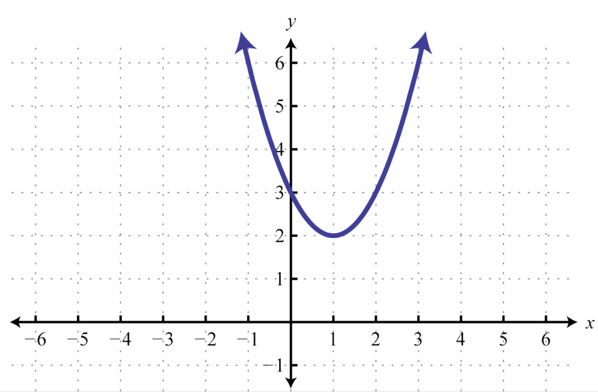


We are looking for the values where $f(x) > 0$; the sign chart implies that any real number for *x* will satisfy this condition.



Answer: $(-\infty, \infty)$

The function in the previous example is graphed below.



We can see that it has no *x*-intercepts and is always above the *x*-axis (positive). If the question was to solve $x^2 - 2x + 3 < 0$, then the answer would have been no solution. The function is never negative.

Try this! Solve: $9x^2 - 12x + 4 \leq 0.$

Answer: One solution, $\frac{2}{3}$.

(click to see video)

Example 7

Find the domain: $f(x) = \sqrt{x^2 - 4}$.

Solution:

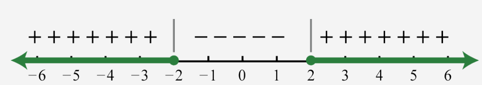
Recall that the argument of a square root function must be nonnegative. Therefore, the domain consists of all real numbers for x such that $x^2 - 4$ is greater than or equal to zero.

$$x^2 - 4 \geq 0$$

It should be clear that $x^2 - 4 = 0$ has two solutions $x = \pm 2$; these are the critical values. Choose test values in each interval and evaluate $f(x) = x^2 - 4$.

$$\begin{aligned} f(-3) &= (-3)^2 - 4 = 9 - 4 = + \\ f(0) &= (0)^2 - 4 \quad = 0 - 4 = - \\ f(3) &= (3)^2 - 4 \quad = 9 - 4 = + \end{aligned}$$

Shade in the x-values that produce positive results.



Answer: Domain: $\left(-\infty, -2\right] \cup \left[2, \infty\right)$

KEY TAKEAWAYS

- Quadratic inequalities can have infinitely many solutions, one solution or no solution.
- We can solve quadratic inequalities graphically by first rewriting the inequality in standard form, with zero on one side. Graph the quadratic function and determine where it is above or below the *x*-axis. If the inequality involves "less than," then determine the *x*-values where the function is below the *x*-axis. If the inequality involves "greater than," then determine the *x*-values where the function is above the *x*-axis.
- We can streamline the process of solving quadratic inequalities by making use of a sign chart. A sign chart gives us a visual reference that indicates where the function is above the *x*-axis using positive signs or below the *x*-axis using negative signs. Shade in the appropriate *x*-values depending on the original inequality.
- To make a sign chart, use the function and test values in each region bounded by the roots. We are only concerned if the function is positive or negative and thus a complete calculation is not necessary.

TOPIC EXERCISES

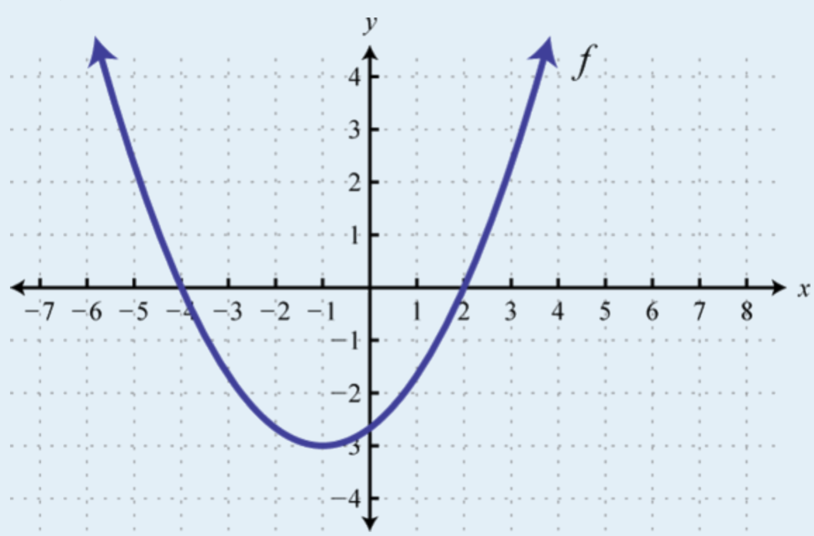
PART A: SOLUTIONS TO QUADRATIC INEQUALITIES

Determine whether or not the given value is a solution.

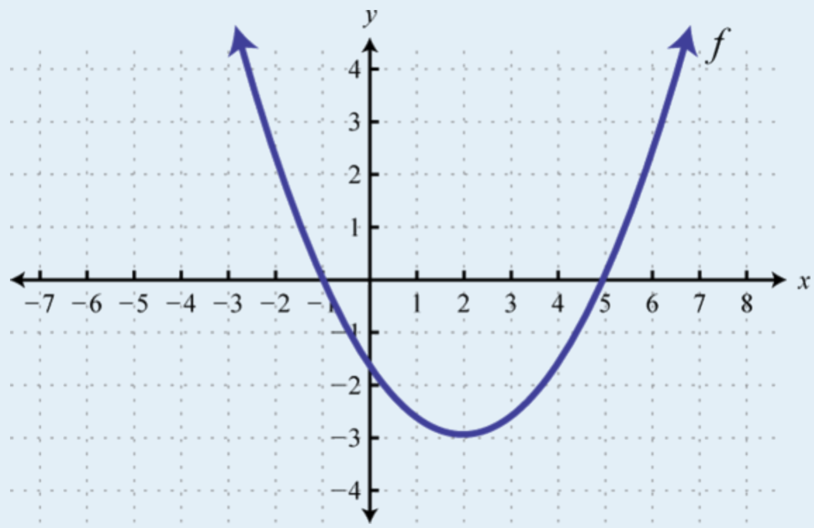
1. $x^2 - x + 1 < 0; x = -1$
2. $x^2 + x - 1 > 0; x = -2$
3. $4x^2 - 12x + 9 \leq 0; x = \frac{3}{2}$
4. $5x^2 - 8x - 4 < 0; x = -\frac{2}{5}$
5. $3x^2 - x - 2 \geq 0; x = 0$
6. $4x^2 - x + 3 \leq 0; x = -1$
7. $2 - 4x - x^2 < 0; x = \frac{1}{2}$
8. $5 - 2x - x^2 > 0; x = 0$
9. $-x^2 - x - 9 < 0; x = -3$
10. $-x^2 + x - 6 \geq 0; x = 6$

Given the graph of *f* determine the solution set.

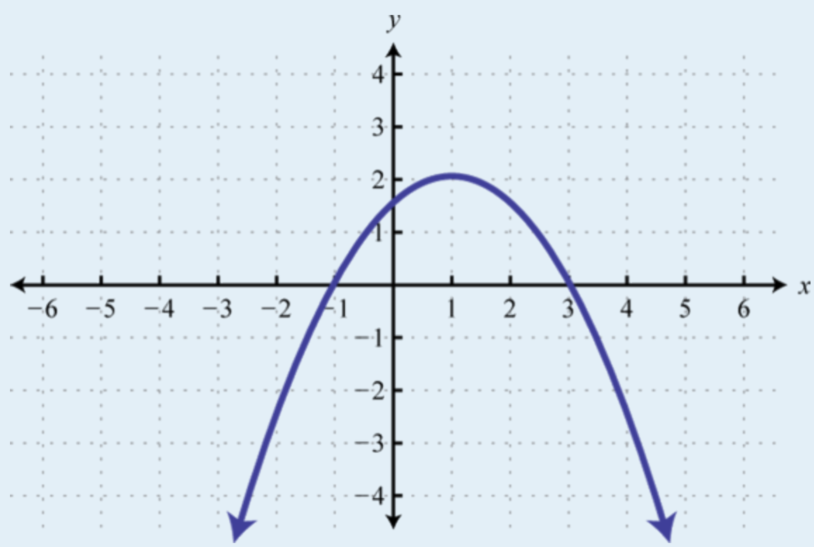
11. $f(x) \leq 0;$



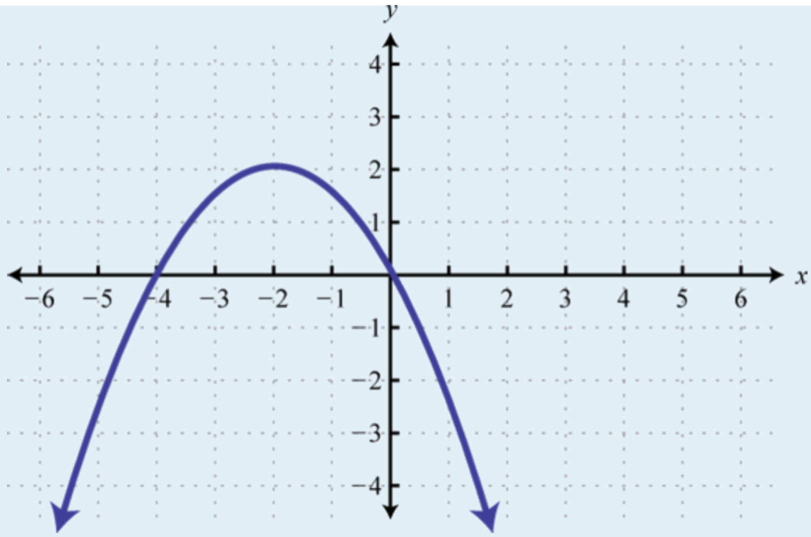
12. $f(x) \geq 0$;



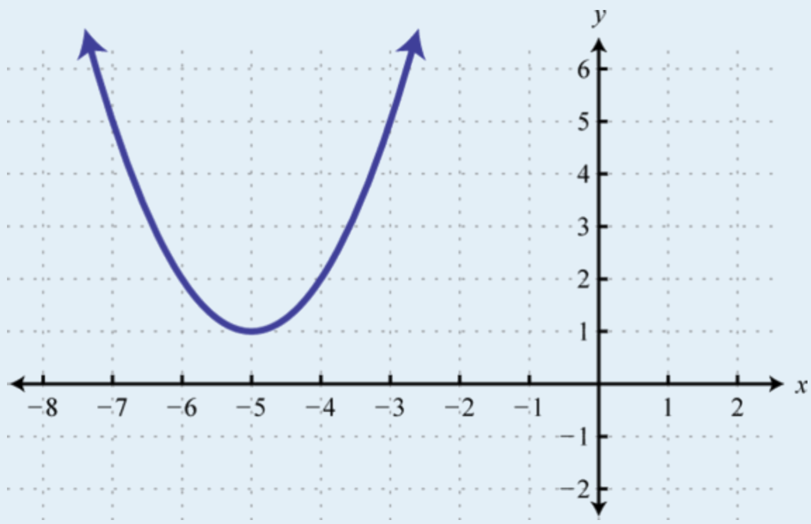
13. $f(x) \geq 0$;



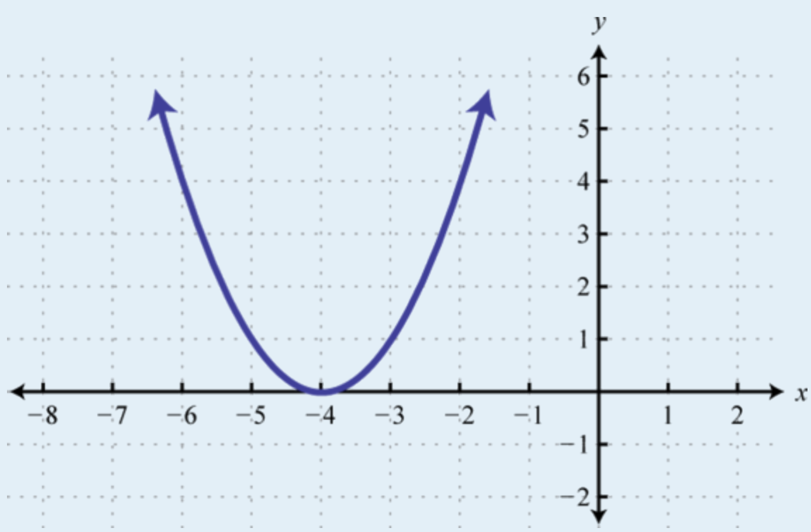
14. $f(x) \leq 0$;


y
x
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6
4 3 2 1 −1 −2 −3 −4

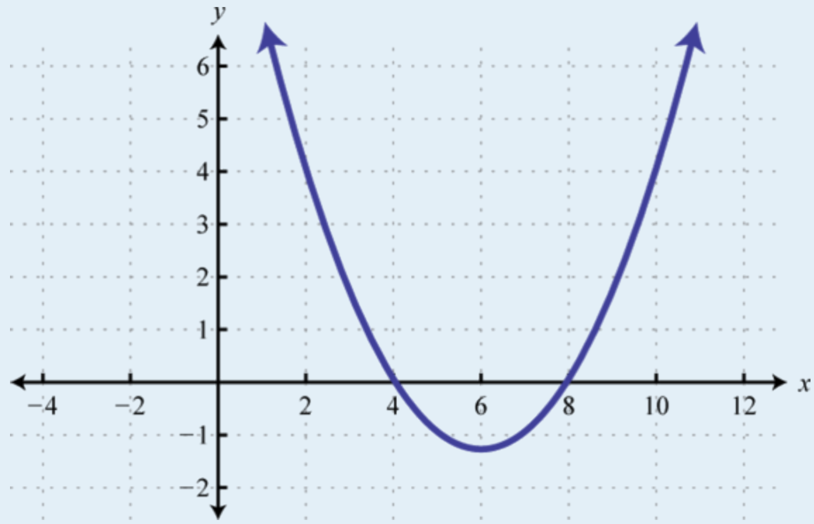
15. $f(x) > 0$;


y
x
−8 −7 −6 −5 −4 −3 −2 −1 1 2
6 5 4 3 2 1 −1 −2

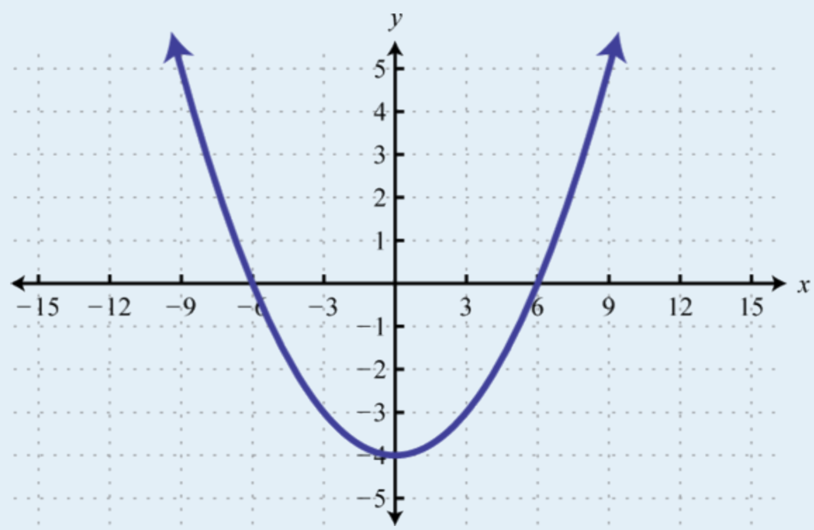
16. $f(x) < 0$;


y
x
−8 −7 −6 −5 −4 −3 −2 −1 1 2
6 5 4 3 2 1 −1 −2

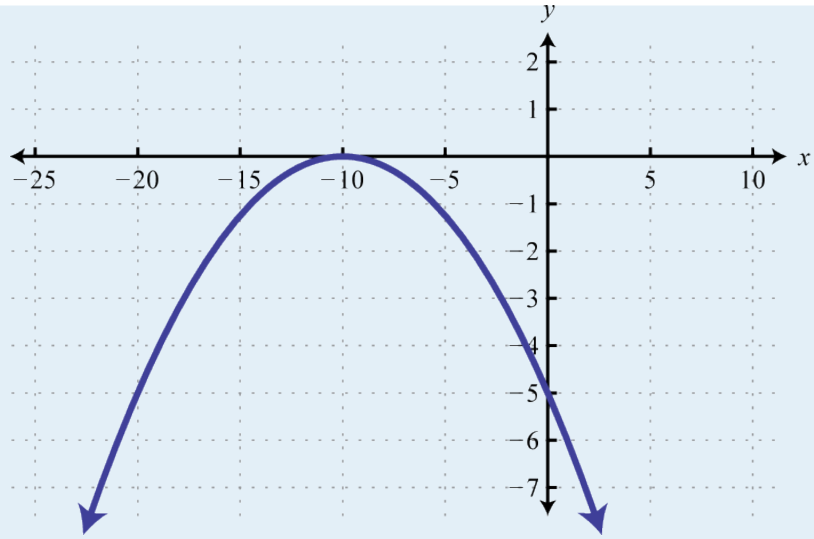
17. $f(x) > 0$;



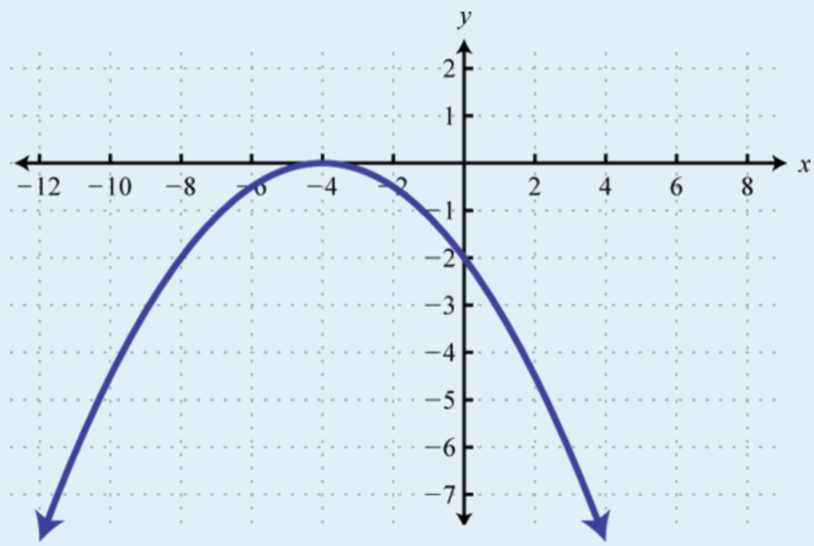
18. $f(x) < 0$;



19. $f(x) \geq 0$;



20. $f(x) < 0$;



Use the transformations to graph the following and then determine the solution set.

21. $x^2 - 1 > 0$

22. $x^2 + 2 > 0$

23. $(x - 1)^2 > 0$

24. $(x + 2)^2 \leq 0$

25. $(x + 2)^2 - 1 \leq 0$

26. $(x + 3)^2 - 4 > 0$

27. $-x^2 + 4 \geq 0$
28. $-(x + 2)^2 > 0$
29. $-(x + 3)^2 + 1 < 0$
30. $-(x - 4)^2 + 9 > 0$

PART B: SOLVING QUADRATIC INEQUALITIES

Use a sign chart to solve and graph the solution set. Present answers using interval notation.

31. $x^2 - x - 12 > 0$
32. $x^2 - 10x + 16 > 0$
33. $x^2 + 2x - 24 < 0$
34. $x^2 + 15x + 54 < 0$
35. $x^2 - 23x - 24 \leq 0$
36. $x^2 - 12x + 20 \leq 0$
37. $2x^2 - 11x - 6 \geq 0$
38. $3x^2 + 17x - 6 \geq 0$
39. $8x^2 - 18x - 5 < 0$
40. $10x^2 + 17x + 6 > 0$
41. $9x^2 + 30x + 25 \leq 0$
42. $16x^2 - 40x + 25 \leq 0$
43. $4x^2 - 4x + 1 > 0$
44. $9x^2 + 12x + 4 > 0$
45. $-x^2 - x + 30 \geq 0$
46. $-x^2 - 6x + 27 \leq 0$
47. $x^2 - 64 < 0$

48. $x^2 - 81 \geq 0$

49. $4x^2 - 9 \geq 0$

50. $16x^2 - 25 < 0$

51. $25 - 4x^2 \geq 0$

52. $1 - 49x^2 < 0$

53. $x^2 - 8 > 0$

54. $x^2 - 75 \leq 0$

55. $2x^2 + 1 > 0$

56. $4x^2 + 3 < 0$

57. $x - x^2 > 0$

58. $3x - x^2 \leq 0$

59. $x^2 - x + 1 < 0$

60. $x^2 + x - 1 > 0$

61. $4x^2 - 12x + 9 \leq 0$

62. $5x^2 - 8x - 4 < 0$

63. $3x^2 - x - 2 \geq 0$

64. $4x^2 - x + 3 \leq 0$

65. $2 - 4x - x^2 < 0$

66. $5 - 2x - x^2 > 0$

67. $-x^2 - x - 9 < 0$

68. $-x^2 + x - 6 \geq 0$

69. $-2x^2 + 4x - 1 \geq 0$

70. $-3x^2 - x + 1 \leq 0$

Find the domain of the function.

71. $f(x) = \sqrt{x^2 - 25}$

72. $f(x) = \sqrt{x^2 + 3x}$

73. $g(x) = \sqrt{3x^2 - x - 2}$

74. $g(x) = \sqrt{12x^2 - 9x - 3}$

75. $h(x) = \sqrt{16 - x^2}$

76. $h(x) = \sqrt{3 - 2x - x^2}$

77. $f(x) = \sqrt{x^2 + 10}$

78. $f(x) = \sqrt{9 + x^2}$

79. A robotics manufacturing company has determined that its weekly profit in thousands of dollars is modeled by $P(n) = -n^2 + 30n - 200$ where n represents the number of units it produces and sells. How many units must the company produce and sell to maintain profitability. (Hint: Profitability occurs when profit is greater than zero.)

80. The height in feet of a projectile shot straight into the air is given by $h(t) = -16t^2 + 400t$ where t represents the time in seconds after it is fired. In what time intervals is the projectile under 1,000 feet? Round to the nearest tenth of a second.

PART C: DISCUSSION BOARD

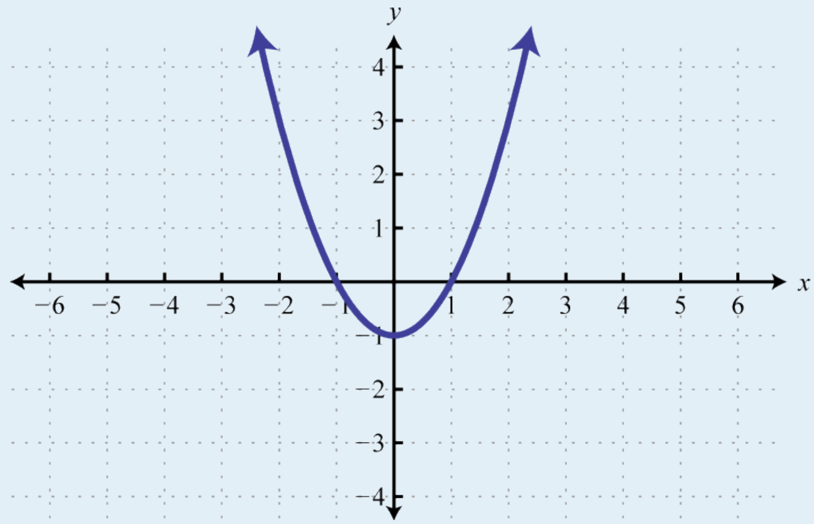
81. Does the sign chart for any given quadratic function always alternate? Explain and illustrate your answer with some examples.

82. Research and discuss other methods for solving a quadratic inequality.

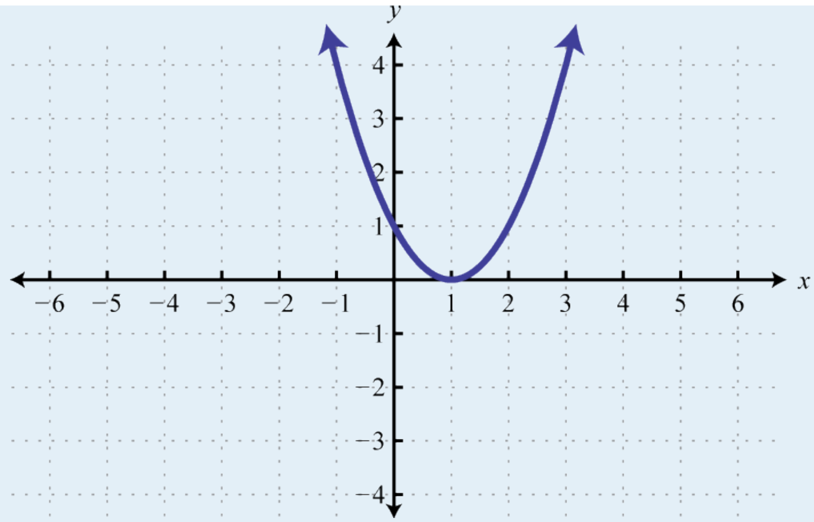
83. Explain the difference between a quadratic equation and a quadratic inequality. How can we identify and solve each? What is the geometric interpretation of each?

ANSWERS

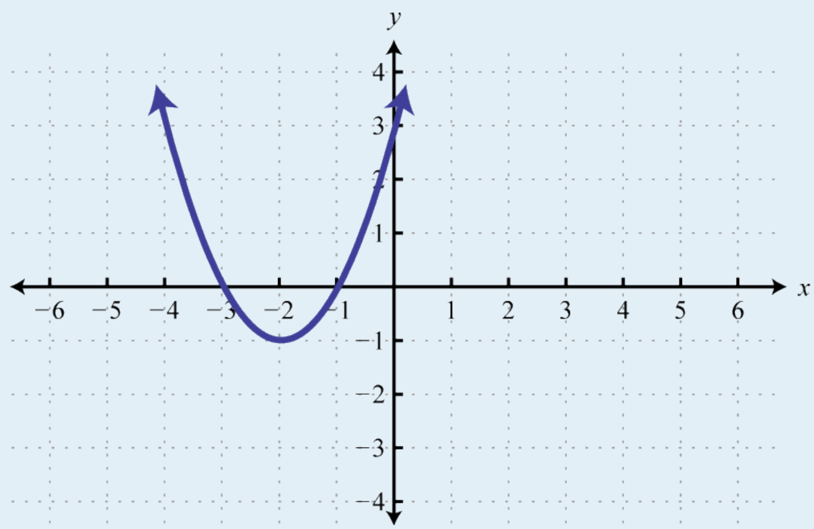
1. No
3. Yes
5. No
7. Yes
9. Yes
11. $[-4, 2]$
13. $[-1, 3]$
15. $(-\infty, \infty)$
17. $(-\infty, 4) \cup (8, \infty)$
19. $\{-10\}$



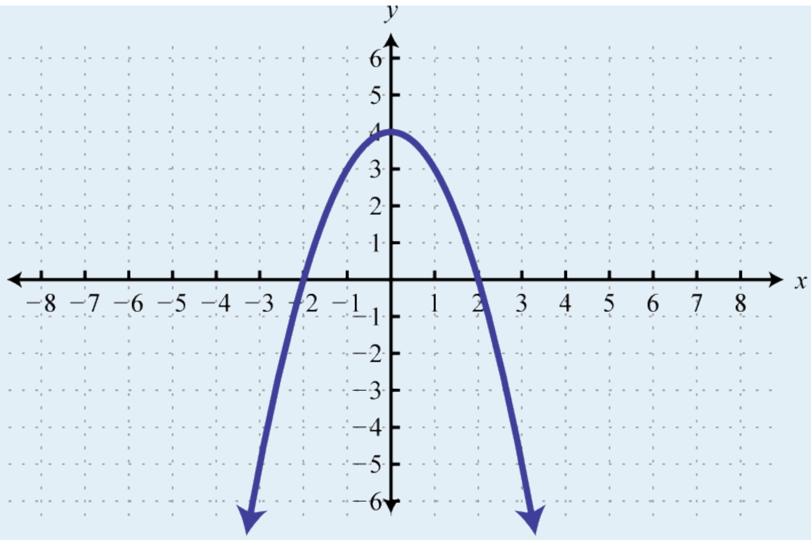
21. $(-\infty, -1) \cup (1, \infty)$



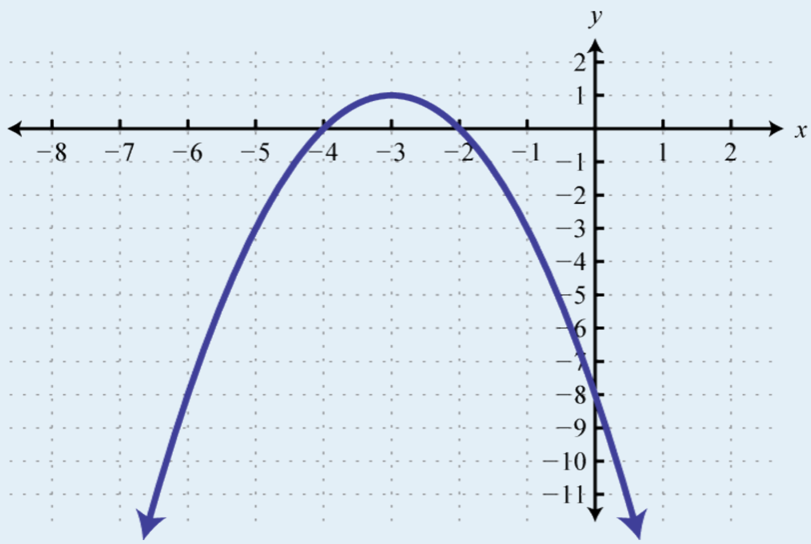
23. $(-\infty, 1) \cup (1, \infty)$



25. $[-3, -1]$



27. $[-2, 2]$



29. $(-\infty, -4) \cup (-2, \infty)$

31. $(-\infty, -3) \cup (4, \infty)$

33. $(-6, 4)$

35. $[-1, 24]$

37. $\left(-\infty, -\frac{1}{2}\right] \cup [6, \infty)$

39. $\left(-\frac{1}{4}, \frac{5}{2}\right)$

41. $-\frac{5}{3}$

43. $\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$

45. $[-6, 5]$

47. $(-8, 8)$

49. $\left(-\infty, -\frac{3}{2}\right] \cup \left[\frac{3}{2}, \infty\right)$

51. $\left[-\frac{5}{2}, \frac{5}{2}\right]$

53. $\left(-\infty, -2\sqrt{2}\right) \cup \left(2\sqrt{2}, \infty\right)$

55. $(-\infty, \infty)$

57. $(0, 1)$

59. Ø

61. $\frac{3}{2}$

63. $\left(-\infty, -\frac{2}{3}\right] \cup [1, \infty)$

65. $\left(-\infty, -2 - \sqrt{6}\right) \cup \left(-2 + \sqrt{6}, \infty\right)$

67. $(-\infty, \infty)$

69. $\left[\frac{2 - \sqrt{2}}{2}, \frac{2 + \sqrt{2}}{2}\right]$

71. $(-\infty, -5] \cup [5, \infty)$

73. $\left(-\infty, -\frac{2}{3}\right] \cup [1, \infty)$

75. $[-4, 4]$

77. $(-\infty, \infty)$

79. The company must produce and sell more than 10 units and fewer than 20 units each week.

81. Answer may vary

83. Answer may vary

6.6 Solving Polynomial and Rational Inequalities

LEARNING OBJECTIVES

1. Solve polynomial inequalities.
2. Solve rational inequalities.

Solving Polynomial Inequalities

A **polynomial inequality**[18] is a mathematical statement that relates a polynomial expression as either less than or greater than another. We can use sign charts to solve polynomial inequalities with one variable.

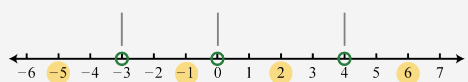
18. A mathematical statement that relates a polynomial expression as either less than or greater than another.

Example 1

Solve: $x(x+3)^2(x-4) < 0$.

Solution:

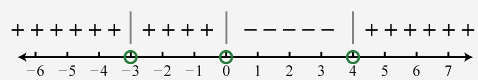
Begin by finding the critical numbers. For a polynomial inequality in standard form, with zero on one side, the critical numbers are the roots. Because $f(x) = x(x+3)^2(x-4)$ is given in its factored form the roots are apparent. Here the roots are: 0, −3, and 4. Because of the strict inequality, plot them using open dots on a number line.



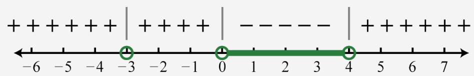
In this case, the critical numbers partition the number line into four regions. Test values in each region to determine if *f* is positive or negative. Here we choose test values −5, −1, 2, and 6. Remember that we are only concerned with the sign (+ or −) of the result.

$$
\begin{aligned}
f(-5) &= (-5)(-5+3)^2(-5-4) = (-)(-)^2(-) = + \quad \textit{Positive} \\
f(-1) &= (-1)(-1+3)^2(-1-4) = (-)(+)^2(-) = + \quad \textit{Positive} \\
f(2) &= (2)(2+3)^2(2-4) = (+)(+)^2(-) = - \quad \textit{Negative} \\
f(6) &= (6)(6+3)^2(6-4) = (+)(+)^2(+) = + \quad \textit{Positive}
\end{aligned}
$$

After testing values we can complete a sign chart.



The question asks us to find the values where $f\left(x\right) < 0$, or where the function is negative. From the sign chart we can see that the function is negative for x-values in between 0 and 4.



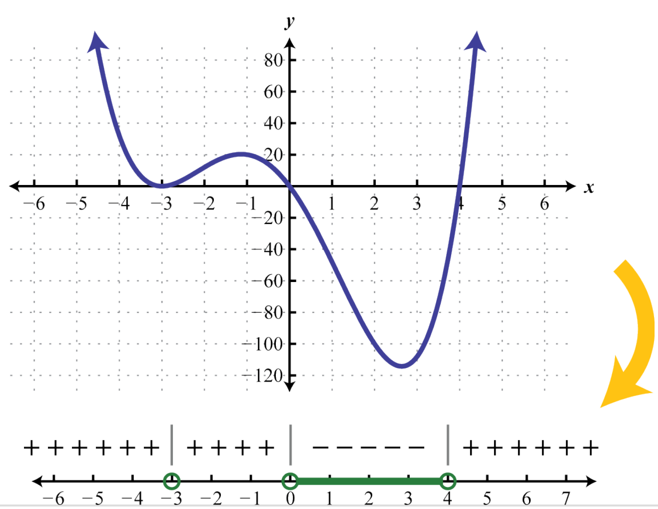
We can express this solution set in two ways:

$$\begin{array}{ll} \{x|0 < x < 4\} & \textit{Set notation} \\ (0, 4) & \textit{Interval notation} \end{array}$$

In this textbook we will continue to present solution sets using interval notation.

Answer: $(0, 4)$

Graphing polynomials such as the one in the previous example is beyond the scope of this textbook. However, the graph of this function is provided below. Compare the graph to its corresponding sign chart.



Certainly it may not be the case that the polynomial is factored nor that it has zero on one side of the inequality. To model a function using a sign chart, all of the terms should be on one side and zero on the other. The general steps for solving a polynomial inequality are listed in the following example.

Example 2

Solve: $2x^4 > 3x^3 + 9x^2$.

Solution:

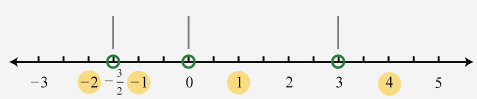
Step 1: Obtain zero on one side of the inequality. In this case, subtract to obtain a polynomial on the left side in standard from.

$$\begin{aligned} 2x^4 &> 3x^3 + 9x^2 \\ 2x^4 - 3x^3 - 9x^2 &> 0 \end{aligned}$$

Step 2: Find the critical numbers. Here we can find the zeros by factoring.

$$\begin{aligned} 2x^4 - 3x^3 - 9x^2 &= 0 \\ x^2 \left(2x^2 - 3x - 9\right) &= 0 \\ x^2 (2x + 3)(x - 3) &= 0 \end{aligned}$$

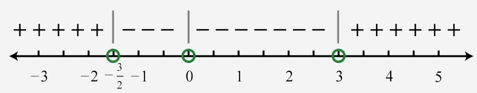
There are three solutions, hence, three critical numbers $-\frac{3}{2}$, 0, and 3.The strict inequality indicates that we should use open dots.



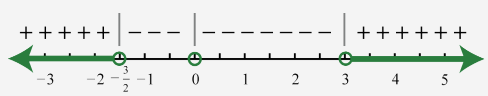
Step 3: Create a sign chart. In this case use $f(x) = x^2 (2x + 3)(x - 3)$ and test values -2, -1, 1, and 4 to determine the sign of the function in each interval.

$$\begin{aligned}
f(-2) &= (-2)^2 \left[2(-2)+3\right](-2-3) = (-)^2(-)(-) = + \\
f(-1) &= (-1)^2 \left[2(-1)+3\right](-1-3) = (-)^2(+)(-) = - \\
f(1) &= (1)^2 \left[2(1)+3\right](1-3) \quad = (+)^2(+)(-) = - \\
f(4) &= (4)^2 \left[2(4)+3\right](4-3) \quad = (+)^2(+)(+) = +
\end{aligned}$$

With this information we can complete the sign chart.



Step 4: Use the sign chart to answer the question. Here the solution consists of all values for which $f(x) > 0$. Shade in the values that produce positive results and then express this set in interval notation.



Answer: $\left(-\infty, -\frac{3}{2}\right) \cup \left(3, \infty\right)$

Example 3

Solve: $x^3 + x^2 \leq 4(x+1)$.

Solution:

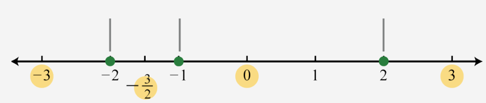
Begin by rewriting the inequality in standard form, with zero on one side.

$$\begin{aligned} x^3 + x^2 &\leq 4(x+1) \\ x^3 + x^2 &\leq 4x + 4 \\ x^3 + x^2 - 4x - 4 &\leq 0 \end{aligned}$$

Next find the critical numbers of $f(x) = x^3 + x^2 - 4x - 4$:

$$\begin{aligned} x^3 + x^2 - 4x - 4 &= 0 \quad \textit{Factor by grouping.} \\ x^2(x+1) - 4(x+1) &= 0 \\ (x+1)\left(x^2 - 4\right) &= 0 \\ (x+1)(x+2)(x-2) &= 0 \end{aligned}$$

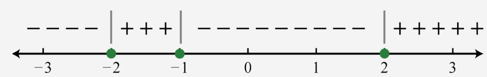
The critical numbers are −2, −1, and 2. Because of the inclusive inequality ($\leq$) we will plot them using closed dots.



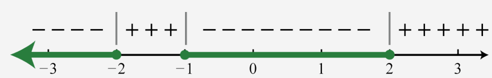
Use test values −3, $-\frac{3}{2}$, 0, and 3 to create a sign chart.

$$
\begin{aligned}
f(-3) &= (-3+1)(-3+2)(-3-2) &&= (-)(-)(-) = -\\
f\left(-\frac{3}{2}\right) &= \left(-\frac{3}{2}+1\right)\left(-\frac{3}{2}+2\right)\left(-\frac{3}{2}-2\right) &&= (-)(+)(-) = +\\
f(0) &= (0+1)(0+2)(0-2) &&= (+)(+)(-) = -\\
f(3) &= (3+1)(3+2)(3-2) &&= (+)(+)(+) = +
\end{aligned}
$$

And we have



Use the sign chart to shade in the values that have negative results ($f(x) \le 0$).



Answer: $(-\infty, -2] \cup [-1, 2]$

Try this! Solve: $-3x^4 + 12x^3 - 9x^2 > 0$.

Answer: $(1, 3)$

(click to see video)

Solving Rational Inequalities

A **rational inequality**[19] is a mathematical statement that relates a rational expression as either less than or greater than another. Because rational functions have restrictions to the domain we must take care when solving rational inequalities. In addition to the zeros, we will include the restrictions to the domain of the function in the set of critical numbers.

19. A mathematical statement that relates a rational expression as either less than or greater than another.

Example 4

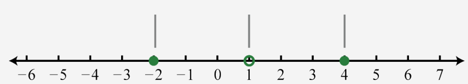
Solve: $\frac{(x-4)(x+2)}{(x-1)} \geq 0.$

Solution:

The zeros of a rational function occur when the numerator is zero and the values that produce zero in the denominator are the restrictions. In this case,

$$\begin{array}{cc|c} \textbf{\textit{Roots}} & \textbf{(\textit{Numerator})} & \textbf{\textit{Restriction}} \ \ \textbf{(\textit{Denominator})} \\ x-4=0 \ \mathit{or} & x+2=0 & x-1=0 \\ x=4 & x=-2 & x=1 \end{array}$$

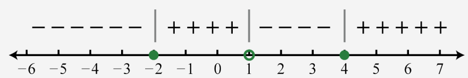
Therefore the critical numbers are −2, 1, and 4. Because of the inclusive inequality (≥) use a closed dot for the roots {−2, 4} and always use an open dot for restrictions {1}. Restrictions are never included in the solution set.



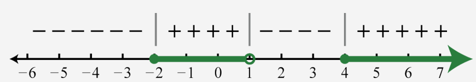
Use test values $x = -4, 0, 2, 6.$

$$\begin{aligned} f(-4) &= \frac{(-4-4)\ (-4+2)}{(-4-1)} = \frac{(-)\ (-)}{(-)} = - \\ f(0) &= \frac{(0-4)\ (0+2)}{(0-1)} = \frac{(-)\ (+)}{(-)} = + \\ f(2) &= \frac{(2-4)\ (2+2)}{(2-1)} = \frac{(-)\ (+)}{(+)} = - \\ f(6) &= \frac{(6-4)\ (6+2)}{(6-1)} = \frac{(+)\ (+)}{(+)} = + \end{aligned}$$

And then complete the sign chart.

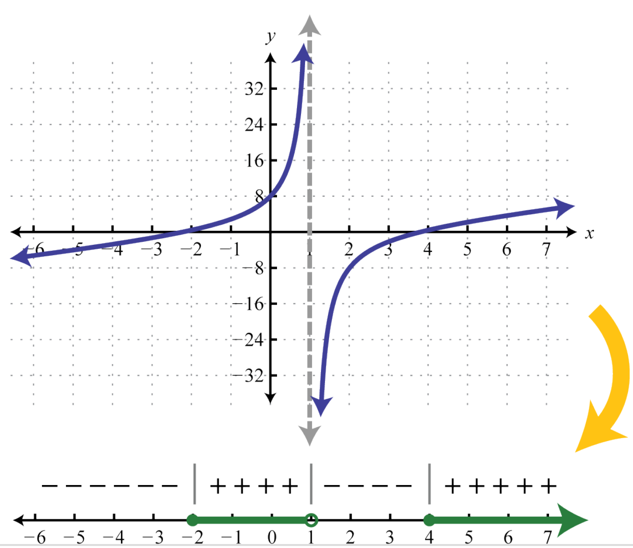


The question asks us to find the values for which $f(x) \geq 0$, in other words, positive or zero. Shade in the appropriate regions and present the solution set in interval notation.



Answer: $[-2, 1) \cup [4, \infty)$

Graphing such rational functions like the one in the previous example is beyond the scope of this textbook. However, the graph of this function is provided below. Compare the graph to its corresponding sign chart.



Notice that the restriction $x = 1$ corresponds to a vertical asymptote which bounds regions where the function changes from positive to negative. While not included in the solution set, the restriction is a critical number. Before creating a sign chart we must ensure the inequality has a zero on one side. The general steps for solving a rational inequality are outlined in the following example.

Example 5

Solve: $\frac{7}{x+3} < 2$.

Solution:

Step 1: Begin by obtaining zero on the right side.

$$\frac{7}{x+3} < 2$$
$$\frac{7}{x+3} - 2 < 0$$

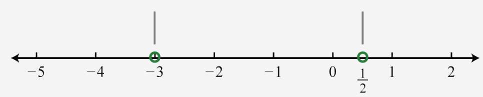
Step 2: Determine the critical numbers. The critical numbers are the zeros and restrictions. Begin by simplifying to a single algebraic fraction.

$$\frac{7}{x+3} - \frac{2}{1} < 0$$
$$\frac{7 - 2(x+3)}{x+3} < 0$$
$$\frac{7 - 2x - 6}{x+3} < 0$$
$$\frac{-2x + 1}{x+3} < 0$$

Next find the critical numbers. Set the numerator and denominator equal to zero and solve.

Root	***Restriction***
$$\begin{aligned} -2x+1&=0\\ -2x&=-1\\ x&=\frac{1}{2}\end{aligned}$$	$$\begin{aligned} x+3&=0\\ x&=-3\end{aligned}$$

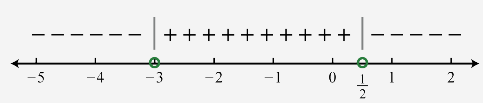
In this case, the strict inequality indicates that we should use an open dot for the root.



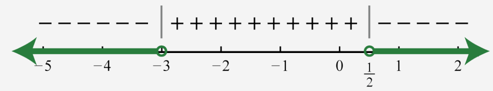
Step 3: Create a sign chart. Choose test values -4, 0, and 1.

$$\begin{aligned} f(-4)&=\frac{-2(-4)+1}{-4+3}=\frac{+}{-}=-\\ f(0)&=\frac{-2(0)+1}{0+3}=\frac{+}{+}=+\\ f(1)&=\frac{-2(1)+1}{1+3}=\frac{-}{+}=-\end{aligned}$$

And we have



Step 4: Use the sign chart to answer the question. In this example we are looking for the values for which the function is negative, $f(x) < 0$. Shade the appropriate values and then present your answer using interval notation.



Answer: $\left(-\infty, -3\right) \cup \left(\frac{1}{2}, \infty\right)$

Example 6

Solve: $\frac{1}{x^2-4} \leq \frac{1}{2-x}$

Solution:

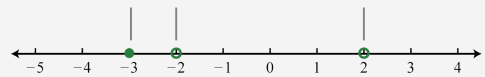
Begin by obtaining zero on the right side.

$$\begin{aligned} \frac{1}{x^2-4} &\leq \frac{1}{2-x} \\ \frac{1}{x^2-4} - \frac{1}{2-x} &\leq 0 \end{aligned}$$

Next simplify the left side to a single algebraic fraction.

$$\begin{aligned} \frac{1}{x^2-4} - \frac{1}{2-x} &\leq 0 \\ \frac{1}{(x+2)\,(x-2)} - \frac{1}{-(x-2)} &\leq 0 \\ \frac{1}{(x+2)\,(x-2)} + \frac{1\,(x+2)}{(x-2)\,(x+2)} &\leq 0 \\ \frac{1+x+2}{(x+2)\,(x-2)} &\leq 0 \\ \frac{x+3}{(x+2)\,(x-2)} &\leq 0 \end{aligned}$$

The critical numbers are −3, −2, and 2. Note that ±2 are restrictions and thus we will use open dots when plotting them on a number line. Because of the inclusive inequality we will use a closed dot at the root −3.



Choose test values -4, $-2\,\frac{1}{2} = -\frac{5}{2}$ 0, and 3.

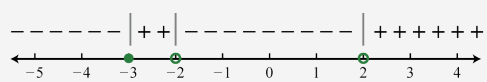
$$f(-4) = \frac{-4+3}{(-4+2)\,(-4-2)} = \frac{(-)}{(-)\,(-)} = -$$

$$f\left(-\frac{5}{2}\right) = \frac{-\frac{5}{2}+3}{\left(-\frac{5}{2}+2\right)\,\left(-\frac{5}{2}-2\right)} = \frac{(+)}{(-)\,(-)} = +$$

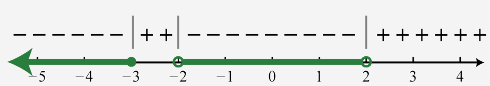
$$f(0) = \frac{0+3}{(0+2)\,(0-2)} = \frac{(+)}{(+)\,(-)} = -$$

$$f(3) = \frac{3+3}{(3+2)\,(3-2)} = \frac{(+)}{(+)\,(+)} = +$$

Construct a sign chart.



Answer the question; in this case, find x where $f(x) \leq 0$.



Answer: $\left(-\infty, -3\right] \cup (-2, 2)$

Try this! Solve: $\frac{2x^2}{2x^2+7x-4} \geq \frac{x}{x+4}$.

Answer: $(-4, 0] \cup \left(\frac{1}{2}, \infty\right)$

(click to see video)

KEY TAKEAWAYS

- When a polynomial inequality is in standard form, with zero on one side, the roots of the polynomial are the critical numbers. Create a sign chart that models the function and then use it to answer the question.
- When a rational inequality is written as a single algebraic fraction, with zero on one side, the roots as well as the restrictions are the critical numbers. The values that produce zero in the numerator are the roots, and the values that produce zero in the denominator are the restrictions. Always use open dots for restrictions, regardless of the given inequality, because restrictions are not part of the domain. Create a sign chart that models the function and then use it to answer the question.

TOPIC EXERCISES

PART A: SOLVING POLYNOMIAL INEQUALITIES

Solve. Present answers using interval notation.

1. $x(x+1)(x-3) > 0$
2. $x(x-1)(x+4) < 0$
3. $(x+2)\left(x-5\right)^2 < 0$
4. $(x-4)(x+1)^2 \geq 0$
5. $(2x-1)(x+3)(x+2) \leq 0$
6. $(3x+2)(x-4)\left(x-5\right) \geq 0$
7. $x(x+2)\left(x-5\right)^2 < 0$
8. $x\left(2x-5\right)(x-1)^2 > 0$
9. $x(4x+3)(x-1)^2 \geq 0$
10. $(x-1)(x+1)(x-4)^2 < 0$
11. $\left(x+5\right)(x-10)\left(x-5\right)^2 \geq 0$
12. $(3x-1)(x-2)(x+2)^2 \leq 0$
13. $-4x(4x+9)(x-8)^2 > 0$
14. $-x(x-10)(x+7)^2 > 0$

Solve.

15. $x^3 + 2x^2 - 24x \geq 0$
16. $x^3 - 3x^2 - 18x \leq 0$
17. $4x^3 - 22x^2 - 12x < 0$
18. $9x^3 + 30x^2 - 24x > 0$

19. $12x^4 + 44x^3 > 80x^2$
20. $6x^4 + 12x^3 < 48x^2$
21. $x\left(x^2 + 25\right) < 10x^2$
22. $x^3 > 12x\,(x - 3)$
23. $x^4 - 5x^2 + 4 \leq 0$
24. $x^4 - 13x^2 + 36 \geq 0$
25. $x^4 > 3x^2 + 4$
26. $4x^4 < 3 - 11x^2$
27. $9x^3 - 3x^2 - 81x + 27 \leq 0$
28. $2x^3 + x^2 - 50x - 25 \geq 0$
29. $x^3 - 3x^2 + 9x - 27 > 0$
30. $3x^3 + 5x^2 + 12x + 20 < 0$

PART B: SOLVING RATIONAL INEQUALITIES

Solve.

31. $\frac{x}{x - 3} > 0$
32. $\frac{x - 5}{x} > 0$
33. $\frac{(x - 3)\,(x + 1)}{x} < 0$
34. $\frac{\left(x + 5\right)\,(x + 4)}{(x - 2)} < 0$
35. $\frac{(2x + 1)\,\left(x + 5\right)}{(x - 3)\,\left(x - 5\right)} \leq 0$
36. $\frac{(3x - 1)\,\left(x + 6\right)}{(x - 1)\,(x + 9)} \geq 0$
37. $\frac{(x - 8)\,(x + 8)}{-2x\,(x - 2)} \geq 0$

38. $\frac{(2x+7)\ (x+4)}{x\left(x+5\right)} \leq 0$

39. $\frac{x^2}{(2x+3)\ (2x-3)} \leq 0$

40. $\frac{(x-4)^2}{-x\,(x+1)} > 0$

41. $\frac{-5x(x-2)^2}{\left(x+5\right)\ \left(x-6\right)} \geq 0$

42. $\frac{(3x-4)\ \left(x+5\right)}{x(x-4)^2} \geq 0$

43. $\frac{1}{\left(x-5\right)^4} > 0$

44. $\frac{1}{\left(x-5\right)^4} < 0$

Solve.

45. $\frac{x^2-11x-12}{x+4} < 0$

46. $\frac{x^2-10x+24}{x-2} > 0$

47. $\frac{x^2+x-30}{2x+1} \geq 0$

48. $\frac{2x^2+x-3}{x-3} \leq 0$

49. $\frac{3x^2-4x+1}{x^2-9} \leq 0$

50. $\frac{x^2-16}{2x^2-3x-2} \geq 0$

51. $\frac{x^2-12x+20}{x^2-10x+25} > 0$

52. $\frac{x^2+15x+36}{x^2-8x+16} < 0$

53. $\frac{8x^2-2x-1}{2x^2-3x-14} \leq 0$

54. $\dfrac{4x^2 - 4x - 15}{x^2 + 4x - 5} \geq 0$
55. $\dfrac{1}{x + 5} + \dfrac{5}{x - 1} > 0$
56. $\dfrac{5}{x + 4} - \dfrac{1}{x - 4} < 0$
57. $\dfrac{1}{x + 7} > 1$
58. $\dfrac{1}{x - 1} < -5$
59. $x \geq \dfrac{30}{x - 1}$
60. $x \leq \dfrac{1 - 2x}{x - 2}$
61. $\dfrac{1}{x - 1} \leq \dfrac{2}{x}$
62. $\dfrac{3}{x + 1} > -\dfrac{1}{x}$
63. $\dfrac{4}{x - 3} \leq \dfrac{1}{x + 3}$
64. $\dfrac{2x - 9}{x} + \dfrac{49}{x - 8} < 0$
65. $\dfrac{x}{2(x + 2)} - \dfrac{1}{x + 2} \leq \dfrac{12}{x(x + 2)}$
66. $\dfrac{1}{2x + 1} - \dfrac{9}{2x - 1} > 2$
67. $\dfrac{3x}{x^2 - 4} - \dfrac{2}{x - 2} < 0$
68. $\dfrac{x}{2x + 1} + \dfrac{4}{2x^2 - 7x - 4} < 0$
69. $\dfrac{x + 1}{2x^2 + 5x - 3} \geq \dfrac{x}{4x^2 - 1}$
70. $\dfrac{x^2 - 14}{2x^2 - 7x - 4} \leq \dfrac{5}{1 + 2x}$

PART C: DISCUSSION BOARD

71. Does the sign chart for any given polynomial or rational function always alternate? Explain and illustrate your answer with some examples.

72. Write down your own steps for solving a rational inequality and illustrate them with an example. Do your steps also work for a polynomial inequality? Explain.

ANSWERS

1. $(-1, 0) \cup (3, \infty)$
3. $(-\infty, -2)$
5. $(-\infty, -3] \cup \left[-2, \frac{1}{2}\right]$
7. $(-2, 0)$
9. $\left(-\infty, -\frac{3}{4}\right] \cup [0, \infty)$
11. $(-\infty, -5] \cup [5, 5] \cup [10, \infty)$
13. $\left(-\frac{9}{4}, 0\right)$
15. $[-6, 0] \cup [4, \infty)$
17. $\left(-\infty, -\frac{1}{2}\right) \cup (0, 6)$
19. $(-\infty, -5) \cup \left(\frac{4}{3}, \infty\right)$
21. $(-\infty, 0)$
23. $[-2, -1] \cup [1, 2]$
25. $(-\infty, -2) \cup (2, \infty)$
27. $(-\infty, -3] \cup \left[\frac{1}{3}, 3\right]$
29. $(3, \infty)$
31. $(-\infty, -0) \cup (3, \infty)$
33. $(-\infty, -1) \cup (0, 3)$
35. $\left[-5, -\frac{1}{2}\right] \cup (3, 5)$
37. $[-8, 0) \cup (2, 8]$

39. $\left(-\dfrac{3}{2}, \dfrac{3}{2}\right)$

41. $(-\infty, -5) \cup [0, 6)$

43. $(-\infty, 5) \cup (5, \infty)$

45. $(-\infty, -4) \cup (-1, 12)$

47. $\left[-6, -\dfrac{1}{2}\right) \cup [5, \infty)$

49. $\left(-3, \dfrac{1}{3}\right] \cup [1, 3)$

51. $(-\infty, 2) \cup (10, \infty)$

53. $\left(-2, -\dfrac{1}{4}\right] \cup \left[\dfrac{1}{2}, \dfrac{7}{2}\right)$

55. $(-5, -4) \cup (1, \infty)$

57. $(-7, -6)$

59. $[-5, 1) \cup [6, \infty)$

61. $(0, 1) \cup [2, \infty)$

63. $(-\infty, 5] \cup (-3, 3)$

65. $[-4, -2) \cup (0, 6]$

67. $(-\infty, -2) \cup (2, 4)$

69. $\left(-3, -\dfrac{1}{2}\right) \cup \left(\dfrac{1}{2}, \infty\right)$

71. Answer may vary

6.7 Review Exercises and Sample Exam

REVIEW EXERCISES

EXTRACTING SQUARE ROOTS AND COMPLETING THE SQUARE

Solve by extracting the roots.

1. $x^2 - 81 = 0$
2. $y^2 - \frac{1}{4} = 0$
3. $9x^2 - 8 = 0$
4. $5x^2 - 12 = 0$
5. $2y^2 - 7 = 0$
6. $3y^2 - 6 = 0$
7. $(2x - 3)^2 - 16 = 0$
8. $4(x - 1)^2 - 5 = 0$
9. $9(x - 3)^2 + 4 = 0$
10. $5(2x + 1)^2 + 1 = 0$
11. $2x^2 + 10 = 0$
12. $x^2 + 64 = 0$
13. The height in feet of an object dropped from a 20-foot stepladder is given by $h(t) = -16t^2 + 20$ where t represents the time in seconds after the object has been dropped. How long does it take the object to hit the ground after it has been dropped? Round to the nearest tenth of a second.
14. A 20-foot ladder, leaning against a building, reaches a height of 19 feet. How far is the base of the ladder from the wall? Round to the nearest tenth of a foot.

Solve by completing the square.

15. $x^2 + 4x - 5 = 0$
16. $x^2 + 2x - 17 = 0$

17. $x^2 - 4x + 1 = 0$

18. $x^2 - 6x - 2 = 0$

19. $x^2 - 3x - 1 = 0$

20. $x^2 + 5x - 6 = 0$

21. $x^2 + x - 2 = 0$

22. $x^2 - x - 4 = 0$

23. $5x^2 - 10x + 1 = 0$

24. $4x^2 + 8x - 3 = 0$

25. $2x^2 - 6x + 1 = 0$

26. $3x^2 + 10x + 6 = 0$

27. $x^2 - x + 3 = 0$

28. $2x^2 + 6x + 5 = 0$

29. $x(x + 9) + 10 = 5x + 2$

30. $(2x + 5)(x + 2) = 8x + 7$

QUADRATIC FORMULA

Solve using the quadratic formula.

31. $2x^2 - x - 6 = 0$

32. $3x^2 + x - 4 = 0$

33. $9x^2 + 12x + 2 = 0$

34. $25x^2 - 10x - 1 = 0$

35. $-x^2 + 8x - 2 = 0$

36. $-x^2 - x + 1 = 0$

37. $5 - 2x - x^2 = 0$

38. $2 + 4x - 3x^2 = 0$

39. $3x^2 - 2x + 4 = 0$

40. $7x^2 - x + 1 = 0$

41. $-x^2 + 2x - 6 = 0$

42. $-3x^2 + 4x - 2 = 0$

43. $36x^2 + 60x + 25 = 0$

44. $72x^2 + 54x - 35 = 0$

45. $1.3x^2 - 2.8x - 4.2 = 0$

46. $5.5x^2 - 4.1x + 2.2 = 0$

47. $(x + 2)^2 - 3x = 4$

48. $(3x + 1)^2 - 6 = 6x - 3$

49. The height in feet of a baseball tossed upward at a speed of 48 feet per second from the ground is given by the function, $h(t) = -16t^2 + 48t$, where t represents the time in seconds after the ball is tossed. At what time does the baseball reach a height of 18 feet? Round off to the nearest hundredth of a second.

50. The height in feet reached by a model rocket launched from a 3-foot platform is given by the function $h(t) = -16t^2 + 256t + 3$ where t represents time in seconds after launch. At what times will the rocket reach 1,000 feet? Round off to the nearest tenth of a second.

Use the discriminant to determine the number and type of solutions.

51. $-x^2 + 6x + 1 = 0$

52. $-x^2 + x - 3 = 0$

53. $4x^2 - 4x + 1 = 0$

54. $16x^2 - 9 = 0$

SOLVING EQUATIONS QUADRATIC IN FORM

Solve using any method.

55. $x^2 - 4x - 96 = 0$
56. $25x^2 + x = 0$
57. $25t^2 - 1 = 0$
58. $t^2 + 25 = 0$
59. $y^2 - y - 7 = 0$
60. $5y^2 - 25y = 0$
61. $2x^2 - 9 = 0$
62. $25x^2 - 10x + 1 = 0$
63. $(2x + 5)^2 - 9 = 0$
64. $(x - 2)(x - 5) = 5$
65. The length of a rectangle is 3 inches less than twice the width. If the area of the rectangle measures 30 square inches, then find the dimensions of the rectangle. Round off to the nearest hundredth of an inch.
66. The value in dollars of a new car is modeled by the function $V(t) = 125t^2 - 2{,}500t + 18{,}000$ where t represents the number of years since it was purchased. Determine the age of the car when its value is $18,000.

Find all solutions.

67. $x^4 - 16x^2 + 48 = 0$
68. $x^{2/3} - x^{1/3} - 20 = 0$
69. $x^{-2} - 5x^{-1} - 50 = 0$
70. $\left(\frac{t+3}{t}\right)^2 + 11\left(\frac{t+3}{t}\right) - 12 = 0$
71. $x + 2\sqrt{x} - 24 = 0$

72. $2x^{1/2} - 3x^{1/4} + 1 = 0$

73. $4\left(\frac{1}{x+1}\right)^2 - 4\left(\frac{1}{x+1}\right) - 3 = 0$

74. $5t^{-2} - 27t^{-1} - 18 = 0$

75. $3x^{2/3} - 5x^{1/3} + 2 = 0$

76. $4x + 4\sqrt{x} + 1 = 0$

77. $16y^4 - 25 = 0$

78. $x^{-2} - 64 = 0$

Find the set of all roots.

79. $f(x) = x^2 - 50$

80. $f(x) = x^3 - 64$

81. $f(x) = x^4 - 81$

82. $f(x) = x^4 + 8x$

Find a quadratic equation with integer coefficients and the given set of solutions.

83. $\left\{\frac{4}{3}, -\frac{1}{2}\right\}$

84. $\left\{\pm\sqrt{5}\right\}$

85. $\left\{\pm 4\sqrt{2}\right\}$

86. $\{\pm 6i\}$

87. $\{2 \pm i\}$

88. $\left\{3 \pm \sqrt{5}\right\}$

QUADRATIC FUNCTIONS AND THEIR GRAPHS

Determine the *x*- and *y*-intercepts.

89. $y = 2x^2 + 5x - 12$
90. $y = x^2 - 18$
91. $y = x^2 + 4x + 7$
92. $y = -9x^2 + 12x - 4$

Find the vertex and the line of symmetry.

93. $y = x^2 - 4x - 12$
94. $y = -x^2 + 8x - 1$
95. $y = x^2 + 3x - 1$
96. $y = 4x^2 - 1$

Graph. Find the vertex and the *y*-intercept. In addition, find the *x*-intercepts if they exist.

97. $y = x^2 + 8x + 12$
98. $y = -x^2 - 6x + 7$
99. $y = -2x^2 - 4$
100. $y = x^2 + 4x$
101. $y = 4x^2 - 4x + 1$
102. $y = -2x^2$
103. $y = -2x^2 + 8x - 7$
104. $y = 3x^2 - 1$

Determine the maximum or minimum *y*-value.

105. $y = x^2 - 10x + 1$
106. $y = -x^2 + 10x - 1$

107. $y = -3x^2 + 2x - 1$

108. $y = 2x^2 - x + 2$

Rewrite in vertex form $y = a(x-h)^2 + k$ and determine the vertex.

109. $y = x^2 - 6x + 13$

110. $y = x^2 + 10x + 24$

111. $y = 2x^2 - 4x - 1$

112. $y = -x^2 - 8x - 11$

Graph. Find the vertex and the *y*-intercept. In addition, find the *x*-intercepts if they exist.

113. $f(x) = (x-4)^2 - 2$

114. $f(x) = -(x+6)^2 + 4$

115. $f(x) = -x^2 + 10$

116. $f(x) = (x+10)^2 - 20$

117. $f(x) = 2(x-1)^2 - 3$

118. $f(x) = -3(x+1)^2 - 2$

119. The value in dollars of a new car is modeled by $V(t) = 125t^2 - 3{,}000t + 22{,}000$ where t represents the number of years since it was purchased. Determine the age of the car when its value is at a minimum.

120. The height in feet of a baseball tossed upward at a speed of 48 feet per second from the ground is given by the function, $h(t) = -16t^2 + 32t$, where t represents the time in seconds after it is tossed. What is the maximum height of the baseball?

121. The rectangular area in square feet that can be enclosed with 200 feet of fencing is given by $A(w) = w(100 - w)$ where w represents the width of the rectangular area in feet. What dimensions will maximize the area that can be enclosed?

122. A manufacturing company has found that production costs in thousands of dollars are modeled by $C(x) = 0.4x^2 - 72x + 8{,}050$ where x represents the number of employees. Determine the number of employees that will minimize production costs.

SOLVING QUADRATIC INEQUALITIES

Solve. Present answers using interval notation.

123. $-2(x-1)(x+3) < 0$
124. $x^2 + 2x - 35 < 0$
125. $x^2 - 6x - 16 \leq 0$
126. $x^2 + 14x + 40 \geq 0$
127. $x^2 - 10x - 24 > 0$
128. $36 - x^2 > 0$
129. $1 - 9x^2 < 0$
130. $8x - 12x^2 \leq 0$
131. $5x^2 + 3 \leq 0$
132. $x^2 - 28 \geq 0$
133. $9x^2 - 30x + 25 \leq 0$
134. $x^2 - 8x + 18 > 0$
135. $x^2 - 2x - 4 < 0$
136. $-x^2 + 3x + 18 > 0$

Find the domain of the function.

137. $f(x) = \sqrt{x^2 - 100}$
138. $f(x) = \sqrt{3x - 6x^2}$
139. $g(x) = \sqrt{3x^2 + 9}$

140. $g\left(x\right) = \sqrt{8 + 2x - x^2}$

SOLVING POLYNOMIAL AND RATIONAL INEQUALITIES

Solve. Present answers using interval notation.

141. $x\left(x - 5\right)\left(x + 2\right) > 0$

142. $(x + 4)^2 (x - 3) < 0$

143. $x^2 (x + 3) \geq 0$

144. $x(x - 1)^2 \leq 0$

145. $x^3 + 4x^2 - 9x - 36 > 0$

146. $2x(4x - 1) \geq 3$

147. $4x^3 - 12x^2 + 9x < 0$

148. $x^3 - 9x^2 + 20x \geq 0$

149. $x^3 - 2x^2 - x + 2 < 0$

150. $6x(x + 1) + 5x \leq 35$

151. $\dfrac{(x - 2)(2x + 1)}{x(x - 1)} \leq 0$

152. $\dfrac{x(x - 3)^2}{x - 4} \leq 0$

153. $\dfrac{x^2 + 4x + 4}{4x^2 - 1} < 0$

154. $\dfrac{x^2 - 10x + 24}{x^2 + 10x + 25} > 0$

155. $\dfrac{1}{x - 2} + \dfrac{3}{x} \geq 0$

156. $\dfrac{2}{x - 1} - \dfrac{1}{x + 1} \leq 0$

157. $\dfrac{3(x + 1)}{x^2 + 2x - 3} \leq \dfrac{2}{x - 1}$

158. $\dfrac{x - 4}{x + 5} \geq \dfrac{x - 2}{x - 5}$

ANSWERS

1. ± 9

3. $\pm \frac{2\sqrt{2}}{3}$

5. $\pm \frac{\sqrt{14}}{2}$

7. $-\frac{1}{2}, \frac{7}{2}$

9. $3 \pm \frac{2}{3}i$

11. $\pm i\sqrt{5}$

13. 1.1 seconds

15. $-5, 1$

17. $2 \pm \sqrt{3}$

19. $\frac{3 \pm \sqrt{13}}{2}$

21. $-2, 1$

23. $\frac{5 \pm 2\sqrt{5}}{5}$

25. $\frac{3 \pm \sqrt{7}}{2}$

27. $\frac{1}{2} \pm \frac{\sqrt{11}}{2}i$

29. $-2 \pm 2i$

31. $-\frac{3}{2}, 2$

33. $\frac{-2 \pm \sqrt{2}}{3}$

35. $4 \pm \sqrt{14}$

37. $-1 \pm \sqrt{6}$

39. $\frac{1}{3} \pm \frac{\sqrt{11}}{3} i$

41. $1 \pm i\sqrt{5}$

43. $-\frac{5}{6}$

45. $x \approx -1.0, x \approx 3.2$

47. −1, 0

49. The ball will reach 18 feet at 0.44 seconds and again at 2.56 seconds.

51. Two irrational solutions

53. One rational solution

55. −8, 12

57. $\pm \frac{1}{5}$

59. $\frac{1 \pm \sqrt{29}}{2}$

61. $\pm \frac{3\sqrt{2}}{2}$

63. −4, −1

65. Length: 6.38 inches; width: 4.69 inches

67. $\pm 2, \pm 2\sqrt{3}$

69. $-\frac{1}{5}, \frac{1}{10}$

71. 16

73. −3, $-\frac{1}{3}$

75. 1, $\frac{8}{27}$

77. $\pm \frac{\sqrt{5}}{2}, \pm \frac{\sqrt{5}}{2} i$

79. $\left\{\pm 5\sqrt{2}\right\}$

81. $\{\pm 3, \pm 3i\}$

83. $6x^2 - 5x - 4 = 0$

85. $x^2 - 32 = 0$

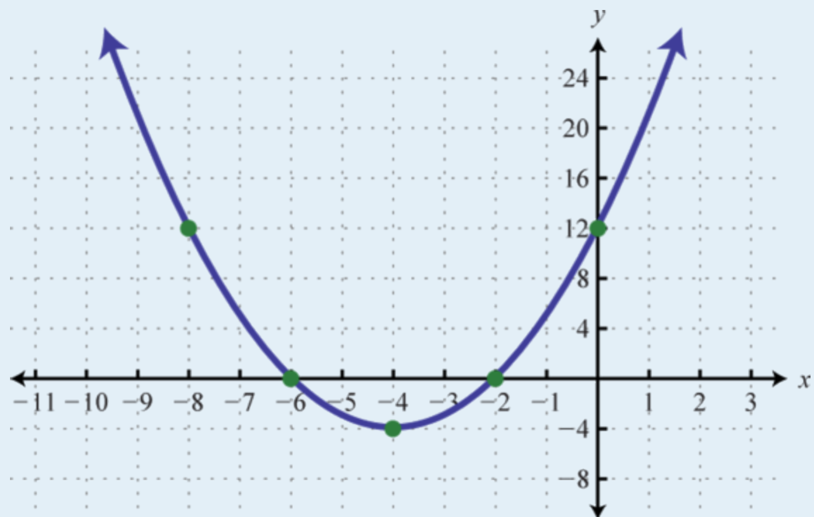
87. $x^2 - 4x + 5 = 0$

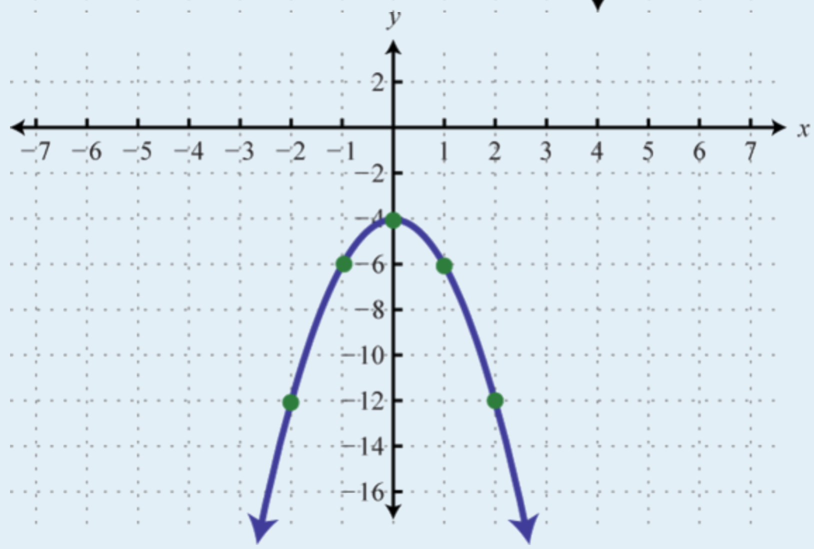
89. x-intercepts: (−4, 0), $\left(\frac{3}{2}, 0\right)$; y-intercept: (0, −12)

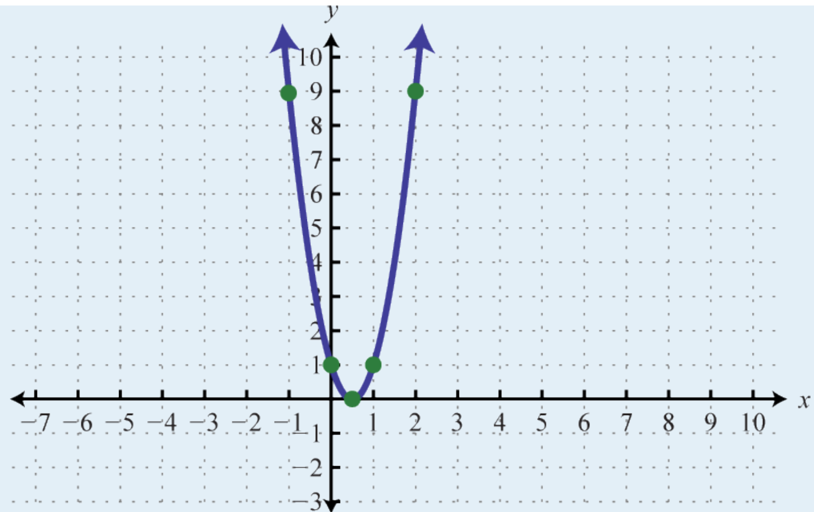
91. x-intercepts: none; y-intercept: (0, 7)

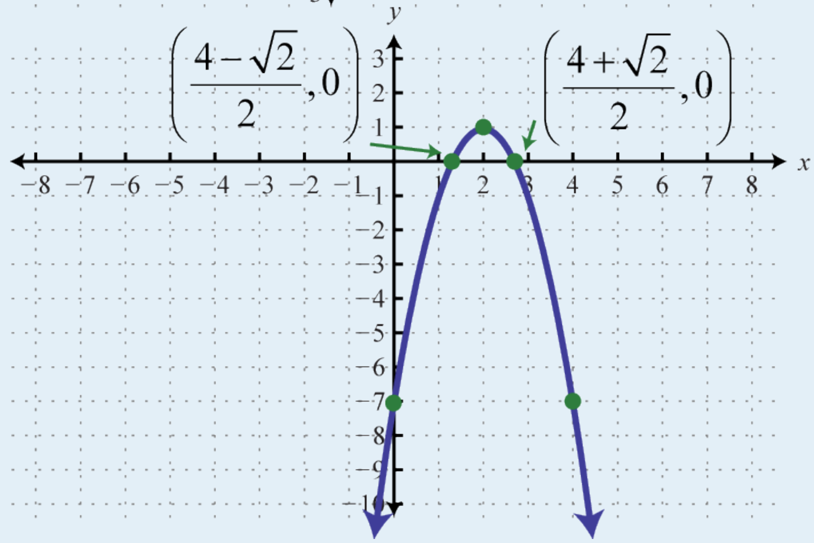
93. Vertex: (2, −16); line of symmetry: $x = 2$

95. Vertex: $\left(-\frac{3}{2}, -\frac{13}{4}\right)$; line of symmetry: $x = -\frac{3}{2}$

97. 

99. 

101. 

103. 

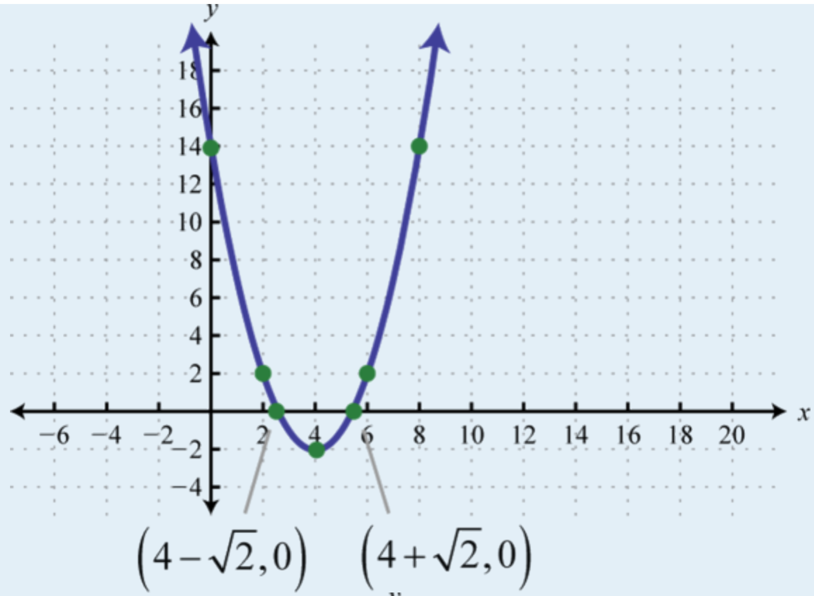
105. Minimum: $y = -24$

107. Maximum: $y = -\frac{2}{3}$

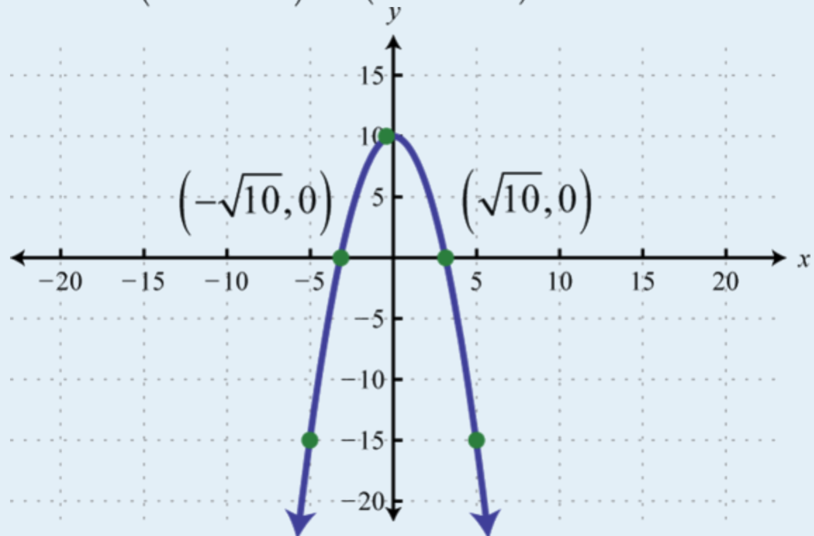
109. $y = (x - 3)^2 + 4$; vertex: $(3, 4)$

111. $y = 2(x - 1)^2 - 3$; vertex: $(1, -3)$

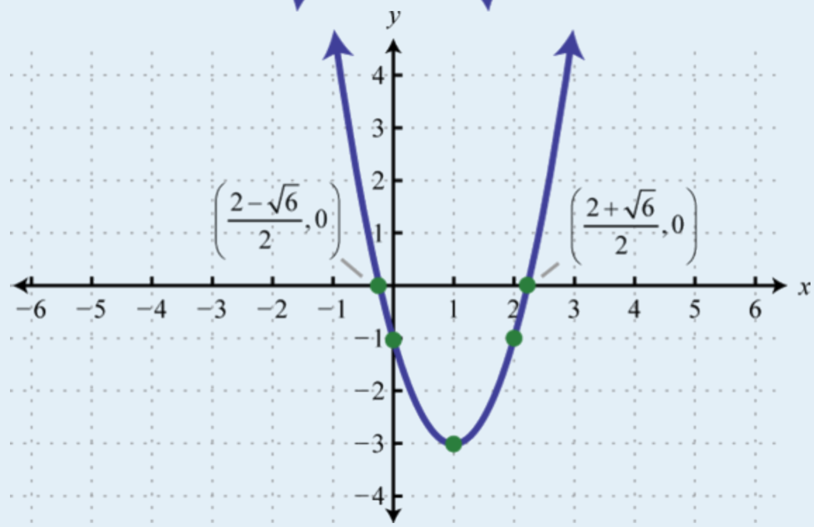
113.


y
x
$\left(4-\sqrt{2},0\right)$
$\left(4+\sqrt{2},0\right)$

115.


y
x
$\left(-\sqrt{10},0\right)$
$\left(\sqrt{10},0\right)$

117.


y
x
$\left(\frac{2-\sqrt{6}}{2},0\right)$
$\left(\frac{2+\sqrt{6}}{2},0\right)$

119. The car will have minimum value 12 years after it is purchased.

121. Length: 50 feet; width: 50 feet

123. $(-\infty, -3) \cup (1, \infty)$

125. $[-2, 8]$

127. $(-\infty, -2) \cup (12, \infty)$

129. $\left(-\infty, -\frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$

131. Ø

133. $\frac{5}{3}$

135. $\left(1 - \sqrt{5}, 1 + \sqrt{5}\right)$

137. $(-\infty, -10] \cup [10, \infty)$

139. $(-\infty, \infty)$

141. $(-2, 0) \cup (5, \infty)$

143. $[-3, \infty)$

145. $(-4, -3) \cup (3, \infty)$

147. $(-\infty, 0)$

149. $(-\infty, -1) \cup (1, 2)$

151. $\left[-\frac{1}{2}, 0\right) \cup (1, 2]$

153. $\left(-\frac{1}{2}, \frac{1}{2}\right)$

155. $\left(0, \frac{3}{2}\right] \cup (2, \infty)$

157. $(-\infty, -3) \cup (1, 3]$

SAMPLE EXAM

1. Solve by extracting the roots: $2x^2 - 5 = 0$.
2. Solve by completing the square: $x^2 - 16x + 1 = 0$.

Solve using the quadratic formula.

3. $x^2 + x + 1 = 0$
4. $2x^2 - x - 4 = 0$
5. $-4x^2 + 2x - 1 = 0$
6. $(x-4)(x-2) = 6$
7. Find a quadratic equation with integer coefficients and solutions $\left\{\pm\sqrt{5}\right\}$.
8. The area of a rectangle is 22 square centimeters. If the length is 5 centimeters less than twice the width, then find the dimensions of the rectangle. Round off to the nearest tenth of a centimeter.
9. Assuming dry road conditions and average reaction times, the safe stopping distance in feet of a certain car is given by $d(x) = \frac{1}{20}x^2 + x$ where x represents the speed of the car in miles per hour. Determine the safe speed of the car if you expect to stop in 100 feet. Round off to the nearest mile per hour.

Find all solutions.

10. $x^4 + x^2 - 12 = 0$
11. $3x^{-2} - 5x^{-1} - 2 = 0$
12. $2x^{2/3} + 3x^{1/3} - 2 = 0$
13. $x - 3\sqrt{x} - 4 = 0$
14. $\left(\frac{t}{t+1}\right)^2 + 4\left(\frac{t}{t+1}\right) - 12 = 0$

Graph. Find the vertex and the *y*-intercept. In addition, find the *x*-intercepts if they exist.

15. $f(x) = x^2 + 4x - 12$

16. $f(x) = -x^2 + 2x + 3$

17. Given the function defined by $y = 3x^2 - 6x - 5$:

    a. Does the function have a minimum or maximum? Explain.
    b. Find the minimum or maximum y-value.

18. The height in feet of a water rocket launched from the ground is given by the function $h(t) = -16t^2 + 96t$ where t represents the number of seconds after launch. What is the maximum height attained by the rocket?

Sketch the graph and use it to solve the given inequality.

19. Graph $f(x) = (x+1)^2 - 4$ and find x where $f(x) \geq 0$.

20. Graph $f(x) = -x^2 + 4$ and find x where $f(x) \geq 0$.

Solve. Present answers using interval notation.

21. $x^2 - 2x - 15 < 0$

22. $x(2x - 1) > 10$

23. $x(x+3)(x-2)^2 \leq 0$

24. $\frac{x^2-10x+25}{x+1} \geq 0$

25. $\frac{x^2-5x+4}{x^2+x} \leq 0$

ANSWERS

1. $\pm \frac{\sqrt{10}}{2}$

3. $-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

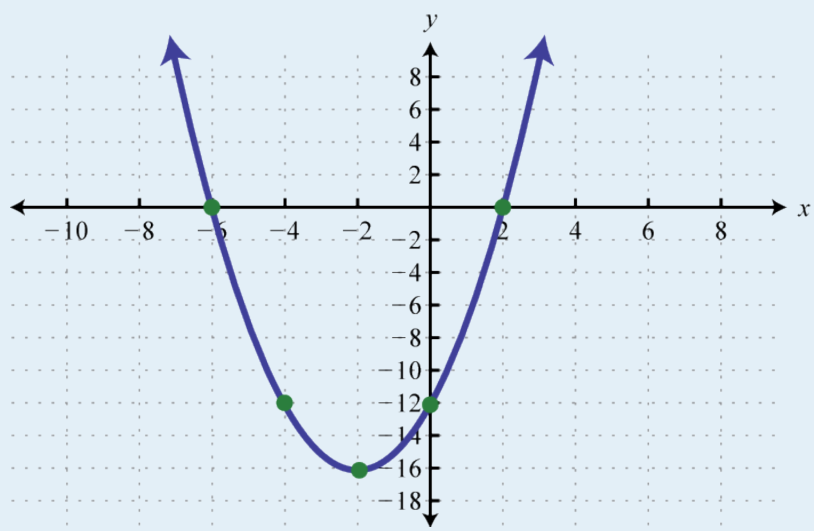
5. $\frac{1}{4} \pm \frac{\sqrt{3}}{4} i$

7. $x^2 - 5 = 0$

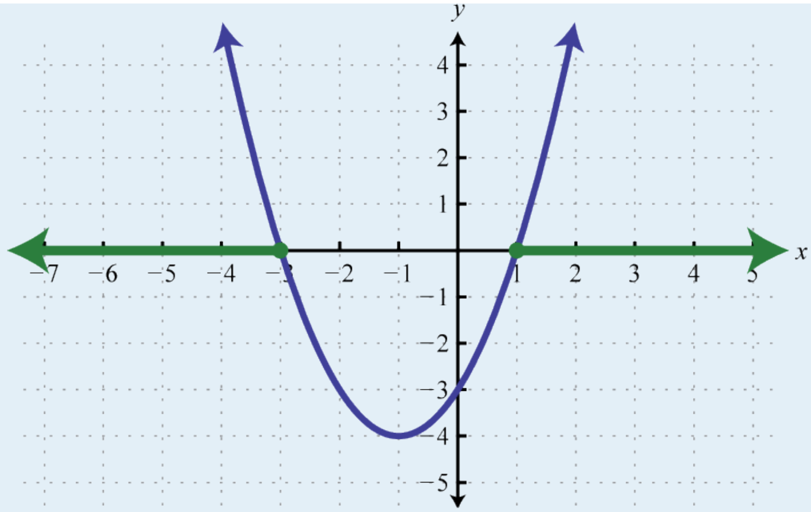
9. 36 miles per hour

11. -3, $\frac{1}{2}$

13. 16

15. 

17. a. Minimum
    b. $y = -8$



19. $(-\infty, -3] \cup [1, \infty)$

21. $(-3, 5)$

23. $[-3, 0] \cup \{2\}$

25. $(-1, 0) \cup [1, 4]$

Chapter 7

Exponential and Logarithmic Functions

7.1 Composition and Inverse Functions

LEARNING OBJECTIVES

1. Perform function composition.
2. Determine whether or not given functions are inverses.
3. Use the horizontal line test.
4. Find the inverse of a one-to-one function algebraically.

Composition of Functions

In mathematics, it is often the case that the result of one function is evaluated by applying a second function. For example, consider the functions defined by $f(x) = x^2$ and $g(x) = 2x + 5$. First, g is evaluated where $x = -1$ and then the result is squared using the second function, f.

$$\begin{aligned} g(-1) &= 3 \\ f(3) &= 9 \end{aligned}$$

This sequential calculation results in 9. We can streamline this process by creating a new function defined by $f\left(g(x)\right)$, which is explicitly obtained by substituting $g(x)$ into $f(x)$.

$$\begin{aligned} f\left(g(x)\right) &= f\left(2x + 5\right) \\ &= \left(2x + 5\right)^2 \\ &= 4x^2 + 20x + 25 \end{aligned}$$

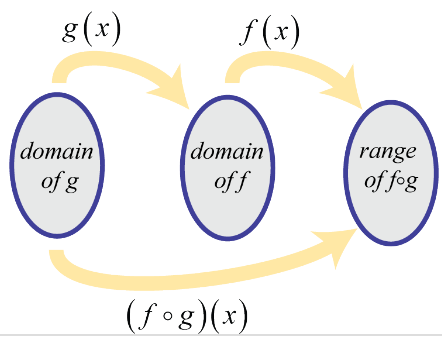
Therefore, $f\left(g(x)\right) = 4x^2 + 20x + 25$ and we can verify that when $x = -1$ the result is 9.

$$
\begin{aligned}
f\left(g\left(-1\right)\right) &= 4(-1)^2 + 20\left(-1\right) + 25 \\
&= 4 - 20 + 25 \\
&= 9
\end{aligned}
$$

The calculation above describes **composition of functions**[1], which is indicated using the **composition operator**[2] ($\bigcirc$). If given functions f and g,

$$\left(f \bigcirc g\right)\ (x) = f\ \left(g\ (x)\right) \qquad \textit{Composition of Functions}$$

The notation $f \bigcirc g$ is read, "f composed with g." This operation is only defined for values, x, in the domain of g such that $g\ (x)$ is in the domain of f.



1. Applying a function to the results of another function.

2. The open dot used to indicate the function composition $\left(f \bigcirc g\right)\ (x) = f\ \left(g\ (x)\right)$.

Example 1

Given $f(x) = x^2 - x + 3$ and $g(x) = 2x - 1$ calculate:

a. $(f \circ g)(x)$.
b. $(g \circ f)(x)$.

Solution:

a. Substitute g into f.

$$
\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f(2x - 1) \\
&= (2x - 1)^2 - (2x - 1) + 3 \\
&= 4x^2 - 4x + 1 - 2x + 1 + 3 \\
&= 4x^2 - 6x + 5
\end{aligned}
$$

b. Substitute f into g.

$$
\begin{aligned}
(g \circ f)(x) &= g(f(x)) \\
&= g(x^2 - x + 3) \\
&= 2(x^2 - x + 3) - 1 \\
&= 2x^2 - 2x + 6 - 1 \\
&= 2x^2 - 2x + 5
\end{aligned}
$$

Answer:

a. $(f \circ g)(x) = 4x^2 - 6x + 5$

b. $\left(g \circ f\right)(x) = 2x^2 - 2x + 5$

The previous example shows that composition of functions is not necessarily commutative.

Example 2

Given $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{3x - 1}$ find $(f \circ g)(4)$.

Solution:

Begin by finding $(f \circ g)(x)$.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f\left(\sqrt[3]{3x-1}\right) \\ &= \left(\sqrt[3]{3x-1}\right)^3 + 1 \\ &= 3x - 1 + 1 \\ &= 3x\end{aligned}$$

Next, substitute 4 in for x.

$$\begin{aligned}(f \circ g)(x) &= 3x \\ (f \circ g)(4) &= 3(4) \\ &= 12\end{aligned}$$

Answer: $(f \circ g)(4) = 12$

Functions can be composed with themselves.

Example 3

Given $f\left(x\right) = x^2 - 2$ find $\left(f \circ f\right)\left(x\right)$.

Solution:

$$\begin{aligned}\left(f \circ f\right)\left(x\right) &= f\left(f\left(x\right)\right)\\ &= f\left(x^2 - 2\right)\\ &= \left(x^2 - 2\right)^2 - 2\\ &= x^4 - 4x^2 + 4 - 2\\ &= x^4 - 4x^2 + 2\end{aligned}$$

Answer: $\left(f \circ f\right)\left(x\right) = x^4 - 4x^2 + 2$

Try this! Given $f\left(x\right) = 2x + 3$ and $g\left(x\right) = \sqrt{x - 1}$ find $\left(f \circ g\right)\left(5\right)$.

Answer: 7

(click to see video)

Inverse Functions

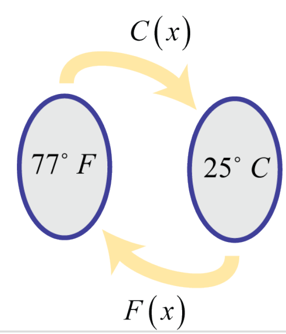
Consider the function that converts degrees Fahrenheit to degrees Celsius: $C\left(x\right) = \frac{5}{9}\left(x - 32\right)$. We can use this function to convert 77°F to degrees Celsius as follows.

$$\begin{aligned} C(77) &= \frac{5}{9}(77 - 32) \\ &= \frac{5}{9}(45) \\ &= 25 \end{aligned}$$

Therefore, 77°F is equivalent to 25°C. If we wish to convert 25°C back to degrees Fahrenheit we would use the formula: $F(x) = \frac{9}{5}x + 32$.

$$\begin{aligned} F(25) &= \frac{9}{5}(25) + 32 \\ &= 45 + 32 \\ &= 77 \end{aligned}$$

Notice that the two functions C and F each reverse the effect of the other.



This describes an inverse relationship. In general, f and g are **inverse functions** if,

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) = x \quad \textit{for all } x \textit{ in the domain of } g \textit{ and} \\ (g \circ f)(x) &= g(f(x)) = x \quad \textit{for all } x \textit{ in the domain of } f. \end{aligned}$$

In this example,

$$C\left(F\left(25\right)\right) = C\left(77\right) = 25$$
$$F\left(C\left(77\right)\right) = F\left(25\right) = 77$$

Example 4

Verify algebraically that the functions defined by $f\left(x\right) = \frac{1}{2}x - 5$ and $g\left(x\right) = 2x + 10$ are inverses.

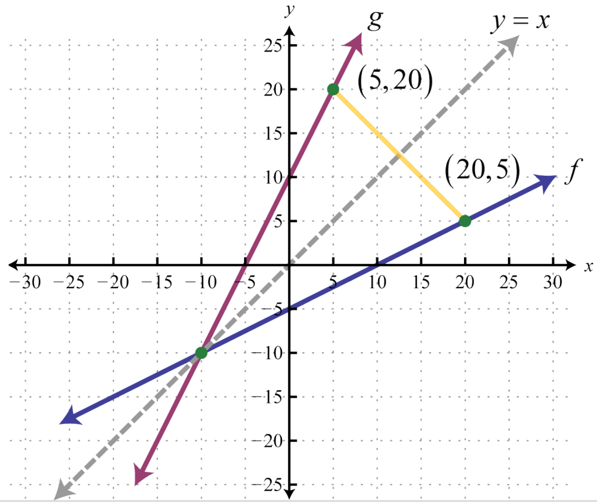
Solution:

Compose the functions both ways and verify that the result is x.

$\begin{aligned}\left(f\circ g\right)\left(x\right) &= f\left(g\left(x\right)\right)\\ &= f\left(2x + 10\right)\\ &= \frac{1}{2}\left(2x + 10\right) - 5\\ &= x + 5 - 5\\ &= x \ \checkmark\end{aligned}$	$\begin{aligned}\left(g\circ f\right)\left(x\right) &= g\left(f\left(x\right)\right)\\ &= g\left(\frac{1}{2}x - 5\right)\\ &= 2\left(\frac{1}{2}x - 5\right) + 10\\ &= x - 10 + 10\\ &= x \ \checkmark\end{aligned}$

Answer: Both $\left(f\circ g\right)\left(x\right) = \left(g\circ f\right)\left(x\right) = x$; therefore, they are inverses.

Next we explore the geometry associated with inverse functions. The graphs of both functions in the previous example are provided on the same set of axes below.



Note that there is symmetry about the line $y = x$; the graphs of f and g are mirror images about this line. Also notice that the point (20, 5) is on the graph of f and that (5, 20) is on the graph of g. Both of these observations are true in general and we have the following properties of inverse functions:

1. The graphs of inverse functions are symmetric about the line $y = x$.
2. If (a, b) is on the graph of a function, then (b, a) is on the graph of its inverse.

Furthermore, if g is the inverse of f we use the notation $g = f^{-1}$. Here f^{-1} is read, "f inverse," and should not be confused with negative exponents. In other words, $f^{-1}(x) \neq \frac{1}{f(x)}$ and we have,

$$\left(f \circ f^{-1}\right)(x) = f\left(f^{-1}(x)\right) = x \text{ and}$$
$$\left(f^{-1} \circ f\right)(x) = f^{-1}\left(f(x)\right) = x$$

Example 5

Verify algebraically that the functions defined by $f(x) = \frac{1}{x} - 2$ and $f^{-1}(x) = \frac{1}{x+2}$ are inverses.

Solution:

Compose the functions both ways to verify that the result is *x*.

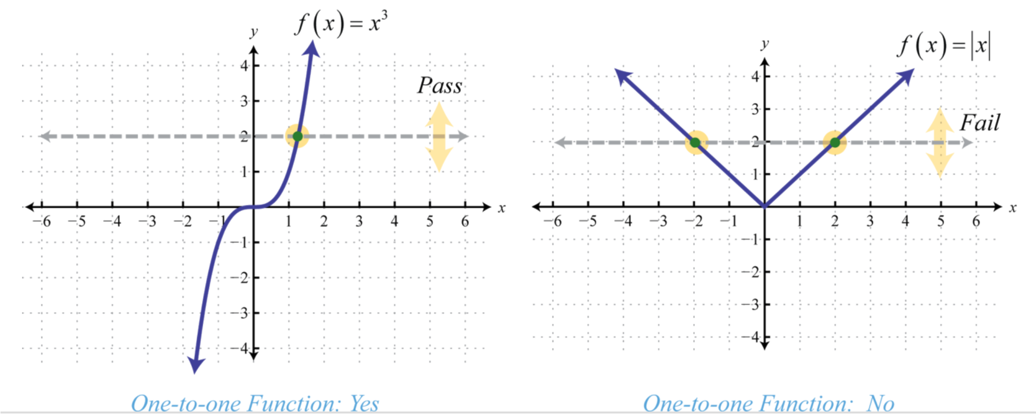
$\begin{aligned}(f \circ f^{-1})(x) &= f(f^{-1}(x)) \\ &= f\left(\frac{1}{x+2}\right) \\ &= \frac{1}{\left(\frac{1}{x+2}\right)} - 2 \\ &= \frac{x+2}{1} - 2 \\ &= x + 2 - 2 \\ &= x \;\checkmark\end{aligned}$	$\begin{aligned}(f^{-1} \circ f)(x) &= f^{-1}(f(x)) \\ &= f^{-1}\left(\frac{1}{x} - 2\right) \\ &= \frac{1}{\left(\frac{1}{x}-2\right)+2} \\ &= \frac{1}{\frac{1}{x}} \\ &= x \;\checkmark\end{aligned}$

Answer: Since $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$ they are inverses.

Recall that a function is a relation where each element in the domain corresponds to exactly one element in the range. We use the vertical line test to determine if a graph represents a function or not. Functions can be further classified using an inverse relationship. **One-to-one functions**[3] are functions where each value in the range corresponds to exactly one element in the domain. The **horizontal line test**[4] is used to determine whether or not a graph represents a one-to-one function. If a horizontal line intersects a graph more than once, then it does not represent a one-to-one function.

3. Functions where each value in the range corresponds to exactly one value in the domain.

4. If a horizontal line intersects the graph of a function more than once, then it is not one-to-one.

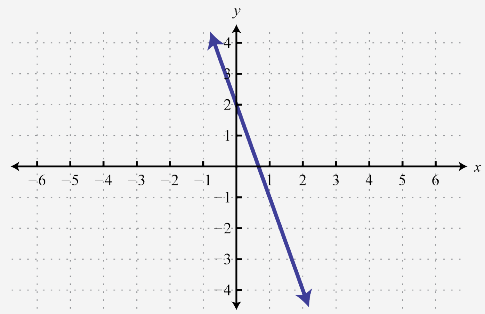


The horizontal line represents a value in the range and the number of intersections with the graph represents the number of values it corresponds to in the domain. The function defined by $f(x) = x^3$ is one-to-one and the function defined by $f(x) = |x|$ is not. Determining whether or not a function is one-to-one is important because a function has an inverse if and only if it is one-to-one. In other words, a function has an inverse if it passes the horizontal line test.

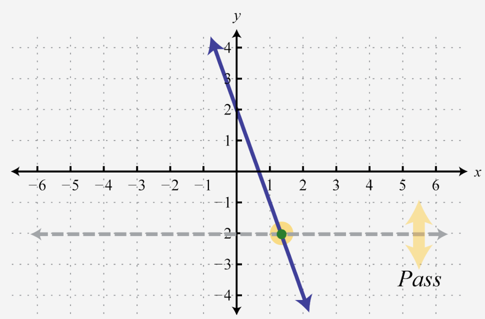
Note: In this text, when we say "*a function has an inverse,*" we mean that there is another function, f^{-1}, such that $\left(f \circ f^{-1}\right)(x) = \left(f^{-1} \circ f\right)(x) = x$.

Example 6

Determine whether or not the given function is one-to-one.



Solution:



Answer: The given function passes the horizontal line test and thus is one-to-one.

In fact, any linear function of the form $f(x) = mx + b$ where $m \neq 0$, is one-to-one and thus has an inverse. The steps for finding the inverse of a one-to-one function are outlined in the following example.

Example 7

Find the inverse of the function defined by $f(x) = \frac{3}{2}x - 5$.

Solution:

Before beginning this process, you should verify that the function is one-to-one. In this case, we have a linear function where $m \neq 0$ and thus it is one-to-one.

- **Step 1:** Replace the function notation $f(x)$ with y.

$$\begin{aligned} f(x) &= \frac{3}{2}x - 5 \\ y &= \frac{3}{2}x - 5 \end{aligned}$$

- **Step 2:** Interchange x and y. We use the fact that if (x, y) is a point on the graph of a function, then (y, x) is a point on the graph of its inverse.

$$x = \frac{3}{2}y - 5$$

- **Step 3:** Solve for y.

$$\begin{aligned} x &= \frac{3}{2}y - 5 \\ x + 5 &= \frac{3}{2}y \\ \frac{2}{3} \cdot (x+5) &= \frac{2}{3} \cdot \frac{3}{2}y \\ \frac{2}{3}x + \frac{10}{3} &= y \end{aligned}$$

- **Step 4:** The resulting function is the inverse of f. Replace y with $f^{-1}(x)$.

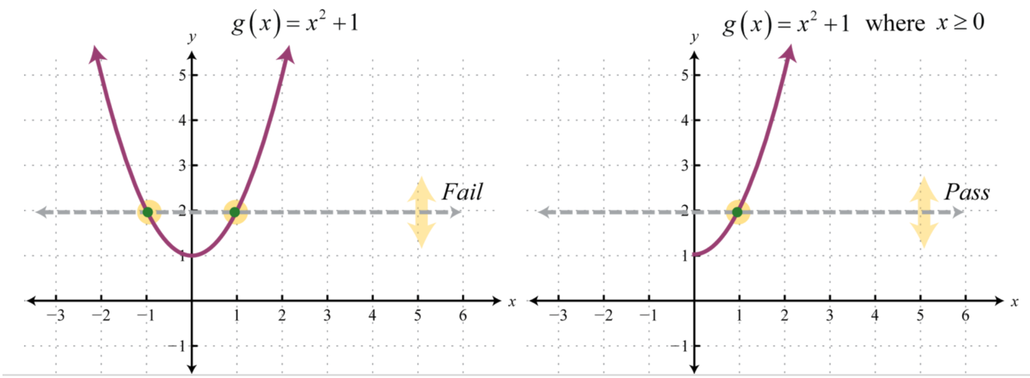
$$f^{-1}(x) = \frac{2}{3}x + \frac{10}{3}$$

- **Step 5:** Check.

$\left(f \circ f^{-1}\right)(x)$ $= f\left(f^{-1}(x)\right)$ $= f\left(\frac{2}{3}x + \frac{10}{3}\right)$ $= \frac{3}{2}\left(\frac{2}{3}x + \frac{10}{3}\right) - 5$ $= x + 5 - 5$ $= x$ ✓	$\left(f^{-1} \circ f\right)(x)$ $= f^{-1}\left(f(x)\right)$ $= f^{-1}\left(\frac{3}{2}x - 5\right)$ $= \frac{2}{3}\left(\frac{3}{2}x - 5\right) + \frac{10}{3}$ $= x - \frac{10}{3} + \frac{10}{3}$ $= x$ ✓

Answer: $f^{-1}(x) = \frac{2}{3}x + \frac{10}{3}$

If a function is not one-to-one, it is often the case that we can restrict the domain in such a way that the resulting graph is one-to-one. For example, consider the squaring function shifted up one unit, $g(x) = x^2 + 1$. Note that it does not pass the horizontal line test and thus is not one-to-one. However, if we restrict the domain to nonnegative values, $x \geq 0$, then the graph does pass the horizontal line test.



On the restricted domain, g is one-to-one and we can find its inverse.

Example 8

Find the inverse of the function defined by $g\left(x\right) = x^2 + 1$ where $x \geq 0$.

Solution:

Begin by replacing the function notation $g\left(x\right)$ with y.

$$\begin{aligned} g\left(x\right) &= x^2 + 1 \\ y &= x^2 + 1 \quad \textit{where} \quad x \geq 0 \end{aligned}$$

Interchange x and y.

$$x = y^2 + 1 \quad \textit{where} \quad y \geq 0$$

Solve for y.

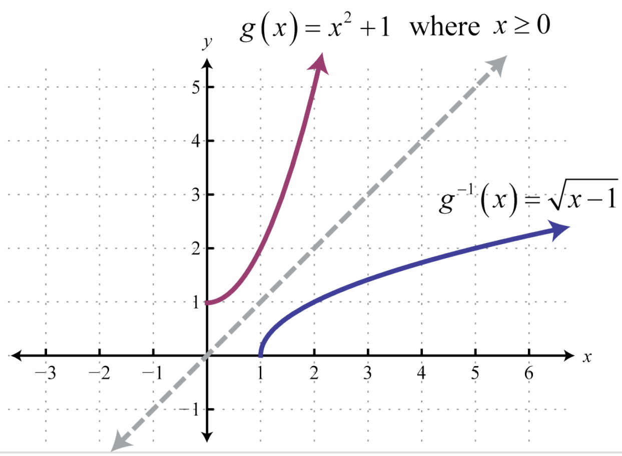
$$\begin{aligned} x &= y^2 + 1 \\ x - 1 &= y^2 \\ \pm\sqrt{x - 1} &= y \end{aligned}$$

Since $y \geq 0$ we only consider the positive result.

$$y = \sqrt{x - 1}$$
$$g^{-1}(x) = \sqrt{x - 1}$$

Answer: $g^{-1}(x) = \sqrt{x - 1}$. The check is left to the reader.

The graphs in the previous example are shown on the same set of axes below. Take note of the symmetry about the line $y = x$.



Example 9

Find the inverse of the function defined by $f\,(x) = \frac{2x+1}{x-3}$.

Solution:

Use a graphing utility to verify that this function is one-to-one. Begin by replacing the function notation $f\,(x)$ with y.

$$\begin{aligned} f\,(x) &= \frac{2x + 1}{x - 3} \\ y &= \frac{2x + 1}{x - 3} \end{aligned}$$

Interchange x and y.

$$x = \frac{2y + 1}{y - 3}$$

Solve for y.

$$\begin{aligned} x &= \frac{2y + 1}{y - 3} \\ x\left(y - 3\right) &= 2y + 1 \\ xy - 3x &= 2y + 1 \end{aligned}$$

Obtain all terms with the variable y on one side of the equation and everything else on the other. This will enable us to treat y as a GCF.

$$\begin{aligned} xy - 3x &= 2y + 1 \\ xy - 2y &= 3x + 1 \\ y(x - 2) &= 3x + 1 \\ y &= \frac{3x + 1}{x - 2} \end{aligned}$$

Answer: $f^{-1}(x) = \frac{3x+1}{x-2}$ .The check is left to the reader.

Try this! Find the inverse of $f(x) = \sqrt[3]{x + 1} - 3$.

Answer: $f^{-1}(x) = (x + 3)^3 - 1$

(click to see video)

KEY TAKEAWAYS

- The composition operator ($\circ$) indicates that we should substitute one function into another. In other words, $(f \circ g)(x) = f(g(x))$ indicates that we substitute $g(x)$ into $f(x)$.
- If two functions are inverses, then each will reverse the effect of the other. Using notation, $(f \circ g)(x) = f(g(x)) = x$ and $(g \circ f)(x) = g(f(x)) = x$.
- Inverse functions have special notation. If g is the inverse of f, then we can write $g(x) = f^{-1}(x)$. This notation is often confused with negative exponents and does not equal one divided by $f(x)$.
- The graphs of inverses are symmetric about the line $y = x$. If (a, b) is a point on the graph of a function, then (b, a) is a point on the graph of its inverse.
- If each point in the range of a function corresponds to exactly one value in the domain then the function is one-to-one. Use the horizontal line test to determine whether or not a function is one-to-one.
- A one-to-one function has an inverse, which can often be found by interchanging *x* and *y*, and solving for *y*. This new function is the inverse of the original function.

TOPIC EXERCISES

PART A: COMPOSITION OF FUNCTIONS

Given the functions defined by *f* and *g* find $\left(f \circ g\right)(x)$ **and** $\left(g \circ f\right)(x)$.

1. $f(x) = 4x - 1, g(x) = 3x$
2. $f(x) = -2x + 5, g(x) = 2x$
3. $f(x) = 3x - 5, g(x) = x - 4$
4. $f(x) = 5x + 1, g(x) = 2x - 3$
5. $f(x) = x^2 - x + 1, g(x) = 2x - 1$
6. $f(x) = x^2 - 3x - 2, g(x) = x - 2$
7. $f(x) = x^2 + 3, g(x) = x^2 - 5$
8. $f(x) = 2x^2, g(x) = x^2 - x$
9. $f(x) = 8x^3 + 5, g(x) = \sqrt[3]{x - 5}$
10. $f(x) = 27x^3 - 1, g(x) = \sqrt[3]{x + 1}$
11. $f(x) = \frac{1}{x+5}, g(x) = \frac{1}{x}$
12. $f(x) = \frac{1}{x} - 3, g(x) = \frac{3}{x+3}$
13. $f(x) = 5\sqrt{x}, g(x) = 3x - 2$
14. $f(x) = \sqrt{2x}, g(x) = 4x + 1$
15. $f(x) = \frac{1}{2x}, g(x) = x^2 + 8$
16. $f(x) = 2x - 1, g(x) = \frac{1}{x+1}$
17. $f(x) = \frac{1-x}{2x}, g(x) = \frac{1}{2x+1}$
18. $f(x) = \frac{2x}{x+1}, g(x) = \frac{x+1}{x}$

Given the functions defined by $f(x) = 3x^2 - 2$, $g(x) = 5x + 1$, **and** $h(x) = \sqrt{x}$, **calculate the following.**

19. $(f \circ g)(2)$

20. $(g \circ f)(-1)$

21. $(g \circ f)(0)$

22. $(f \circ g)(0)$

23. $(f \circ h)(3)$

24. $(g \circ h)(16)$

25. $(h \circ g)\left(\frac{3}{5}\right)$

26. $(h \circ f)(-3)$

Given the functions defined by $f(x) = \sqrt[3]{x+3}$, $g(x) = 8x^3 - 3$, **and** $h(x) = 2x - 1$, **calculate the following.**

27. $(f \circ g)(1)$

28. $(g \circ f)(-2)$

29. $(g \circ f)(0)$

30. $(f \circ g)(-2)$

31. $(f \circ h)(-1)$

32. $(h \circ g)\left(-\frac{1}{2}\right)$

33. $(h \circ f)(24)$

34. $(g \circ h)(0)$

Given the function, determine $(f \circ f)(x)$.

35. $f(x) = 3x - 1$

36. $f(x) = \frac{2}{5}x + 1$

37. $f(x) = x^2 + 5$

38. $f(x) = x^2 - x + 6$

39. $f(x) = x^3 + 2$

40. $f(x) = x^3 - x$

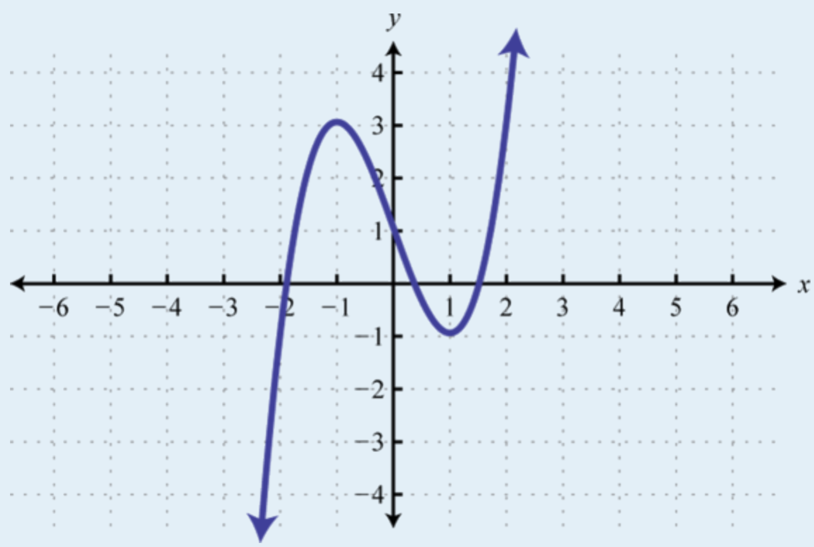
41. $f(x) = \frac{1}{x+1}$

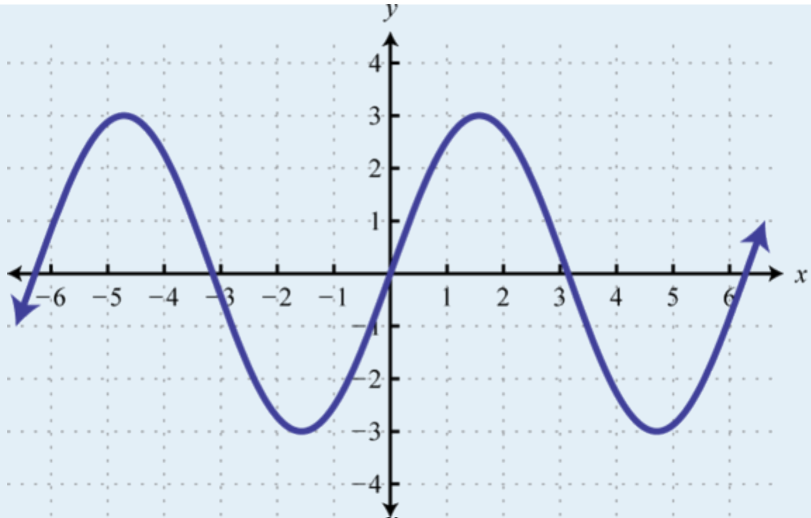
42. $f(x) = \frac{x+1}{2x}$

PART B: INVERSE FUNCTIONS

Are the given functions one-to-one? Explain.

43.





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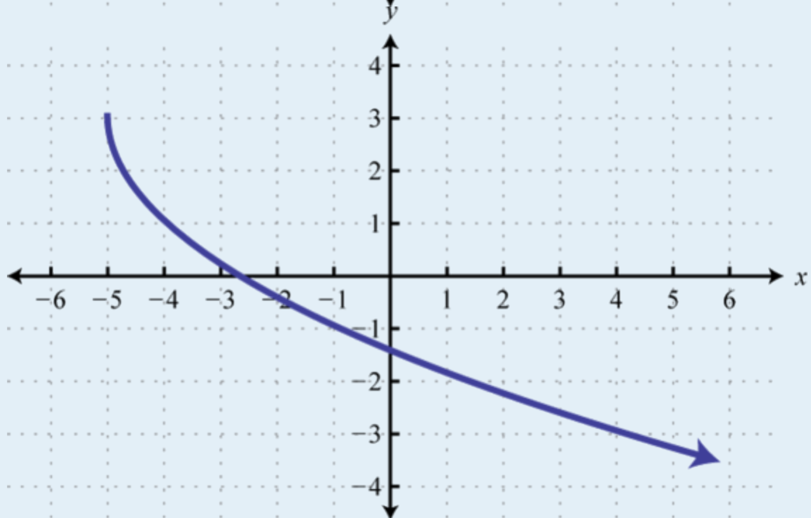

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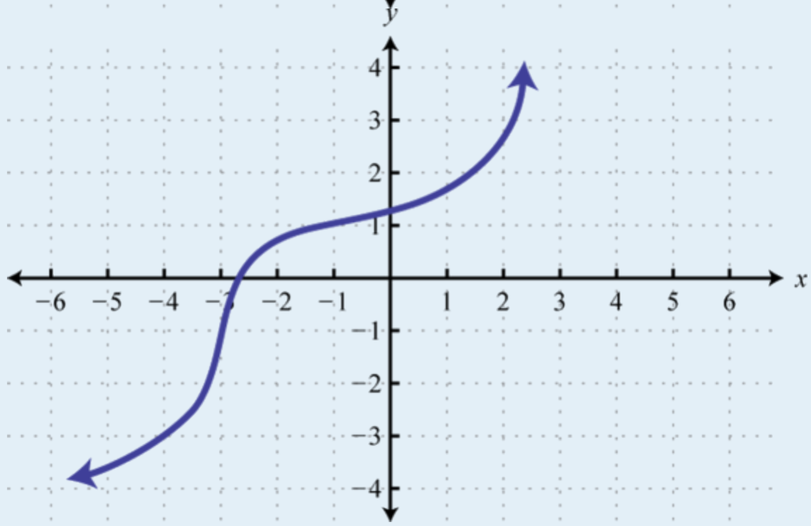


46.

47. $f(x) = x + 1$

48. $g(x) = x^2 + 1$

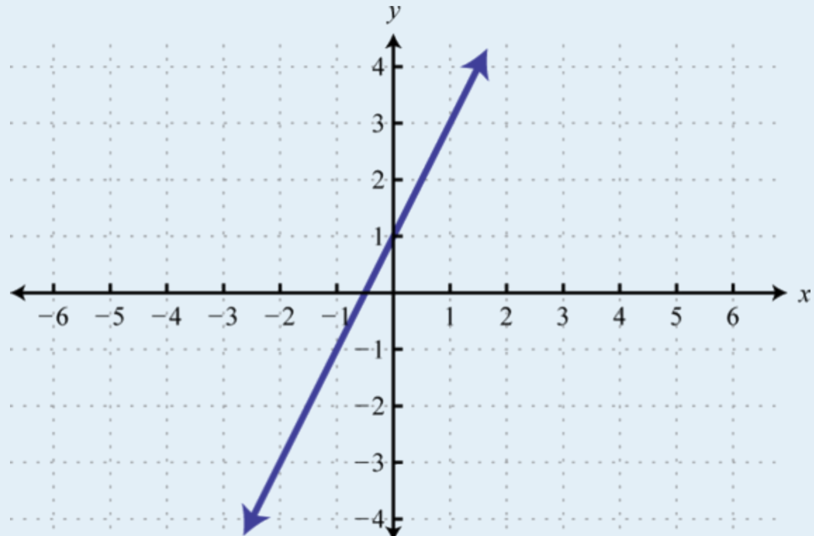
49. $h(x) = |x| + 1$

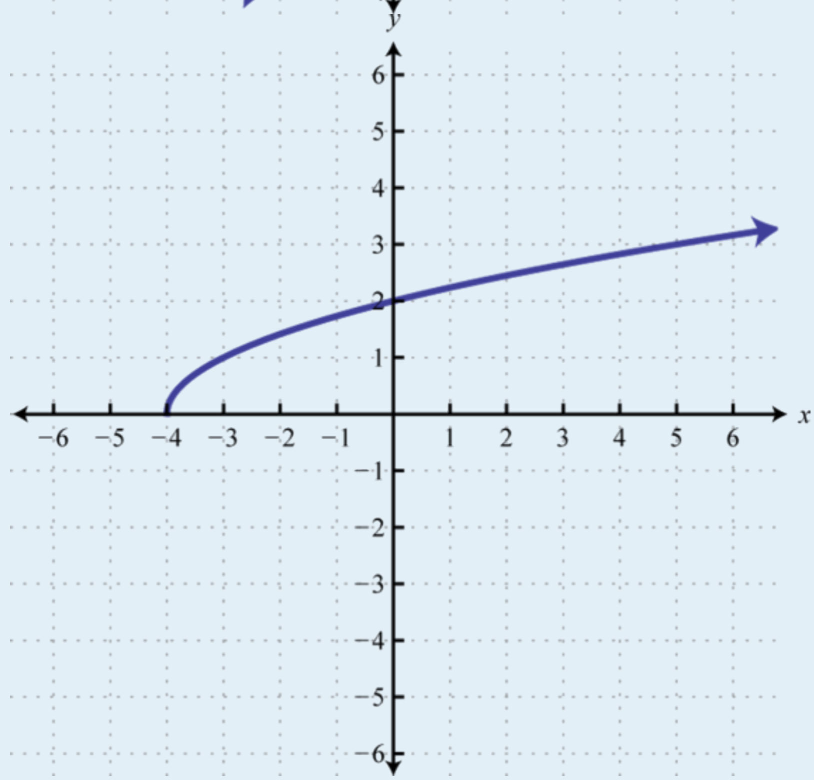
50. $r(x) = x^3 + 1$

51. $f(x) = \sqrt{x+1}$

52. $g(x) = 3$

Given the graph of a one-to-one function, graph its inverse.

53. 

54. 

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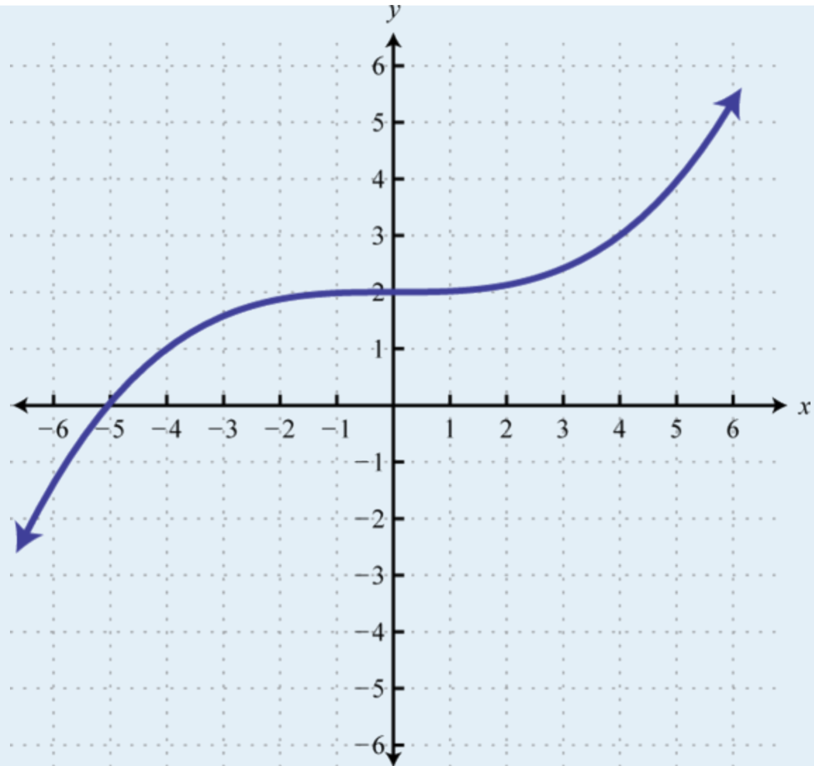

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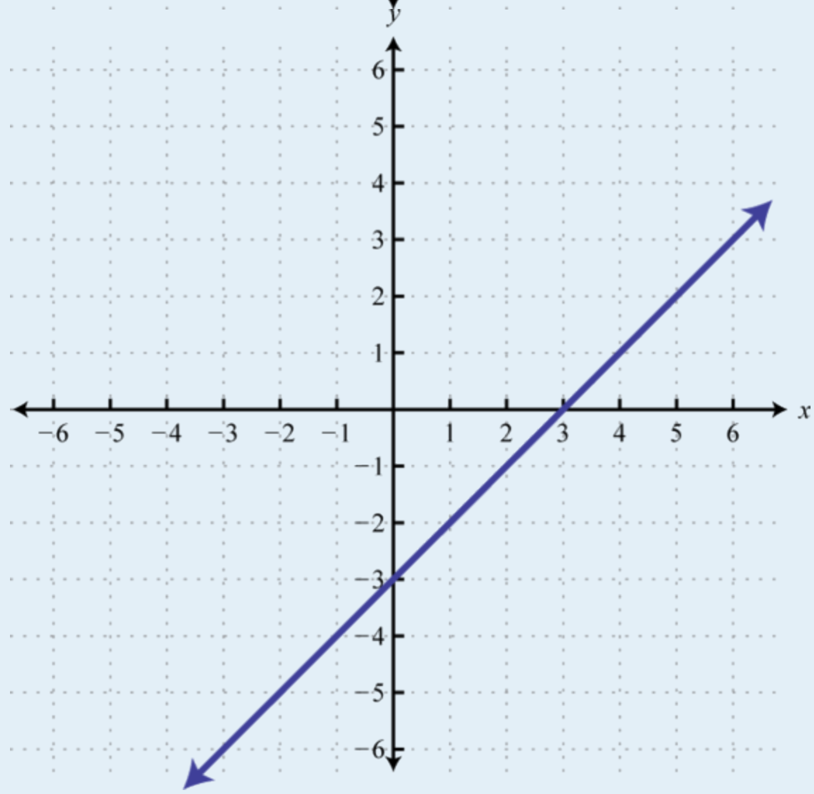
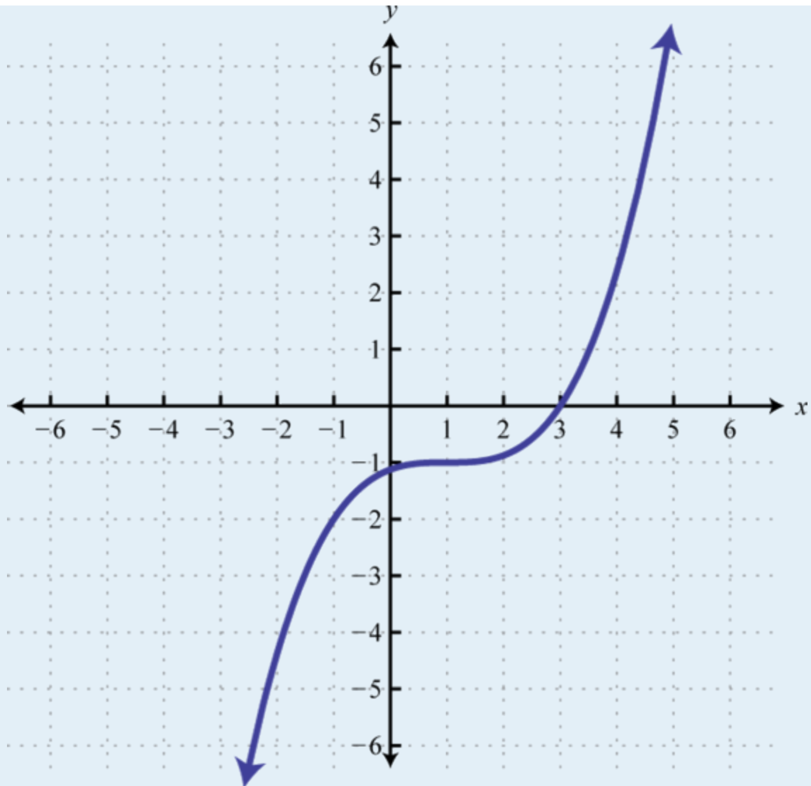
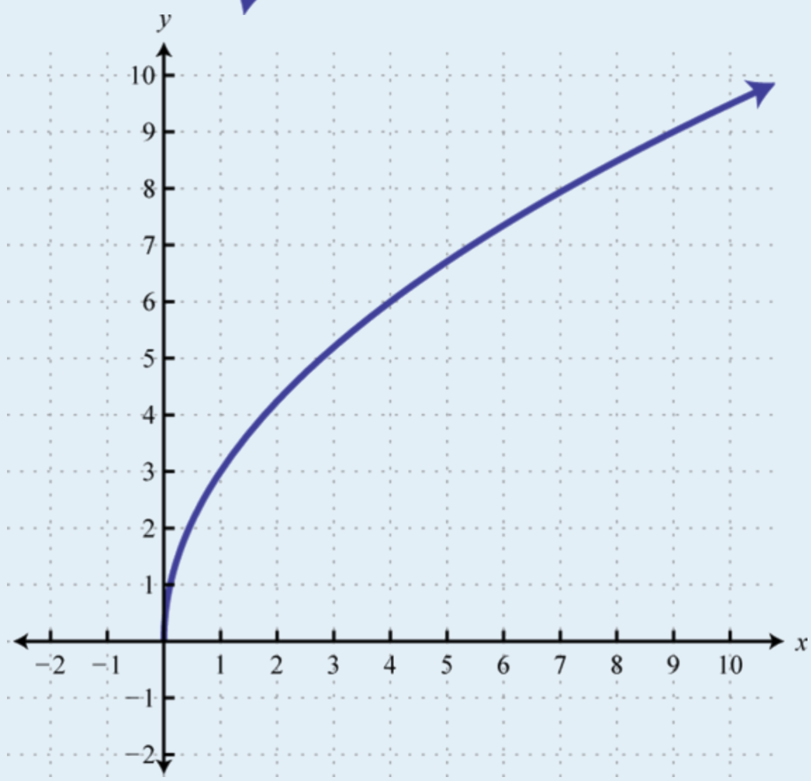


57. 

58. 

Verify algebraically that the two given functions are inverses. In other words, show that $\left(f \circ f^{-1}\right)(x) = x$ **and** $\left(f^{-1} \circ f\right)(x) = x$.

59. $f(x) = 3x - 4,\, f^{-1}(x) = \frac{x+4}{3}$

60. $f(x) = -5x + 1,\, f^{-1}(x) = \frac{1-x}{5}$

61. $f(x) = -\frac{2}{3}x + 1,\, f^{-1}(x) = -\frac{3}{2}x + \frac{3}{2}$

62. $f(x) = 4x - \frac{1}{3},\, f^{-1}(x) = \frac{1}{4}x + \frac{1}{12}$

63. $f(x) = \sqrt{x - 8},\, f^{-1}(x) = x^2 + 8, x \geq 0$

64. $f(x) = \sqrt[3]{6x} - 3,\, f^{-1}(x) = \frac{(x+3)^3}{6}$

65. $f(x) = \frac{x}{x+1},\, f^{-1}(x) = \frac{x}{1-x}$

66. $f(x) = \frac{x-3}{3x},\, f^{-1}(x) = \frac{3}{1-3x}$

67. $f(x) = 2(x-1)^3 + 3,\, f^{-1}(x) = 1 + \sqrt[3]{\frac{x-3}{2}}$

68. $f(x) = \sqrt[3]{5x-1} + 4,\, f^{-1}(x) = \frac{(x-4)^3+1}{5}$

Find the inverses of the following functions.

69. $f(x) = 5x$

70. $f(x) = \frac{1}{2}x$

71. $f(x) = 2x + 5$

72. $f(x) = -4x + 3$

73. $f(x) = -\frac{2}{3}x + \frac{1}{3}$

74. $f(x) = -\frac{1}{2}x + \frac{3}{4}$

75. $g(x) = x^2 + 5, x \geq 0$

76. $g(x) = x^2 - 7, x \geq 0$

77. $f(x) = (x - 5)^2, x \geq 5$

78. $f(x) = (x + 1)^2, x \geq -1$

79. $h(x) = 3x^3 + 5$

80. $h(x) = 2x^3 - 1$

81. $f(x) = (2x - 3)^3$

82. $f(x) = (x + 4)^3 - 1$

83. $g(x) = \frac{2}{x^3 + 1}$

84. $g(x) = \frac{1}{x^3} - 2$

85. $f(x) = \frac{5}{x + 1}$

86. $f(x) = \frac{1}{2x - 9}$

87. $f(x) = \frac{x + 5}{x - 5}$

88. $f(x) = \frac{3x - 4}{2x - 1}$

89. $h(x) = \frac{x - 5}{10x}$

90. $h(x) = \frac{9x + 1}{3x}$

91. $g(x) = \sqrt[3]{5x + 2}$

92. $g(x) = \sqrt[3]{4x - 3}$

93. $f(x) = \sqrt[3]{x - 6} - 4$

94. $f(x) = 2\sqrt[3]{x + 2} + 5$

95. $h(x) = \sqrt[5]{x + 1} - 3$

96. $h(x) = \sqrt[5]{x - 8} + 1$

97. $f(x) = mx + b, m \neq 0$

98. $f(x) = ax^2 + c, x \geq 0$

99. $f(x) = ax^3 + d$

100. $f(x) = a(x - h)^2 + k, x \geq h$

Graph the function and its inverse on the same set of axes.

101. $f(x) = x + 2$

102. $f(x) = \frac{2}{3}x - 4$

103. $f(x) = -2x + 2$

104. $f(x) = -\frac{1}{3}x + 4$

105. $g(x) = x^2 - 2, x \geq 0$

106. $g(x) = (x - 2)^2, x \geq 2$

107. $h(x) = x^3 + 1$

108. $h(x) = (x + 2)^3 - 2$

109. $f(x) = 2 - \sqrt{x}$

110. $f(x) = \sqrt{-x} + 1$

PART C: DISCUSSION BOARD

111. Is composition of functions associative? Explain.

112. Explain why $C(x) = \frac{5}{9}(x - 32)$ and $F(x) = \frac{9}{5}x + 32$ define inverse functions. Prove it algebraically.

113. Do the graphs of all straight lines represent one-to-one functions? Explain.

114. If the graphs of inverse functions intersect, then how can we find the point of intersection? Explain.

ANSWERS

1. $(f \circ g)(x) = 12x - 1$; $(g \circ f)(x) = 12x - 3$

3. $(f \circ g)(x) = 3x - 17$; $(g \circ f)(x) = 3x - 9$

5. $(f \circ g)(x) = 4x^2 - 6x + 3$; $(g \circ f)(x) = 2x^2 - 2x + 1$

7. $(f \circ g)(x) = x^4 - 10x^2 + 28$; $(g \circ f)(x) = x^4 + 6x^2 + 4$

9. $(f \circ g)(x) = 8x - 35$; $(g \circ f)(x) = 2x$

11. $(f \circ g)(x) = \frac{x}{5x+1}$; $(g \circ f)(x) = x + 5$

13. $(f \circ g)(x) = 5\sqrt{3x-2}$; $(g \circ f)(x) = 15\sqrt{x} - 2$

15. $(f \circ g)(x) = \dfrac{1}{2x^2 + 16}$; $(g \circ f)(x) = \dfrac{1 + 32x^2}{4x^2}$

17. $(f \circ g)(x) = x$; $(g \circ f)(x) = x$

19. 361

21. -9

23. 7

25. 2

27. 2

29. 21

31. 0

33. 5

35. $(f \circ f)(x) = 9x - 4$

37. $(f \circ f)(x) = x^4 + 10x^2 + 30$

39. $(f \circ f)(x) = x^9 + 6x^6 + 12x^3 + 10$

41. $(f \circ f)(x) = \frac{x+1}{x+2}$

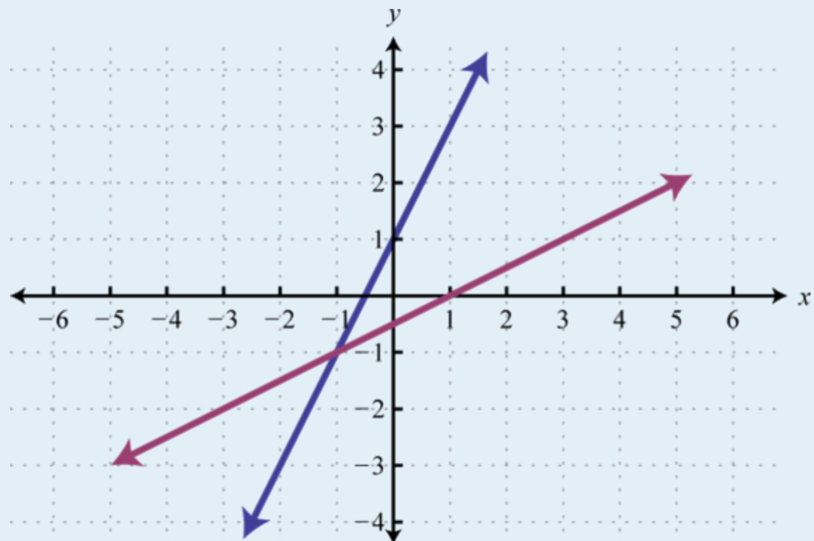
43. No, fails the HLT

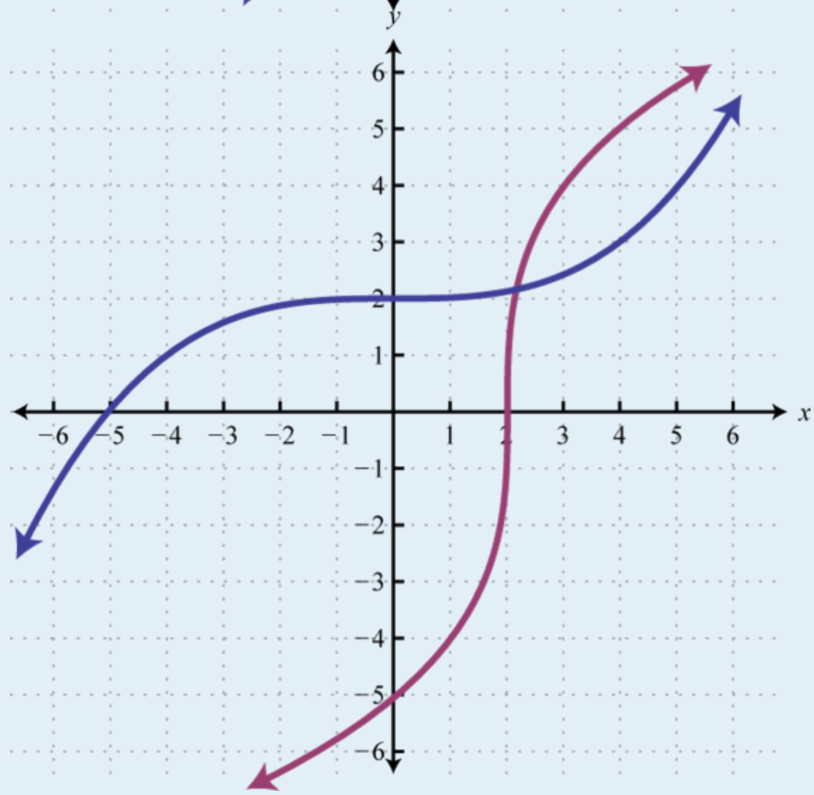
45. Yes, passes the HLT

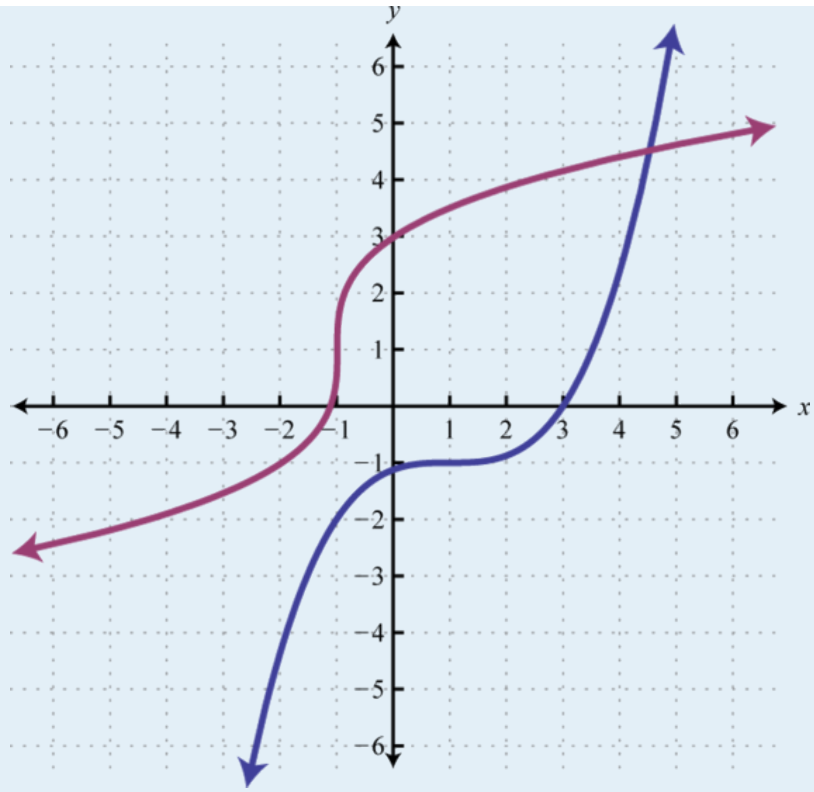
47. Yes, its graph passes the HLT.

49. No, its graph fails the HLT.

51. Yes, its graph passes the HLT.

53. 

55. 

57. 

59. Proof

61. Proof

63. Proof

65. Proof

67. Proof

69. $f^{-1}(x) = \frac{x}{5}$

71. $f^{-1}(x) = \frac{1}{2}x - \frac{5}{2}$

73. $f^{-1}(x) = -\frac{3}{2}x + \frac{1}{2}$

75. $g^{-1}(x) = \sqrt{x - 5}$

77. $f^{-1}(x) = \sqrt{x} + 5$

79. $h^{-1}(x) = \sqrt[3]{\frac{x - 5}{3}}$

81. $f^{-1}(x) = \frac{\sqrt[3]{x} + 3}{2}$

83. $g^{-1}(x) = \sqrt[3]{\frac{2-x}{x}}$

85. $f^{-1}(x) = \frac{5-x}{x}$

87. $f^{-1}(x) = \frac{5(x+1)}{x-1}$

89. $h^{-1}(x) = -\frac{5}{10x-1}$

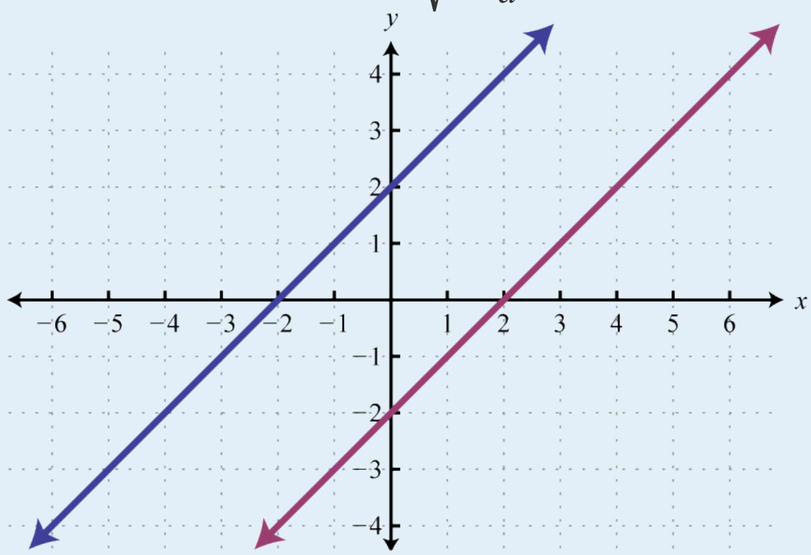
91. $g^{-1}(x) = \frac{x^3-2}{5}$

93. $f^{-1}(x) = (x+4)^3 + 6$

95. $h^{-1}(x) = (x+3)^5 - 1$

97. $f^{-1}(x) = \frac{x-b}{m}$

99. $f^{-1}(x) = \sqrt[3]{\frac{x-d}{a}}$

101. 

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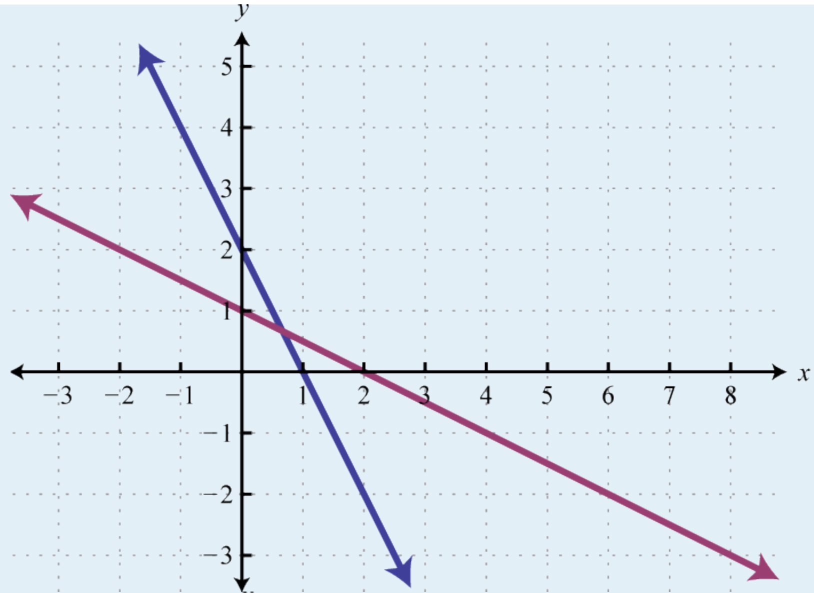

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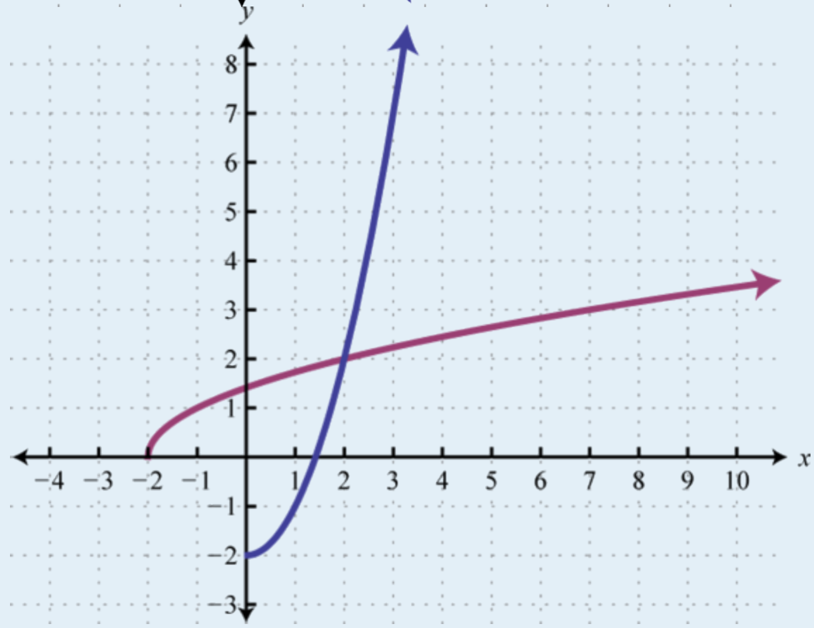
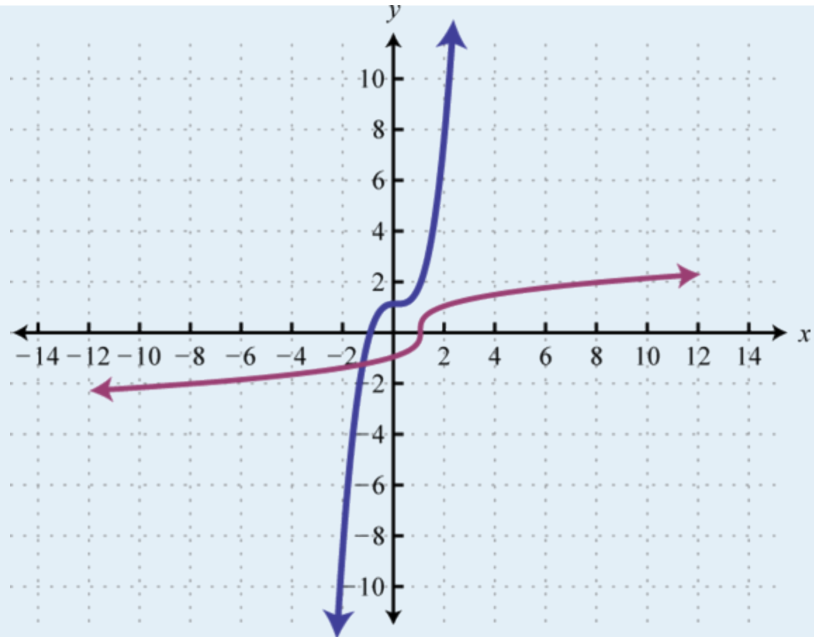
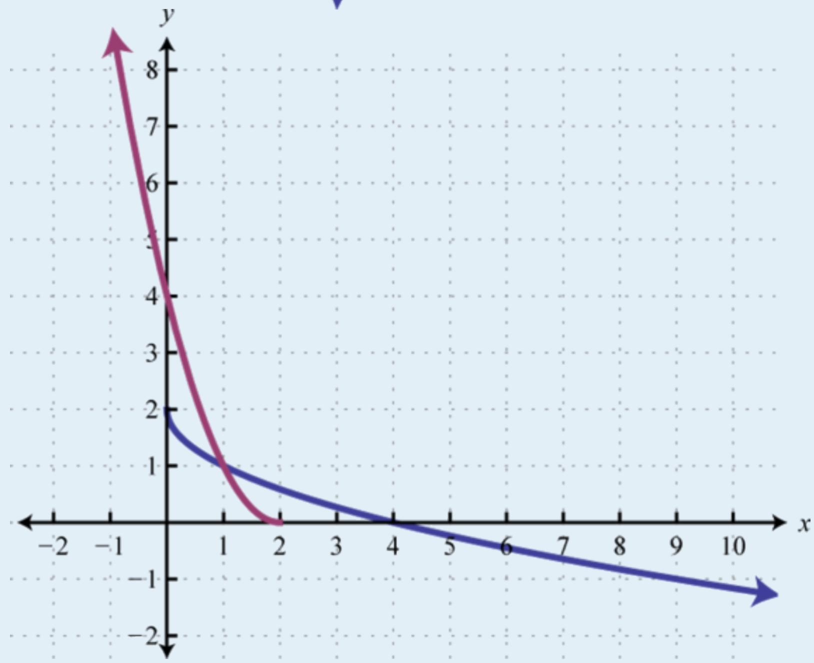


107. 

109. 

111. Answer may vary

113. Answer may vary

7.2 Exponential Functions and Their Graphs

LEARNING OBJECTIVES

1. Identify and evaluate exponential functions.
2. Sketch the graph of exponential functions and determine the domain and range.
3. Identify and graph the natural exponential function.
4. Apply the formulas for compound interest.

Exponential Functions

At this point in our study of algebra we begin to look at transcendental functions or functions that seem to "transcend" algebra. We have studied functions with variable bases and constant exponents such as x^2 or y^{-3}. In this section we explore functions with a constant base and variable exponents. Given a real number $b > 0$ where $b \neq 1$ an **exponential function**[5] has the form,

$$f(x) = b^x \quad \textit{Exponential Function}$$

For example, if the base b is equal to 2, then we have the exponential function defined by $f(x) = 2^x$. Here we can see the exponent is the variable. Up to this point, rational exponents have been defined but irrational exponents have not. Consider $2^{\sqrt{7}}$, where the exponent is an irrational number in the range,

$$2.64 < \sqrt{7} < 2.65$$

We can use these bounds to estimate $2^{\sqrt{7}}$,

5. Any function with a definition of the form $f(x) = b^x$ where $b > 0$ and $b \neq 1$.

$$2^{2.64} < 2^{\sqrt{7}} < 2^{2.65}$$
$$6.23 < 2^{\sqrt{7}} < 6.28$$

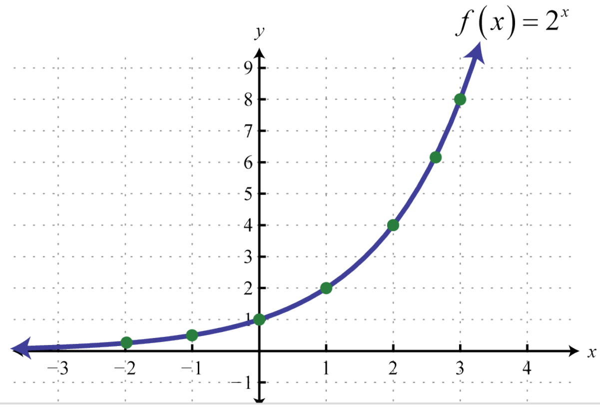
Using rational exponents in this manner, an approximation of $2^{\sqrt{7}}$ can be obtained to any level of accuracy. On a calculator,

$$2 \wedge \sqrt{7} \approx 6.26$$

Therefore the domain of any exponential function consists of all real numbers $(-\infty, \infty)$. Choose some values for *x* and then determine the corresponding *y*-values.

x	y	$f(x) = 2^x$	*Solutions*
-2	$\frac{1}{4}$	$y = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$
-1	$\frac{1}{2}$	$y = 2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$
0	1	$y = 2^0 = 1$	$(0, 1)$
1	2	$y = 2^1 = 2$	$(1, 2)$
2	4	$y = 2^2 = 4$	$(2, 4)$
$\sqrt{7}$	6.26	$y = 2^{\sqrt{7}} \approx 6.26$	$(2.65, 6.26)$

Because exponents are defined for any real number we can sketch the graph using a continuous curve through these given points:

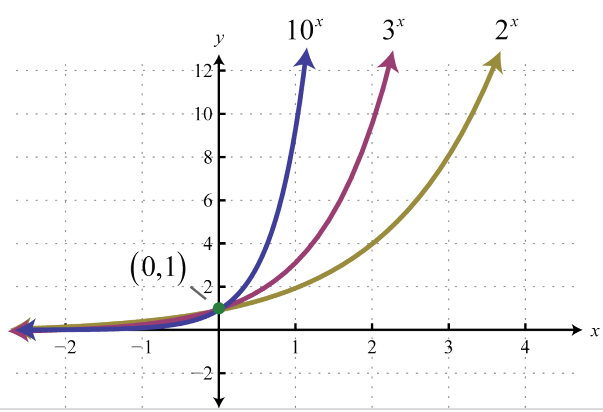


It is important to point out that as *x* approaches negative infinity, the results become very small but never actually attain zero. For example,

$$f(-5)=2^{-5} = \frac{1}{2^5} \approx 0.03125$$
$$f(-10)=2^{-10} = \frac{1}{2^{10}} \approx 0.0009766$$
$$f(-15)=2^{-15} = \frac{1}{2^{-15}} \approx .00003052$$

This describes a horizontal asymptote at $y = 0$, the *x*-axis, and defines a lower bound for the range of the function: $(0, \infty)$.

The base *b* of an exponential function affects the rate at which it grows. Below we have graphed $y = 2^x$, $y = 3^x$, and $y = 10^x$ on the same set of axes.



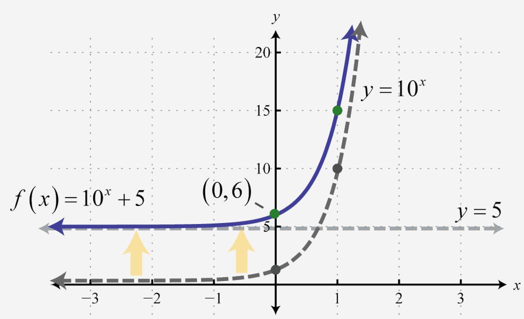
Note that all of these exponential functions have the same y-intercept, namely $(0, 1)$. This is because $f(0) = b^0 = 1$ for any function defined using the form $f(x) = b^x$. As the functions are read from left to right, they are interpreted as increasing or growing exponentially. Furthermore, any exponential function of this form will have a domain that consists of all real numbers $(-\infty, \infty)$ and a range that consists of positive values $(0, \infty)$ bounded by a horizontal asymptote at $y = 0$.

Example 1

Sketch the graph and determine the domain and range: $f(x) = 10^x + 5$.

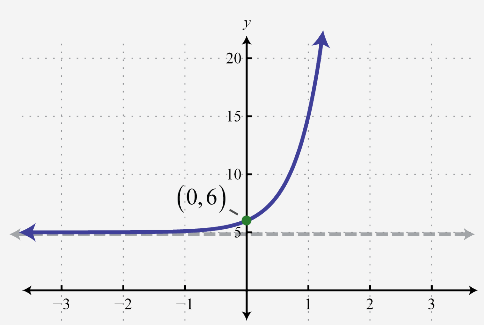
Solution:

The base 10 is used often, most notably with scientific notation. Hence, 10 is called the **common base**. In fact, the exponential function $y = 10^x$ is so important that you will find a button $\boxed{10^x}$ dedicated to it on most modern scientific calculators. In this example, we will sketch the basic graph $y = 10^x$ and then shift it up 5 units.



Note that the horizontal asymptote of the basic graph $y = 10^x$ was shifted up 5 units to $y = 5$ (shown dashed). Take a minute to evaluate a few values of x with your calculator and convince yourself that the result will never be less than 5.

Answer:

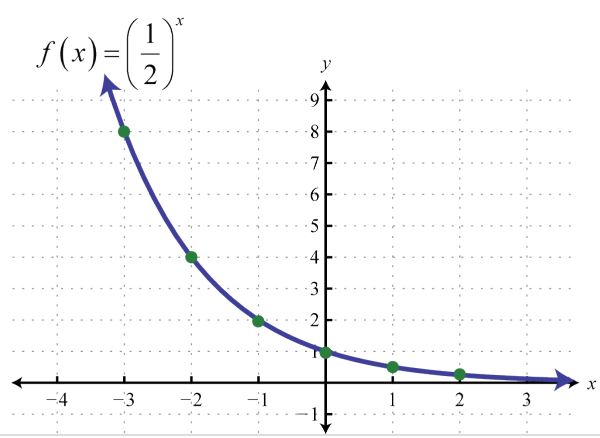


Domain: $(-\infty, \infty)$; Range: $(5, \infty)$

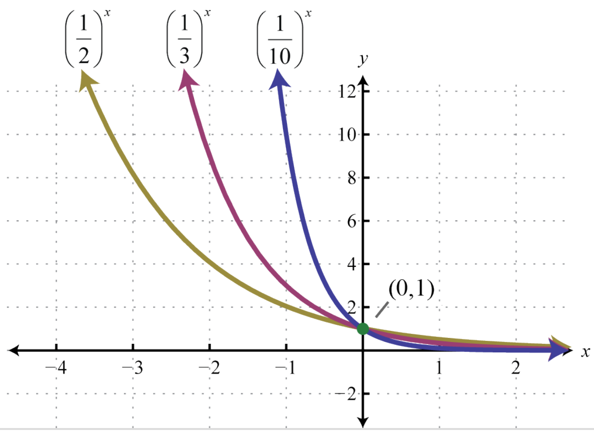
Next consider exponential functions with fractional bases $0 < b < 1$. For example, $f(x) = \left(\frac{1}{2}\right)^x$ is an exponential function with base $b = \frac{1}{2}$.

x	y	$f(x) = \left(\frac{1}{2}\right)^x$	*Solutions*
-2	4	$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{-2} = \frac{1^{-2}}{2^{-2}} = \frac{2^2}{1^2} = 4$	$(-2, 4)$
-1	2	$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{-1} = \frac{1^{-1}}{2^{-1}} = \frac{2^1}{1^1} = 2$	$(-1, 2)$
0	1	$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{0} = 1$	$(0, 1)$
1	$\frac{1}{2}$	$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{1} = \frac{1}{2}$	$\left(1, \frac{1}{2}\right)$
2	$\frac{1}{4}$	$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{2} = \frac{1}{4}$	$\left(2, \frac{1}{4}\right)$

Plotting points we have,



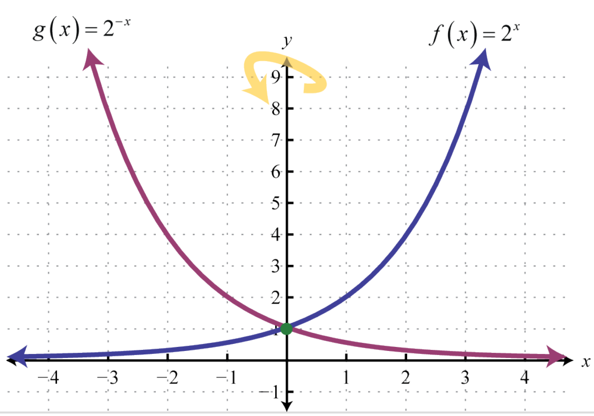
Reading the graph from left to right, it is interpreted as decreasing exponentially. The base affects the rate at which the exponential function decreases or decays. Below we have graphed $y = \left(\frac{1}{2}\right)^x$, $y = \left(\frac{1}{3}\right)^x$, and $y = \left(\frac{1}{10}\right)^x$ on the same set of axes.



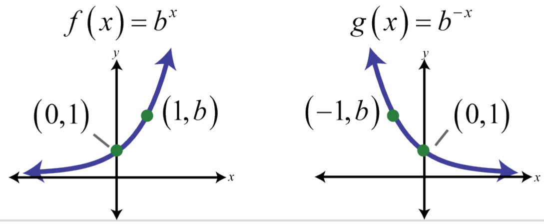
Recall that $x^{-1} = \frac{1}{x}$ and so we can express exponential functions with fractional bases using negative exponents. For example,

$$g(x) = \left(\frac{1}{2}\right)^x = \frac{1^x}{2^x} = \frac{1}{2^x} = 2^{-x}.$$

Furthermore, given that $f(x) = 2^x$ we can see $g(x) = f(-x) = 2^{-x}$ and can consider g to be a reflection of f about the y-axis.



In summary, given $b > 0$



And for both cases,

$$\begin{aligned} \textit{Domain}: & \ (-\infty, \infty) \\ \textit{Range}: & \ (0, \infty) \\ y-\textit{intercept}: & \ (0, 1) \\ \textit{Asymptote}: & \ y = 0 \end{aligned}$$

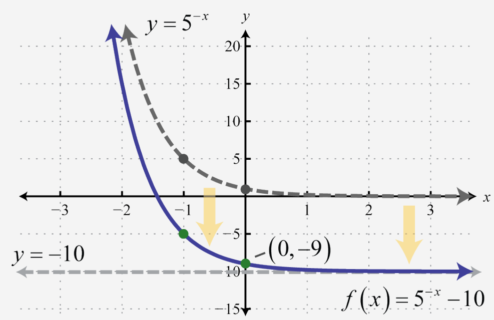
Furthermore, note that the graphs pass the horizontal line test and thus exponential functions are one-to-one. We use these basic graphs, along with the transformations, to sketch the graphs of exponential functions.

Example 2

Sketch the graph and determine the domain and range: $f(x) = 5^{-x} - 10$.

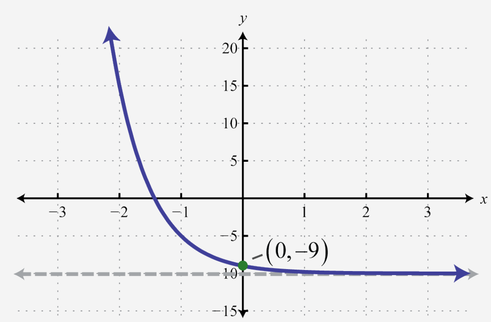
Solution:

Begin with the basic graph $y = 5^{-x}$ and shift it down 10 units.



The y-intercept is $(0, -9)$ and the horizontal asymptote is $y = -10$.

Answer:



Domain: $(-\infty, \infty)$; Range: $(-10, \infty)$

Note: Finding the x-intercept of the graph in the previous example is left for a later section in this chapter. For now, we are more concerned with the general shape of exponential functions.

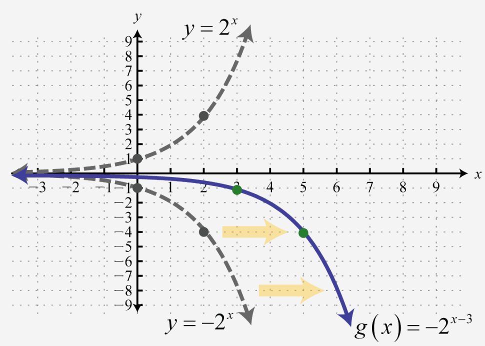
Example 3

Sketch the graph and determine the domain and range: $g(x) = -2^{x-3}$.

Solution:

Begin with the basic graph $y = 2^x$ and identify the transformations.

$$\begin{aligned} y &= 2^x && \textit{Basic graph} \\ y &= -2^x && \textit{Reflection about the x-axis} \\ y &= -2^{x-3} && \textit{Shift right 3 units} \end{aligned}$$

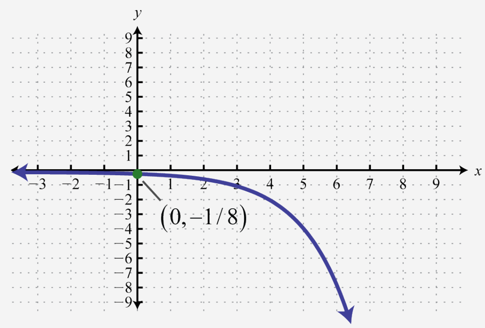


Note that the horizontal asymptote remains the same for all of the transformations. To finish we usually want to include the y-intercept. Remember that to find the y-intercept we set $x = 0$.

$$\begin{aligned} g(0) &= -2^{0-3} \\ &= -2^{-3} \\ &= -\frac{1}{2^3} \\ &= -\frac{1}{8} \end{aligned}$$

Therefore the y-intercept is $\left(0, \ -\frac{1}{8}\right)$.

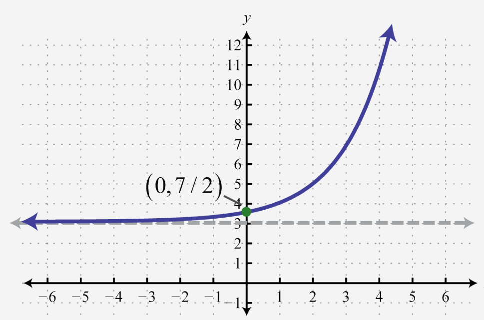
Answer:



Domain: $(-\infty, \infty)$; Range: $(-\infty, 0)$

Try this! Sketch the graph and determine the domain and range: $f(x) = 2^{x-1} + 3$.

Answer:



Domain: $(-\infty, \infty)$; Range: $(3, \infty)$

(click to see video)

Natural Base e

Some numbers occur often in common applications. One such familiar number is pi (π), which we know occurs when working with circles. This irrational number has a dedicated button on most calculators $\boxed{\pi}$ and approximated to five decimal places, $\pi \approx 3.14159$. Another important number *e* occurs when working with exponential growth and decay models. It is an irrational number and approximated to five decimal places, $e \approx 2.71828$. This constant occurs naturally in many real-world applications and thus is called the **natural base**. Sometimes *e* is called Euler's constant in honor of Leonhard Euler (pronounced "Oiler").

Figure 7.1

Leonhard Euler (1707–1783)

In fact, the **natural exponential function**,

$$f(x) = e^x$$

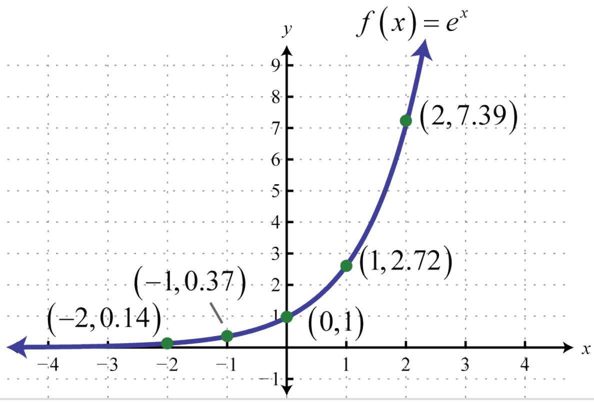
is so important that you will find a button $\boxed{e^x}$ dedicated to it on any modern scientific calculator. In this section, we are interested in evaluating the natural exponential function for given real numbers and sketching its graph. To evaluate the natural exponential function, defined by $f(x) = e^x$ where $x = -2$ using a calculator, you may need to apply the shift button. On many scientific calculators the caret will display as follows,

$$f(-2) = e \text{ ^ } (-2) \approx 0.13534$$

After learning how to use your particular calculator, you can now sketch the graph by plotting points. (Round off to the nearest hundredth.)

x	y	$f(x) = e^x$	*Solutions*
-2	0.14	$f(-2) = e^{-2} = 0.14$	$(-2,\ 0.14)$
-1	0.37	$f(-1) = e^{-1} = 0.37$	$(-1,\ 0.37)$
0	1	$f(0) = e^{0} = 1$	$(0,\ 1)$
1	2.72	$f(1) = e^{1} = 2.72$	$(1,\ 2.72)$
2	7.39	$f(2) = e^{2} = 7.39$	$(2,\ 7.39)$

Plot the points and sketch the graph.



Note that the function is similar to the graph of $y = 3^x$. The domain consists of all real numbers and the range consists of all positive real numbers. There is an asymptote at $y = 0$ and a y-intercept at $(0, 1)$. We can use the transformations to sketch the graph of more complicated exponential functions.

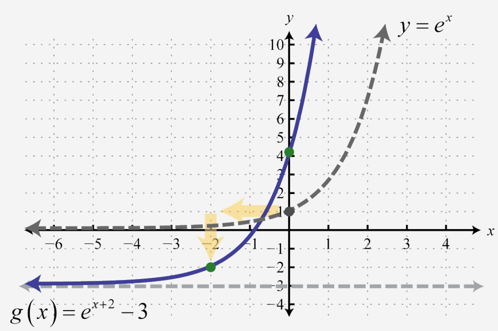
Example 4

Sketch the graph and determine the domain and range: $g(x) = e^{x+2} - 3$.

Solution:

Identify the basic transformations.

$$\begin{aligned} y &= e^{x} && \textit{Basic graph} \\ y &= e^{x+2} && \textit{Shift left 2 units} \\ y &= e^{x+2} - 3 && \textit{Shift down 3 units} \end{aligned}$$

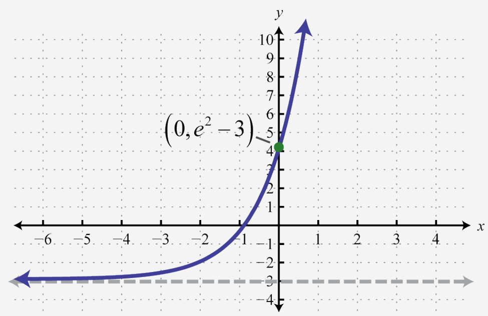


To determine the y-intercept set $x = 0$.

$$\begin{aligned} g(0) &= e^{0+2} - 3 \\ &= e^{2} - 3 \\ &\approx 4.39 \end{aligned}$$

Therefore the y-intercept is $\left(0, e^{2} - 3\right)$.

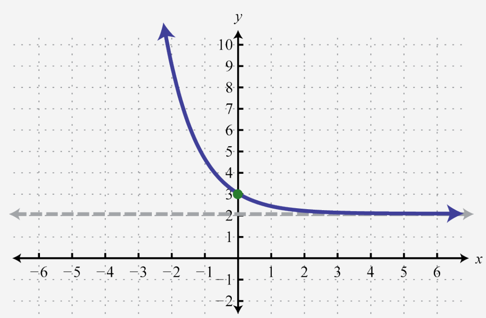
Answer:



Domain: $(-\infty, \infty)$; Range: $(-3, \infty)$

Try this! Sketch the graph and determine the domain and range: $f(x) = e^{-x} + 2$.

Answer:



Domain: $(-\infty, \infty)$; Range: $(2, \infty)$

(click to see video)

Compound Interest Formulas

Exponential functions appear in formulas used to calculate interest earned in most regular savings accounts. Compound interest occurs when interest accumulated for one period is added to the principal investment before calculating interest for the next period. The amount accrued in this manner over time is modeled by the **compound interest formula**[6]:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

Here the amount A depends on the time t in years the principal P is accumulating compound interest at an annual interest rate r. The value n represents the number of times the interest is compounded in a year.

6. A formula that gives the amount accumulated by earning interest on principal and interest over time: $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$.

Example 5

An investment of \$500 is made in a 6-year CD that earns $4\frac{1}{2}\%$ annual interest that is compounded monthly. How much will the CD be worth at the end of the 6-year term?

Solution:

Here the principal $P = \$500$, the interest rate $r = 4\frac{1}{2}\% = 0.045$, and because the interest is compounded monthly, $n = 12$. The investment is modeled by the following,

$$A(t) = 500\left(1 + \frac{0.045}{12}\right)^{12t}$$

To determine the amount in the account after 6 years evaluate $A(6)$ and round off to the nearest cent.

$$\begin{aligned} A(6) &= 500\left(1 + \frac{0.045}{12}\right)^{12(6)} \\ &= 500(1.00375)^{72} \\ &= 654.65 \end{aligned}$$

Answer: The CD will be worth \$654.65 at the end of the 6-year term.

Next we explore the effects of increasing *n* in the formula. For the sake of clarity we let *P* and *r* equal 1 and calculate accordingly.

Annual compounding	$\left(1+\frac{1}{n}\right)^{n}$
Yearly $(n = 1)$	$\left(1+\frac{1}{1}\right)^{1} = 2$
Semi-annually $(n = 2)$	$\left(1+\frac{1}{2}\right)^{2} = 2.25$
Quarterly $(n = 4)$	$\left(1+\frac{1}{4}\right)^{4} \approx 2.44140$
Monthly $(n = 12)$	$\left(1+\frac{1}{12}\right)^{12} \approx 2.61304$
Weekly $(n = 52)$	$\left(1+\frac{1}{52}\right)^{52} \approx 2.69260$
Daily $(n = 365)$	$\left(1+\frac{1}{365}\right)^{365} \approx 2.71457$
Hourly $(n = 8760)$	$\left(1+\frac{1}{8760}\right)^{8760} \approx 2.71813$

Continuing this pattern, as n increases to say compounding every minute or even every second, we can see that the result tends toward the natural base $e \approx 2.71828$. Compounding interest every instant leads to the **continuously compounding interest formula**[7],

7. A formula that gives the amount accumulated by earning continuously compounded interest: $A(t) = Pe^{rt}$.

$$A\,(t) = Pe^{rt}$$

Here P represents the initial principal amount invested, r represents the annual interest rate, and t represents the time in years the investment is allowed to accrue continuously compounded interest.

Example 6

An investment of $500 is made in a 6-year CD that earns $4\,\frac{1}{2}$% annual interest that is compounded continuously. How much will the CD be worth at the end of the 6-year term?

Solution:

Here the principal $P = \$500$, and the interest rate $r = 4\,\frac{1}{2}\,\% = 0.045$ Since the interest is compounded continuously we will use the formula $A\,(t) = Pe^{rt}$. The investment is modeled by the following,

$$A\,(t) = 500e^{0.045t}$$

To determine the amount in the account after 6 years, evaluate $A\,(6)$ and round off to the nearest cent.

$$\begin{aligned} A\,(6) &= 500e^{0.045(6)} \\ &= 500e^{0.27} \\ &= 654.98 \end{aligned}$$

Answer: The CD will be worth $654.98 at the end of the 6-year term.

Compare the previous two examples and note that compounding continuously may not be as beneficial as it sounds. While it is better to compound interest more often, the difference is not that profound. Certainly, the interest rate is a much greater factor in the end result.

Try this! How much will a \$1,200 CD, earning 5.2% annual interest compounded continuously, be worth at the end of a 10-year term?

Answer: \$2,018.43

(click to see video)

KEY TAKEAWAYS

- Exponential functions have definitions of the form $f(x) = b^x$ where $b > 0$ and $b \neq 1$. The domain consists of all real numbers $(-\infty, \infty)$ and the range consists of positive numbers $(0, \infty)$. Also, all exponential functions of this form have a *y*-intercept of $(0, 1)$ and are asymptotic to the *x*-axis.
- If the base of an exponential function is greater than 1 ($b > 1$), then its graph increases or grows as it is read from left to right.
- If the base of an exponential function is a proper fraction ($0 < b < 1$), then its graph decreases or decays as it is read from left to right.
- The number 10 is called the common base and the number *e* is called the natural base.
- The natural exponential function defined by $f(x) = e^x$ has a graph that is very similar to the graph of $g(x) = 3^x$.
- Exponential functions are one-to-one.

TOPIC EXERCISES

PART A: EXPONENTIAL FUNCTIONS

Evaluate.

1. $f(x) = 3^x$ where $f(-2), f(0)$, and $f(2)$.
2. $f(x) = 10^x$ where $f(-1), f(0)$, and $f(1)$.
3. $g(x) = \left(\frac{1}{3}\right)^x$ where $g(-1), g(0)$, and $g(3)$.
4. $g(x) = \left(\frac{3}{4}\right)^x$ where $g(-2), g(-1)$, and $g(0)$.
5. $h(x) = 9^{-x}$ where $h(-1), h(0)$, and $h\left(\frac{1}{2}\right)$.
6. $h(x) = 4^{-x}$ where $h(-1), h\left(-\frac{1}{2}\right)$, and $h(0)$.
7. $f(x) = -2^x + 1$ where $f(-1), f(0)$, and $f(3)$.
8. $f(x) = 2 - 3^x$ where $f(-1), f(0)$, and $f(2)$.
9. $g(x) = 10^{-x} + 20$ where $g(-2), g(-1)$, and $g(0)$.
10. $g(x) = 1 - 2^{-x}$ where $g(-1), g(0)$, and $g(1)$.

Use a calculator to approximate the following to the nearest hundredth.

11. $f(x) = 2^x + 5$ where $f(2.5)$.
12. $f(x) = 3^x - 10$ where $f(3.2)$.
13. $g(x) = 4^x$ where $g\left(\sqrt{2}\right)$.
14. $g(x) = 5^x - 1$ where $g\left(\sqrt{3}\right)$.
15. $h(x) = 10^x$ where $h(\pi)$.
16. $h(x) = 10^x + 1$ where $h\left(\frac{\pi}{3}\right)$
17. $f(x) = 10^{-x} - 2$ where $f(1.5)$.

18. $f(x) = 5^{-x} + 3$ where $f(1.3)$.

19. $f(x) = \left(\frac{2}{3}\right)^{x} + 1$ where $f(-2.7)$.

20. $f(x) = \left(\frac{3}{5}\right)^{-x} - 1$ where $f(1.4)$.

Sketch the function and determine the domain and range. Draw the horizontal asymptote with a dashed line.

21. $f(x) = 4^{x}$

22. $g(x) = 3^{x}$

23. $f(x) = 4^{x} + 2$

24. $f(x) = 3^{x} - 6$

25. $f(x) = 2^{x-2}$

26. $f(x) = 4^{x+2}$

27. $f(x) = 3^{x+1} - 4$

28. $f(x) = 10^{x-4} + 2$

29. $h(x) = 2^{x-3} - 2$

30. $h(x) = 3^{x+2} + 4$

31. $f(x) = \left(\dfrac{1}{4}\right)^{x}$

32. $h(x) = \left(\dfrac{1}{3}\right)^{x}$

33. $f(x) = \left(\dfrac{1}{4}\right)^{x} - 2$

34. $h(x) = \left(\dfrac{1}{3}\right)^{x} + 2$

35. $g(x) = 2^{-x} - 3$

36. $g(x) = 3^{-x} + 1$

37. $f(x) = 6 - 10^{-x}$

38. $g(x) = 5 - 4^{-x}$

39. $f(x) = 5 - 2^x$
40. $f(x) = 3 - 3^x$

PART B: NATURAL BASE *E*

Find $f(-1)$, $f(0)$, and $f\left(\frac{3}{2}\right)$ for the given function. Use a calculator where appropriate to approximate to the nearest hundredth.

41. $f(x) = e^x + 2$
42. $f(x) = e^x - 4$
43. $f(x) = 5 - 3e^x$
44. $f(x) = e^{-x} + 3$
45. $f(x) = 1 + e^{-x}$
46. $f(x) = 3 - 2e^{-x}$
47. $f(x) = e^{-2x} + 2$
48. $f(x) = e^{-x^2} - 1$

Sketch the function and determine the domain and range. Draw the horizontal asymptote with a dashed line.

49. $f(x) = e^x - 3$
50. $f(x) = e^x + 2$
51. $f(x) = e^{x+1}$
52. $f(x) = e^{x-3}$
53. $f(x) = e^{x-2} + 1$
54. $f(x) = e^{x+2} - 1$
55. $g(x) = -e^x$
56. $g(x) = e^{-x}$
57. $h(x) = -e^{x+1}$
58. $h(x) = -e^x + 3$

PART C: COMPOUND INTEREST FORMULAS

59. Jim invested \$750 in a 3-year CD that earns 4.2% annual interest that is compounded monthly. How much will the CD be worth at the end of the 3-year term?

60. Jose invested \$2,450 in a 4-year CD that earns 3.6% annual interest that is compounded semi-annually. How much will the CD be worth at the end of the 4-year term?

61. Jane has her \$5,350 savings in an account earning $3\frac{5}{8}$% annual interest that is compounded quarterly. How much will be in the account at the end of 5 years?

62. Bill has \$12,400 in a regular savings account earning $4\frac{2}{3}$% annual interest that is compounded monthly. How much will be in the account at the end of 3 years?

63. If \$85,200 is invested in an account earning 5.8% annual interest compounded quarterly, then how much interest is accrued in the first 3 years?

64. If \$124,000 is invested in an account earning 4.6% annual interest compounded monthly, then how much interest is accrued in the first 2 years?

65. Bill invested \$1,400 in a 3-year CD that earns 4.2% annual interest that is compounded continuously. How much will the CD be worth at the end of the 3-year term?

66. Brooklyn invested \$2,850 in a 5-year CD that earns 5.3% annual interest that is compounded continuously. How much will the CD be worth at the end of the 5-year term?

67. Omar has his \$4,200 savings in an account earning $4\frac{3}{8}$% annual interest that is compounded continuously. How much will be in the account at the end of $2\frac{1}{2}$ years?

68. Nancy has her \$8,325 savings in an account earning $5\frac{7}{8}$% annual interest that is compounded continuously. How much will be in the account at the end of $5\frac{1}{2}$ years?

69. If \$12,500 is invested in an account earning 3.8% annual interest compounded continuously, then how much interest is accrued in the first 10 years?

70. If \$220,000 is invested in an account earning 4.5% annual interest compounded continuously, then how much interest is accrued in the first 2 years?

71. The population of a certain small town is growing according to the function $P(t) = 12{,}500(1.02)^t$ where *t* represents time in years since the last census. Use the function to determine the population on the day of the census (when *t* = 0) and estimate the population in 6 years from that time.

72. The population of a certain small town is decreasing according to the function $P(t) = 22{,}300(0.982)^t$ where *t* represents time in years since the last census. Use the function to determine the population on the day of the census (when *t* = 0) and estimate the population in 6 years from that time.

73. The decreasing value, in dollars, of a new car is modeled by the formula $V(t) = 28{,}000(0.84)^t$ where *t* represents the number of years after the car was purchased. Use the formula to determine the value of the car when it was new (*t* = 0) and the value after 4 years.

74. The number of unique visitors to the college website can be approximated by the formula $N(t) = 410(1.32)^t$ where *t* represents the number of years after 1997 when the website was created. Approximate the number of unique visitors to the college website in the year 2020.

75. If left unchecked, a new strain of flu virus can spread from a single person to others very quickly. The number of people affected can be modeled using the formula $P(t) = e^{0.22t}$ where *t* represents the number of days the virus is allowed to spread unchecked. Estimate the number of people infected with the virus after 30 days and after 60 days.

76. If left unchecked, a population of 24 wild English rabbits can grow according to the formula $P(t) = 24e^{0.19t}$ where the time *t* is measured in months. How many rabbits would be present $3\,\frac{1}{2}$ years later?

77. The population of a certain city in 1975 was 65,000 people and was growing exponentially at an annual rate of 1.7%. At the time, the population growth was modeled by the formula $P(t) = 65{,}000e^{0.017t}$ where *t* represented the number of years since 1975. In the year 2000, the census determined that the actual population was 104,250 people. What population did the model predict for the year 2000 and what was the actual error?

78. Because of radioactive decay, the amount of a 10 milligram sample of Iodine-131 decreases according to the formula $A(t) = 10e^{-0.087t}$ where *t* represents time measured in days. How much of the sample remains after 10 days?

79. The number of cells in a bacteria sample is approximated by the logistic growth model $N(t) = \frac{1.2\times10^5}{1+9e^{-0.32t}}$ where *t* represents time in hours.

Determine the initial number of cells and then determine the number of cells 6 hours later.

80. The market share of a product, as a percentage, is approximated by the formula $P(t) = \frac{100}{2+e^{-0.44t}}$ where *t* represents the number of months after an aggressive advertising campaign is launched. By how much can we expect the market share to increase after the first three months of advertising?

PART D: DISCUSSION BOARD

81. Why is $b = 1$ excluded as a base in the definition of exponential functions? Explain.
82. Explain why an exponential function of the form $y = b^x$ can never be negative.
83. Research and discuss the derivation of the compound interest formula.
84. Research and discuss the logistic growth model. Provide a link to more information on this topic.
85. Research and discuss the life and contributions of Leonhard Euler.

ANSWERS

1. $f(-2) = \frac{1}{9}, f(0) = 1, f(2) = 9$

3. $g(-1) = 3, g(0) = 1, g(3) = \frac{1}{27}$

5. $h(-1) = 9, h(0) = 1, h\left(\frac{1}{2}\right) = \frac{1}{3}$

7. $f(-1) = \frac{1}{2}, f(0) = 0, f(3) = -7$

9. $g(-2) = 120, g(-1) = 30, g(0) = 21$

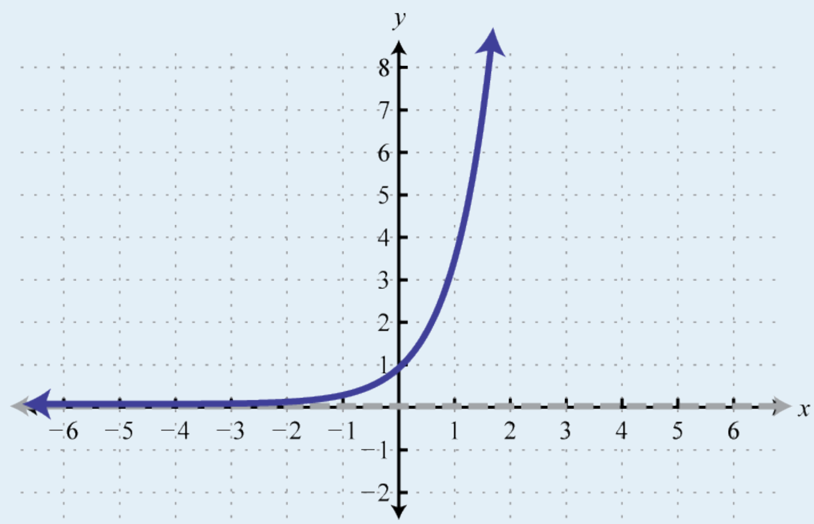
11. 10.66

13. 7.10

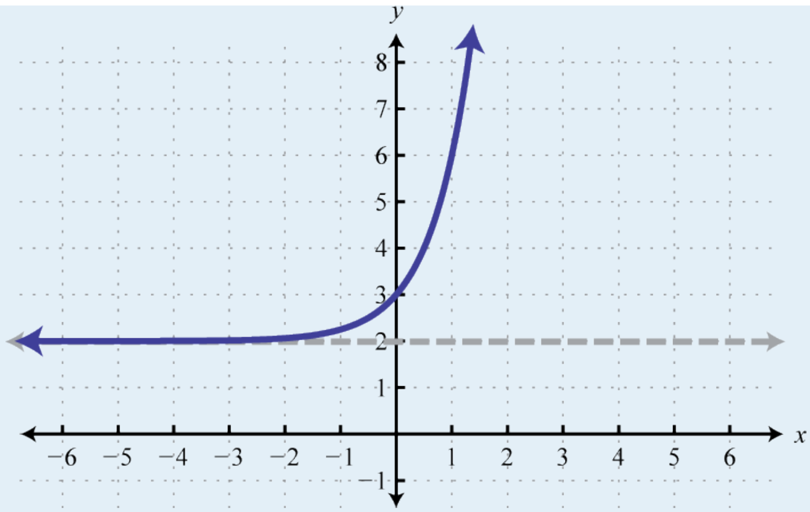
15. 1385.46

17. -1.97

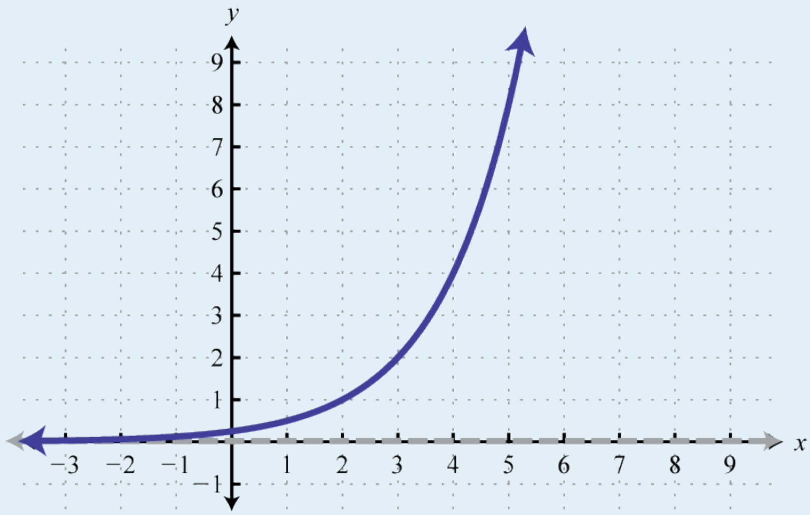
19. 3.99



21. Domain: $(-\infty, \infty)$; Range: $(0, \infty)$

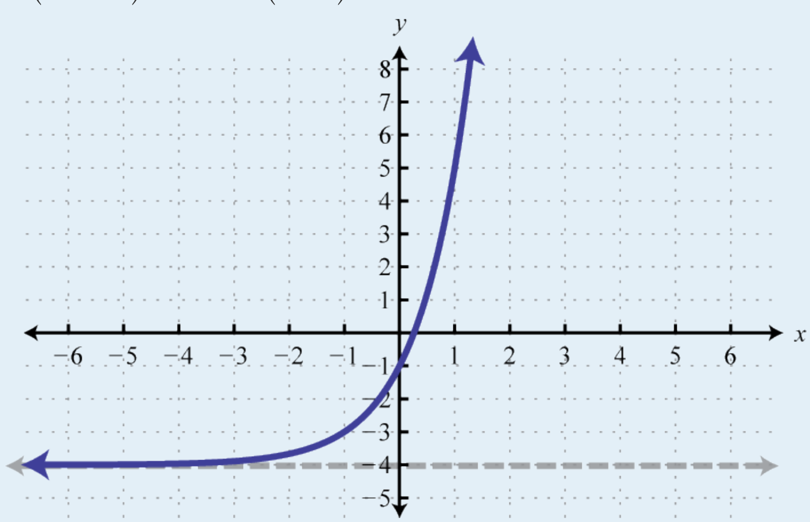


23. Domain: $(-\infty, \infty)$; Range: $(2, \infty)$

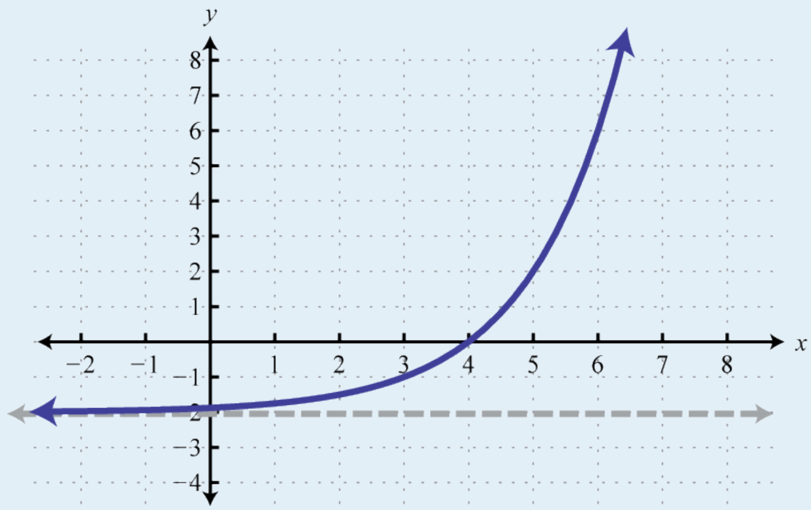


25. Domain: $(-\infty, \infty)$; Range: $(0, \infty)$

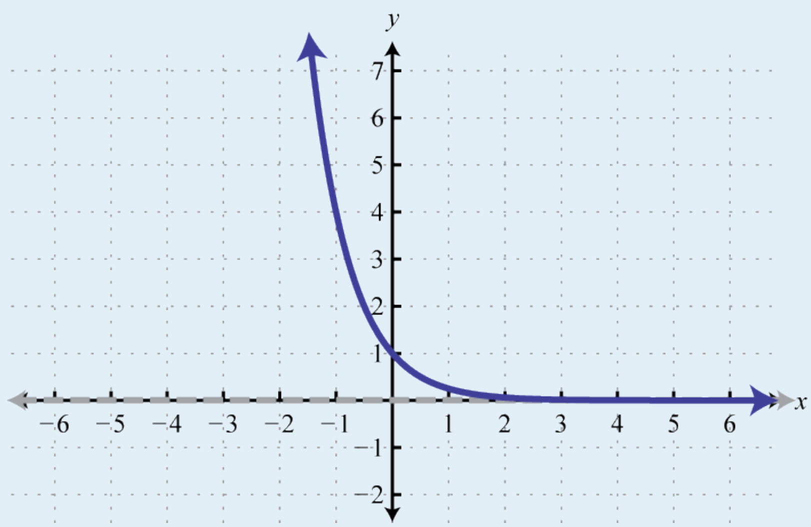
27.



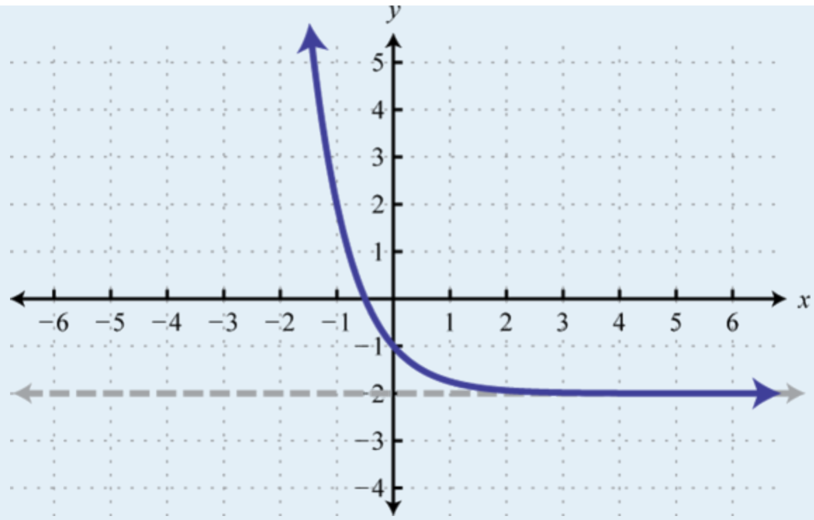
Domain: $(-\infty, \infty)$; Range: $(-4, \infty)$



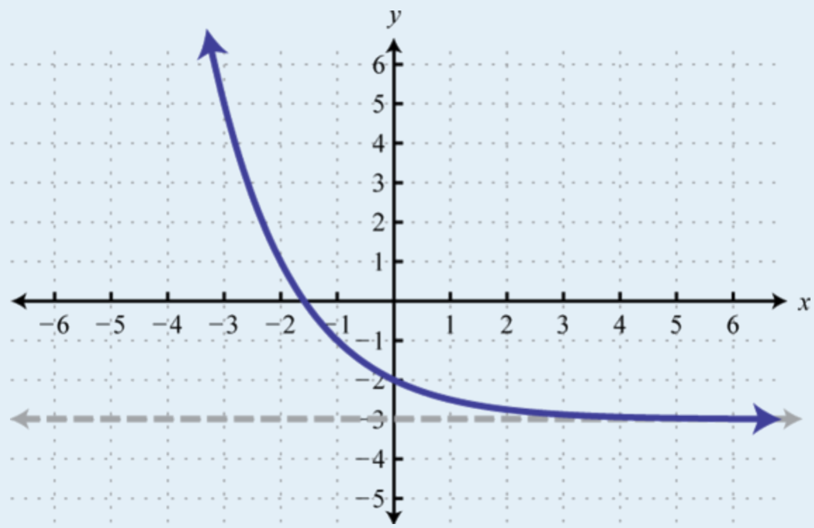
29. Domain: $(-\infty, \infty)$; Range: $(-2, \infty)$



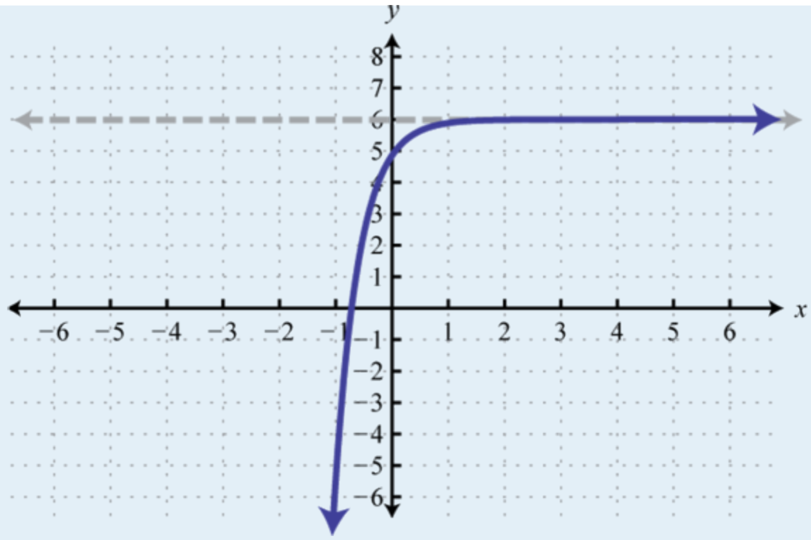
31. Domain: $(-\infty, \infty)$; Range: $(0, \infty)$



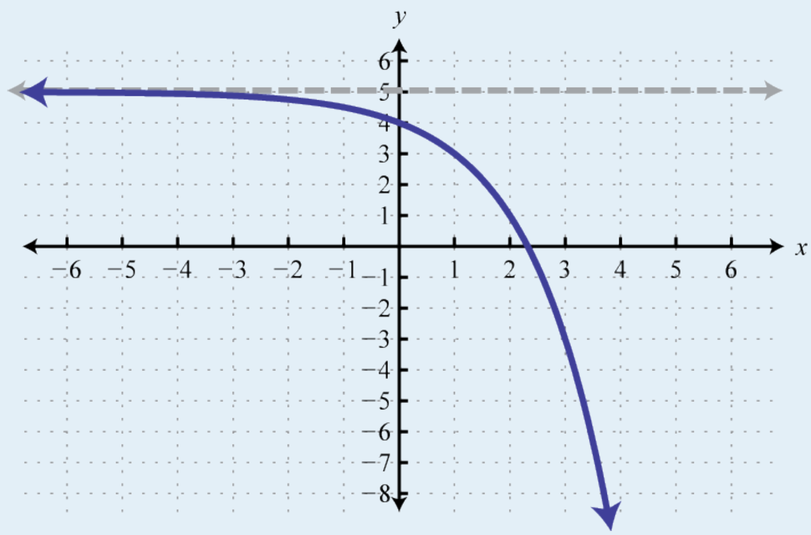
33. Domain: $(-\infty, \infty)$; Range: $(-2, \infty)$



35. Domain: $(-\infty, \infty)$; Range: $(-3, \infty)$



37. Domain: $(-\infty, \infty)$; Range: $(-\infty, 6)$



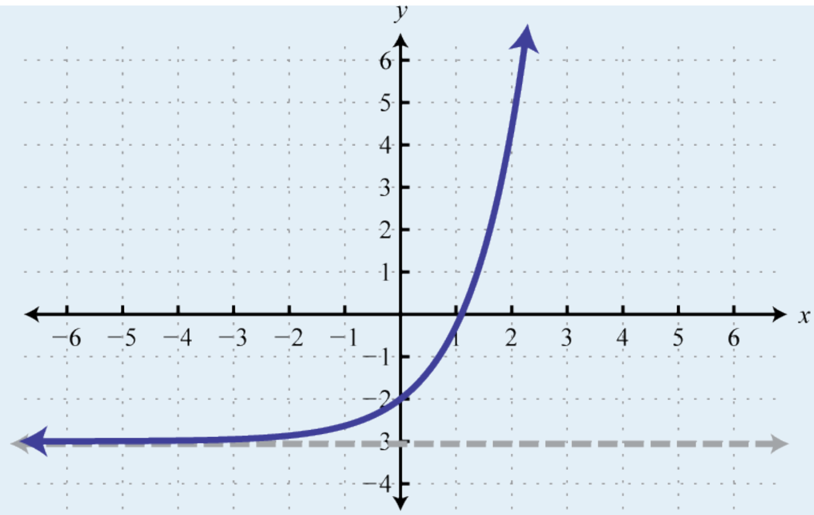
39. Domain: $(-\infty, \infty)$; Range: $(-\infty, 5)$

41. $f(-1) \approx 2.37, f(0) = 3, f\left(\frac{3}{2}\right) \approx 6.48$

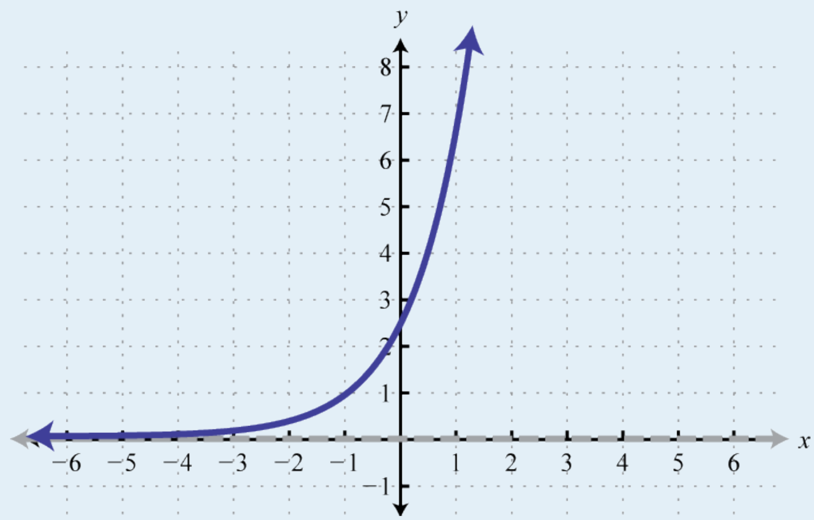
43. $f(-1) \approx 3.90, f(0) = 2, f\left(\frac{3}{2}\right) \approx -8.45$

45. $f(-1) \approx 3.72, f(0) = 2, f\left(\frac{3}{2}\right) \approx 1.22$

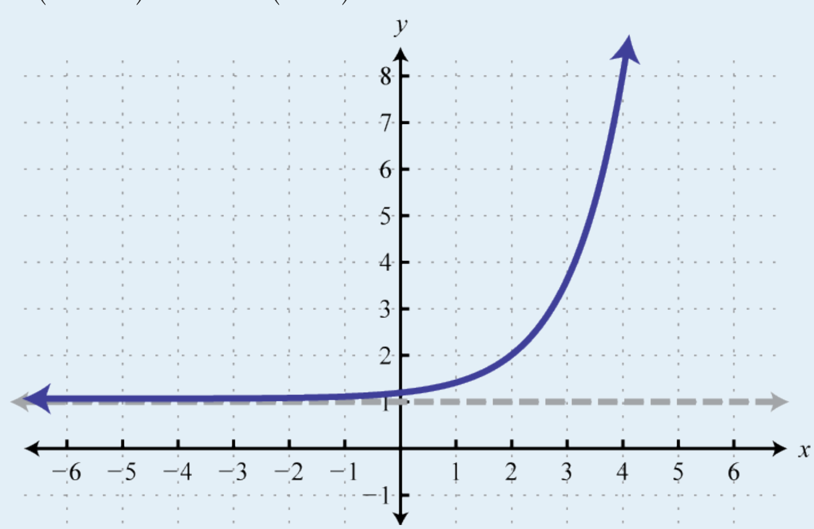
47. $f(-1) \approx 9.39, f(0) = 3, f\left(\frac{3}{2}\right) \approx 2.05$



49. Domain: $(-\infty, \infty)$; Range: $(-3, \infty)$

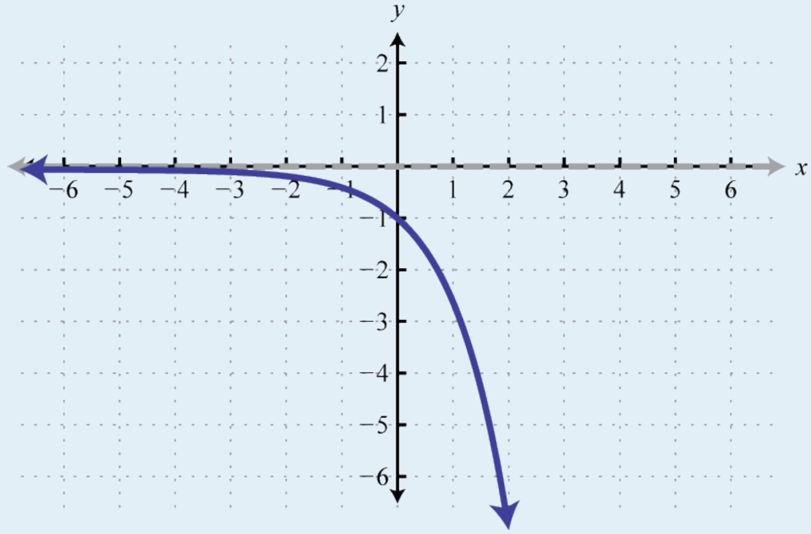


51. Domain: $(-\infty, \infty)$; Range: $(0, \infty)$

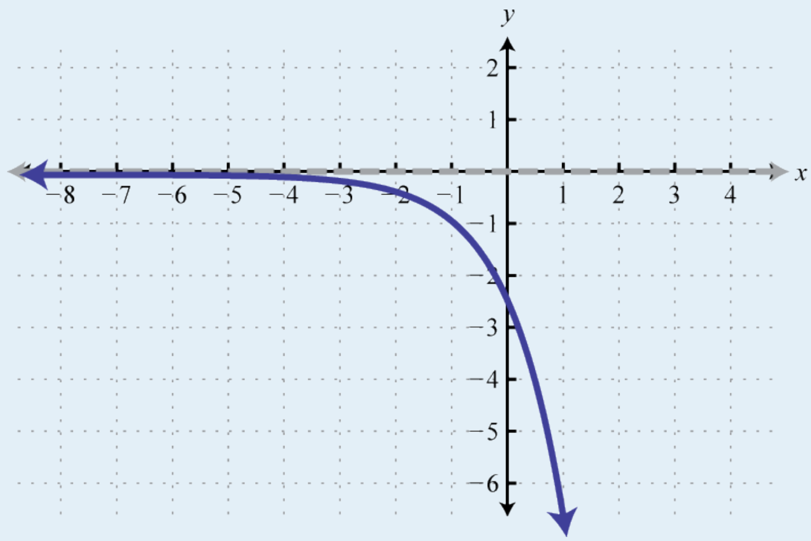


53.

Domain: $(-\infty, \infty)$; Range: $(1, \infty)$



55. Domain: $(-\infty, \infty)$; Range: $(-\infty, 0)$



57. Domain: $(-\infty, \infty)$; Range: $(-\infty, 0)$

59. $850.52

61. $6,407.89

63. $16,066.13

65. $1,588.00

67. $4,685.44

69. $5,778.56

71. Initial population: 12,500; Population 6 years later: 14,077

73. New: $28,000; In 4 years: $13,940.40

75. After 30 days: 735 people; After 60 days: 540,365 people

77. Model: 99,423 people; error: 4,827 people

79. Initially there are 12,000 cells and 6 hours later there are 51,736 cells.

81. Answer may vary

83. Answer may vary

85. Answer may vary

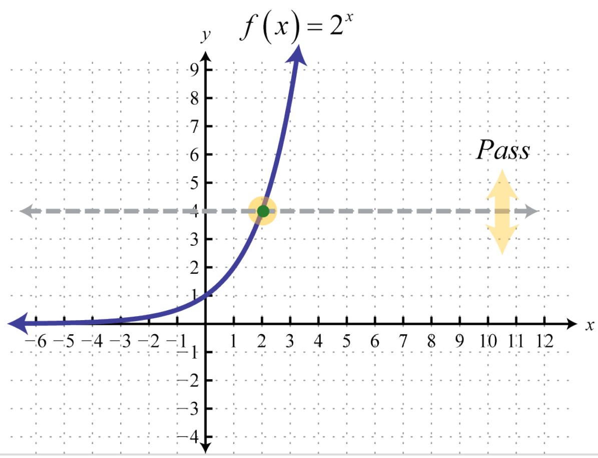
7.3 Logarithmic Functions and Their Graphs

LEARNING OBJECTIVES

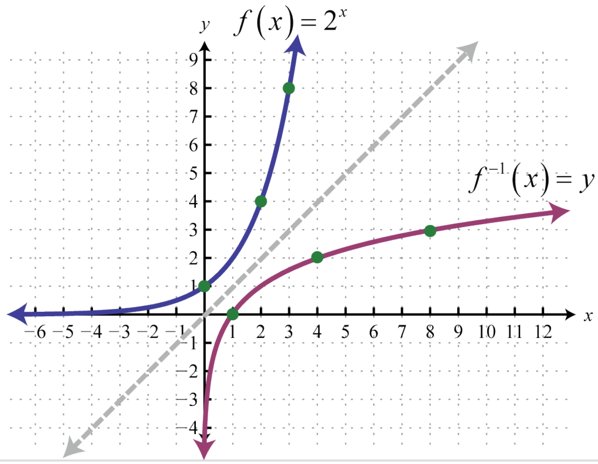
1. Define and evaluate logarithms.
2. Identify the common and natural logarithm.
3. Sketch the graph of logarithmic functions.

Definition of the Logarithm

We begin with the exponential function defined by $f(x) = 2^x$ and note that it passes the horizontal line test.



Therefore it is one-to-one and has an inverse. Reflecting $y = 2^x$ about the line $y = x$ we can sketch the graph of its inverse. Recall that if (x, y) is a point on the graph of a function, then (y, x) will be a point on the graph of its inverse.



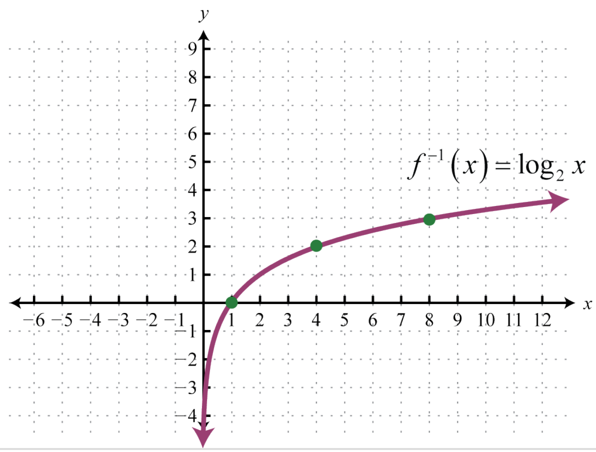
To find the inverse algebraically, begin by interchanging x and y and then try to solve for y.

$$f(x)=2^x$$
$$y=2^x \quad \Rightarrow \quad x = 2^y$$

We quickly realize that there is no method for solving for y. This function seems to "transcend" algebra. Therefore, we define the inverse to be the base-2 logarithm, denoted $\log_2 x$. The following are equivalent:

$$y = \log_2 x \quad \Leftrightarrow \quad x = 2^y$$

This gives us another transcendental function defined by $f^{-1}(x) = \log_2 x$, which is the inverse of the exponential function defined by $f(x) = 2^x$.



The domain consists of all positive real numbers $(0, \infty)$ and the range consists of all real numbers $(-\infty, \infty)$. John Napier is widely credited for inventing the term logarithm.

Figure 7.2

John Napier (1550–1617)

In general, given base $b > 0$ where $b \neq 1$, the **logarithm base *b***[8] is defined as follows:

$$y = \log_b x \quad \textit{if and only if} \quad x = b^y$$

Use this definition to convert logarithms to exponential form and back.

8. The exponent to which the base b is raised in order to obtain a specific value. In other words, $y = \log_b x$ is equivalent to $b^y = x$.

Logarithmic Form	Exponential Form
$\log_2 16 = 4$	$2^4 = 16$
$\log_5 25 = 2$	$5^2 = 25$
$\log_6 1 = 0$	$6^0 = 1$
$\log_3 \sqrt{3} = \frac{1}{2}$	$3^{1/2} = \sqrt{3}$
$\log_7 \left(\frac{1}{49}\right) = -2$	$7^{-2} = \frac{1}{49}$

It is useful to note that the logarithm is actually the exponent y to which the base b is raised to obtain the argument x.

equals x

$$\log_b x = y \quad \Leftrightarrow \quad b^y = x$$

b raised to the y power

Example 1

Evaluate:

a. $\log_5 125$
b. $\log_2 \left(\frac{1}{8}\right)$
c. $\log_4 2$
d. $\log_{11} 1$

Solution:

a. $\log_5 125 = 3$ because $5^3 = 125$.
b. $\log_2 \left(\frac{1}{8}\right) = -3$because $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$.
c. $\log_4 2 = \frac{1}{2}$because $4^{1/2} = \sqrt{4} = 2$.
d. $\log_{11} 1 = 0$ because $11^0 = 1$.

Note that the result of a logarithm can be negative or even zero. However, the argument of a logarithm is not defined for negative numbers or zero:

$$\log_2(-4) = ? \Longrightarrow 2^? = -4$$
$$\log_2(0) = ? \Longrightarrow 2^? = 0$$

There is no power of two that results in -4 or 0. Negative numbers and zero are not in the domain of the logarithm. At this point it may be useful to go back and review all of the rules of exponents.

Example 2

Find *x*:

a. $\log_7 x = 2$
b. $\log_{16} x = \frac{1}{2}$
c. $\log_{1/2} x = -5$

Solution:

Convert each to exponential form and then simplify using the rules of exponents.

a. $\log_7 x = 2$ is equivalent to $7^2 = x$ and thus $x = 49$.
b. $\log_{16} x = \frac{1}{2}$ is equivalent to $16^{1/2} = x$ or $\sqrt{16} = x$ and thus $x = 4$.
c. $\log_{1/2} x = -5$ is equivalent to $\left(\frac{1}{2}\right)^{-5} = x$ or $2^5 = x$ and thus $x = 32$

Try this! Evaluate: $\log_5\left(\frac{1}{\sqrt[3]{5}}\right)$.

Answer: $-\frac{1}{3}$

(click to see video)

The Common and Natural Logarithm

A logarithm can have any positive real number, other than 1, as its base. If the base is 10, the logarithm is called the **common logarithm**[9].

9. The logarithm base 10, denoted $\log x$.

$$f(x) = \log_{10} x = \log\ x \qquad \textit{Common logarithm}$$

When a logarithm is written without a base it is assumed to be the common logarithm. (**Note:** This convention varies with respect to the subject in which it appears. For example, computer scientists often let $\log\ x$ represent the logarithm base 2.)

Example 3

Evaluate:

a. $\log\ 10^5$
b. $\log\ \sqrt{10}$
c. $\log\ 0.01$

Solution:

a. $\log\ 10^5 = 5$ because $10^5 = 10^5$.
b. $\log\ \sqrt{10} = \frac{1}{2}$ because $10^{1/2} = \sqrt{10}$.
c. $\log\ 0.01 = \log\ \left(\frac{1}{100}\right) = -2$ because $10^{-2} = \frac{1}{10^2} = \frac{1}{100} = 0.01$.

The result of a logarithm is not always apparent. For example, consider $\log\ 75$.

$$\begin{aligned} \log\ 10 &= 1 \\ \log\ 75 &= ? \\ \log\ 100 &= 2 \end{aligned}$$

We can see that the result of $\log\ 75$ is somewhere between 1 and 2. On most scientific calculators there is a common logarithm button $\boxed{LOG}$. Use it to find the $\log\ 75$ as follows:

$$\boxed{LOG}\ 75\ \boxed{=}\ 1.87506$$

Therefore, rounded off to the nearest thousandth, $\log\ 75 \approx 1.875$. As a check, we can use a calculator to verify that $10\ \hat{}\ 1.875 \approx 75$.

If the base of a logarithm is e, the logarithm is called the **natural logarithm**[10].

$$f(x) = \log_e x = \ln\ x \qquad \textit{Natural logarithm}$$

The natural logarithm is widely used and is often abbreviated $\ln\ x$.

10. The logarithm base e, denoted $\ln\ x$.

Example 4

Evaluate:

a. $\ln\ e$
b. $\ln\ \sqrt[3]{e^2}$
c. $\ln\ \left(\frac{1}{e^4}\right)$

Solution:

a. $\ln\ e = 1$ because $\ln\ e = \log_e\ e = 1$ and $e^1 = e$.
b. $\ln\ \left(\sqrt[3]{e^2}\right) = \frac{2}{3}$because $e^{2/3} = \sqrt[3]{e^2}$.
c. $\ln\ \left(\frac{1}{e^4}\right) = -4$because $e^{-4} = \frac{1}{e^4}$

On a calculator you will find a button for the natural logarithm $\boxed{LN}$.

$$\boxed{LN}\quad 75\quad \boxed{=}\quad 4.317488$$

Therefore, rounded off to the nearest thousandth, $\ln\ (75) \approx 4.317.$As a check, we can use a calculator to verify that e ^ $4.317 \approx 75$.

Example 5

Find *x*. Round answers to the nearest thousandth.

a. $\log\ x = 3.2$
b. $\ln\ x = -4$
c. $\log\ x = -\frac{2}{3}$

Solution:

Convert each to exponential form and then use a calculator to approximate the answer.

a. $\log\ x = 3.2$ is equivalent to $10^{3.2} = x$ and thus $x \approx 1584.893.$
b. $\ln\ x = -4$ is equivalent to $e^{-4} = x$ and thus $x \approx 0.018.$
c. $\log\ x = -\frac{2}{3}$ is equivalent to $10^{-2/3} = x$ and thus $x \approx 0.215$

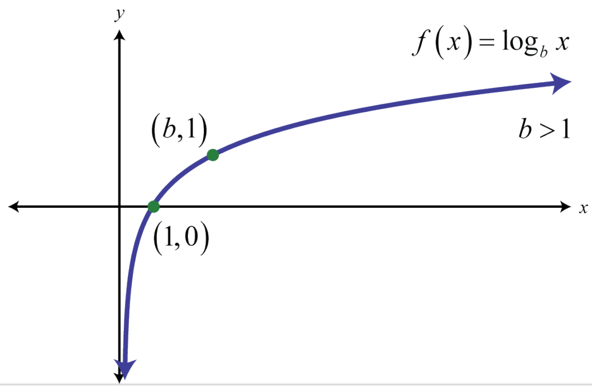
Try this! Evaluate: $\ln\left(\frac{1}{\sqrt{e}}\right)$

Answer: $-\frac{1}{2}$

(click to see video)

Graphing Logarithmic Functions

We can use the translations to graph logarithmic functions. When the base $b > 1$, the graph of $f\left(x\right) = \log_b x$ has the following general shape:



The domain consists of positive real numbers, $(0, \infty)$ and the range consists of all real numbers, $(-\infty, \infty)$. The *y*-axis, or $x = 0$, is a vertical asymptote and the *x*-intercept is $(1, 0)$. In addition, $f(b) = \log_b b = 1$ and so $(b, 1)$ is a point on the graph no matter what the base is.

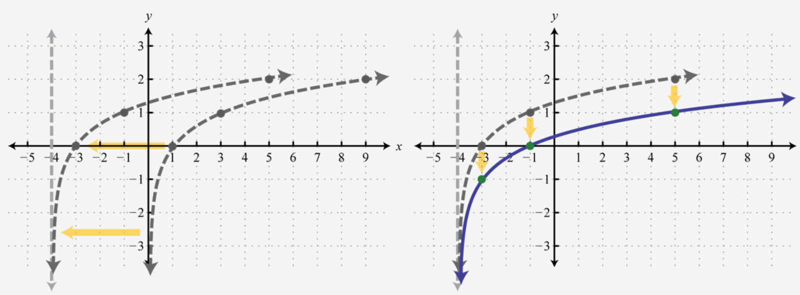
Example 6

Sketch the graph and determine the domain and range:
$f(x) = \log_3 (x + 4) - 1.$

Solution:

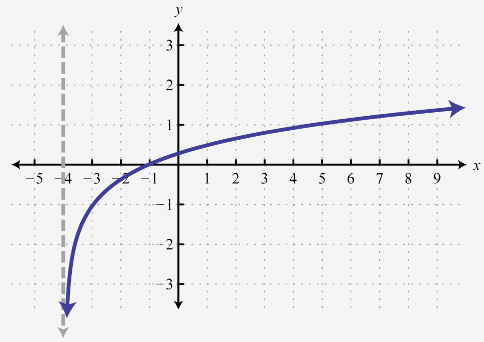
Begin by identifying the basic graph and the transformations.

$$
\begin{array}{ll}
y=\log_3 x & \textit{Basic graph} \\
y=\log_3 (x+4) & \textit{Shift left 4 units.} \\
y=\log_3 (x+4)-1 & \textit{Shift down 1 unit.}
\end{array}
$$



Notice that the asymptote was shifted 4 units to the left as well. This defines the lower bound of the domain. The final graph is presented without the intermediate steps.

Answer:



Domain: $(-4, \infty)$; Range: $(-\infty, \infty)$

Note: Finding the intercepts of the graph in the previous example is left for a later section in this chapter. For now, we are more concerned with the general shape of logarithmic functions.

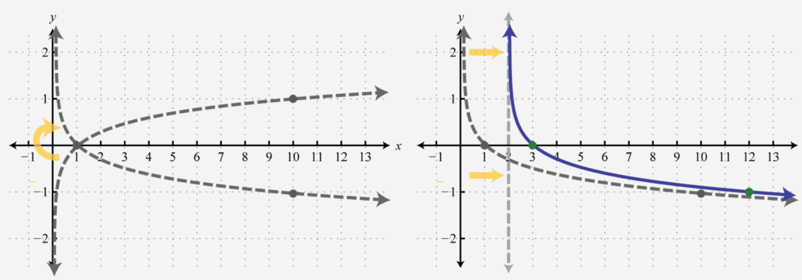
Example 7

Sketch the graph and determine the domain and range:
$f\,(x) = -\log\ (x - 2)\,.$

Solution:

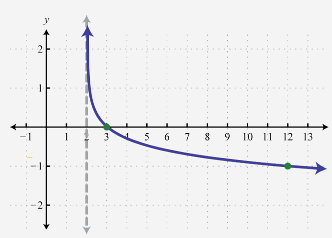
Begin by identifying the basic graph and the transformations.

$y = \log\ x$ *Basic graph*
$y = -\log\ x$ *Reflection about the x-axis*
$y = -\log\ (x - 2)$ *Shift right 2 units.*



Here the vertical asymptote was shifted two units to the right. This defines the lower bound of the domain.

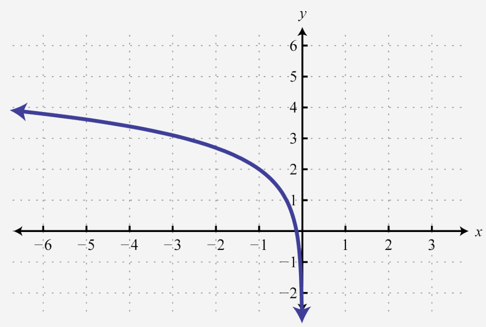
Answer:



Domain: $(2, \infty)$; Range: $(-\infty, \infty)$

Try this! Sketch the graph and determine the domain and range: $g(x) = \ln(-x) + 2$.

Answer:



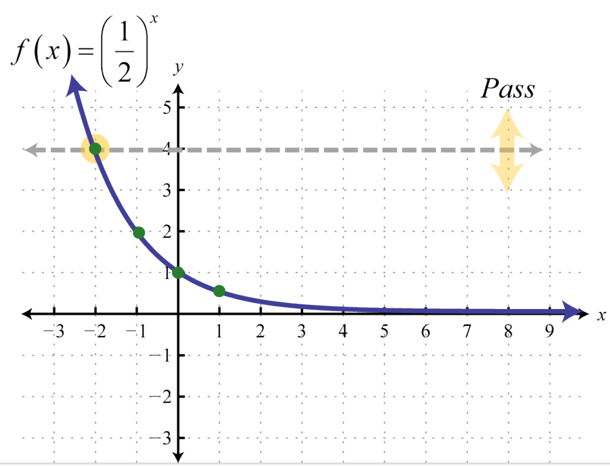
Domain: (−∞, 0); Range: (−∞, ∞)

(click to see video)

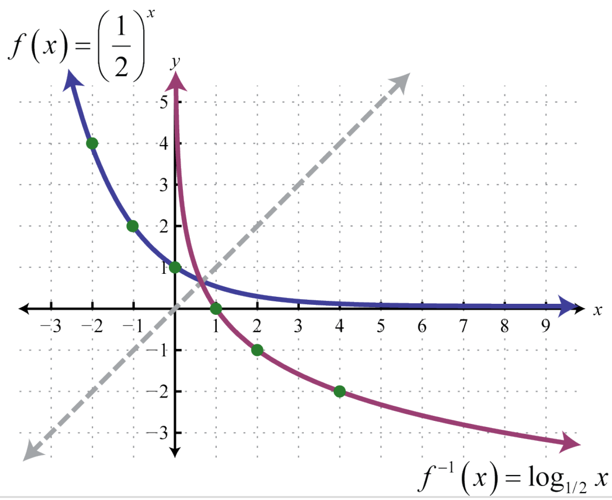
Next, consider exponential functions with fractional bases, such as the function defined by $f(x) = \left(\frac{1}{2}\right)^x$. The domain consists of all real numbers. Choose some values for *x* and then find the corresponding *y*-values.

x	y		*Solutions*
-2	4	$f(-2) = \left(\frac{1}{2}\right)^{-2} = 2^2 = 4$	$(-2, 4)$
-1	2	$f(-1) = \left(\frac{1}{2}\right)^{-1} = 2^1 = 2$	$(-1, 2)$
0	1	$f(0) = \left(\frac{1}{2}\right)^{0} = 1$	$(0, 1)$
1	$\frac{1}{2}$	$f(1) = \left(\frac{1}{2}\right)^{1} = \frac{1}{2}$	$\left(1, \frac{1}{2}\right)$
2	$\frac{1}{4}$	$f(2) = \left(\frac{1}{2}\right)^{2} = \frac{1}{4}$	$\left(2, \frac{1}{4}\right)$

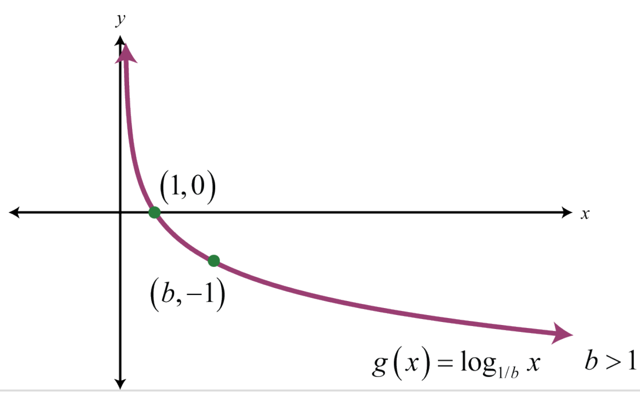
Use these points to sketch the graph and note that it passes the horizontal line test.



Therefore this function is one-to-one and has an inverse. Reflecting the graph about the line $y = x$ we have:



which gives us a picture of the graph of $f^{-1}(x) = \log_{1/2} x$. In general, when the base $b > 1$, the graph of the function defined by $g(x) = \log_{1/b} x$ has the following shape.



The domain consists of positive real numbers, $(0, \infty)$ and the range consists of all real numbers, $(-\infty, \infty)$. The y-axis, or $x = 0$, is a vertical asymptote and the x-intercept is $(1, 0)$. In addition, $f(b) = \log_{1/b} b = -1$ and so $(b, -1)$ is a point on the graph.

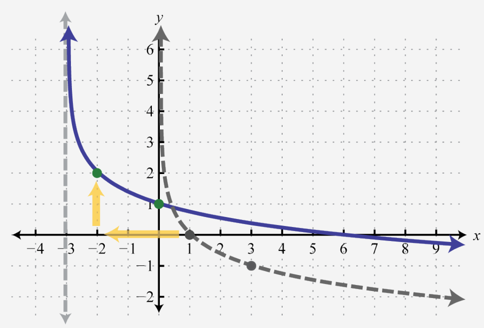
Example 8

Sketch the graph and determine the domain and range:
$f(x) = \log_{1/3}(x+3)+2$.

Solution:

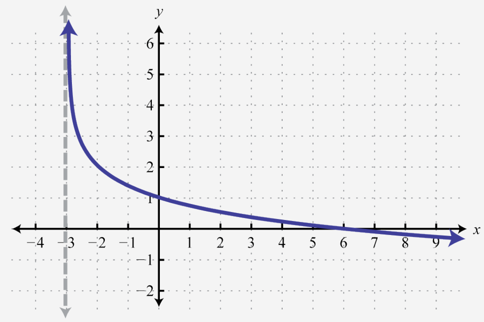
Begin by identifying the basic graph and the transformations.

$$\begin{aligned} y &= \log_{1/3} x && \textit{Basic graph} \\ y &= \log_{1/3}(x+3) && \textit{Shift left 3 units.} \\ y &= \log_{1/3}(x+3)+2 && \textit{Shift up 2 units.} \end{aligned}$$



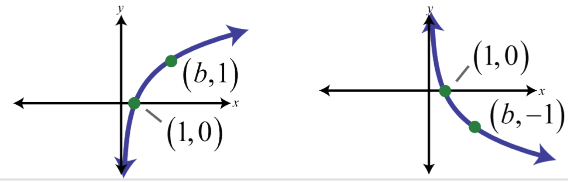
In this case the shift left 3 units moved the vertical asymptote to $x=-3$ which defines the lower bound of the domain.

Answer:



Domain: $(-3, \infty)$; Range: $(-\infty, \infty)$

In summary, if $b > 1$

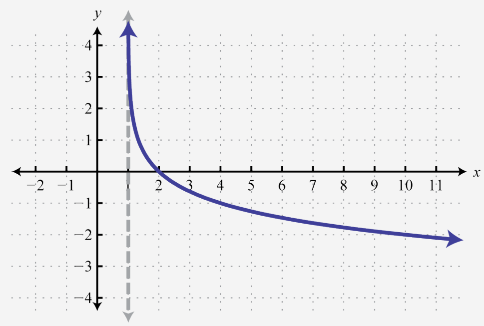


And for both cases,

$$\begin{aligned} \textit{Domain} &: (0, \infty) \\ \textit{Range} &: (-\infty, \infty) \\ \textit{x-intercept} &: (1, 0) \\ \textit{Asymptote} &: x = 0 \end{aligned}$$

Try this! Sketch the graph and determine the domain and range: $f(x) = \log_{1/3}(x-1)$.

Answer:



(click to see video)

KEY TAKEAWAYS

- The base-*b* logarithmic function is defined to be the inverse of the base-*b* exponential function. In other words, $y = \log_b x$ if and only if $b^y = x$ where $b > 0$ and $b \neq 1$.
- The logarithm is actually the exponent to which the base is raised to obtain its argument.
- The logarithm base 10 is called the common logarithm and is denoted $\log x$.
- The logarithm base *e* is called the natural logarithm and is denoted $\ln x$.
- Logarithmic functions with definitions of the form $f(x) = \log_b x$ have a domain consisting of positive real numbers $(0, \infty)$ and a range consisting of all real numbers $(-\infty, \infty)$. The *y*-axis, or $x = 0$, is a vertical asymptote and the *x*-intercept is $(1, 0)$.
- To graph logarithmic functions we can plot points or identify the basic function and use the transformations. Be sure to indicate that there is a vertical asymptote by using a dashed line. This asymptote defines the boundary of the domain.

TOPIC EXERCISES

PART A: DEFINITION OF THE LOGARITHM

Evaluate.

1. $\log_3 9$
2. $\log_7 49$
3. $\log_4 4$
4. $\log_5 1$
5. $\log_5 625$
6. $\log_3 243$
7. $\log_2 \left(\frac{1}{16}\right)$
8. $\log_3 \left(\frac{1}{9}\right)$
9. $\log_5 \left(\frac{1}{125}\right)$
10. $\log_2 \left(\frac{1}{64}\right)$
11. $\log_4 4^{10}$
12. $\log_9 9^5$
13. $\log_5 \sqrt[3]{5}$
14. $\log_2 \sqrt{2}$
15. $\log_7 \left(\frac{1}{\sqrt{7}}\right)$
16. $\log_9 \left(\frac{1}{\sqrt[3]{9}}\right)$

17. $\log_{1/2} 4$

18. $\log_{1/3} 27$

19. $\log_{2/3} \left(\frac{2}{3}\right)$

20. $\log_{3/4} \left(\frac{9}{16}\right)$

21. $\log_{25} 5$

22. $\log_{8} 2$

23. $\log_{4} \left(\frac{1}{2}\right)$

24. $\log_{27} \left(\frac{1}{3}\right)$

25. $\log_{1/9} 1$

26. $\log_{3/5} \left(\frac{5}{3}\right)$

Find *x*.

27. $\log_3 x = 4$

28. $\log_2 x = 5$

29. $\log_5 x = -3$

30. $\log_6 x = -2$

31. $\log_{12} x = 0$

32. $\log_7 x = -1$

33. $\log_{1/4} x = -2$

34. $\log_{2/5} x = 2$

35. $\log_{1/9} x = \frac{1}{2}$

36. $\log_{1/4} x = \frac{3}{2}$

37. $\log_{1/3} x = -1$

38. $\log_{1/5} x = 0$

PART B: THE COMMON AND NATURAL LOGARITHM

Evaluate. Round off to the nearest hundredth where appropriate.

39. $\log 1000$

40. $\log 100$

41. $\log 0.1$

42. $\log 0.0001$

43. $\log 162$

44. $\log 23$

45. $\log 0.025$

46. $\log 0.235$

47. $\ln e^4$

48. $\ln 1$

49. $\ln \left(\frac{1}{e}\right)$

50. $\ln \left(\frac{1}{e^5}\right)$

51. $\ln (25)$

52. $\ln (100)$

53. $\ln (0.125)$

54. $\ln (0.001)$

Find *x*. Round off to the nearest hundredth.

55. $\log x = 2.5$

56. $\log x = 1.8$

57. $\log x = -1.22$

58. $\log x = -0.8$

59. $\ln x = 3.1$

60. $\ln x = 1.01$

61. $\ln x = -0.69$

62. $\ln x = -1$

Find *a* without using a calculator.

63. $\log_3 \left(\frac{1}{27}\right) = a$

64. $\ln e = a$

65. $\log_2 a = 8$

66. $\log_2 \sqrt[5]{2} = a$

67. $\log 10^{12} = a$

68. $\ln a = 9$

69. $\log_{1/8} \left(\frac{1}{64}\right) = a$

70. $\log_6 a = -3$

71. $\ln a = \dfrac{1}{5}$

72. $\log_{4/9} \left(\dfrac{2}{3}\right) = a$

In 1935 Charles Richter developed a scale used to measure earthquakes on a seismograph. The magnitude M of an earthquake is given by the formula,

$$M = \log \left(\frac{I}{I_0}\right)$$

Here I represents the intensity of the earthquake as measured on the seismograph 100 km from the epicenter and I_0 is the minimum intensity used for comparison. For example, if an earthquake intensity is measured to be 100 times that of the minimum, then $I = 100I_0$ and

$$M = \log\left(\frac{100I_0}{I_0}\right) = \log(100) = 2$$

The earthquake would be said to have a magnitude 2 on the Richter scale. Determine the magnitudes of the following intensities on the Richter scale. Round off to the nearest tenth.

73. I is 3 million times that of the minimum intensity.
74. I is 6 million times that of the minimum intensity.
75. I is the same as the minimum intensity.
76. I is 30 million times that of the minimum intensity.

In chemistry, pH is a measure of acidity and is given by the formula,

$$\text{pH} = -\log\left(H^+\right)$$

Here H^+ represents the hydrogen ion concentration (measured in moles of hydrogen per liter of solution.) Determine the pH given the following hydrogen ion concentrations.

77. Pure water: $H^+ = 0.0000001$
78. Blueberry: $H^+ = 0.0003162$
79. Lemon Juice: $H^+ = 0.01$
80. Battery Acid: $H^+ = 0.1$

PART C: GRAPHING LOGARITHMIC FUNCTIONS

Sketch the function and determine the domain and range. Draw the vertical asymptote with a dashed line.

81. $f(x) = \log_2(x+1)$
82. $f(x) = \log_3(x-2)$
83. $f(x) = \log_2 x - 2$
84. $f(x) = \log_3 x + 3$
85. $f(x) = \log_2(x-2) + 4$
86. $f(x) = \log_3(x+1) - 2$
87. $f(x) = -\log_2 x + 1$
88. $f(x) = -\log_3(x+3)$
89. $f(x) = \log_2(-x) + 1$
90. $f(x) = 2 - \log_3(-x)$
91. $f(x) = \log x + 5$
92. $f(x) = \log x - 1$
93. $f(x) = \log(x+4) - 8$
94. $f(x) = \log\left(x-5\right) + 10$
95. $f(x) = -\log(x+2)$
96. $f(x) = -\log(x-1) + 2$
97. $f(x) = \ln(x-3)$
98. $f(x) = \ln x + 3$
99. $f(x) = \ln(x-2) + 4$
100. $f(x) = \ln\left(x+5\right)$
101. $f(x) = 2 - \ln x$

102. $f(x) = -\ln(x - 1)$

103. $f(x) = \log_{1/2} x$

104. $f(x) = \log_{1/3} x + 2$

105. $f(x) = \log_{1/2}(x - 2)$

106. $f(x) = \log_{1/3}(x + 1) - 1$

107. $f(x) = 2 - \log_{1/4} x$

108. $f(x) = 1 + \log_{1/4}(-x)$

109. $f(x) = 1 - \log_{1/3}(x - 2)$

110. $f(x) = 1 + \log_{1/2}(-x)$

PART D: DISCUSSION BOARD

111. Research and discuss the origins and history of the logarithm. How did students work with them before the common availability of calculators?

112. Research and discuss the history and use of the Richter scale. What does each unit on the Richter scale represent?

113. Research and discuss the life and contributions of John Napier.

ANSWERS

1. 2

3. 1

5. 4

7. −4

9. −3

11. 10

13. $\frac{1}{3}$

15. $-\frac{1}{2}$

17. −2

19. 1

21. $\frac{1}{2}$

23. $-\frac{1}{2}$

25. 0

27. 81

29. $\frac{1}{125}$

31. 1

33. 16

35. $\frac{1}{3}$

37. 3

39. 3

41. −1

43. 2.21

45. −1.60

47. 4

49. −1

51. 3.22

53. −2.08

55. 316.23

57. 0.06

59. 22.20

61. 0.50

63. −3

65. 256

67. 12

69. 2

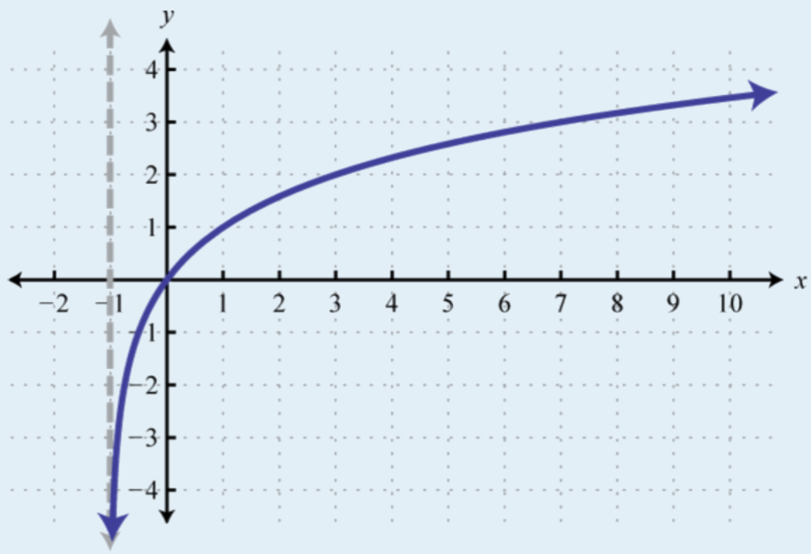
71. $\sqrt[5]{e}$

73. 6.5

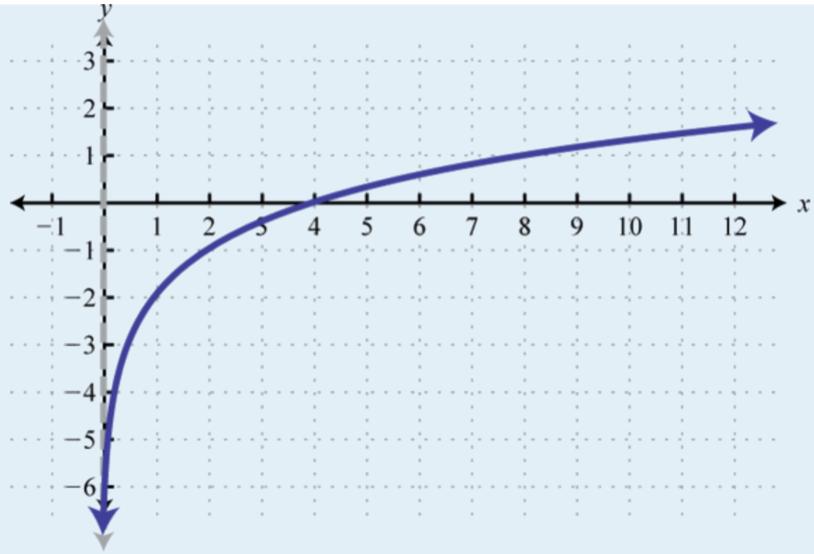
75. 0

77. 7

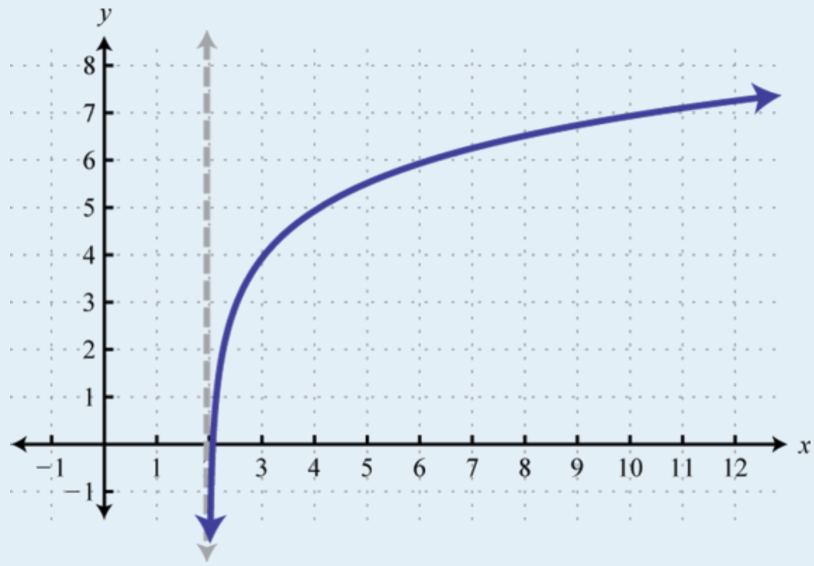
79. 2



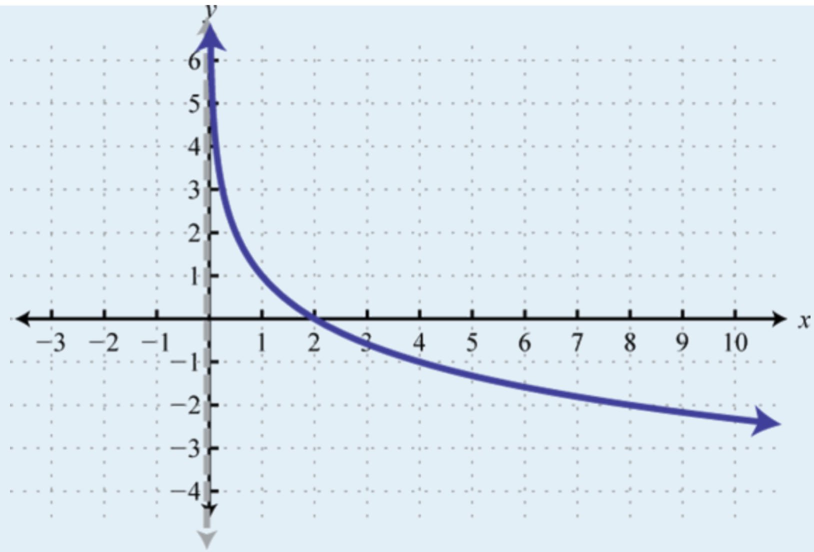
81. Domain: $(-1, \infty)$; Range: $(-\infty, \infty)$



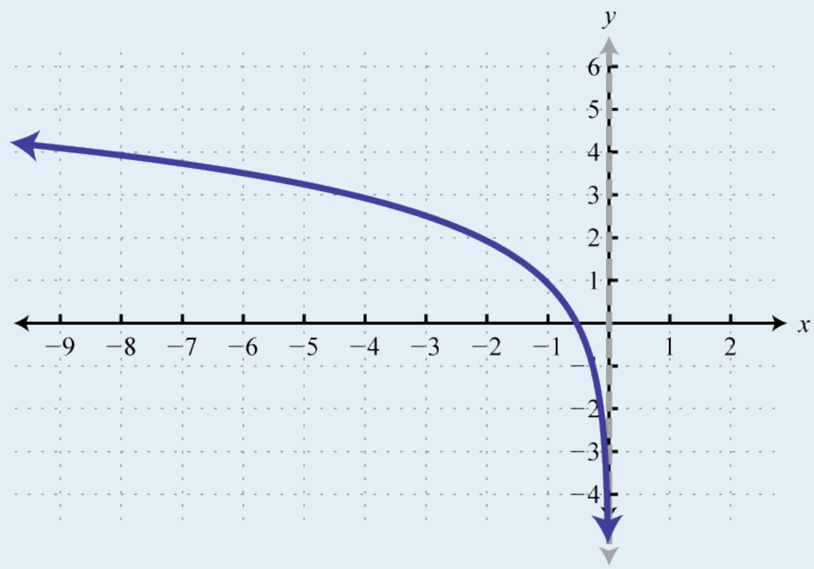
83. Domain: $(0, \infty)$; Range: $(-\infty, \infty)$



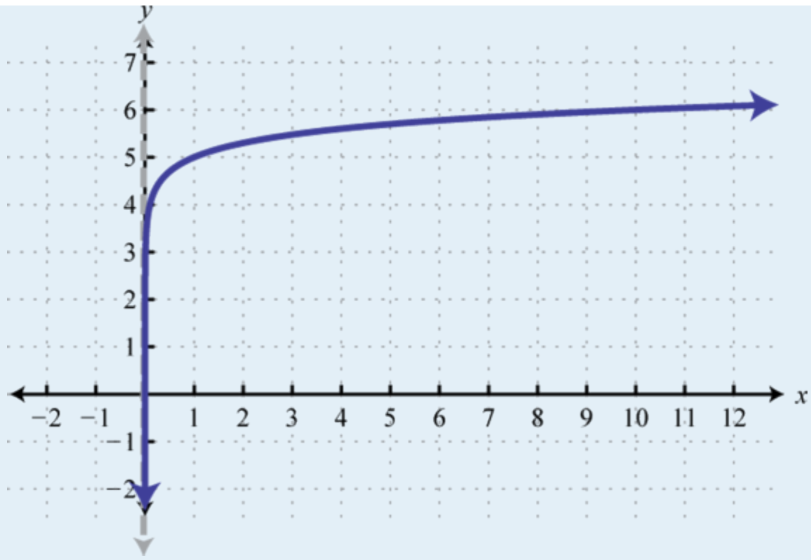
85. Domain: $(2, \infty)$; Range: $(-\infty, \infty)$



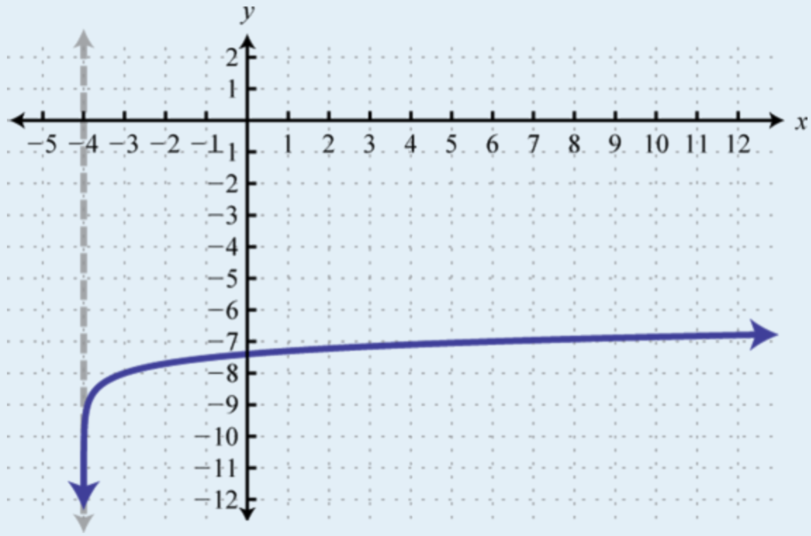
87. Domain: $(0, \infty)$; Range: $(-\infty, \infty)$



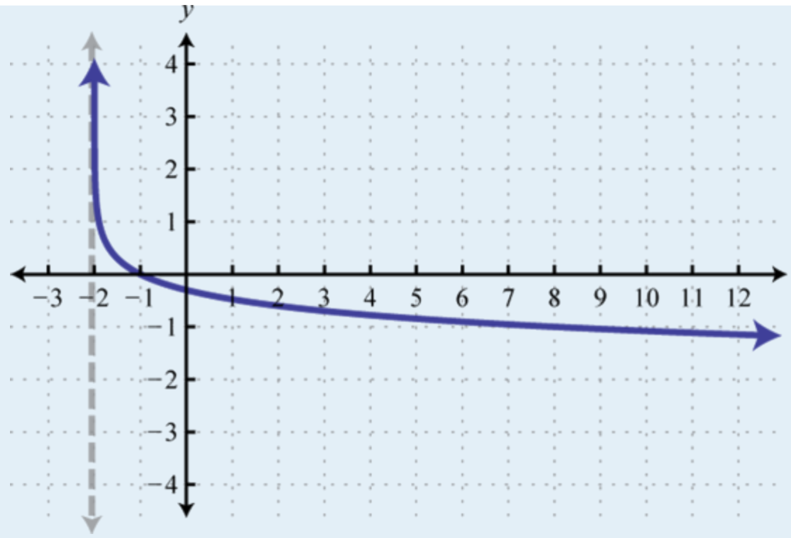
89. Domain: $(-\infty, 0)$; Range: $(-\infty, \infty)$



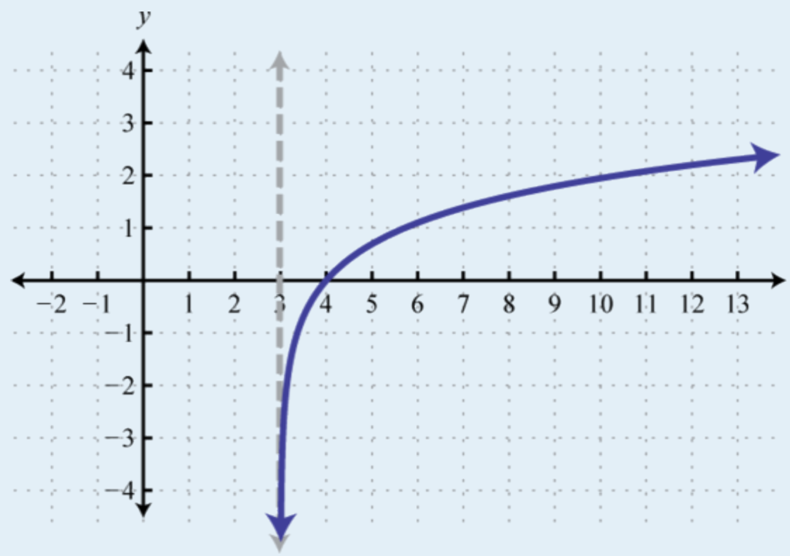
91. Domain: $(0, \infty)$; Range: $(-\infty, \infty)$



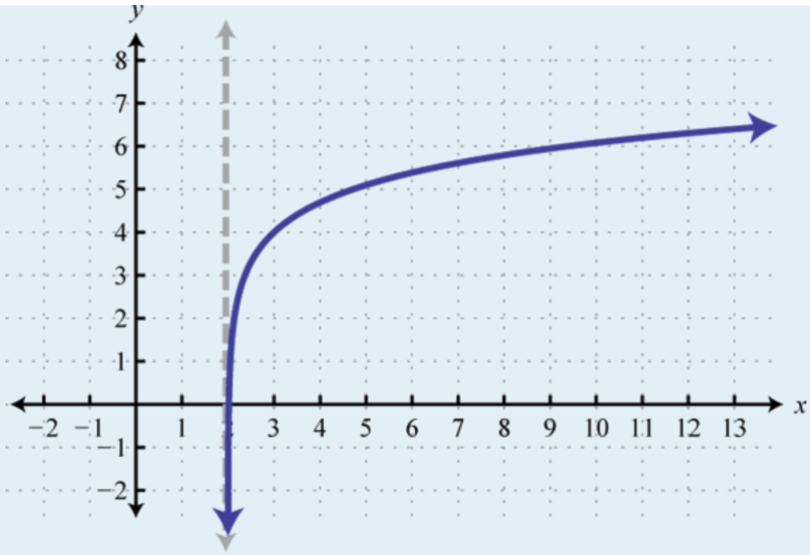
93. Domain: $(-4, \infty)$; Range: $(-\infty, \infty)$



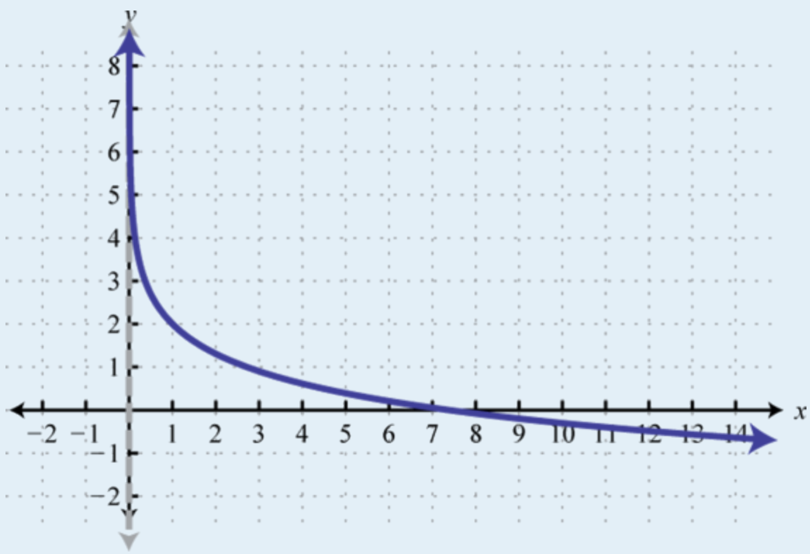
95. Domain: $(-2, \infty)$; Range: $(-\infty, \infty)$



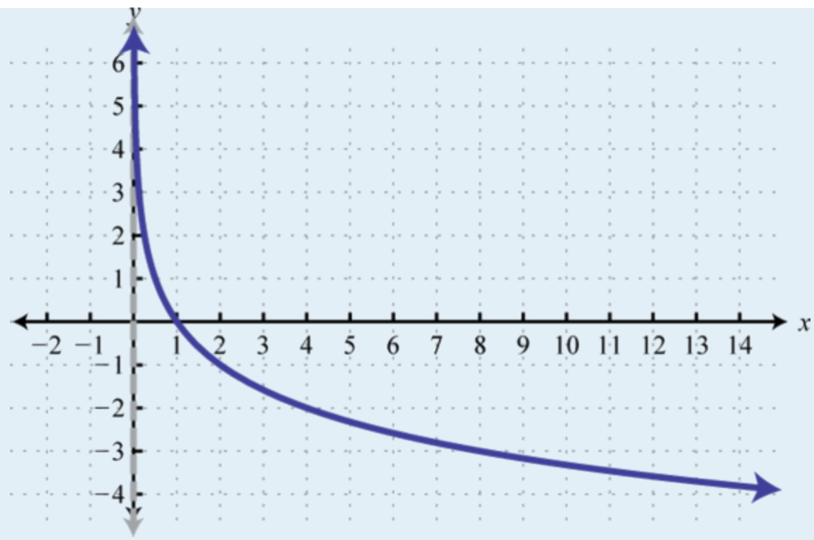
97. Domain: $(3, \infty)$; Range: $(-\infty, \infty)$



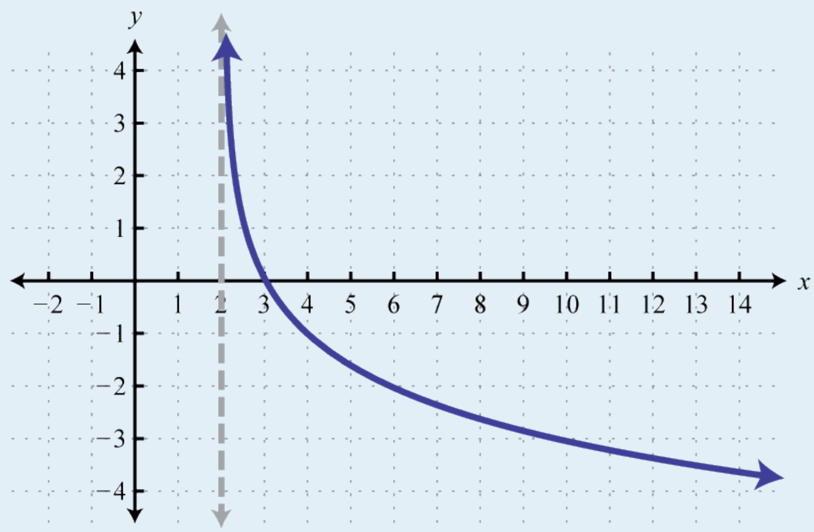
99. Domain: $(2, \infty)$; Range: $(-\infty, \infty)$



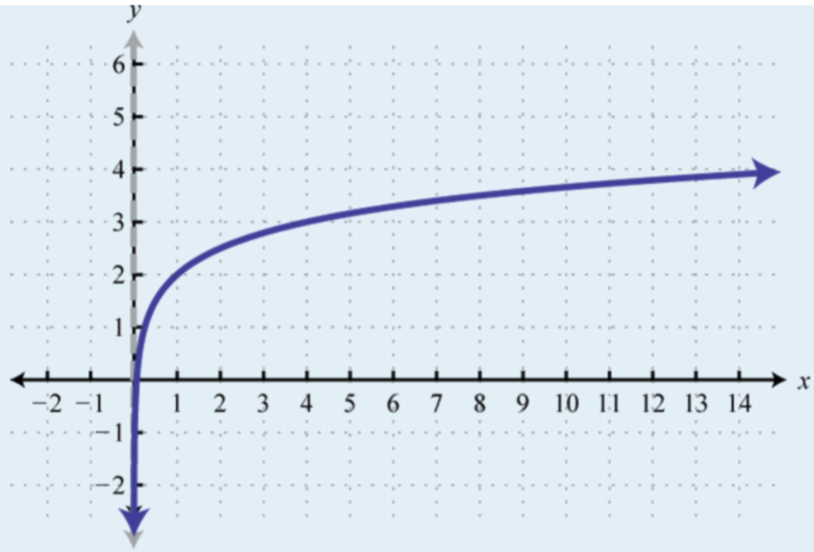
101. Domain: $(0, \infty)$; Range: $(-\infty, \infty)$



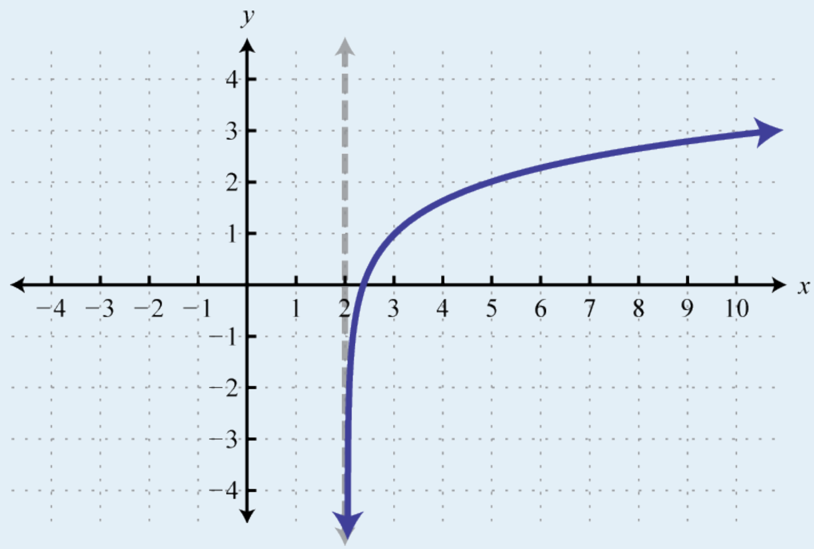
103. Domain: $(0, \infty)$; Range: $(-\infty, \infty)$



105. Domain: $(2, \infty)$; Range: $(-\infty, \infty)$



107. Domain: $(0, \infty)$; Range: $(-\infty, \infty)$



109. Domain: $(2, \infty)$; Range: $(-\infty, \infty)$

111. Answer may vary

113. Answer may vary

7.4 Properties of the Logarithm

LEARNING OBJECTIVES

1. Apply the inverse properties of the logarithm.
2. Expand logarithms using the product, quotient, and power rule for logarithms.
3. Combine logarithms into a single logarithm with coefficient 1.

Logarithms and Their Inverse Properties

Recall the definition of the base-b logarithm: given $b > 0$ where $b \neq 1$,

$$y = \log_b x \quad \text{if and only if} \quad x = b^y$$

Use this definition to convert logarithms to exponential form. Doing this, we can derive a few properties:

$$\begin{aligned} \log_b\ 1 &= 0 && \text{because} \quad b^0 = 1 \\ \log_b\ b &= 1 && \text{because} \quad b^1 = b \\ \log_b\ \left(\frac{1}{b}\right) &= -1 && \text{because} \quad b^{-1} = \frac{1}{b} \end{aligned}$$

Example 1

Evaluate:

a. $\log\ 1$
b. $\ln\ e$
c. $\log_5\ \left(\frac{1}{5}\right)$

Solution:

a. When the base is not written, it is assumed to be 10. This is the common logarithm,

$$\log\ 1 = \log_{10}\ 1 = 0$$

b. The natural logarithm, by definition, has base e,

$$\ln\ e = \log_e\ e = 1$$

c. Because $5^{-1} = \frac{1}{5}$ we have,

$$\log_5\ \left(\frac{1}{5}\right) = -1$$

Furthermore, consider fractional bases of the form $1/b$ where $b > 1$.

$$\log_{1/b}\ b = -1 \quad \text{because} \quad \left(\frac{1}{b}\right)^{-1} = \frac{1^{-1}}{b^{-1}} = \frac{b}{1} = b$$

Example 2

Evaluate:

a. $\log_{1/4} 4$
b. $\log_{2/3} \left(\frac{3}{2}\right)$

Solution:

a. $\log_{1/4} 4 = -1 \quad \text{because} \quad \left(\frac{1}{4}\right)^{-1} = 4$
b. $\log_{2/3} \left(\frac{3}{2}\right) = -1 \quad \text{because} \quad \left(\frac{2}{3}\right)^{-1} = \frac{3}{2}$

Given an exponential function defined by $f(x) = b^x$, where $b > 0$ and $b \neq 1$, its inverse is the base-b logarithm, $f^{-1}(x) = \log_b x$. And because $f\left(f^{-1}(x)\right) = x$ and $f^{-1}\left(f(x)\right) = x$, we have the following **inverse properties of the logarithm**[11]:

$$f^{-1}\left(f(x)\right) = \log_b b^x = x \quad \text{and}$$
$$f\left(f^{-1}(x)\right) = b^{\log_b x} = x\,,\ x > 0$$

Since $f^{-1}(x) = \log_b x$ has a domain consisting of positive values $(0, \infty)$, the property $b^{\log_b x} = x$ is restricted to values where $x > 0$.

11. Given $b > 0$ we have $\log_b b^x = x$ and $b^{\log_b x} = x$ when $x > 0$.

Example 3

Evaluate:

a. $\log_5 625$
b. $5^{\log_5 3}$
c. $e^{\ln\ 5}$

Solution:

Apply the inverse properties of the logarithm.

a. $\log_5 625 = \log_5 5^4 = 4$
b. $5^{\log_5 3} = 3$
c. $e^{\ln\ 5} = 5$

In summary, when $b > 0$ and $b \neq 1$, we have the following properties:

$\log_b 1 = 0$	$\log_b b = 1$
$\log_{1/b} b = -1$	$\log_b \left(\frac{1}{b}\right) = -1$
$\log_b b^x = x$	$b^{\log_b x} = x,\ x > 0$

Try this! Evaluate: $\log\ 0.00001$

Answer: -5

(click to see video)

Product, Quotient, and Power Properties of Logarithms

In this section, three very important properties of the logarithm are developed. These properties will allow us to expand our ability to solve many more equations. We begin by assigning *u* and *v* to the following logarithms and then write them in exponential form:

$$\begin{aligned}\log_b x = u &\Longrightarrow\quad b^u = x\\ \log_b y = v &\Longrightarrow\quad b^v = y\end{aligned}$$

Substitute $x = b^u$ and $y = b^v$ into the logarithm of a product $\log_b\ (xy)$ and the logarithm of a quotient $\log_b\ \left(\frac{x}{y}\right)$.Then simplify using the rules of exponents and the inverse properties of the logarithm.

Logarithm of a Product	Logarithm of a Quotient
$\begin{aligned}\log_b\ (xy) &= \log_b\ (b^u b^v)\\ &= \log_b b^{u+v}\\ &= u + v\\ &= \log_b x + \log_b y\end{aligned}$	$\begin{aligned}\log_b\ \left(\frac{x}{y}\right) &= \log_b\ \left(\frac{b^u}{b^v}\right)\\ &= \log_b b^{u-v}\\ &= u - v\\ &= \log_b x - \log_b y\end{aligned}$

This gives us two essential properties: the **product property of logarithms**[12],

$$\log_b (xy) = \log_b x + \log_b y$$

and the **quotient property of logarithms**[13],

$$\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$$

In words, the logarithm of a product is equal to the sum of the logarithm of the factors. Similarly, the logarithm of a quotient is equal to the difference of the logarithm of the numerator and the logarithm of the denominator.

Example 4

Write as a sum: $\log_2 (8x)$.

Solution:

Apply the product property of logarithms and then simplify.

$$\begin{aligned}\log_2 (8x) &= \log_2 8 + \log_2 x \\ &= \log_2 2^3 + \log_2 x \\ &= 3 + \log_2 x\end{aligned}$$

Answer: $3 + \log_2 x$

12. $\log_b (xy) = \log_b x + \log_b y$; the logarithm of a product is equal to the sum of the logarithm of the factors.

13. $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$; the logarithm of a quotient is equal to the difference of the logarithm of the numerator and the logarithm of the denominator.

Example 5

Write as a difference: $\log\left(\frac{x}{10}\right)$.

Solution:

Apply the quotient property of logarithms and then simplify.

$$\begin{aligned}\log\left(\frac{x}{10}\right) &= \log x - \log 10\\ &= \log x - 1\end{aligned}$$

Answer: $\log x - 1$

Next we begin with $\log_b x = u$ and rewrite it in exponential form. After raising both sides to the nth power, convert back to logarithmic form, and then back substitute.

$$\begin{aligned}\log_b x = u \quad &\Longrightarrow \quad b^u = x\\ &\qquad (b^u)^n = (x)^n\\ \log_b x^n = nu \quad &\Longleftarrow \quad b^{nu} = x^n\\ \log_b x^n = n\log_b x &\end{aligned}$$

This leads us to the **power property of logarithms**[14],

$$\log_b x^n = n\log_b x$$

14. $\log_b x^n = n\log_b x$; the logarithm of a quantity raised to a power is equal to that power times the logarithm of the quantity.

In words, the logarithm of a quantity raised to a power is equal to that power times the logarithm of the quantity.

Example 6

Write as a product:

a. $\log_2 x^4$
b. $\log_5 \left(\sqrt{x}\right)$.

Solution:

a. Apply the power property of logarithms.

$$\log_2 x^4 = 4\log_2 x$$

b. Recall that a square root can be expressed using rational exponents, $\sqrt{x} = x^{1/2}$. Make this replacement and then apply the power property of logarithms.

$$\begin{aligned}\log_5 \left(\sqrt{x}\right) &= \log_5 x^{1/2} \\ &= \frac{1}{2}\log_5 x\end{aligned}$$

In summary,

Product property of logarithms	$\log_b (xy) = \log_b x + \log_b y$

Quotient property of logarithms	$\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$
Power property of logarithms	$\log_b x^n = n\log_b x$

We can use these properties to expand logarithms involving products, quotients, and powers using sums, differences and coefficients. A logarithmic expression is completely expanded when the properties of the logarithm can no further be applied.

Caution: It is important to point out the following:

$$\log\ (xy) \neq \log\ x \cdot \log\ y\ \ and\ \ \log\ \left(\frac{x}{y}\right) \neq \frac{\log\ x}{\log\ y}$$

Example 7

Expand completely: $\ln\left(2x^3\right)$.

Solution:

Recall that the natural logarithm is a logarithm base e, $\ln x = \log_e x$. Therefore, all of the properties of the logarithm apply.

$$\begin{aligned} \ln\left(2x^3\right) &= \ln 2 + \ln x^3 & \textit{Product rule for logarithms} \\ &= \ln 2 + 3\ln x & \textit{Power rule for logarithms} \end{aligned}$$

Answer: $\ln 2 + 3\ln x$

Example 8

Expand completely: $\log \sqrt[3]{10xy^2}$.

Solution:

Begin by rewriting the cube root using the rational exponent $\frac{1}{3}$ and then apply the properties of the logarithm.

$$\begin{aligned}\log \sqrt[3]{10xy^2} &= \log \left(10xy^2\right)^{1/3}\\ &= \frac{1}{3}\log \left(10xy^2\right)\\ &= \frac{1}{3}\left(\log 10 + \log x + \log y^2\right)\\ &= \frac{1}{3}\left(1 + \log x + 2\log y\right)\\ &= \frac{1}{3} + \frac{1}{3}\log x + \frac{2}{3}\log y\end{aligned}$$

Answer: $\frac{1}{3} + \frac{1}{3}\log x + \frac{2}{3}\log y$

Example 9

Expand completely: $\log_2\left(\frac{(x+1)^2}{5y}\right)$.

Solution:

When applying the product property to the denominator, take care to distribute the negative obtained from applying the quotient property.

$$\begin{aligned}\log_2\left(\frac{(x+1)^2}{5y}\right) &= \log_2(x+1)^2 - \log_2(5y)\\ &= \log_2(x+1)^2 - (\log_2 5 + \log_2 y) \quad \textit{Distribute.}\\ &= \log_2(x+1)^2 - \log_2 5 - \log_2 y\\ &= 2\log_2(x+1) - \log_2 5 - \log_2 y\end{aligned}$$

Answer: $2\log_2(x+1) - \log_2 5 - \log_2 y$

Caution: There is no rule that allows us to expand the logarithm of a sum or difference. In other words,

$$\log(x \pm y) \neq \log x \pm \log y$$

Try this! Expand completely: $\ln\left(\frac{5y^4}{\sqrt{x}}\right)$

Answer: $\ln\ 5 + 4\ln\ y - \frac{1}{2}\ln\ x$

(click to see video)

Example 10

Given that $\log_2 x = a$, $\log_2 y = b$, and that $\log_2 z = c$, write the following in terms of a, b and c:

a. $\log_2 \left(8x^2 y\right)$

b. $\log_2 \left(\frac{2x^4}{\sqrt{z}}\right)$

Solution:

a. Begin by expanding using sums and coefficients and then replace a and b with the appropriate logarithm.

$$\begin{aligned} \log_2 \left(8x^2 y\right) &= \log_2 8 + \log_2 x^2 + \log_2 y \\ &= \log_2 8 + 2\log_2 x + \log_2 y \\ &= 3 + 2a + b \end{aligned}$$

b. Expand and then replace a, b, and c where appropriate.

$$\begin{aligned} \log_2 \left(\frac{2x^4}{\sqrt{z}}\right) &= \log_2 \left(2x^4\right) - \log_2 z^{1/2} \\ &= \log_2 2 + \log_2 x^4 - \log_2 z^{1/2} \\ &= \log_2 2 + 4\log_2 x - \frac{1}{2}\log_2 z \\ &= 1 + 4a - \frac{1}{2}b \end{aligned}$$

Next we will condense logarithmic expressions. As we will see, it is important to be able to combine an expression involving logarithms into a single logarithm with coefficient 1. This will be one of the first steps when solving logarithmic equations.

Example 11

Write as a single logarithm with coefficient 1: $3\log_3 x - \log_3 y + 2\log_3 5$.

Solution:

Begin by rewriting all of the logarithmic terms with coefficient 1. Use the power rule to do this. Then use the product and quotient rules to simplify further.

$$\begin{aligned} 3\log_3 x - \log_3 y + 2\log_3 5 &= \left\{\log_3 x^3 - \log_3 y\right\} + \log_3 5^2 && \textit{quotient p} \\ &= \left\{\log_3\left(\frac{x^3}{y}\right) + \log_3 25\right\} && \textit{produ} \\ &= \log_3\left(\frac{x^3}{y}\cdot 25\right) \\ &= \log_3\left(\frac{25x^3}{y}\right) \end{aligned}$$

Answer: $\log_3\left(\frac{25x^3}{y}\right)$

Example 12

Write as a single logarithm with coefficient 1: $\frac{1}{2}\ln x - 3\ln y - \ln z$.

Solution:

Begin by writing the coefficients of the logarithms as powers of their argument, after which we will apply the quotient rule twice working from left to right.

$$\begin{aligned}\frac{1}{2}\ln x - 3\ln y - \ln z &= \ln x^{1/2} - \ln y^3 - \ln z \\ &= \ln\left(\frac{x^{1/2}}{y^3}\right) - \ln z \\ &= \ln\left(\frac{x^{1/2}}{y^3} \div z\right) \\ &= \ln\left(\frac{x^{1/2}}{y^3} \cdot \frac{1}{z}\right) \\ &= \ln\left(\frac{x^{1/2}}{y^3 z}\right) \quad or \quad = \ln\left(\frac{\sqrt{x}}{y^3 z}\right)\end{aligned}$$

Answer: $\ln\left(\frac{\sqrt{x}}{y^3 z}\right)$

Try this! Write as a single logarithm with coefficient $1{:}\ 3\log\ (x+y) - 6\log\ z + 2\log\ 5.$

Answer: $\log\ \left(\frac{25(x+y)^3}{z^6}\right)$

(click to see video)

KEY TAKEAWAYS

- Given any base $b > 0$ and $b \neq 1$, we can say that $\log_b 1 = 0$, $\log_b b = 1$, $\log_{1/b} b = -1$ and that $\log_b \left(\frac{1}{b}\right) = -1$.
- The inverse properties of the logarithm are $\log_b b^x = x$ and $b^{\log_b x} = x$ where $x > 0$.
- The product property of the logarithm allows us to write a product as a sum: $\log_b\ (xy) = \log_b x + \log_b y$.
- The quotient property of the logarithm allows us to write a quotient as a difference: $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$.
- The power property of the logarithm allows us to write exponents as coefficients: $\log_b x^n = n\log_b x$.
- Since the natural logarithm is a base-e logarithm, $\ln\ x = \log_e x$, all of the properties of the logarithm apply to it.
- We can use the properties of the logarithm to expand logarithmic expressions using sums, differences, and coefficients. A logarithmic expression is completely expanded when the properties of the logarithm can no further be applied.
- We can use the properties of the logarithm to combine expressions involving logarithms into a single logarithm with coefficient 1. This is an essential skill to be learned in this chapter.

TOPIC EXERCISES

PART A: LOGARITHMS AND THEIR INVERSE PROPERTIES

Evaluate:

1. $\log_7 1$
2. $\log_{1/2} 2$
3. $\log 10^{14}$
4. $\log 10^{-23}$
5. $\log_3 3^{10}$
6. $\log_6 6$
7. $\ln e^7$
8. $\ln \left(\frac{1}{e}\right)$
9. $\log_{1/2} \left(\frac{1}{2}\right)$
10. $\log_{1/5} 5$
11. $\log_{3/4} \left(\frac{4}{3}\right)$
12. $\log_{2/3} 1$
13. $2^{\log_2 100}$
14. $3^{\log_3 1}$
15. $10^{\log 18}$
16. $e^{\ln 23}$
17. $e^{\ln x^2}$
18. $e^{\ln e^x}$

Find a:

19. $\ln a = 1$
20. $\log a = -1$
21. $\log_9 a = -1$
22. $\log_{12} a = 1$
23. $\log_2 a = 5$
24. $\log a = 13$
25. $2^a = 7$
26. $e^a = 23$
27. $\log_a 4^5 = 5$
28. $\log_a 10 = 1$

PART B: PRODUCT, QUOTIENT, AND POWER PROPERTIES OF LOGARITHMS

Expand completely.

29. $\log_4 (xy)$
30. $\log (6x)$
31. $\log_3 (9x^2)$
32. $\log_2 (32x^7)$
33. $\ln (3y^2)$
34. $\log (100x^2)$
35. $\log_2 \left(\frac{x}{y^2}\right)$
36. $\log_5 \left(\frac{25}{x}\right)$

37. $\log\left(10x^2y^3\right)$

38. $\log_2\left(2x^4y^5\right)$

39. $\log_3\left(\frac{x^3}{yz^2}\right)$

40. $\log\left(\frac{x}{y^3z^2}\right)$

41. $\log_5\left(\frac{1}{x^2yz}\right)$

42. $\log_4\left(\frac{1}{16x^2z^3}\right)$

43. $\log_6\left[36(x+y)^4\right]$

44. $\ln\left[e^4(x-y)^3\right]$

45. $\log_7\left(2\sqrt{xy}\right)$

46. $\ln\left(2x\sqrt{y}\right)$

47. $\log_3\left(\frac{x^2\sqrt[3]{y}}{z}\right)$

48. $\log\left(\frac{2(x+y)^3}{z^2}\right)$

49. $\log\left(\frac{100x^3}{(y+10)^3}\right)$

50. $\log_7\left(\frac{x}{\sqrt[5]{(y+z)^3}}\right)$

51. $\log_5\left(\frac{x^3}{\sqrt[3]{yz^2}}\right)$

52. $\log\left(\dfrac{x^2}{\sqrt[5]{y^3z^2}}\right)$

Given $\log_3 x = a$, $\log_3 y = b$, and $\log_3 z = c$, write the following logarithms in terms of *a*, *b*, and *c*.

53. $\log_3\left(27x^2y^3z\right)$
54. $\log_3\left(xy^3\sqrt{z}\right)$
55. $\log_3\left(\dfrac{9x^2y}{z^3}\right)$
56. $\log_3\left(\dfrac{\sqrt[3]{x}}{yz^2}\right)$

Given $\log_b 2 = 0.43$, $\log_b 3 = 0.68$, and $\log_b 7 = 1.21$, calculate the following. (Hint: Expand using sums, differences, and quotients of the factors 2, 3, and 7.)

57. $\log_b 42$
58. $\log_b\left(36\right)$
59. $\log_b\left(\frac{28}{9}\right)$
60. $\log_b\sqrt{21}$

Expand using the properties of the logarithm and then approximate using a calculator to the nearest tenth.

61. $\log\left(3.10\times 10^{25}\right)$
62. $\log\left(1.40\times 10^{-33}\right)$
63. $\ln\left(6.2e^{-15}\right)$
64. $\ln\left(1.4e^{22}\right)$

Write as a single logarithm with coefficient 1.

65. $\log x + \log y$

66. $\log_3 x - \log_3 y$

67. $\log_2 5 + 2\log_2 x + \log_2 y$

68. $\log_3 4 + 3\log_3 x + \frac{1}{2} \log_3 y$

69. $3\log_2 x - 2\log_2 y + \frac{1}{2} \log_2 z$

70. $4 \log x - \log y - \log 2$

71. $\log 5 + 3 \log \left(x + y\right)$

72. $4\log_5 \left(x + 5\right) + \log_5 y$

73. $\ln x - 6 \ln y + \ln z$

74. $\log_3 x - 2\log_3 y + 5\log_3 z$

75. $7 \log x - \log y - 2 \log z$

76. $2 \ln x - 3 \ln y - \ln z$

77. $\frac{2}{3} \log_3 x - \frac{1}{2} \left(\log_3 y + \log_3 z\right)$

78. $\frac{1}{5} \left(\log_7 x + 2\log_7 y\right) - 2\log_7 (z + 1)$

79. $1 + \log_2 x - \frac{1}{2} \log_2 y$

80. $2 - 3\log_3 x + \frac{1}{3} \log_3 y$

81. $\frac{1}{3} \log_2 x + \frac{2}{3} \log_2 y$

82. $-2\log_5 x + \frac{3}{5} \log_5 y$

83. $- \ln 2 + 2 \ln \left(x + y\right) - \ln z$

84. $-3 \ln \left(x - y\right) - \ln z + \ln 5$

85. $\frac{1}{3} \left(\ln x + 2 \ln y\right) - \left(3 \ln 2 + \ln z\right)$

86. $4 \log 2 + \frac{2}{3} \log x - 4 \log \left(y + z\right)$

87. $\log_2 3 - 2\log_2 x + \frac{1}{2} \log_2 y - 4\log_2 z$

88. $2\log_5 4 - \log_5 x - 3\log_5 y + \frac{2}{3}\log_5 z$

Express as a single logarithm and simplify.

89. $\log\ (x+1) + \log\ (x-1)$

90. $\log_2\ (x+2) + \log_2\ (x+1)$

91. $\ln\ \left(x^2+2x+1\right) - \ln\ (x+1)$

92. $\ln\ \left(x^2-9\right) - \ln\ (x+3)$

93. $\log_5\ \left(x^3-8\right) - \log_5\ (x-2)$

94. $\log_3\ \left(x^3+1\right) - \log_3\ (x+1)$

95. $\log x + \log\ \left(x+5\right) - \log\ \left(x^2-25\right)$

96. $\log\ (2x+1) + \log\ (x-3) - \log\ \left(2x^2-5x-3\right)$

ANSWERS

1. 0
3. 14
5. 10
7. 7
9. 1
11. -1
13. 100
15. 18
17. x^2
19. e
21. $\frac{1}{9}$
23. $2^5 = 32$
25. $\log_2 7$
27. 4
29. $\log_4 x + \log_4 y$
31. $2 + 2\log_3 x$
33. $\ln 3 + 2\ln y$
35. $\log_2 x - 2\log_2 y$
37. $1 + 2\log x + 3\log y$
39. $3\log_3 x - \log_3 y - 2\log_3 z$
41. $-2\log_5 x - \log_5 y - \log_5 z$
43. $2 + 4\log_6 (x + y)$
45. $\log_7 2 + \frac{1}{2}\log_7 x + \frac{1}{2}\log_7 y$

47. $2\log_3 x + \frac{1}{3}\log_3 y - \log_3 z$

49. $2 + 3\log x - 3\log\left(y + 10\right)$

51. $3\log_5 x - \frac{1}{3}\log_5 y - \frac{2}{3}\log_5 z$

53. $3 + 2a + 3b + c$

55. $2 + 2a + b - 3c$

57. 2.32

59. 0.71

61. $\log(3.1) + 25 \approx 25.5$

63. $\ln\left(6.2\right) - 15 \approx -13.2$

65. $\log\left(xy\right)$

67. $\log_2\left(5x^2 y\right)$

69. $\log_2\left(\frac{x^3\sqrt{z}}{y^2}\right)$

71. $\log\left[5\left(x+y\right)^3\right]$

73. $\ln\left(\frac{xz}{y^6}\right)$

75. $\log\left(\frac{x^7}{yz^2}\right)$

77. $\log_3\left(\frac{\sqrt[3]{x^2}}{\sqrt{yz}}\right)$

79. $\log_2\left(\frac{2x}{\sqrt{y}}\right)$

81. $\log_2\left(\sqrt[3]{xy^2}\right)$

83. $\ln\left(\frac{(x+y)^2}{2z}\right)$

85. $\ln\left(\frac{\sqrt[3]{xy^2}}{8z}\right)$

87. $\log_2\left(\frac{3\sqrt{y}}{x^2 z^4}\right)$

89. $\log\left(x^2-1\right)$

91. $\ln(x+1)$

93. $\log_5\left(x^2+2x+4\right)$

95. $\log\left(\frac{x}{x-5}\right)$

7.5 Solving Exponential and Logarithmic Equations

LEARNING OBJECTIVES

1. Solve exponential equations.
2. Use the change of base formula to approximate logarithms.
3. Solve logarithmic equations.

Solving Exponential Equations

An **exponential equation**[15] is an equation that includes a variable as one of its exponents. In this section we describe two methods for solving exponential equations. First, recall that exponential functions defined by $f(x) = b^x$ where $b > 0$ and $b \neq 1$, are one-to-one; each value in the range corresponds to exactly one element in the domain. Therefore, $f(x) = f(y)$ implies $x = y$. The converse is true because *f* is a function. This leads to the very important **one-to-one property of exponential functions**[16]:

$$b^x = b^y \quad \text{if and only if} \quad x = y$$

Use this property to solve special exponential equations where each side can be written in terms of the same base.

15. An equation which includes a variable as an exponent.

16. Given $b > 0$ and $b \neq 1$ we have $b^x = b^y$ if and only if $x = y$.

Example 1

Solve: $3^{2x-1} = 27$.

Solution:

Begin by writing 27 as a power of 3.

$$\begin{aligned} 3^{2x-1} &= 27 \\ 3^{2x-1} &= 3^3 \end{aligned}$$

Next apply the one-to-one property of exponential functions. In other words, set the exponents equal to each other and then simplify.

$$\begin{aligned} 2x - 1 &= 3 \\ 2x &= 4 \\ x &= 2 \end{aligned}$$

Answer: 2

Example 2

Solve: $16^{1-3x} = 2$.

Solution:

Begin by writing 16 as a power of 2 and then apply the power rule for exponents.

$$\begin{aligned} 16^{1-3x} &= 2 \\ \left(2^4\right)^{1-3x} &= 2 \\ 2^{4(1-3x)} &= 2^1 \end{aligned}$$

Now that the bases are the same we can set the exponents equal to each other and simplify.

$$\begin{aligned} 4\,(1-3x) &= 1 \\ 4-12x &= 1 \\ -12x &= -3 \\ x &= \frac{-3}{-12} = \frac{1}{4} \end{aligned}$$

Answer: $\frac{1}{4}$

Try this! Solve: $25^{2x+3} = 125$.

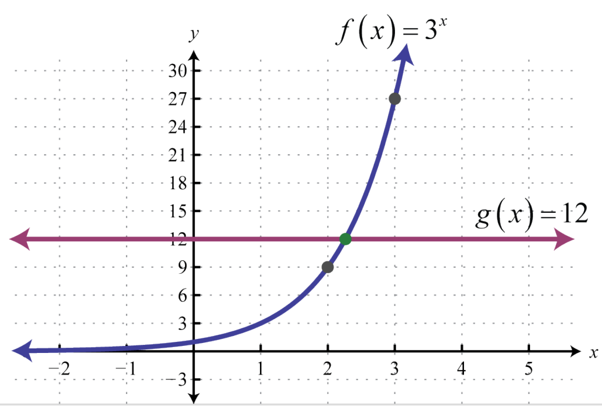
Answer: $-\frac{3}{4}$

(click to see video)

In many cases we will not be able to equate the bases. For this reason we develop a second method for solving exponential equations. Consider the following equations:

$$\begin{aligned} 3^2 &= 9 \\ 3^{?} &= 12 \\ 3^3 &= 27 \end{aligned}$$

We can see that the solution to $3^x = 12$ should be somewhere between 2 and 3. A graphical interpretation follows.



To solve this we make use of fact that logarithms are one-to-one functions. Given $x, y > 0$ the **one-to-one property of logarithms**[17] follows:

$$\log_b x = \log_b y \quad \text{if and only if} \quad x = y$$

17. Given $b > 0$ and $b \neq 1$ where $x, y > 0$ we have $\log_b x = \log_b y$ if and only if $x = y$.

This property, as well as the properties of the logarithm, allows us to solve exponential equations. For example, to solve $3^x = 12$ apply the common logarithm to both sides and then use the properties of the logarithm to isolate the variable.

$$\begin{aligned} 3^x &= 12 \\ \log\ 3^x &= \log\ 12 && \textit{One-to-one property of logarithms} \\ x \log\ 3 &= \log\ 12 && \textit{Power rule for logarithms} \\ x &= \frac{\log\ 12}{\log\ 3} \end{aligned}$$

Approximating to four decimal places on a calculator.

$$x = \log\ (12)/\log\ (3) \approx 2.2619$$

An answer between 2 and 3 is what we expected. Certainly we can check by raising 3 to this power to verify that we obtain a good approximation of 12.

$$3 \wedge 2.2618 \approx 12 \checkmark$$

Note that we are *not* multiplying both sides by "log"; we are applying the one-to-one property of logarithmic functions — which is often expressed as "*taking the log of both sides*." The general steps for solving exponential equations are outlined in the following example.

Example 3

Solve: $5^{2x-1} + 2 = 9$.

Solution:

- **Step 1:** Isolate the exponential expression.

$$\begin{aligned} 5^{2x-1} + 2 &= 9 \\ 5^{2x-1} &= 7 \end{aligned}$$

- **Step 2:** Take the logarithm of both sides. In this case, we will take the common logarithm of both sides so that we can approximate our result on a calculator.

$$\log 5^{2x-1} = \log 7$$

- **Step 3:** Apply the power rule for logarithms and then solve.

$$\begin{aligned} \log 5^{2x-1} &= \log 7 \\ (2x-1)\log 5 &= \log 7 && \textit{Distribute.} \\ 2x\log 5 - \log 5 &= \log 7 \\ 2x\log 5 &= \log 5 + \log 7 \\ x &= \frac{\log 5 + \log 7}{2\log 5} \end{aligned}$$

This is an irrational number which can be approximated using a calculator. Take care to group the numerator and the product in the denominator when entering this into your calculator. To do this, make use of the parenthesis buttons $\boxed{(}$ and $\boxed{)}$:

$$x = \left(\log\ 5 + \log\ (7)\right) / \left(2 * \log\ (5)\right) \approx 1.1045$$

Answer: $\frac{\log\ 5+\log\ 7}{2\log\ 5} \approx 1.1045$

Example 4

Solve: $e^{5x+3} = 1$.

Solution:

The exponential function is already isolated and the base is e. Therefore, we choose to apply the natural logarithm to both sides.

$$\begin{aligned} e^{5x+3} &= 1 \\ \ln e^{5x+3} &= \ln 1 \end{aligned}$$

Apply the power rule for logarithms and then simplify.

$$\begin{aligned} \ln e^{5x+3} &= \ln 1 \\ (5x+3)\ln e &= \ln 1 \quad \textit{Recall } \ln e = 1 \textit{ and } \ln 1 = 0. \\ (5x+3)\cdot 1 &= 0 \\ 5x+3 &= 0 \\ x &= -\frac{3}{5} \end{aligned}$$

Answer: $-\frac{3}{5}$

On most calculators there are only two logarithm buttons, the common logarithm $\boxed{LOG}$ and the natural logarithm $\boxed{LN}$. If we want to approximate $\log_3 10$ we have to somehow change this base to 10 or e. The idea begins by rewriting the logarithmic function $y = \log_a x$, in exponential form.

$$\log_a x = y \Longrightarrow x = a^y$$

Here $x > 0$ and so we can apply the one-to-one property of logarithms. Apply the logarithm base b to both sides of the function in exponential form.

$$\begin{aligned} x &= a^y \\ \log_b x &= \log_b a^y \end{aligned}$$

And then solve for y.

$$\begin{aligned} \log_b x &= y \log_b a \\ \frac{\log_b x}{\log_b a} &= y \end{aligned}$$

Replace y into the original function and we have the very important **change of base formula**[18]:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

We can use this to approximate $\log_3 10$ as follows.

$$\log_3 10 = \frac{\log 10}{\log 3} \approx 2.0959 \quad or \quad \log_3 10 = \frac{\ln 10}{\ln 3} \approx 2.0959$$

18. $\log_a x = \frac{\log_b x}{\log_b a}$; we can write any base-$a$ logarithm in terms of base-b logarithms using this formula.

Notice that the result is independent of the choice of base. In words, we can approximate the logarithm of any given base on a calculator by dividing the logarithm of the argument by the logarithm of that given base.

Example 5

Approximate $\log_7 120$ the nearest hundredth.

Solution:

Apply the change of base formula and use a calculator.

$$\log_7 120 = \frac{\log\ 120}{\log\ 7}$$

On a calculator,

$$\log\ (120) / \log\ (7) \approx 2.46$$

Answer: 2.46

Try this! Solve: $2^{3x+1} - 4 = 1$. Give the exact and approximate answer rounded to four decimal places.

Answer: $\frac{\log\ 5 - \log\ 2}{3\log\ 2} \approx 0.4406$

(click to see video)

Solving Logarithmic Equations

A **logarithmic equation**[19] is an equation that involves a logarithm with a variable argument. Some logarithmic equations can be solved using the one-to-one property of logarithms. This is true when a single logarithm with the same base can be obtained on both sides of the equal sign.

19. An equation that involves a logarithm with a variable argument.

Example 6

Solve: $\log_2\left(2x-5\right)-\log_2(x-2)=0$.

Solution:

We can obtain two equal logarithms base 2 by adding $\log_2(x-2)$ to both sides of the equation.

$$\begin{aligned}\log_2\left(2x-5\right)-\log_2(x-2)&=0\\ \log_2\left(2x-5\right)&=\log_2(x-2)\end{aligned}$$

Here the bases are the same and so we can apply the one-to-one property and set the arguments equal to each other.

$$\begin{aligned}\log_2\left(2x-5\right)&=\log_2(x-2)\\ 2x-5&=x-2\\ x&=3\end{aligned}$$

Checking $x=3$ in the original equation:

$$\begin{aligned}\log_2\left(2(3)-5\right)&=\log_2((3)-2)\\ \log_2 1&=\log_2 1\\ 0&=0\quad\checkmark\end{aligned}$$

Answer: 3

When solving logarithmic equations the check is very important because extraneous solutions can be obtained. The properties of the logarithm only apply for values in the domain of the given logarithm. And when working with variable arguments, such as $\log\ (x-2)$, the value of *x* is not known until the end of this process. The logarithmic expression $\log\ (x-2)$ is only defined for values $x>2$.

Example 7

Solve: $\log(3x - 4) = \log(x - 2)$.

Solution:

Apply the one-to-one property of logarithms (set the arguments equal to each other) and then solve for *x*.

$$\begin{aligned} \log(3x-4) &= \log(x-2) \\ 3x-4 &= x-2 \\ 2x &= 2 \\ x &= 1 \end{aligned}$$

When performing the check we encounter a logarithm of a negative number:

$$\begin{aligned} \log(x-2) &= \log(1-2) \\ &= \log(-1) \quad \mathit{Undefined} \end{aligned}$$

Try this on a calculator, what does it say? Here $x = 1$ is not in the domain of $\log(x-2)$. Therefore our only possible solution is extraneous and we conclude that there are no solutions to this equation.

Answer: No solution, Ø.

Caution: Solving logarithmic equations sometimes leads to extraneous solutions — we must check our answers.

Try this! Solve: $\ln\left(x^2 - 15\right) - \ln(2x) = 0$.

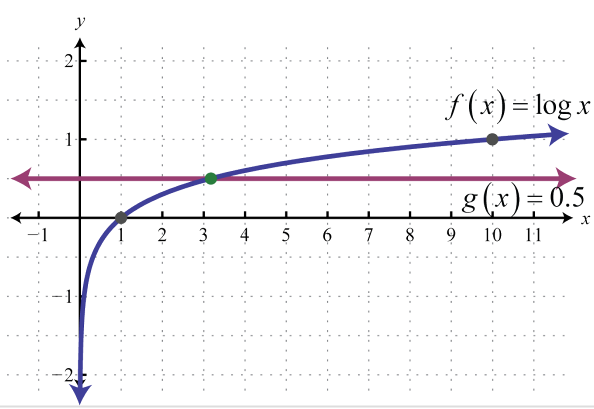
Answer: 5

(click to see video)

In many cases we will not be able to obtain two equal logarithms. To solve such equations we make use of the definition of the logarithm. If $b > 0$, where $b \neq 1$, then $\log_b x = y$ implies that $b^y = x$. Consider the following common logarithmic equations (base 10),

$$\begin{aligned} \log x &= 0 \implies x = 1 \quad \textit{Because } 10^0 = 1. \\ \log x &= 0.5 \implies x = ? \\ \log x &= 1 \implies x = 10 \quad \textit{Because } 10^1 = 10. \end{aligned}$$

We can see that the solution to $\log x = 0.5$ will be somewhere between 1 and 10. A graphical interpretation follows.



To find *x* we can apply the definition as follows.

$$\log_{10} x = 0.5 \implies 10^{0.5} = x$$

This can be approximated using a calculator,

$$x = 10^{0.5} = 10 \wedge 0.5 \approx 3.1623$$

An answer between 1 and 10 is what we expected. Check this on a calculator.

$$\log\ 3.1623 \approx 5 \checkmark$$

Example 8

Solve: $\log_3(2x-5)=2$.

Solution:

Apply the definition of the logarithm.

$$\log_3(2x-5)=2 \Longrightarrow 2x-5=3^2$$

Solve the resulting equation.

$$\begin{aligned} 2x-5&=9 \\ 2x&=14 \\ x&=7 \end{aligned}$$

Check.

$$\begin{aligned} \log_3(2(7)-5)&\stackrel{?}{=}2 \\ \log_3(9)&=2 \checkmark \end{aligned}$$

Answer: 7

In order to apply the definition, we will need to rewrite logarithmic expressions as a single logarithm with coefficient 1.The general steps for solving logarithmic equations are outlined in the following example.

Example 9

Solve: $\log_2 (x-2) + \log_2 (x-3) = 1$.

Solution:

- **Step 1:** Write all logarithmic expressions as a single logarithm with coefficient 1. In this case, apply the product rule for logarithms.

$$\begin{aligned} \log_2 (x-2) + \log_2 (x-3) &= 1 \\ \log_2 \left[(x-2)(x-3)\right] &= 1 \end{aligned}$$

- **Step 2:** Use the definition and rewrite the logarithm in exponential form.

$$\log_2 \left[(x-2)(x-3)\right] = 1 \implies (x-2)(x-3) = 2^1$$

- **Step 3:** Solve the resulting equation. Here we can solve by factoring.

$$\begin{aligned} (x-2)(x-3) &= 2 \\ x^2 - 5x + 6 &= 2 \\ x^2 - 5x + 4 &= 0 \\ (x-4)(x-1) &= 0 \\ x - 4 &= 0 \quad \text{or} \quad x - 1 = 0 \\ x &= 4 \qquad\qquad\quad x = 1 \end{aligned}$$

- **Step 4:** Check. This step is required.

Check $x = 4$	*Check* $x = 1$
$\log_2 (x-2) + \log_2 (x-3) = 1$ $\log_2 ({\color{blue}4}-2) + \log_2 ({\color{blue}4}-3) = 1$ $\log_2 (2) + \log_2 (1) = 1$ $1 + 0 = 1$ ✓	$\log_2 (x-2) + \log_2 (x-3)$ $\log_2 ({\color{blue}1}-2) + \log_2 ({\color{blue}1}-3)$ $\log_2 (-1) + \log_2 (-2)$

In this example, $x = 1$ is not in the domain of the given logarithmic expression and is extraneous. The only solution is $x = 4$.

Answer: 4

Example 10

Solve: $\log(x + 15) - 1 = \log(x + 6)$.

Solution:

Begin by writing all logarithmic expressions on one side and constants on the other.

$$\begin{aligned} \log(x + 15) - 1 &= \log(x + 6) \\ \log(x + 15) - \log(x + 6) &= 1 \end{aligned}$$

Apply the quotient rule for logarithms as a means to obtain a single logarithm with coefficient 1.

$$\begin{aligned} \log(x + 15) - \log(x + 6) &= 1 \\ \log\left(\frac{x + 15}{x + 6}\right) &= 1 \end{aligned}$$

This is a common logarithm; therefore use 10 as the base when applying the definition.

$$\begin{aligned} \frac{x+15}{x+6} &= 10^1 \\ x+15 &= 10\left(x+6\right) \\ x+15 &= 10x+60 \\ -9x &= 45 \\ x &= -5 \end{aligned}$$

Check.

$$\begin{aligned} \log\left(x+15\right)-1 &= \log\left(x+6\right) \\ \log\left(-5+15\right)-1 &= \log\left(-5+6\right) \\ \log 10 - 1 &= \log 1 \\ 1-1 &= 0 \\ 0 &= 0 \quad ✓ \end{aligned}$$

Answer: -5

Try this! Solve: $\log_2(x) + \log_2(x-1) = 1$.

Answer: 2

(click to see video)

Example 11

Find the inverse: $f(x) = \log_2(3x - 4)$.

Solution:

Begin by replacing the function notation $f(x)$ with y.

$$\begin{aligned} f(x) &= \log_2(3x - 4) \\ y &= \log_2(3x - 4) \end{aligned}$$

Interchange x and y and then solve for y.

$$x = \log_2\left(3y - 4\right) \Longrightarrow \begin{aligned} 3y - 4 &= 2^x \\ 3y &= 2^x + 4 \\ y &= \frac{2^x + 4}{3} \end{aligned}$$

The resulting function is the inverse of f. Present the answer using function notation.

Answer: $f^{-1}(x) = \frac{2^x+4}{3}$

KEY TAKEAWAYS

- If each side of an exponential equation can be expressed using the same base, then equate the exponents and solve.
- To solve a general exponential equation, first isolate the exponential expression and then apply the appropriate logarithm to both sides. This allows us to use the properties of logarithms to solve for the variable.
- The change of base formula allows us to use a calculator to calculate logarithms. The logarithm of a number is equal to the common logarithm of the number divided by the common logarithm of the given base.
- If a single logarithm with the same base can be isolated on each side of an equation, then equate the arguments and solve.
- To solve a general logarithmic equation, first isolate the logarithm with coefficient 1 and then apply the definition. Solve the resulting equation.
- The steps for solving logarithmic equations sometimes produce extraneous solutions. Therefore, the check is required.

TOPIC EXERCISES

PART A: SOLVING EXPONENTIAL EQUATIONS

Solve using the one-to-one property of exponential functions.

1. $3^x = 81$
2. $2^{-x} = 16$
3. $5^{x-1} = 25$
4. $3^{x+4} = 27$
5. $2^{5x-2} = 16$
6. $2^{3x+7} = 8$
7. $81^{2x+1} = 3$
8. $64^{3x-2} = 2$
9. $9^{2-3x} - 27 = 0$
10. $8^{1-5x} - 32 = 0$
11. $16^{x^2} - 2 = 0$
12. $4^{x^2-1} - 64 = 0$
13. $9^{x(x+1)} = 81$
14. $4^{x(2x+5)} = 64$
15. $100^{x^2} - 10^{7x-3} = 0$
16. $e^{3(3x^2-1)} - e = 0$

Solve. Give the exact answer and the approximate answer rounded to the nearest thousandth.

17. $3^x = 5$
18. $7^x = 2$

19. $4^x = 9$

20. $2^x = 10$

21. $5^{x-3} = 13$

22. $3^{x+5} = 17$

23. $7^{2x+5} = 2$

24. $3^{5x-9} = 11$

25. $5^{4x+3} + 6 = 4$

26. $10^{7x-1} - 2 = 1$

27. $e^{2x-3} - 5 = 0$

28. $e^{5x+1} - 10 = 0$

29. $6^{3x+1} - 3 = 7$

30. $8 - 10^{9x+2} = 9$

31. $15 - e^{3x} = 2$

32. $7 + e^{4x+1} = 10$

33. $7 - 9e^{-x} = 4$

34. $3 - 6e^{-x} = 0$

35. $5^{x^2} = 2$

36. $3^{2x^2-x} = 1$

37. $100e^{27x} = 50$

38. $6e^{12x} = 2$

39. $\dfrac{3}{1 + e^{-x}} = 1$

40. $\dfrac{2}{1 + 3e^{-x}} = 1$

Find the *x*- and *y*-intercepts of the given function.

41. $f(x) = 3^{x+1} - 4$

42. $f(x) = 2^{3x-1} - 1$

43. $f(x) = 10^{x+1} + 2$

44. $f(x) = 10^{4x} - 5$

45. $f(x) = e^{x-2} + 1$

46. $f(x) = e^{x+4} - 4$

Use a *u*-substitution to solve the following.

47. $3^{2x} - 3^{x} - 6 = 0$(Hint: Let $u = 3^{x}$)

48. $2^{2x} + 2^{x} - 20 = 0$

49. $10^{2x} + 10^{x} - 12 = 0$

50. $10^{2x} - 10^{x} - 30 = 0$

51. $e^{2x} - 3e^{x} + 2 = 0$

52. $e^{2x} - 8e^{x} + 15 = 0$

Use the change of base formula to approximate the following to the nearest hundredth.

53. $\log_2 5$

54. $\log_3 7$

55. $\log_5 \left(\frac{2}{3}\right)$

56. $\log_7 \left(\frac{1}{5}\right)$

57. $\log_{1/2} 10$

58. $\log_{2/3} 30$

59. $\log_2 \sqrt{5}$

60. $\log_2 \sqrt[3]{6}$

61. If left unchecked, a new strain of flu virus can spread from a single person to others very quickly. The number of people affected can be modeled using the formula $P(t) = e^{0.22t}$, where t represents the number of days the virus is allowed to spread unchecked. Estimate the number of days it will take 1,000 people to become infected.

62. The population of a certain small town is growing according to the function $P(t) = 12{,}500(1.02)^t$, where t represents time in years since the last census. Use the function to determine number of years it will take the population to grow to 25,000 people.

PART B: SOLVING LOGARITHMIC EQUATIONS

Solve using the one-to-one property of logarithms.

63. $\log_5 (2x + 4) = \log_5 (3x - 6)$

64. $\log_4 (7x) = \log_4 (5x + 14)$

65. $\log_2 (x - 2) - \log_2 (6x - 5) = 0$

66. $\ln (2x - 1) = \ln (3x)$

67. $\log (x + 5) - \log (2x + 7) = 0$

68. $\ln (x^2 + 4x) = 2 \ln (x + 1)$

69. $\log_3 2 + 2\log_3 x = \log_3 (7x - 3)$

70. $2 \log x - \log 36 = 0$

71. $\ln (x + 3) + \ln (x + 1) = \ln 8$

72. $\log_5 (x - 2) + \log_5 (x - 5) = \log_5 10$

Solve.

73. $\log_2 (3x - 7) = 5$

74. $\log_3 (2x + 1) = 2$

75. $\log (2x + 20) = 1$

76. $\log_4 (3x + 5) = \frac{1}{2}$

77. $\log_3 x^2 = 2$

78. $\log\left(x^2 + 3x + 10\right) = 1$

79. $\ln\left(x^2 - 1\right) = 0$

80. $\log_5\left(x^2 + 20\right) - 2 = 0$

81. $\log_2\left(x - 5\right) + \log_2 (x - 9) = 5$

82. $\log_2\left(x + 5\right) + \log_2 (x + 1) = 5$

83. $\log_4 x + \log_4\left(x - 6\right) = 2$

84. $\log_6 x + \log_6 (2x - 1) = 2$

85. $\log_3\left(2x + 5\right) - \log_3 (x - 1) = 2$

86. $\log_2 (x + 1) - \log_2 (x - 2) = 4$

87. $\ln x - \ln (x - 1) = 1$

88. $\ln (2x + 1) - \ln x = 2$

89. $2\log_3 x = 2 + \log_3 (2x - 9)$

90. $2\log_2 x = 3 + \log_2 (x - 2)$

91. $\log_2 (x - 2) = 2 - \log_2 x$

92. $\log_2 (x + 3) + \log_2 (x + 1) - 1 = 0$

93. $\log x - \log (x + 1) = 1$

94. $\log_2 (x + 2) + \log_2 (1 - x) = 1 + \log_2 (x + 1)$

Find the *x*- and *y*-intercepts of the given function.

95. $f(x) = \log (x + 3) - 1$

96. $f(x) = \log (x - 2) + 1$

97. $f(x) = \log_2 (3x) - 4$

98. $f(x) = \log_3 (x + 4) - 3$

99. $f(x) = \ln\left(2x + 5\right) - 6$

100. $f(x) = \ln(x + 1) + 2$

Find the inverse of the following functions.

101. $f(x) = \log_2\left(x + 5\right)$

102. $f(x) = 4 + \log_3 x$

103. $f(x) = \log(x + 2) - 3$

104. $f(x) = \ln(x - 4) + 1$

105. $f(x) = \ln(9x - 2) + 5$

106. $f(x) = \log_6(2x + 7) - 1$

107. $g(x) = e^{3x}$

108. $g(x) = 10^{-2x}$

109. $g(x) = 2^{x+3}$

110. $g(x) = 3^{2x} + 5$

111. $g(x) = 10^{x+4} - 3$

112. $g(x) = e^{2x-1} + 1$

Solve.

113. $\log\left(9x + 5\right) = 1 + \log\left(x - 5\right)$

114. $2 + \log_2\left(x^2 + 1\right) = \log_2 13$

115. $e^{5x-2} - e^{3x} = 0$

116. $3^{x^2} - 11 = 70$

117. $2^{3x} - 5 = 0$

118. $\log_7(x + 1) + \log_7(x - 1) = 1$

119. $\ln(4x - 1) - 1 = \ln x$

120. $\log(20x+1)=\log x+2$

121. $\frac{3}{1+e^{2x}}=2$

122. $2e^{-3x}=4$

123. $2e^{3x}=e^{4x+1}$

124. $2\log x+\log x-1=0$

125. $3\log x=\log(x-2)+2\log x$

126. $2\ln 3+\ln x^2=\ln\left(x^2+1\right)$

127. In chemistry, pH is a measure of acidity and is given by the formula $\text{pH}=-\log\left(H^+\right)$, where H^+ is the hydrogen ion concentration (measured in moles of hydrogen per liter of solution.) Determine the hydrogen ion concentration if the pH of a solution is 4.

128. The volume of sound, L in decibels (dB), is given by the formula $L=10\log\left(I/10^{-12}\right)$ where I represents the intensity of the sound in watts per square meter. Determine the intensity of an alarm that emits 120 dB of sound.

PART C: DISCUSSION BOARD

129. Research and discuss the history and use of the slide rule.

130. Research and discuss real-world applications involving logarithms.

ANSWERS

1. 4
3. 3
5. $\frac{6}{5}$
7. $-\frac{3}{8}$
9. $\frac{1}{6}$
11. $\pm\frac{1}{2}$
13. $-2, 1$
15. $\frac{1}{2}, 3$
17. $\frac{\log 5}{\log 3} \approx 1.465$
19. $\frac{\log 3}{\log 2} \approx 1.585$
21. $\dfrac{3\log 5 + \log 13}{\log 5} \approx 4.594$
23. $\dfrac{\log 2 - 5\log 7}{2\log 7} \approx -2.322$
25. $\varnothing$
27. $\dfrac{3 + \ln 5}{2} \approx 2.305$
29. $\dfrac{1 - \log 6}{3\log 6} \approx 0.095$
31. $\dfrac{\ln 13}{3} \approx 0.855$
33. $\ln 3 \approx 1.099$
35. $\pm\sqrt{\frac{\log 2}{\log 5}} \approx \pm 0.656$
37. $-\frac{\ln 2}{27} \approx -0.026$

39. $-\ln 2 \approx -0.693$

41. x-intercept: $\left(\frac{2\log 2 - \log 3}{\log 3}, 0\right)$; y-intercept: $(0, -1)$

43. x-intercept: None; y-intercept: $(0, 12)$

45. x-intercept: None; y-intercept: $\left(0, \frac{1+e^2}{e^2}\right)$

47. 1

49. $\log 3$

51. 0, $\ln 2$

53. 2.32

55. −0.25

57. −3.32

59. 1.16

61. Approximately 31 days

63. 10

65. $\frac{3}{5}$

67. −2

69. $\frac{1}{2}$, 3

71. 1

73. 13

75. −5

77. ±3

79. $\pm\sqrt{2}$

81. 13

83. 8

85. 2

87. $\dfrac{e}{e-1}$

89. 9

91. $1+\sqrt{5}$

93. $\varnothing$

95. x-intercept: $(7, 0)$; y-intercept: $(0, \log\ 3 - 1)$

97. x-intercept: $\left(\frac{16}{3}\ ,\ 0\right)$; y-intercept: None

99. x-intercept: $\left(\frac{e^6-5}{2}\ , 0\right)$; y-intercept: $(0, \ln\ 5 - 6)$

101. $f^{-1}(x) = 2^x - 5$

103. $f^{-1}(x) = 10^{x+3} - 2$

105. $f^{-1}(x) = \dfrac{e^{x-5} + 2}{9}$

107. $g^{-1}(x) = \dfrac{\ln\ x}{3}$

109. $g^{-1}(x) = \log_2 x - 3$

111. $g^{-1}(x) = \log\ (x+3) - 4$

113. 55

115. 1

117. $\dfrac{\log_2 5}{3}$

119. $\frac{1}{4-e}$

121. $\dfrac{\ln\ (1/2)}{2}$

123. $\ln\ 2 - 1$

125. $\varnothing$

127. 10^{-4} moles per liter

129. Answer may vary

7.6 Applications

LEARNING OBJECTIVES

1. Use the compound and continuous interest formulas.
2. Calculate doubling time.
3. Use the exponential growth/decay model.
4. Calculate the rate of decay given half-life.

Compound and Continuous Interest Formulas

Recall that compound interest occurs when interest accumulated for one period is added to the principal investment before calculating interest for the next period. The amount *A* accrued in this manner over time *t* is modeled by the compound interest formula:

$$A\left(t\right) = P\left(1 + \frac{r}{n}\right)^{nt}$$

Here the initial principal *P* is accumulating compound interest at an annual rate *r* where the value *n* represents the number of times the interest is compounded in a year.

Example 1

Susan invested \$500 in an account earning $4\frac{1}{2}$% annual interest that is compounded monthly.

a. How much will be in the account after 3 years?
b. How long will it take for the amount to grow to \$750?

Solution:

In this example, the principal $P = \$500$, the interest rate $r = 4\frac{1}{2}\% = 0.045$, and because the interest is compounded monthly, $n = 12$. The investment can be modeled by the following function:

$$A(t) = 500\left(1 + \frac{0.045}{12}\right)^{12t}$$
$$A(t) = 500(1.00375)^{12t}$$

a. Use this model to calculate the amount in the account after $t = 3$ years.

$$\begin{aligned} A(3) &= 500(1.00375)^{12(3)} \\ &= 500(1.00375)^{36} \\ &\approx 572.12 \end{aligned}$$

Rounded off to the nearest cent, after 3 years, the amount accumulated will be \$572.12.

b. To calculate the time it takes to accumulate \$750, set $A(t) = 750$ and solve for t.

$$A(t) = 500(1.00375)^{12t}$$
$$750 = 500(1.00375)^{12t}$$

This results in an exponential equation that can be solved by first isolating the exponential expression.

$$750 = 500\left(1.00375\right)^{12t}$$
$$\frac{750}{500} = \left(1.00375\right)^{12t}$$
$$1.5 = \left(1.00375\right)^{12t}$$

At this point take the common logarithm of both sides, apply the power rule for logarithms, and then solve for *t*.

$$\log\left(1.5\right) = \log\left(1.00375\right)^{12t}$$
$$\log\left(1.5\right) = 12t\log\left(1.00375\right)$$
$$\frac{\log\left(1.5\right)}{12\log\left(1.00375\right)} = \frac{\cancel{12}\,t\,\cancel{\log\left(1.00375\right)}}{\cancel{12}\,\cancel{\log\left(1.00375\right)}}$$
$$\frac{\log\left(1.5\right)}{12\log\left(1.00375\right)} = t$$

Using a calculator we can approximate the time it takes.

$$t = \log\left(1.5\right) / \left(12 * \log\left(1.00375\right)\right) \approx 9 \text{ } years$$

Answer:

a. $572.12
b. Approximately 9 years.

The period of time it takes a quantity to double is called the **doubling time**[20]. We next outline a technique for calculating the time it takes to double an initial investment earning compound interest.

20. The period of time it takes a quantity to double.

Example 2

Mario invested $1,000 in an account earning 6.3% annual interest that is compounded semi-annually. How long will it take the investment to double?

Solution:

Here the principal $P = \$1{,}000$, the interest rate $r = 6.3\% = 0.063$, and because the interest is compounded semi-annually $n = 2$. This investment can be modeled as follows:

$$\begin{aligned} A(t) &= 1{,}000\left(1 + \frac{0.063}{2}\right)^{2t} \\ A(t) &= 1{,}000(1.0315)^{2t} \end{aligned}$$

Since we are looking for the time it takes to double $1,000, substitute $2,000 for the resulting amount $A(t)$ and then solve for t.

$$\begin{aligned} 2{,}000 &= 1{,}000(1.0315)^{2t} \\ \frac{2{,}000}{1{,}000} &= (1.0315)^{2t} \\ 2 &= (1.0315)^{2t} \end{aligned}$$

At this point we take the common logarithm of both sides.

$$2=(1.0315)^{2t}$$
$$\log\ 2=\log\ (1.0315)^{2t}$$
$$\log\ 2=2t\log\ (1.0315)$$
$$\frac{\log\ 2}{2\log\ (1.0315)}=t$$

Using a calculator we can approximate the time it takes:

$$t = \log\ (2)/(2 * \log\ (1.0315)) \approx 11.17\ \text{years}$$

Answer: Approximately 11.17 years to double at 6.3%.

If the investment in the previous example was one million dollars, how long would it take to double? To answer this we would use $P = \$1{,}000{,}000$ and $A(t) = \$2{,}000{,}000$:

$$A(t)=1{,}000(1.0315)^{2t}$$
$$2{,}000{,}000=1{,}000{,}000(1.0315)^{2t}$$

Dividing both sides by 1,000,000 we obtain the same exponential function as before.

$$2 = (1.0315)^{2t}$$

Hence, the result will be the same, about 11.17 years. In fact, doubling time is independent of the initial investment *P*.

Interest is typically compounded semi-annually ($n = 2$), quarterly ($n = 4$), monthly ($n = 12$), or daily ($n = 365$). However if interest is compounded every instant we obtain a formula for continuously compounding interest:

$$A(t) = Pe^{rt}$$

Here *P* represents the initial principal amount invested, *r* represents the annual interest rate, and *t* represents the time in years the investment is allowed to accrue continuously compounded interest.

Example 3

Mary invested $200 in an account earning $5\,\frac{3}{4}\%$ annual interest that is compounded continuously. How long will it take the investment to grow to $350?

Solution:

Here the principal $P = \$200$ and the interest rate $r = 5\,\frac{3}{4}\,\% = 5.75\% = 0.0575$Since the interest is compounded continuously, use the formula $A\,(t) = Pe^{rt}$. Hence, the investment can be modeled by the following,

$$A\,(t) = 200e^{0.0575t}$$

To calculate the time it takes to accumulate to $350, set $A\,(t) = 350$ and solve for t.

$$\begin{aligned} A\,(t) &= 200e^{0.0575t} \\ 350 &= 200e^{0.0575t} \end{aligned}$$

Begin by isolating the exponential expression.

$$\begin{aligned} \frac{350}{200} &= e^{0.0575t} \\ \frac{7}{4} &= e^{0.0575t} \\ 1.75 &= e^{0.0575t} \end{aligned}$$

Because this exponential has base e, we choose to take the natural logarithm of both sides and then solve for t.

$$\begin{aligned} \ln\left(1.75\right) &= \ln e^{0.0575t} && \textit{Apply the power rule for logarithms.} \\ \ln\left(1.75\right) &= 0.0575t \ln e && \textit{Recall that } \ln e = 1. \\ \ln\left(1.75\right) &= 0.0575t \cdot 1 \\ \frac{\ln\left(1.75\right)}{0.0575} &= t \end{aligned}$$

Using a calculator we can approximate the time it takes:

$$t = \ln\left(1.75\right)/0.0575 \approx 9.73 \quad \textit{years}$$

Answer: It will take approximately 9.73 years.

When solving applications involving compound interest, look for the keyword "continuous," or the keywords that indicate the number of annual compoundings. It is these keywords that determine which formula to choose.

Try this! Mario invested $1,000 in an account earning 6.3% annual interest that is compounded continuously. How long will it take the investment to double?

Answer: Approximately 11 years.

(click to see video)

Modeling Exponential Growth and Decay

In the sciences, when a quantity is said to grow or decay exponentially, it is specifically meant to be modeled using the **exponential growth/decay formula**[21]:

$$P(t) = P_0 e^{kt}$$

Here P_0, read "P naught," or "P zero," represents the initial amount, k represents the growth rate, and t represents the time the initial amount grows or decays exponentially. If k is negative, then the function models exponential decay. Notice that the function looks very similar to that of continuously compounding interest formula. We can use this formula to model population growth when conditions are optimal.

21. A formula that models exponential growth or decay: $P(t) = P_0 e^{kt}$.

Example 4

It is estimated that the population of a certain small town is 93,000 people with an annual growth rate of 2.6%. If the population continues to increase exponentially at this rate:

a. Estimate the population in 7 years' time.
b. Estimate the time it will take for the population to reach 120,000 people.

Solution:

We begin by constructing a mathematical model based on the given information. Here the initial population $P_0 = 93{,}000$ people and the growth rate $r = 2.6\% = 0.026$.The following model gives population in terms of time measured in years:

$$P(t) = 93{,}000e^{0.026t}$$

a. Use this function to estimate the population in $t = 7$ years.

$$\begin{aligned} P(t) &= 93{,}000e^{0.026(7)} \\ &= 93{,}000e^{0.182} \\ &\approx 111{,}564 \quad \mathit{people} \end{aligned}$$

b. Use the model to determine the time it takes to reach $P(t) = 120{,}000$people.

$$\begin{aligned} P(t) &= 93{,}000e^{0.026t} \\ 120{,}000 &= 93{,}000e^{0.026t} \\ \frac{120{,}000}{93{,}000} &= e^{0.026t} \\ \frac{40}{31} &= e^{0.026t} \end{aligned}$$

Take the natural logarithm of both sides and then solve for t.

$$\begin{aligned} \ln\left(\frac{40}{31}\right) &= \ln e^{0.026t} \\ \ln\left(\frac{40}{31}\right) &= 0.026t \ln e \\ \ln\left(\frac{40}{31}\right) &= 0.026t \cdot 1 \\ \frac{\ln\left(\frac{40}{31}\right)}{0.026} &= t \end{aligned}$$

Using a calculator,

$$t = \ln(40/31)/0.026 \approx 9.8 \ \ years$$

Answer:

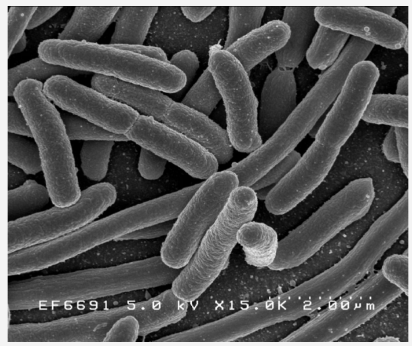
a. 111,564 people
b. 9.8 years

Often the growth rate k is not given. In this case, we look for some other information so that we can determine it and then construct a mathematical model. The general steps are outlined in the following example.

Example 5

Under optimal conditions *Escherichia coli (E. coli)* bacteria will grow exponentially with a doubling time of 20 minutes. If 1,000 *E. coli* cells are placed in a Petri dish and maintained under optimal conditions, how many *E. coli* cells will be present in 2 hours?

Figure 7.3



Escherichia coli (E. coli) (Wikipedia)

Solution:

The goal is to use the given information to construct a mathematical model based on the formula $P(t) = P_0 e^{kt}$.

- **Step 1**: Find the growth rate k. Use the fact that the initial amount, $P_0 = 1{,}000$ cells, doubles in 20 minutes. That is, $P(t) = 2{,}000$ cells when $t = 20$ minutes.

$$\begin{aligned} P(t) &= P_0 e^{kt} \\ 2{,}000 &= 1{,}000 e^{k20} \end{aligned}$$

Solve for the only variable k.

$$\begin{aligned}
2{,}000 &= 1{,}000e^{k20} \\
\frac{2{,}000}{1{,}000} &= e^{k20} \\
2 &= e^{k20} \\
\ln\ (2) &= \ln\ e^{k20} \\
\ln\ (2) &= k20 \ln\ e \\
\ln\ (2) &= k20 \cdot 1 \\
\frac{\ln\ (2)}{20} &= k
\end{aligned}$$

- **Step 2**: Write a mathematical model based on the given information. Here $k \approx 0.0347$, which is about 3.5% growth rate per minute. However, we will use the exact value for k in our model. This will allow us to avoid round-off error in the final result. Use $P_0 = 1{,}000$ and $k = \ln\ (2)/20$:

$$P(t) = 1{,}000e^{(\ln\ (2)/20)t}$$

This equation models the number of *E. coli* cells in terms of time in minutes.

- **Step 3**: Use the function to answer the questions. In this case, we are asked to find the number of cells present in 2 hours. Because time is measured in minutes, use $t = 120$ minutes to calculate the number of *E. coli* cells.

$$\begin{aligned}
P(120) &= 1{,}000e^{(\ln\ (2)/20)(120)} \\
&= 1{,}000e^{\ln\ (2)\cdot 6} \\
&= 1{,}000e^{\ln\ 2^6} \\
&= 1{,}000 \cdot 2^6 \\
&= 64{,}000\ \ cells
\end{aligned}$$

Answer: In two hours 64,000 cells will be present.

When the growth rate is negative the function models exponential decay. We can describe decreasing quantities using a **half-life**[22], or the time it takes to decay to one-half of a given quantity.

22. The period of time it takes a quantity to decay to one-half of the initial amount.

Example 6

Due to radioactive decay, caesium-137 has a half-life of 30 years. How long will it take a 50-milligram sample to decay to 10 milligrams?

Solution:

Use the half-life information to determine the rate of decay k. In $t = 30$ years the initial amount $P_0 = 50$ milligrams will decay to half $P(30) = 25$ milligrams.

$$\begin{aligned} P(t) &= P_0 e^{kt} \\ 25 &= 50e^{k30} \end{aligned}$$

Solve for the only variable, k.

$$\begin{aligned} 25 &= 50e^{k30} \\ \frac{25}{50} &= e^{30k} \\ \ln\left(\frac{1}{2}\right) &= \ln\ e^{30k} \\ \ln\left(\frac{1}{2}\right) &= 30k \ln\ e \\ \frac{\ln\ 1 - \ln\ 2}{30} &= k \qquad \textit{Recall that } \ln\ 1 = 0. \\ -\frac{\ln\ 2}{30} &= k \end{aligned}$$

Note that $k = -\ln \frac{2}{30} \approx -0.0231$ is negative. However, we will use the exact value to construct a model that gives the amount of cesium-137 with respect to time in years.

$$P(t) = 50e^{(-\ln 2/30)t}$$

Use this model to find t when $P(t) = 10$ milligrams.

$$\begin{aligned} 10 &= 50e^{(-\ln 2/30)t} \\ \frac{10}{50} &= e^{(-\ln 2/30)t} \\ \ln\left(\frac{1}{5}\right) &= \ln e^{(-\ln 2/30)t} \\ \ln 1 - \ln 5 &= \left(-\frac{\ln 2}{30}\right) t \ln e \quad \textit{Recall that } \ln e = 1. \\ -\frac{30(\ln 1 - \ln 5)}{\ln 2} &= t \\ \frac{-30(0 - \ln 5)}{\ln 2} &= t \\ \frac{30 \ln 5}{\ln 2} &= t \end{aligned}$$

Answer: Using a calculator, it will take $t \approx 69.66$ years to decay to 10 milligrams.

Radiocarbon dating is a method used to estimate the age of artifacts based on the relative amount of carbon-14 present in it. When an organism dies, it stops absorbing this naturally occurring radioactive isotope, and the carbon-14 begins to decay at a known rate. Therefore, the amount of carbon-14 present in an artifact can be used to estimate the age of the artifact.

Example 7

An ancient bone tool is found to contain 25% of the carbon-14 normally found in bone. Given that carbon-14 has a half-life of 5,730 years, estimate the age of the tool.

Solution:

Begin by using the half-life information to find *k*. Here the initial amount P_0 of carbon-14 is not given, however, we know that in $t = 5{,}730$ years, this amount decays to half, $\frac{1}{2}P_0$.

$$\begin{aligned} P(t) &= P_0 e^{kt} \\ \frac{1}{2}P_0 &= P_0 e^{k5{,}730} \end{aligned}$$

Dividing both sides by P_0 leaves us with an exponential equation in terms of *k*. This shows that half-life is independent of the initial amount.

$$\frac{1}{2} = e^{k5{,}730}$$

Solve for *k*.

$$\begin{aligned}\ln\left(\frac{1}{2}\right) &= \ln e^{k5{,}730}\\ \ln 1 - \ln 2 &= 5{,}730k \ln e\\ \frac{0-\ln 2}{5{,}730} &= k\\ -\frac{\ln 2}{5{,}730} &= k\end{aligned}$$

Therefore we have the model,

$$P(t) = P_0 e^{(-\ln 2/5{,}730)t}$$

Next we wish to the find time it takes the carbon-14 to decay to 25% of the initial amount, or $P(t) = 0.25P_0$.

$$0.25P_0 = P_0 e^{(-\ln 2/5{,}730)t}$$

Divide both sides by P_0 and solve for t.

$$
\begin{aligned}
0.25 &= e^{(-\ln\ 2/5{,}730)t}\\
\ln\ (0.25) &= \ln\ e^{(-\ln\ 2/5{,}730)t}\\
\ln\ (0.25) &= \left(-\frac{\ln\ 2}{5{,}730}\right)t \ln\ e\\
-\frac{5{,}730\ \ln\ (0.25)}{\ln\ 2} &= t\\
11{,}460 &\approx t
\end{aligned}
$$

Answer: The tool is approximately 11,460 years old.

Try this! The half-life of strontium-90 is about 28 years. How long will it take a 36 milligram sample of strontium-90 to decay to 30 milligrams?

Answer: 7.4 years

(click to see video)

KEY TAKEAWAYS

- When interest is compounded a given number of times per year use the formula $A\left(t\right) = P\left(1 + \frac{r}{n}\right)^{nt}$.
- When interest is to be compounded continuously use the formula $A\left(t\right) = Pe^{rt}$.
- Doubling time is the period of time it takes a given amount to double. Doubling time is independent of the principal.
- When amounts are said to be increasing or decaying exponentially, use the formula $P\left(t\right) = P_0 e^{kt}$.
- Half-life is the period of time it takes a given amount to decrease to one-half. Half-life is independent of the initial amount.
- To model data using the exponential growth/decay formula, use the given information to determine the growth/decay rate *k*. Once *k* is determined, a formula can be written to model the problem. Use the formula to answer the questions.

TOPIC EXERCISES

PART A: COMPOUND AND CONTINUOUS INTEREST

1. Jill invested $1,450 in an account earning $4\frac{5}{8}$% annual interest that is compounded monthly.
   a. How much will be in the account after 6 years?
   b. How long will it take the account to grow to $2,200?
2. James invested $825 in an account earning $5\frac{2}{5}$% annual interest that is compounded monthly.
   a. How much will be in the account after 4 years?
   b. How long will it take the account to grow to $1,500?
3. Raul invested $8,500 in an online money market fund earning 4.8% annual interest that is compounded continuously.
   a. How much will be in the account after 2 years?
   b. How long will it take the account to grow to $10,000?
4. Ian deposited $500 in an account earning 3.9% annual interest that is compounded continuously.
   a. How much will be in the account after 3 years?
   b. How long will it take the account to grow to $1,500?
5. Bill wants to grow his $75,000 inheritance to $100,000 before spending any of it. How long will this take if the bank is offering 5.2% annual interest compounded quarterly?
6. Mary needs $25,000 for a down payment on a new home. If she invests her savings of $21,350 in an account earning 4.6% annual interest that is compounded semi-annually, how long will it take to grow to the amount that she needs?
7. Joe invested his $8,700 savings in an account earning $6\frac{3}{4}$% annual interest that is compounded continuously. How long will it take to earn $300 in interest?
8. Miriam invested $12,800 in an account earning $5\frac{1}{4}$% annual interest that is compounded monthly. How long will it take to earn $1,200 in interest?

9. Given that the bank is offering 4.2% annual interest compounded monthly, what principal is needed to earn $25,000 in interest for one year?

10. Given that the bank is offering 3.5% annual interest compounded continuously, what principal is needed to earn $12,000 in interest for one year?

11. Jose invested his $3,500 bonus in an account earning $5\,\frac{1}{2}$% annual interest that is compounded quarterly. How long will it take to double his investment?

12. Maria invested her $4,200 savings in an account earning $6\,\frac{3}{4}$% annual interest that is compounded semi-annually. How long will it take to double her savings?

13. If money is invested in an account earning 3.85% annual interest that is compounded continuously, how long will it take the amount to double?

14. If money is invested in an account earning 6.82% annual interest that is compounded continuously, how long will it take the amount to double?

15. Find the annual interest rate at which an account earning continuously compounding interest has a doubling time of 9 years.

16. Find the annual interest rate at which an account earning interest that is compounded monthly has a doubling time of 10 years.

17. Alice invested her savings of $7,000 in an account earning 4.5% annual interest that is compounded monthly. How long will it take the account to triple in value?

18. Mary invested her $42,000 bonus in an account earning 7.2% annual interest that is compounded continuously. How long will it take the account to triple in value?

19. Calculate the doubling time of an investment made at 7% annual interest that is compounded:

    a. monthly
    b. continuously

20. Calculate the doubling time of an investment that is earning continuously compounding interest at an annual interest rate of:

    a. 4%
    b. 6%

21. Billy's grandfather invested in a savings bond that earned 5.5% annual interest that was compounded annually. Currently, 30 years later, the savings bond is valued at $10,000. Determine what the initial investment was.

22. In 1935 Frank opened an account earning 3.8% annual interest that was compounded quarterly. He rediscovered this account while cleaning out his garage in 2005. If the account is now worth $11,294.30, how much was his initial deposit in 1935?

PART B: MODELING EXPONENTIAL GROWTH AND DECAY

23. The population of a small town of 24,000 people is expected grow exponentially at a rate of 1.6% per year. Construct an exponential growth model and use it to:

    a. Estimate the population in 3 years' time.
    b. Estimate the time it will take for the population to reach 30,000 people.

24. During the exponential growth phase, certain bacteria can grow at a rate of 4.1% per hour. If 10,000 cells are initially present in a sample, construct an exponential growth model and use it to:

    a. Estimate the population in 5 hours.
    b. Estimate the time it will take for the population to reach 25,000 cells.

25. In 2000, the world population was estimated to be 6.115 billion people and in 2010 the estimate was 6.909 billion people. If the world population continues to grow exponentially, estimate the total world population in 2020.

26. In 2000, the population of the United States was estimated to be 282 million people and in 2010 the estimate was 309 million people. If the population of the United States grows exponentially, estimate the population in 2020.

27. An automobile was purchased new for $42,500 and 2 years later it was valued at $33,400. Estimate the value of the automobile in 5 years if it continues to decrease exponentially.

28. A new PC was purchased for $1,200 and in 1.5 years it was worth $520. Assume the value is decreasing exponentially and estimate the value of the PC four years after it is purchased.

29. The population of the downtown area of a certain city decreased from 12,500 people to 10,200 people in two years. If the population continues to decrease exponentially at this rate, what would we expect the population to be in two more years?

30. A new MP3 player was purchased for $320 and in 1 year it was selling used online for $210. If the value continues to decrease exponentially at this rate, determine the value of the MP3 player 3 years after it was purchased.

31. The half-life of radium-226 is about 1,600 years. How long will a 5-milligram sample of radium-226 take to decay to 1 milligram?
32. The half-life of plutonium-239 is about 24,000 years. How long will a 5-milligram sample of plutonium-239 take to decay to 1 milligram?
33. The half-life of radioactive iodine-131 is about 8 days. How long will it take a 28-gram initial sample of iodine-131 to decay to 12 grams?
34. The half-life of caesium-137 is about 30 years. How long will it take a 15-milligram sample of caesium-137 to decay to 5 milligrams?
35. The Rhind Mathematical Papyrus is considered to be the best example of Egyptian mathematics found to date. This ancient papyrus was found to contain 64% of the carbon-14 normally found in papyrus. Given that carbon-14 has a half-life of 5,730 years, estimate the age of the papyrus.
36. A wooden bowl artifact carved from oak was found to contain 55% of the carbon-14 normally found in oak. Given that carbon-14 has a half-life of 5,730 years, estimate the age of the bowl.
37. The half-life of radioactive iodine-131 is about 8 days. How long will it take a sample of iodine-131 to decay to 10% of the original amount?
38. The half-life of caesium-137 is about 30 years. How long will it take a sample of caesium-137 to decay to 25% of the original amount?
39. The half-life of caesium-137 is about 30 years. What percent of an initial sample will remain in 100 years?
40. The half-life of radioactive iodine-131 is about 8 days. What percent of an initial sample will remain in 30 days?
41. If a bone is 100 years old, what percent of its original amount of carbon-14 do we expect to find in it?
42. The half-life of plutonium-239 is about 24,000 years. What percent of an initial sample will remain in 1,000 years?
43. Find the amount of time it will take for 10% of an initial sample of plutonium-239 to decay. (Hint: If 10% decays, then 90% will remain.)
44. Find the amount of time it will take for 10% of an initial sample of carbon-14 to decay.

Solve for the given variable:

45. Solve for *t*: $A = Pe^{rt}$

46. Solve for t: $A = P(1+r)^t$
47. Solve for I: $M = \log\left(\frac{I}{I_0}\right)$
48. Solve for H^+: $pH = -\log\left(H^+\right)$
49. Solve for t: $P = \frac{1}{1+e^{-t}}$
50. Solve for I: $L = 10\log\left(I/10^{-12}\right)$
51. The number of cells in a certain bacteria sample is approximated by the logistic growth model $N(t) = \frac{1.2\times10^5}{1+9e^{-0.32t}}$, where t represents time in hours. Determine the time it takes the sample to grow to 24,000 cells.
52. The market share of a product, as a percentage, is approximated by the formula $P(t) = \frac{100}{3+e^{-0.44t}}$ where t represents the number of months after an aggressive advertising campaign is launched.

    a. What was the initial market share?
    b. How long would we expect to see a 3.5% increase in market share?
53. In chemistry, pH is a measure of acidity and is given by the formula $\text{pH} = -\log\left(H^+\right)$, where H^+ is the hydrogen ion concentration (measured in moles of hydrogen per liter of solution.) What is the hydrogen ion concentration of seawater with a pH of 8?
54. Determine the hydrogen ion concentration of milk with a pH of 6.6.
55. The volume of sound, L in decibels (dB), is given by the formula $L = 10\log\left(I/10^{-12}\right)$ where I represents the intensity of the sound in watts per square meter. Determine the sound intensity of a hair dryer that emits 70 dB of sound.
56. The volume of a chainsaw measures 110 dB. Determine the intensity of this sound.

PART C: DISCUSSION BOARD

57. Which factor affects the doubling time the most, the annual compounding n or the interest rate r? Explain.

58. Research and discuss radiocarbon dating. Post something interesting you have learned as well as a link to more information.
59. Is exponential growth sustainable over an indefinite amount of time? Explain.
60. Research and discuss the half-life of radioactive materials.

ANSWERS

1.
   a. $1,912.73
   b. 9 years

3.
   a. $9,356.45
   b. 3.4 years

5. 5.6 years

7. $\frac{1}{2}$ year

9. $583,867

11. 12.7 years

13. 18 years

15. 7.7%

17. 24.5 years

19.
   a. 9.93 years
   b. 9.90 years

21. $2,006.44

23.
   a. About 25,180 people
   b. About 14 years

25. About 7.806 billion people

27. About $23,269.27

29. 8,323 people

31. 3,715 years

33. 9.8 days

35. About 3,689 years old

37. 26.6 days

39. 9.9%

41. 98.8%

43. 3,648 years

45. $t = \dfrac{\ln\ (A) - \ln\ (P)}{r}$

47. $I = I_0 \cdot 10^M$

49. $t = \ln\left(\dfrac{P}{1-P}\right)$

51. Approximately 2.5 hours

53. 10^{-8} moles per liter

55. 10^{-5} watts per square meter

57. Answer may vary

59. Answer may vary

7.7 Review Exercises and Sample Exam

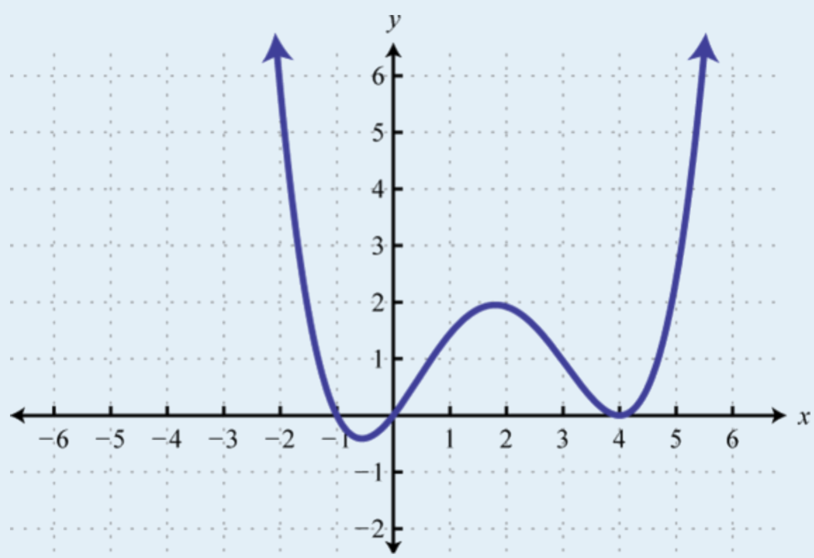
REVIEW EXERCISES

COMPOSITION AND INVERSE FUNCTIONS

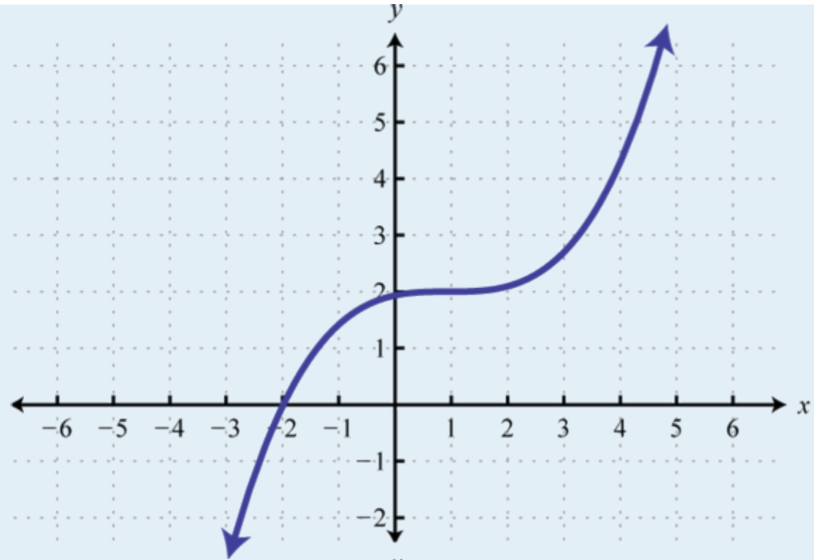
Given f and g find $(f \circ g)(x)$ and $(g \circ f)(x)$.

1. $f(x) = 6x - 5,\ g(x) = 2x + 1$
2. $f(x) = 5 - 6x,\ g(x) = \frac{3}{2}x$
3. $f(x) = 2x^2 + x - 2,\ g(x) = 5x$
4. $f(x) = x^2 - x - 6,\ g(x) = x - 3$
5. $f(x) = \sqrt{x+2},\ g(x) = 8x - 2$
6. $f(x) = \frac{x-1}{3x-1},\ g(x) = \frac{1}{x}$
7. $f(x) = x^2 + 3x - 1,\ g(x) = \frac{1}{x-2}$
8. $f(x) = \sqrt[3]{3(x+2)},\ g(x) = 9x^3 - 2$

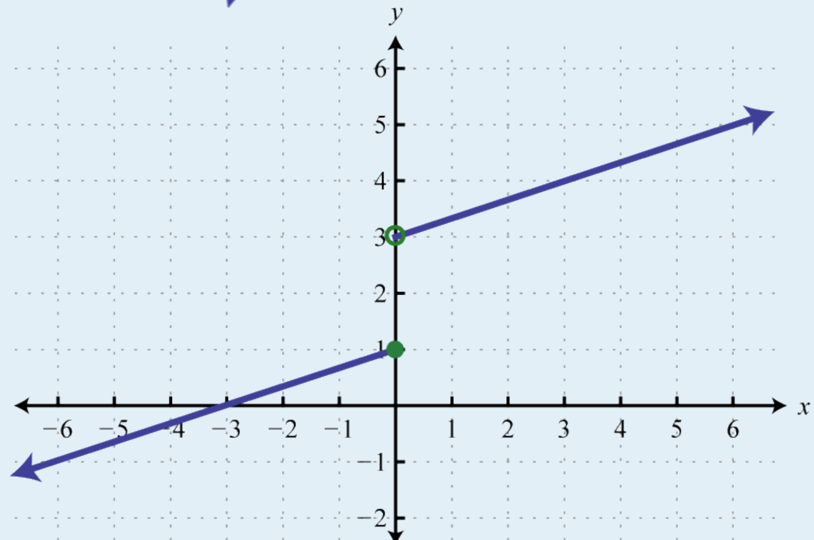
Are the given functions one-to-one? Explain.

9. 

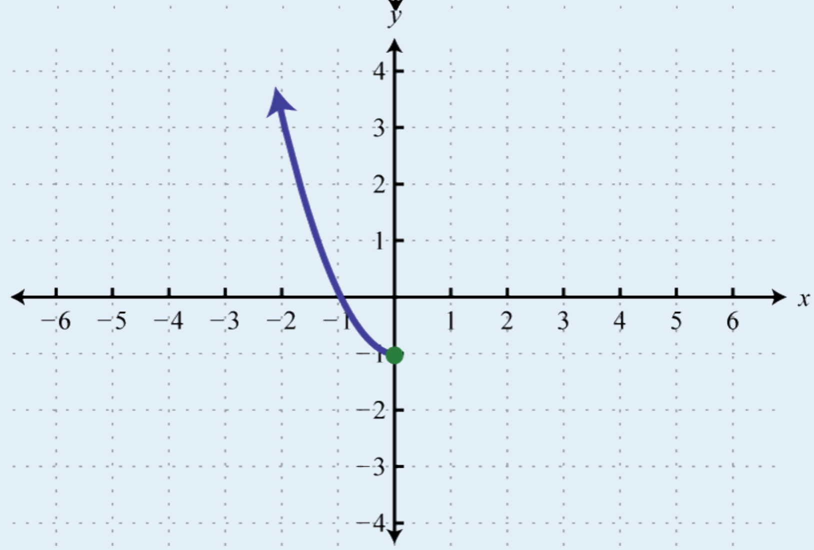
10.


y
x
6
5
4
3
2
1
−1
−2
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6

11.


y
x
6
5
4
3
2
1
−1
−2
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6

12.


y
x
4
3
2
1
−1
−2
−3
−4
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6

Verify algebraically that the two given functions are inverses. In other words, show that $\left(f \circ f^{-1}\right)(x) = x$ and $\left(f^{-1} \circ f\right)(x) = x$.

13. $f(x) = 6x - 5,\ f^{-1}(x) = \frac{1}{6}x + \frac{5}{6}$
14. $f(x) = \sqrt{2x+3},\ f^{-1}(x) = \frac{x^2-3}{2}, x \geq 0$
15. $f(x) = \frac{x}{3x-2},\ f^{-1}(x) = \frac{2x}{3x-1}$
16. $f(x) = \sqrt[3]{x+3} - 4,\ f^{-1}(x) = (x+4)^3 - 3$

Find the inverses of each function defined as follows:

17. $f(x) = -7x + 3$
18. $f(x) = \frac{2}{3}x - \frac{1}{2}$
19. $g(x) = x^2 - 12, x \geq 0$
20. $g(x) = (x-1)^3 + 5$
21. $g(x) = \frac{2}{x-1}$
22. $h(x) = \frac{x+5}{x-5}$
23. $h(x) = \frac{3x-1}{x}$
24. $p(x) = \sqrt[3]{5x} + 3$
25. $h(x) = \sqrt[3]{2x-7} + 2$
26. $h(x) = \sqrt[5]{x+2} - 3$

EXPONENTIAL FUNCTIONS AND THEIR GRAPHS

Evaluate.

27. $f(x) = 5^x$; find $f(-1), f(0)$, and $f(3)$.
28. $f(x) = \left(\frac{1}{2}\right)^x$; find $f(-4), f(0)$, and $f(-3)$.
29. $g(x) = 10^{-x}$; find $g(-5), g(0)$, and $g(2)$.

30. $g(x) = 1 - 3^x$; find $g(-2)$, $g(0)$, and $g(3)$.

Sketch the exponential function. Draw the horizontal asymptote with a dashed line.

31. $f(x) = 5^x + 10$
32. $f(x) = 5^{x-4}$
33. $f(x) = -3^x - 9$
34. $f(x) = 3^{x+2} + 6$
35. $f(x) = \left(\frac{1}{3}\right)^x$
36. $f(x) = \left(\frac{1}{2}\right)^x - 4$
37. $f(x) = 2^{-x} + 3$
38. $f(x) = 1 - 3^{-x}$

Use a calculator to evaluate the following. Round off to the nearest hundredth.

39. $f(x) = e^x + 1$; find $f(-3)$, $f(-1)$, and $f\left(\frac{1}{2}\right)$.
40. $g(x) = 2 - 3e^x$; find $g(-1)$, $g(0)$, and $g\left(\frac{2}{3}\right)$.
41. $p(x) = 1 - 5e^{-x}$; find $p(-4)$, $p\left(-\frac{1}{2}\right)$, and $p(0)$.
42. $r(x) = e^{-2x} - 1$; find $r(-1)$, $r\left(\frac{1}{4}\right)$, and $r(2)$.

Sketch the function. Draw the horizontal asymptote with a dashed line.

43. $f(x) = e^x + 4$
44. $f(x) = e^{x-4}$
45. $f(x) = e^{x+3} + 2$
46. $f(x) = e^{-x} + 5$
47. Jerry invested $6,250 in an account earning $3\frac{5}{8}$% annual interest that is compounded monthly. How much will be in the account after 4 years?

48. Jose invested \$7,500 in an account earning $4\frac{1}{4}$% annual interest that is compounded continuously. How much will be in the account after $3\frac{1}{2}$ years?

49. A 14-gram sample of radioactive iodine is accidently released into the atmosphere. The amount of the substance in grams is given by the formula $P(t) = 14e^{-0.087t}$, where t represents the time in days after the sample was released. How much radioactive iodine will be present in the atmosphere 30 days after it was released?

50. The number of cells in a bacteria sample is given by the formula $N(t) = \frac{2.4\times10^5}{1+9e^{-0.28t}}$, where t represents the time in hours since the initial placement of 24,000 cells. Use the formula to calculate the number of cells in the sample 20 hours later.

LOGARITHMIC FUNCTIONS AND THEIR GRAPHS

Evaluate.

51. $\log_4 16$

52. $\log_3 27$

53. $\log_2 \left(\frac{1}{32}\right)$

54. $\log \left(\frac{1}{10}\right)$

55. $\log_{1/3} 9$

56. $\log_{3/4} \left(\frac{4}{3}\right)$

57. $\log_7 1$

58. $\log_3 (-3)$

59. $\log_4 0$

60. $\log_3 81$

61. $\log_6 \sqrt{6}$

62. $\log_5 \sqrt[3]{25}$

63. $\ln e^8$

64. $\ln\left(\frac{1}{e^5}\right)$

65. $\log\ (0.00001)$

66. $\log\ 1{,}000{,}000$

Find *x*.

67. $\log_5 x = 3$

68. $\log_3 x = -4$

69. $\log_{2/3} x = 3$

70. $\log_3 x = \frac{2}{5}$

71. $\log x = -3$

72. $\ln x = \frac{1}{2}$

Sketch the graph of the logarithmic function. Draw the vertical asymptote with a dashed line.

73. $f(x) = \log_2\ (x - 5)$

74. $f(x) = \log_2 x - 5$

75. $g(x) = \log_3\ (x + 5) + 15$

76. $g(x) = \log_3\ (x - 5) - 5$

77. $h(x) = \log_4\ (-x) + 1$

78. $h(x) = 3 - \log_4 x$

79. $g(x) = \ln\ (x - 2) + 3$

80. $g(x) = \ln\ (x + 3) - 1$

81. The population of a certain small town is growing according to the function $P(t) = 89{,}000(1.035)^t$, where *t* represents time in years since the last

census. Use the function to estimate the population $8\,\frac{1}{2}$ years after the census was taken.

82. The volume of sound L in decibels (dB) is given by the formula $L = 10\log\left(I/10^{-12}\right)$, where I represents the intensity of the sound in watts per square meter. Determine the volume of a sound with an intensity of 0.5 watts per square meter.

PROPERTIES OF THE LOGARITHM

Evaluate without using a calculator.

83. $\log_9 9$
84. $\log_8 1$
85. $\log_{1/3} 3$
86. $\log\left(\frac{1}{10}\right)$
87. $e^{\ln 17}$
88. $10^{\log 27}$
89. $\ln e^{63}$
90. $\log 10^{33}$

Expand completely.

91. $\log\left(100x^2\right)$
92. $\log_5\left(5x^3\right)$
93. $\log_3\left(\frac{3x^5}{5}\right)$
94. $\ln\left(\frac{10}{3x^2}\right)$
95. $\log_2\left(\frac{8x^2}{y^2 z}\right)$

96. $\log\left(\frac{x^{10}}{10y^3z^4}\right)$

97. $\ln\left(\frac{3b\sqrt{a}}{c^4}\right)$

98. $\log\left(\frac{20y^3}{\sqrt[3]{x^2}}\right)$

Write as a single logarithm with coefficient 1.

99. $\log x + 2\log y - 3\log z$

100. $\log_2 5 - 3\log_2 x + 4\log_2 y$

101. $-2\log_5 x + \log_5 y - 5\log_5 (x-1)$

102. $\ln x - \ln (x-1) - \ln (x+1)$

103. $3\log_2 x + \frac{1}{2}\log_2 y - \frac{2}{3}\log_2 z$

104. $\frac{1}{3}\log x - 3\log y - \frac{3}{5}\log z$

105. $\log_5 4 + 5\log_5 x - \frac{1}{3}\left(\log_5 y + 2\log_5 z\right)$

106. $\ln x - \frac{1}{2}\left(\ln y - 4\ln z\right)$

SOLVING EXPONENTIAL AND LOGARITHMIC EQUATIONS

Solve. Give the exact answer and the approximate answer rounded to the nearest hundredth where appropriate.

107. $5^{2x+1} = 125$

108. $10^{3x-2} = 100$

109. $9^{x-3} = 81$

110. $16^{2x+3} = 8$

111. $5^x = 7$

112. $3^{2x} = 5$

113. $10^{x+2} - 3 = 7$

114. $e^{2x-1} + 2 = 3$

115. $7^{4x-1} - 2 = 9$

116. $3^{5x-2} + 5 = 7$

117. $3 - e^{4x} = 2$

118. $5 + e^{3x} = 4$

119. $\dfrac{4}{1 + e^{5x}} = 2$

120. $\dfrac{100}{1 + e^{3x}} = \dfrac{1}{2}$

Use the change of base formula to approximate the following to the nearest tenth.

121. $\log_5 13$

122. $\log_2 27$

123. $\log_4 5$

124. $\log_9 0.81$

125. $\log_{1/4} 21$

126. $\log_2 \sqrt[3]{5}$

Solve.

127. $\log_2 \left(3x - 5\right) = \log_2 (2x + 7)$

128. $\ln (7x) = \ln (x + 8)$

129. $\log_5 8 - 2\log_5 x = \log_5 2$

130. $\log_3 (x + 2) + \log_3 (x) = \log_3 8$

131. $\log_5 (2x - 1) = 2$

132. $2\log_4 (3x - 2) = 4$

133. $2 = \log_2 \left(x^2 - 4\right) - \log_2 3$

134. $\log_2 (x - 1) + \log_2 (x + 1) = 3$

135. $\log_2 x + \log_2 (x - 1) = 1$

136. $\log_4 \left(x + 5\right) + \log_4 (x + 11) = 2$

137. $\log \left(2x + 5\right) - \log (x - 1) = 1$

138. $\ln x - \ln (2x - 1) = 1$

139. $2\log_2 (x + 4) = \log_2 (x + 2) + 3$

140. $2\log_3 x = 1 + \log_3 \left(x + 6\right)$

141. $\log_3 (x + 1) - 2\log_3 x = 1$

142. $\log_5 (2x) + \log_5 (x - 1) = 1$

APPLICATIONS

Solve.

143. An amount of $3,250 is invested in an account that earns 4.6% annual interest that is compounded monthly. Estimate the number of years for the amount in the account to reach $4,000.

144. An amount of $2,500 is invested in an account that earns 5.5% annual interest that is compounded continuously. Estimate the number of years for the amount in the account to reach $3,000.

145. How long does it take to double an investment made in an account that earns $6\frac{3}{4}$% annual interest that is compounded continuously?

146. How long does it take to double an investment made in an account that earns $6\frac{3}{4}$% annual interest that is compounded semi-annually?

147. In the year 2000 a certain small town had a population of 46,000 people. In the year 2010 the population was estimated to have grown to 92,000 people. If the population continues to grow exponentially at this rate, estimate the population in the year 2016.

148. A fleet van was purchased new for $28,000 and 2 years later it was valued at $20,000. If the value of the van continues to decrease exponentially at this rate, determine its value 7 years after it is purchased new.

149. A website that has been in decline registered 4,200 unique visitors last month and 3,600 unique visitors this month. If the number of unique visitors continues to decline exponentially, how many unique visitors would you expect next month?

150. An initial population of 18 rabbits was introduced into a wildlife preserve. The number of rabbits doubled in the first year. If the rabbit population continues to grow exponentially at this rate, how many rabbits will be present 5 years after they were introduced?

151. The half-life of sodium-24 is about 15 hours. How long will it take a 50-milligram sample to decay to 10 milligrams?

152. The half-life of radium-226 is about 1,600 years. How long will it take an initial sample to decay to 30% of the original amount?

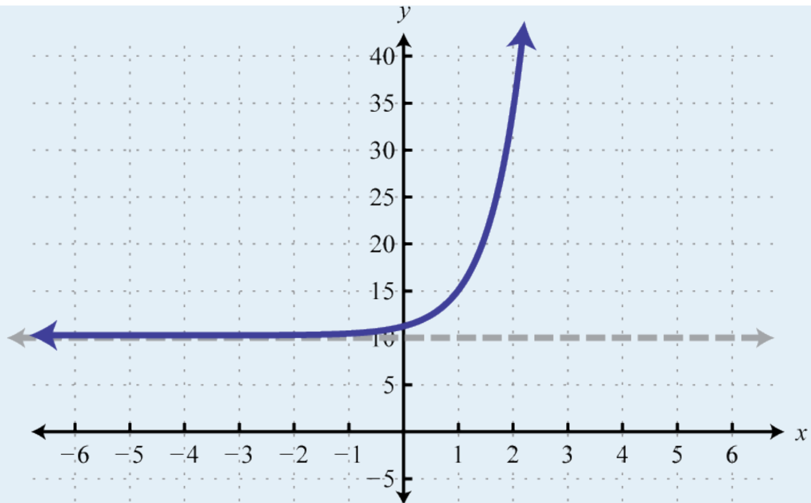
153. An archeologist discovered a bone tool artifact. After analysis, the artifact was found to contain 62% of the carbon-14 normally found in bone from the same animal. Given that carbon-14 has a half-life of 5,730 years, estimate the age of the artifact.

154. The half-life of radioactive iodine-131 is about 8 days. What percentage of an initial sample accidentally released into the atmosphere do we expect to remain after 53 days?

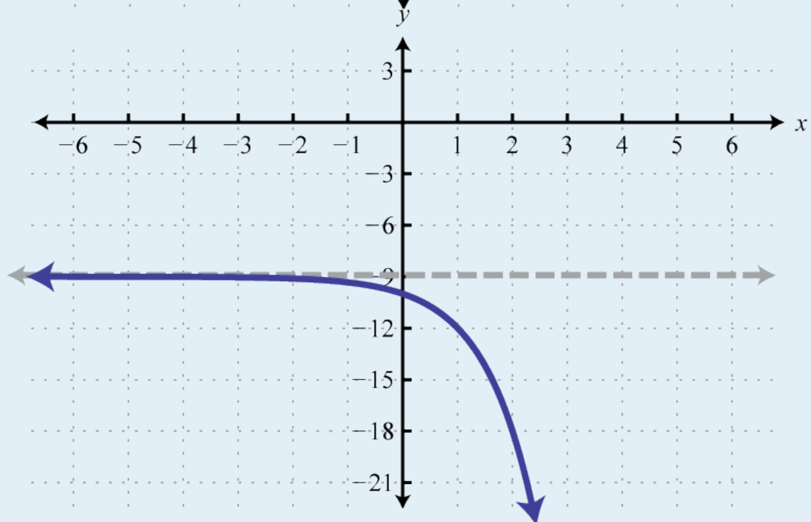
ANSWERS

1. $\left(f \circ g\right)(x) = 12x + 1$; $\left(g \circ f\right)(x) = 12x - 9$
3. $\left(f \circ g\right)(x) = 50x^2 + 5x - 2$; $\left(g \circ f\right)(x) = 10x^2 + 5x - 10$
5. $\left(f \circ g\right)(x) = 2\sqrt{2x}$; $\left(g \circ f\right)(x) = 8\sqrt{x + 2} - 2$
7. $\left(f \circ g\right)(x) = -\dfrac{x^2 - 7x + 9}{(x - 2)^2}$; $\left(g \circ f\right)(x) = \dfrac{1}{x^2 + 3x - 3}$
9. No, fails the HLT
11. Yes, passes the HLT
13. Proof
15. Proof
17. $f^{-1}(x) = -\frac{1}{7}x + \frac{3}{7}$
19. $g^{-1}(x) = \sqrt{x + 12}$
21. $g^{-1}(x) = \frac{x+2}{x}$
23. $h^{-1}(x) = -\frac{1}{x-3}$
25. $h^{-1}(x) = \frac{(x-2)^3+7}{2}$
27. $f(-1) = \frac{1}{5}, f(0) = 1, f(3) = 125$
29. $g\left(-5\right) = 100{,}000$, $g(0) = 1, g(2) = \frac{1}{100}$

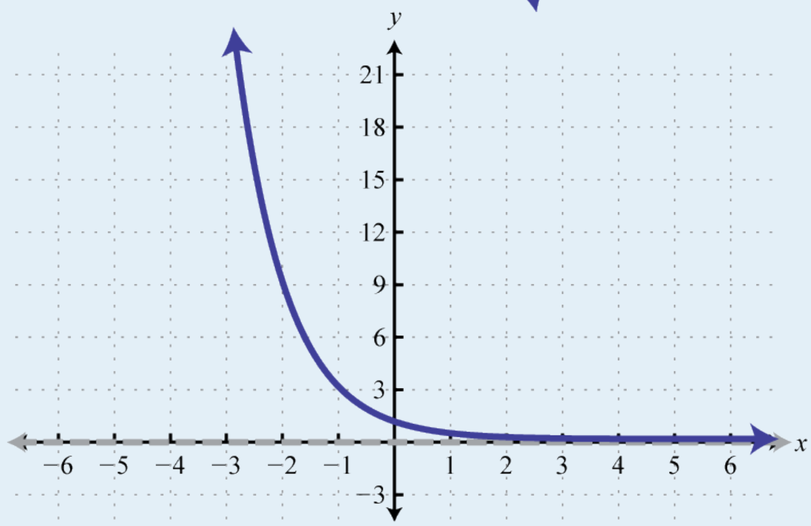
31.

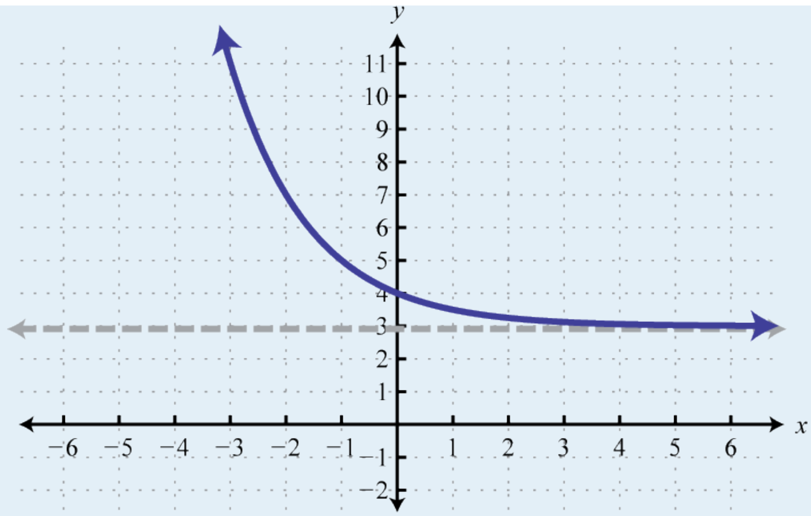

y
40
35
30
25
20
15
10
5
−5
x
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6

33.


y
3
−3
−6
−9
−12
−15
−18
−21
x
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6

35.

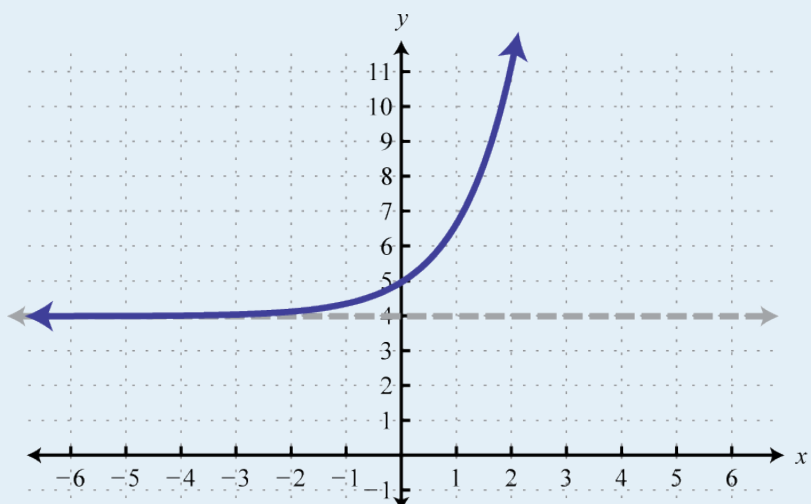

y
21
18
15
12
9
6
3
−3
x
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6



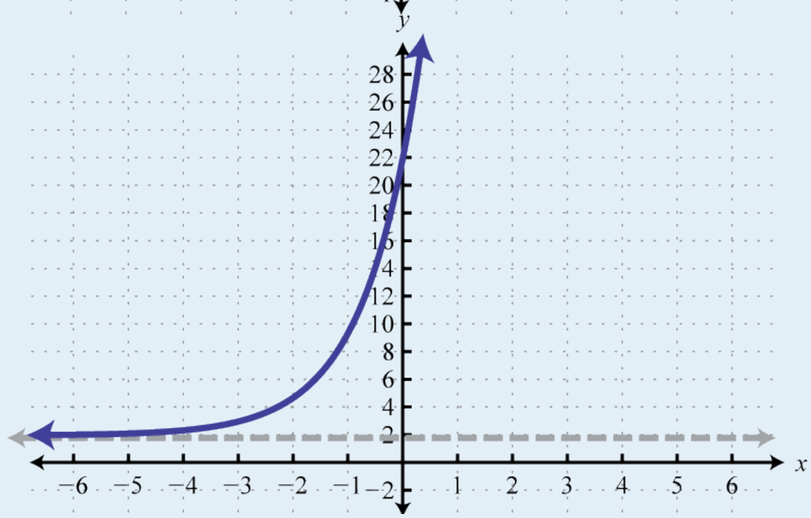
37.

39. $f(-3) \approx 1.05, f(-1) \approx 1.37, f\left(\frac{1}{2}\right) \approx 2.65$

41. $p(-4) \approx -271.99, p\left(-\frac{1}{2}\right) \approx -7.24, p(0) = -4$



43.



45.

47. $7,223.67

49. Approximately 1 gram

51. 2

53. −5

55. −2

57. 0

59. Undefined

61. $\frac{1}{2}$

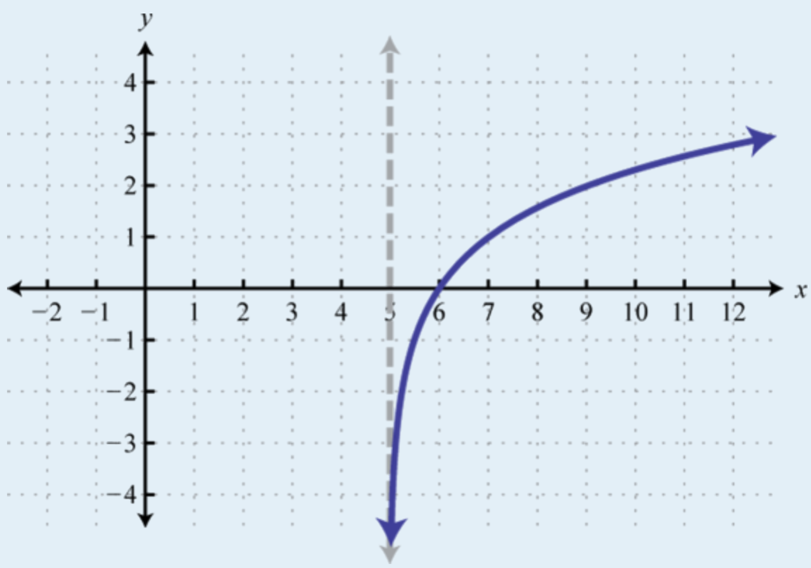
63. 8

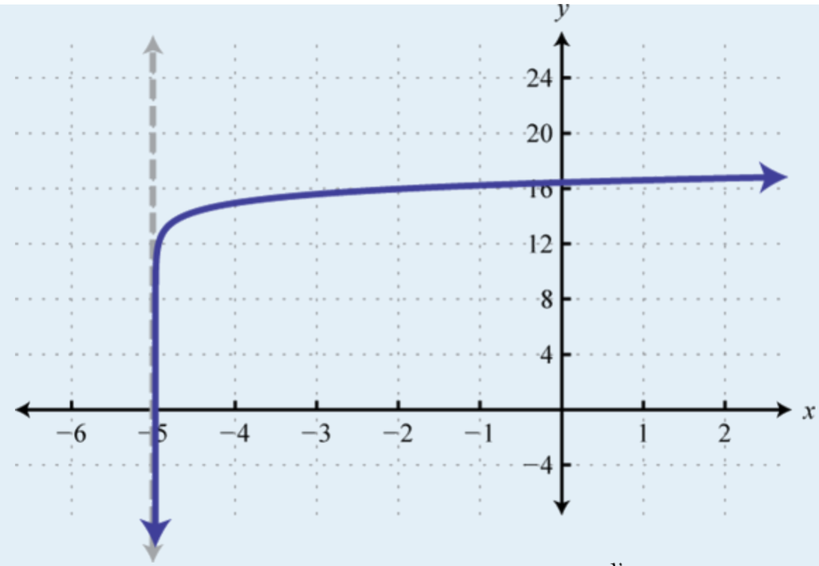
65. −5

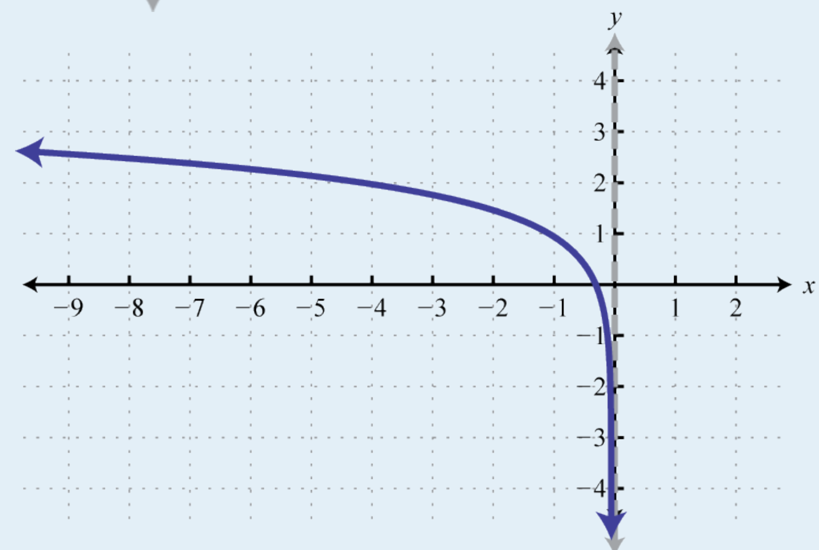
67. 125

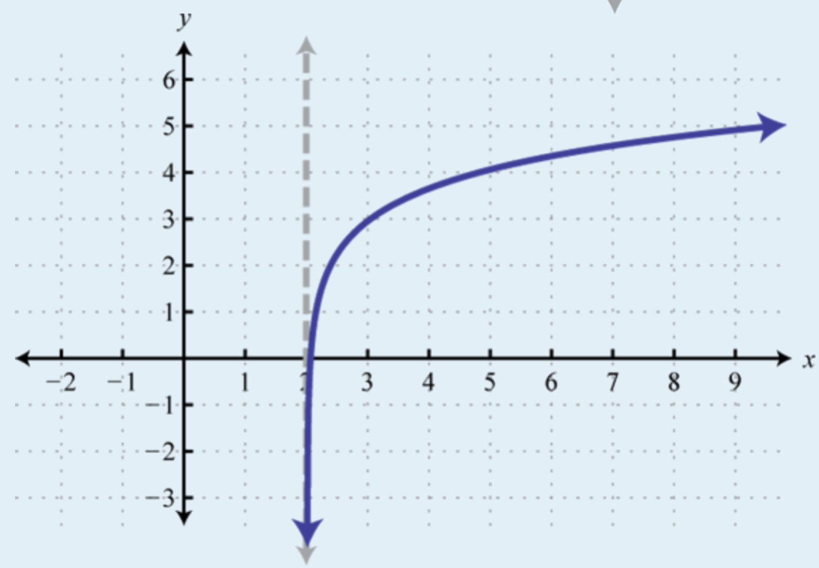
69. $\frac{8}{27}$

71. 0.001

73. 

75.

y
24
20
16
12
8
4
−6
−5
−4
−3
−2
−1
1
2
x
−4

77.

y
4
3
2
1
−9
−8
−7
−6
−5
−4
−3
−2
−1
1
2
x
−1
−2
−3
−4

79.

y
6
5
4
3
2
1
−2
−1
1
2
3
4
5
6
7
8
9
x
−1
−2
−3

81. 119,229 people

83. 1

85. −1

87. 17

89. 63

91. $2 + 2\log x$

93. $1 + 5\log_3 x - \log_3 5$

95. $3 + 2\log_2 x - 2\log_2 y - \log_2 z$

97. $\ln 3 + \ln b + \frac{1}{2}\ln a - 4\ln c$

99. $\log\left(\frac{xy^2}{z^3}\right)$

101. $\log_5\left(\frac{y}{x^2(x-1)^5}\right)$

103. $\log_2\left(\frac{x^3\sqrt{y}}{\sqrt[3]{z^2}}\right)$

105. $\log_5\left(\frac{4x^5}{\sqrt[3]{yz^2}}\right)$

107. 1

109. 5

111. $\frac{\log(7)}{\log(5)} \approx 1.21$

113. -1

115. $\frac{\log 7 + \log 11}{4\log 7} \approx 0.56$

117. 0

119. 0

121. 1.6

123. 1.2

125. −2.2

127. 12

129. 2

131. 13

133. ±4

135. 2

137. $\frac{15}{8}$

139. 0

141. $$\frac{1+\sqrt{13}}{6}$$

143. 4.5 years

145. 10.27 years

147. About 139,446 people

149. 3,086 unique visitors

151. 35 hours

153. About 3,952 years old

SAMPLE EXAM

1. Given $f(x) = x^2 - x + 3$ and $g(x) = 3x - 1$ find $(f \circ g)(x)$.
2. Show that $f(x) = \sqrt[3]{7x - 2}$ and $g(x) = \frac{x^3+2}{7}$ are inverses.

Find the inverse of the following functions:

3. $f(x) = \frac{1}{2}x - 3$
4. $h(x) = x^2 + 3$ where $x \geq 0$

Sketch the graph.

5. $f(x) = e^x - 5$
6. $g(x) = 10^{-x}$
7. Joe invested \$5,200 in an account earning 3.8% annual interest that is compounded monthly. How much will be in the account at the end of 4 years?
8. Mary has \$3,500 in a savings account earning $4\frac{1}{2}$% annual interest that is compounded continuously. How much will be in the account at the end of 3 years?

Evaluate.

9.
   a. $\log_3 81$
   b. $\log_2\left(\frac{1}{4}\right)$
   c. $\log 1{,}000$
   d. $\ln e$
10.
   a. $\log_4 2$
   b. $\log_9\left(\frac{1}{3}\right)$
   c. $\ln e^3$
   d. $\log_{1/5} 25$

Sketch the graph.

11. $f(x) = \log_4(x + 5) + 2$
12. $f(x) = -\ln(x - 2)$

13. Expand: $\log\left(\dfrac{100x^2y}{\sqrt{z}}\right)$.

14. Write as a single logarithm with coefficient 1: $2\log_2 x + \frac{1}{3}\log_2 y - 3\log_2 z$.

Evaluate. Round off to the nearest tenth.

15.
 a. $\log_2 10$
 b. $\ln 1$
 c. $\log_3\left(\frac{1}{5}\right)$

Solve:

16. $2^{3x-1} = 16$

17. $3^{7x+1} = 5$

18. $\log_5(3x-4) = \log_5(2x+7)$

19. $\log_3\left(x^2+26\right) = 3$

20. $\log_2 x + \log_2(2x+7) = 2$

21. $\log(2x+3) = 1 + \log(x+1)$

22. Joe invested $5,200 in an account earning 3.8% annual interest that is compounded monthly. How long will it take to accumulate a total of $6,200 in the account?

23. Mary has $3,500 in a savings account earning $4\frac{1}{2}$% annual interest that is compounded continuously. How long will it take to double the amount in the account?

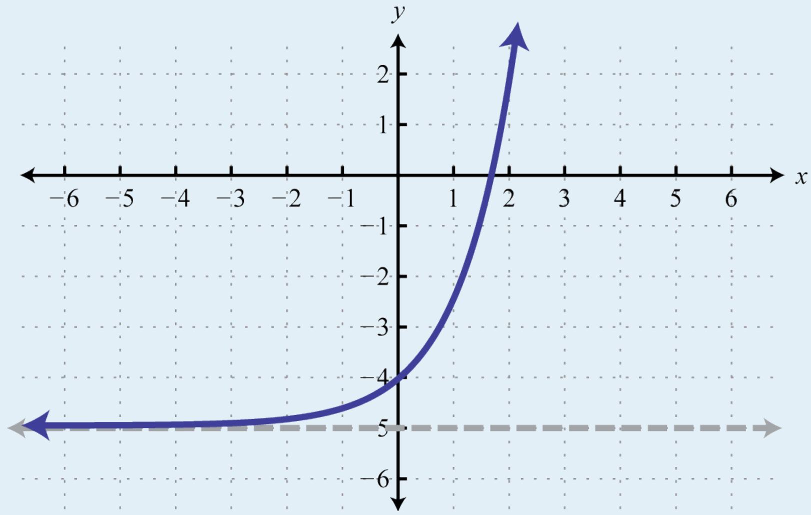
24. During the exponential growth phase, certain bacteria can grow at a rate of 5.3% per hour. If 12,000 cells are initially present in a sample, construct an exponential growth model and use it to: a. Estimate the population of bacteria in 3.5 hours. b. Estimate the time it will take the population to double.

25. The half-life of caesium-137 is about 30 years. Approximate the time it will take a 20-milligram sample of caesium-137 to decay to 8 milligrams.

ANSWERS

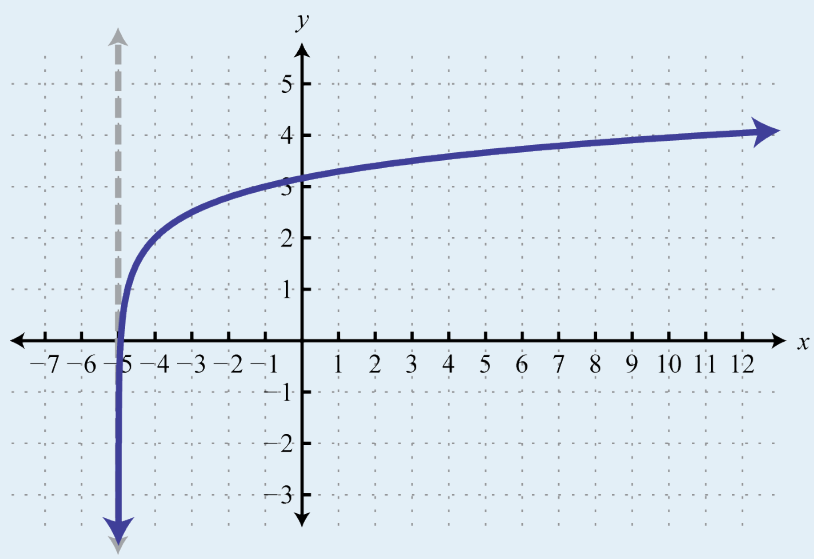
1. $\left(f \bigcirc g\right)(x) = 9x^2 - 9x + 5$

3. $f^{-1}(x) = 2x + 6$

5. 

7. $6,052.18

9.
   a. 4
   b. -2
   c. 3
   d. 1

11. 

13. $2 + 2\log x + \log y - \frac{1}{2}\log z$

15.
   a. 3.3

b. 0
c. −1.5

17. $\frac{\log\ 5-\log\ 3}{7\log\ 3}$

19. ±1

21. $-\frac{7}{8}$

23. 15.4 years

25. 40 years

Chapter 8

Conic Sections

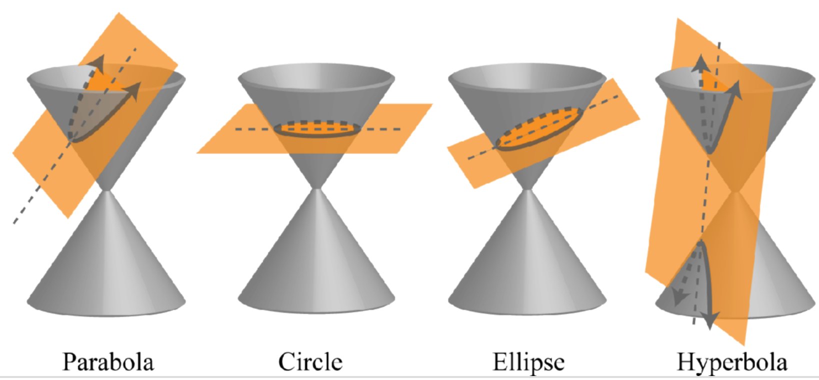
8.1 Distance, Midpoint, and the Parabola

LEARNING OBJECTIVES

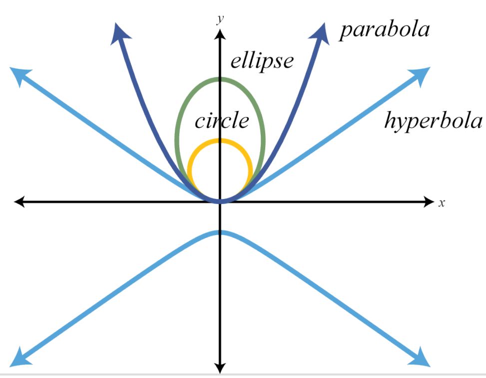
1. Apply the distance and midpoint formulas.
2. Graph a parabola using its equation given in standard from.
3. Determine standard form for the equation of a parabola given general form.

Conic Sections

A **conic section**[1] is a curve obtained from the intersection of a right circular cone and a plane. The conic sections are the parabola, circle, ellipse, and hyperbola.



The goal is to sketch these graphs on a rectangular coordinate plane.



1. A curve obtained from the intersection of a right circular cone and a plane.

The Distance and Midpoint Formulas

We begin with a review of the **distance formula**[2]. Given two points (x_1, y_1) and (x_2, y_2) in a rectangular coordinate plane, the distance *d* between them is given by the distance formula,

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Furthermore, the point that bisects the line segment formed by these two points is called the **midpoint**[3] and is given by the formula,

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

The midpoint is an ordered pair formed by the average of the *x*-values and the average of the *y*-values.

2. Given two points (x_1, y_1) and (x_2, y_2), the distance *d* between them is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

3. Given two points (x_1, y_1) and (x_2, y_2), the midpoint is an ordered pair given by $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

Example 1

Given $(-2, -5)$ and $(-4, -3)$ calculate the distance and midpoint between them.

Solution:

In this case, we will use the formulas with the following points:

$$\begin{array}{cc} (x_1, \ y_1) & (x_2, \ y_2) \\ (-2, -5) & (-4, -3) \end{array}$$

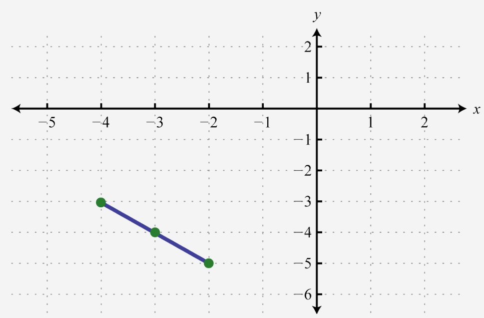
It is a good practice to include the formula in its general form before substituting values for the variables; this improves readability and reduces the probability of making errors.

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{[-4 - (-2)]^2 + [-3 - (-5)]^2} \\ &= \sqrt{(-4 + 2)^2 + (-3 + 5)^2} \\ &= \sqrt{(-2)^2 + (2)^2} \\ &= \sqrt{4 + 4} \\ &= \sqrt{8} \\ &= 2\sqrt{2} \end{aligned}$$

Next determine the midpoint.

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-2 + (-4)}{2}, \frac{-5 + (-3)}{2}\right)$$
$$= \left(\frac{-6}{2}, \frac{-8}{2}\right)$$
$$= (-3, -4)$$

Plotting these points on a graph we have,



Answer: Distance: $2\sqrt{2}$ units; midpoint: $(-3, -4)$

Example 2

The diameter of a circle is defined by the two points $(-1, 2)$ and $(1, -2)$. Determine the radius of the circle and use it to calculate its area.

Solution:

Find the diameter using the distance formula.

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{[1 - (-1)]^2 + (-2 - 2)^2} \\ &= \sqrt{(2)^2 + (-4)^2} \\ &= \sqrt{4 + 16} \\ &= \sqrt{20} \\ &= 2\sqrt{5} \end{aligned}$$

Recall that the radius of a circle is one-half of the circle's diameter. Therefore, if $d = 2\sqrt{5}$ units, then

$$r = \frac{d}{2} = \frac{2\sqrt{5}}{2} = \sqrt{5}$$

The area of a circle is given by the formula $A = \pi r^2$ and we have

$$\begin{aligned} A &= \pi\left(\sqrt{5}\right)^2 \\ &= \pi \cdot 5 \\ &= 5\pi \end{aligned}$$

Area is measured in square units.

Answer: Radius: $\sqrt{5}$ units; area: 5π square units

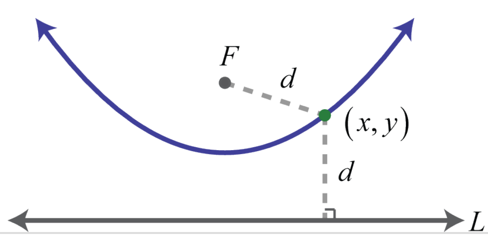
Try this! Given $(0, 0)$ and $(9, -3)$ calculate the distance and midpoint between them.

Answer: Distance: $3\sqrt{10}$ units; midpoint: $\left(\frac{9}{2}, -\frac{3}{2}\right)$

(click to see video)

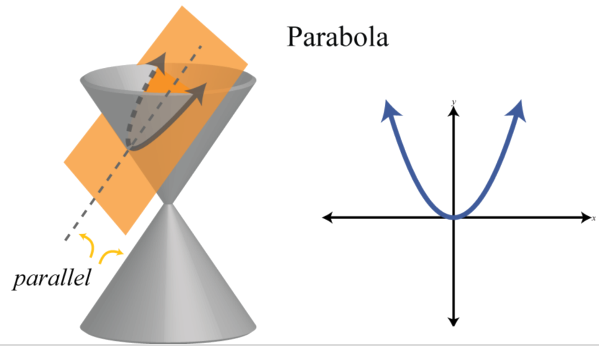
The Parabola

A **parabola**[4] is the set of points in a plane equidistant from a given line, called the directrix, and a point not on the line, called the focus. In other words, if given a line L the directrix, and a point F the focus, then (x, y) is a point on the parabola if the shortest distance from it to the focus and from it to the line is equal as pictured below:



4. The set of points in a plane equidistant from a given line, called the directrix, and a point not on the line, called the focus.

The vertex of the parabola is the point where the shortest distance to the directrix is at a minimum. In addition, a parabola is formed by the intersection of a cone with an oblique plane that is parallel to the side of the cone:



Recall that the graph of a quadratic function, a polynomial function of degree 2, is parabolic. We can write the equation of a **parabola in general form**[5] or we can write the equation of a **parabola in standard form**[6]:

$$\begin{array}{ll} \textit{General Form} & \textit{Standard Form} \\ y = ax^2 + bx + c & y = a(x-h)^2 + k \end{array}$$

Here *a*, *b*, and *c* are real numbers, $a \neq 0$. Both forms are useful in determining the general shape of the graph. However, in this section we will focus on obtaining standard form, which is often called **vertex form**[7]. Given a quadratic function in standard form, the vertex is (h, k). To see that this is the case, consider graphing $y = (x+3)^2 + 2$ using transformations.

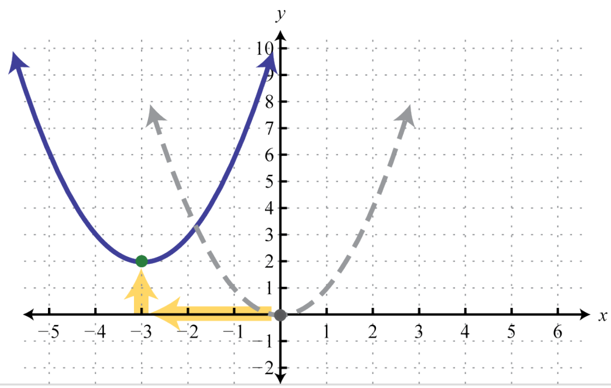
$$\begin{array}{ll} y = x^2 & \textit{Basic squaring function.} \\ y = (x+3)^2 & \textit{Horizontal shift left 3 units.} \\ y = (x+3)^2 + 2 & \textit{Vertical shift up 2 units.} \end{array}$$

Use these translations to sketch the graph,

5. The equation of a parabola written in the form $y = ax^2 + bx + c$ or $x = ay^2 + by + c$, where *a*, *b*, and *c* are real numbers and $a \neq 0$.

6. The equation of a parabola written in the form $y = a(x-h)^2 + k$ or $x = a(y-k)^2 + h$.

7. The equation of a parabola written in standard form is often called vertex form. In this form the vertex is apparent: (h, k).



Here we can see that the vertex is $(-3, 2)$. This can be determined directly from the equation in standard form,

$$\begin{aligned} y &= \quad a(x-h)^2+k \\ & \qquad\qquad \downarrow \quad\; \downarrow \\ y &= [x - \ (-3)]^2 + 2 \end{aligned}$$

Written in this form we can see that the vertex is $(-3, 2)$. However, the equation is typically not given in standard form. Transforming general form to standard form, by completing the square, is the main process by which we will sketch all of the conic sections.

Example 3

Rewrite the equation in standard form and determine the vertex of its graph: $y = x^2 - 8x + 15$.

Solution:

Begin by making room for the constant term that completes the square.

$$\begin{aligned} y &= x^2 - 8x + 15 \\ &= x^2 - 8x + \_\_\_ + 15 - \_\_\_ \end{aligned}$$

The idea is to add and subtract the value that completes the square, $\left(\frac{b}{2}\right)^2$, and then factor. In this case, add and subtract $\left(\frac{b}{2}\right)^2 = \left(\frac{-8}{2}\right)^2 = (-4)^2 = 16$.

$$\begin{aligned} y &= x^2 - 8x + 15 && \textit{Add and subtract 16.} \\ &= \left(x^2 - 8x + 16\right) + 15 - 16 && \textit{Factor.} \\ &= (x-4)\,(x-4) - 1 \\ &= (x-4)^2 - 1 \end{aligned}$$

Adding and subtracting the same value within an expression does not change it. Doing so is equivalent to adding 0. Once the equation is in this form, we can easily determine the vertex.

$$\begin{array}{l} y = a(x - h)^2 + \quad k \\ \qquad\qquad\quad \downarrow \qquad\ \ \downarrow \\ y = \quad (x - 4)^2 + (-1) \end{array}$$

Here we have a translation to the right 4 units and down 1 unit. Hence, $h = 4$ and $k = -1$.

Answer: $y = (x - 4)^2 - 1$; vertex: $(4, -1)$

If there is a leading coefficient other than 1, then begin by factoring out that leading coefficient from the first two terms of the trinomial.

Example 4

Rewrite the equation in standard form and determine the vertex of the graph: $y = -2x^2 + 12x - 16$.

Solution:

Since $a = -2$, factor this out of the first two terms in order to complete the square. Leave room inside the parentheses to add and subtract the value that completes the square.

$$\begin{aligned} y &= -2x^2 + 12x - 16 \\ &= -2\left(x^2 - 6x + \_\_\_ - \_\_\_\right) - 16 \end{aligned}$$

Now use -6 to determine the value that completes the square. In this case, $\left(\frac{b}{2}\right)^2 = \left(\frac{-6}{2}\right)^2 = (-3)^2 = 9$. Add and subtract 9 and factor as follows:

$$\begin{aligned} y &= -2x^2 + 12x - 16 \\ &= -2\left(x^2 - 6x + \_\_\_ - \_\_\_\right) - 16 && \textit{Add and subtract 9.} \\ &= -2(x^2 - 6x + 9 - 9) - 16 && \textit{Factor.} \\ &= -2\left[(x-3)\,(x-3) - 9\right] - 16 \\ &= -2\left[(x-3)^2 - 9\right] - 16 && \textit{Distribute the } -2. \\ &= -2(x-3)^2 + 18 - 16 \\ &= -2(x-3)^2 + 2 \end{aligned}$$

In this form, we can easily determine the vertex.

$$\begin{array}{l} y = a(x - h)^2 + k \\ \qquad\qquad\quad \downarrow \quad\; \downarrow \\ y = -2(x - 3)^2 + 2 \end{array}$$

Here $h = 3$ and $k = 2$.

Answer: $y = -2(x - 3)^2 + 2$; vertex: $(3, 2)$

Make use of both general form and standard form when sketching the graph of a parabola.

Example 5

Graph: $y = -2x^2 + 12x - 16$.

Solution:

From the previous example we have two equivalent forms of this equation,

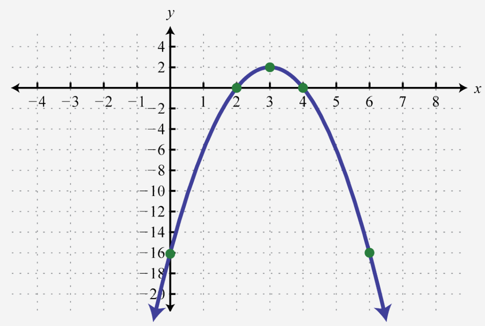
$$\begin{array}{ll} \textit{General Form} & \textit{Standard Form} \\ y = -2x^2 + 12x - 16 & y = -2(x-3)^2 + 2 \end{array}$$

Recall that if the leading coefficient $a > 0$ the parabola opens upward and if $a < 0$ the parabola opens downward. In this case, $a = -2$ and we conclude the parabola opens downward. Use general form to determine the y-intercept. When $x = 0$ we can see that the y-intercept is $(0, -16)$. From the equation in standard form, we can see that the vertex is $(3, 2)$. To find the x-intercept we could use either form. In this case, we will use standard form to determine the x-values where $y = 0$,

$$\begin{aligned} y &= -2(x-3)^2 + 2 && \textit{Set } y = 0 \textit{ and solve.} \\ 0 &= -2(x-3)^2 + 2 \\ -2 &= -2(x-3)^2 \\ 1 &= (x-3)^2 && \textit{Apply the square root property.} \\ \pm 1 &= x - 3 \\ 3 \pm 1 &= x \end{aligned}$$

Here $x = 3 - 1 = 2$ or $x = 3 + 1 = 4$ and therefore the x-intercepts are $(2, 0)$ and $(4, 0)$. Use this information to sketch the graph.

Answer:



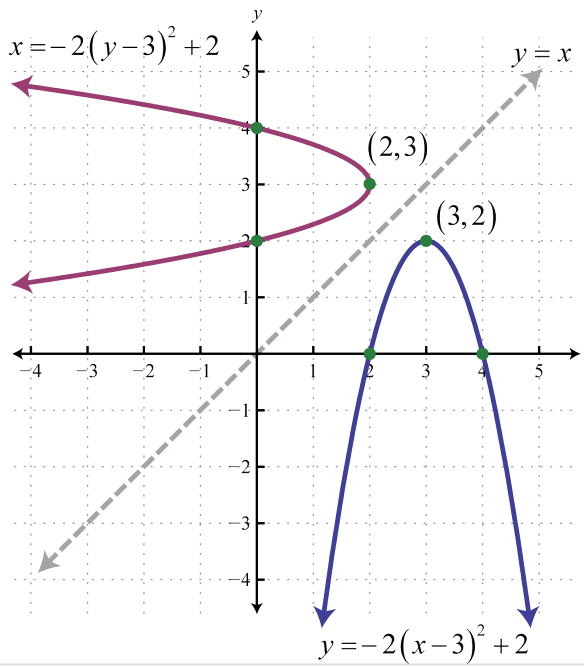
So far we have been sketching parabolas that open upward or downward because these graphs represent functions. At this point we extend our study to include parabolas that open right or left. If we take the equation that defines the parabola in the previous example,

$$y = -2(x-3)^2 + 2$$

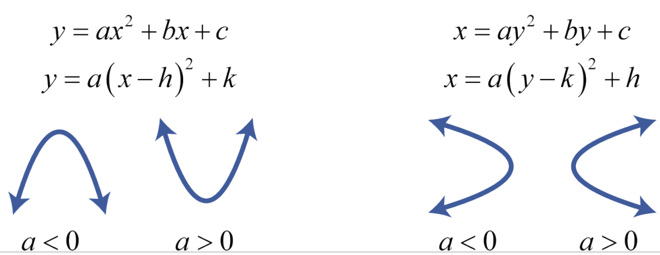
and switch the *x* and *y* values we obtain

$$x = -2\left(y-3\right)^2 + 2$$

This produces a new graph with symmetry about the line $y = x$.



Note that the resulting graph is not a function. However, it does have the same general parabolic shape that opens left. We can recognize equations of parabolas that open left or right by noticing that they are quadratic in *y* instead of *x*. Graphing parabolas that open left or right is similar to graphing parabolas that open upward and downward. In general, we have



In all cases, the vertex is (h, k). Take care to note the placement of *h* and *k* in each equation.

Example 6

Graph: $x = y^2 + 10y + 13.$

Solution:

Because the coefficient of y^2 is positive, $a = 1$, we conclude that the graph is a parabola that opens to the right. Furthermore, when $y = 0$ it is clear that $x = 13$ and therefore the x-intercept is $(13, 0)$. Complete the square to obtain standard form. Here we will add and subtract $\left(\frac{b}{2}\right)^2 = \left(\frac{10}{2}\right)^2 = (5)^2 = 25.$

$$\begin{aligned} x &= y^2 + 10y + 13 \\ &= y^2 + 10y + 25 - 25 + 13 \\ &= (y+5)\ (y+5) - 12 \\ &= (y+5)^2 - 12 \end{aligned}$$

Therefore,

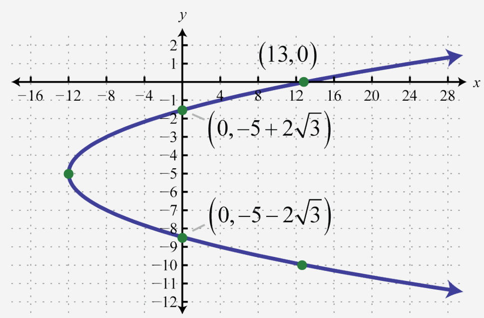
$$\begin{array}{l} x = a\ (y-k)^2 + h \\ \qquad\qquad\quad \downarrow \qquad\ \ \downarrow \\ x = (y - (-5))^2 + (-12) \end{array}$$

From this we can see that the vertex $(h, k) = (-12, -5)$ .Next use standard form to find the y-intercepts by setting $x = 0$.

$$x = (y+5)^2 - 12$$
$$0 = (y+5)^2 - 12$$
$$12 = (y+5)^2$$
$$\pm\sqrt{12} = y + 5$$
$$\pm 2\sqrt{3} = y + 5$$
$$-5 \pm 2\sqrt{3} = y$$

The y-intercepts are $\left(0, -5 - 2\sqrt{3}\right)$ and $\left(0, -5 + 2\sqrt{3}\right)$. Use this information to sketch the graph.

Answer:



Example 7

Graph: $x = -2y^2 + 4y - 5$.

Solution:

Because the coefficient of y^2 is $a = -2$, we conclude that the graph is a parabola that opens to the left. Furthermore, when $y = 0$ it is clear that $x = -5$ and therefore the x-intercept is $(-5, 0)$. Begin by factoring out the leading coefficient as follows:

$$\begin{aligned} x &= -2y^2 + 4y - 5 \\ &= -2\left(y^2 - 2y + \_\_\_ - \_\_\_\right) - 5 \end{aligned}$$

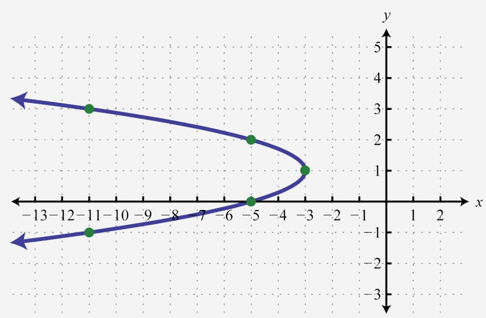
Here we will add and subtract $\left(\frac{b}{2}\right)^2 = \left(\frac{-2}{2}\right)^2 = (-1)^2 = 1$.

$$\begin{aligned} x &= -2y^2 + 4y - 5 \\ &= -2(y^2 - 2y + 1 - 1) - 5 \\ &= -2\left[(y-1)^2 - 1\right] - 5 \\ &= -2(y-1)^2 + 2 - 5 \\ &= -2(y-1)^2 - 3 \end{aligned}$$

Therefore, from vertex form, $x = -2(y-1)^2 - 3$, we can see that the vertex is $(h, k) = (-3, 1)$. Because the vertex is at $(-3, 1)$ and the parabola opens to the left, we can conclude that there are no y-intercepts. Since we only have two points, choose some y-values and find the corresponding x-values.

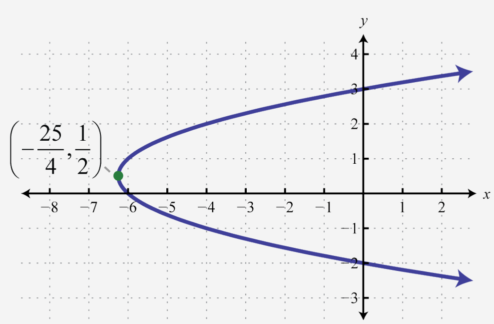
x	y	$x = -2(y-1)^2 - 3$
-11	-1	$x = -2(-1-1)^2 - 3 = -2(-2)^2 - 3 = -11$
-5	2	$x = -2(2-1)^2 - 3 = -2(1)^2 - 3 = -5$
-11	3	$x = -2(3-1)^2 - 3 = -2(2)^2 - 3 = -11$

Answer:



Try this! Graph: $x = y^2 - y - 6$.

Answer:



(click to see video)

KEY TAKEAWAYS

- Use the distance formula to determine the distance between any two given points. Use the midpoint formula to determine the midpoint between any two given points.
- A parabola can open upward or downward, in which case, it is a function. In this section, we extend our study of parabolas to include those that open left or right. Such graphs do not represent functions.
- The equation of a parabola that opens upward or downward is quadratic in *x*, $y = ax^2 + bx + c$. If $a > 0$, then the parabola opens upward and if $a < 0$, then the parabola opens downward.
- The equation of a parabola that opens left or right is quadratic in *y*, $x = ay^2 + by + c$. If $a > 0$, then the parabola opens to the right and if $a < 0$, then the parabola opens to the left.
- The equation of a parabola in general form $y = ax^2 + bx + c$ or $x = ay^2 + by + c$ can be transformed to standard form $y = a(x - h)^2 + k$ or $x = a(y - k)^2 + h$ by completing the square.
- When completing the square, ensure that the leading coefficient of the variable grouping is 1 before adding and subtracting the value that completes the square.
- Both general and standard forms are useful when graphing parabolas. Given standard form the vertex is apparent (h, k). To find the *x*-intercept set $y = 0$ and solve for *x* and to find the *y*-intercept set $x = 0$ and solve for *y*.

TOPIC EXERCISES

PART A: THE DISTANCE AND MIDPOINT FORMULAS

Calculate the distance and midpoint between the given two points.

1. $(-1, -3)$ and $(5, -11)$
2. $(-3, 2)$ and $(1, -1)$
3. $(4, -2)$ and $(-2, -6)$
4. $(-5, -6)$ and $(-3, -4)$
5. $(10, -1)$ and $(9, 6)$
6. $(-6, -4)$ and $(-12, 1)$
7. $(0, 0)$ and $\left(\sqrt{2}, \sqrt{3}\right)$
8. $(0, 0)$ and $\left(2\sqrt{2}, -\sqrt{3}\right)$
9. $\left(\sqrt{5}, -\sqrt{3}\right)$ and $\left(2\sqrt{5}, -\sqrt{3}\right)$
10. $\left(3\sqrt{10}, \sqrt{6}\right)$ and $\left(\sqrt{10}, -5\sqrt{6}\right)$
11. $\left(\frac{1}{2}, -1\right)$ and $\left(-2, \frac{3}{2}\right)$
12. $\left(-\frac{4}{3}, 2\right)$ and $\left(-\frac{1}{3}, -\frac{1}{2}\right)$
13. $\left(\frac{1}{5}, -\frac{9}{5}\right)$ and $\left(\frac{3}{10}, -\frac{5}{2}\right)$
14. $\left(-\frac{1}{2}, \frac{4}{3}\right)$ and $\left(-\frac{2}{3}, \frac{5}{6}\right)$
15. (a, b) and $(0, 0)$
16. $(0, 0)$ and $\left(a\sqrt{2}, 2\sqrt{a}\right)$

Determine the area of a circle whose diameter is defined by the given two points.

17. $(-8, 12)$ and $(-6, 8)$
18. $(9, 5)$ and $(9, -1)$
19. $(7, -8)$ and $(5, -10)$
20. $(0, -5)$ and $(6, 1)$
21. $\left(\sqrt{6}, 0\right)$ and $\left(0, 2\sqrt{3}\right)$
22. $\left(0, \sqrt{7}\right)$ and $\left(\sqrt{5}, 0\right)$

Determine the perimeter of the triangle given the coordinates of the vertices.

23. $(5, 3)$, $(2, -3)$, and $(8, -3)$
24. $(-3, 2)$, $(-4, -1)$, and $(-1, 0)$
25. $(3, 3)$, $\left(5, 3 - 2\sqrt{3}\right)$, and $(7, 3)$
26. $(0, 0)$, $\left(0, 2\sqrt{2}\right)$, and $\left(\sqrt{2}, 0\right)$

Find *a* so that the distance *d* between the points is equal to the given quantity.

27. $(1, 2)$ and $(4, a)$; $d = 5$ units
28. $(-3, a)$ and $(5, 6)$; $d = 10$ units
29. $(3, 1)$ and $(a, 0)$; $d = \sqrt{2}$ units
30. $(a, 1)$ and $(5, 3)$; $d = \sqrt{13}$ units

PART B: THE PARABOLA

Graph. Be sure to find the vertex and all intercepts.

31. $y = x^2 + 3$

32. $y = \frac{1}{2}(x - 4)^2$

33. $y = -2(x + 1)^2 - 1$

34. $y = -(x - 2)^2 + 1$

35. $y = -x^2 + 3$

36. $y = -(x + 1)^2 + 5$

37. $x = y^2 + 1$

38. $x = y^2 - 4$

39. $x = \left(y + 2\right)^2$

40. $x = \left(y - 3\right)^2$

41. $x = -y^2 + 2$

42. $x = -\left(y + 1\right)^2$

43. $x = \frac{1}{3}\left(y - 3\right)^2 - 1$

44. $x = -\frac{1}{3}\left(y + 3\right)^2 - 1$

Rewrite in standard form and give the vertex.

45. $y = x^2 - 6x + 18$

46. $y = x^2 + 8x + 36$

47. $x = y^2 + 20y + 87$

48. $x = y^2 - 10y + 21$

49. $y = x^2 - 14x + 49$

50. $x = y^2 + 16y + 64$

51. $x = 2y^2 - 4y + 5$

52. $y = 3x^2 - 30x + 67$

53. $y = 6x^2 + 36x + 54$

54. $x = 3y^2 + 6y - 1$

55. $y = 2x^2 - 2x - 1$

56. $x = 5y^2 + 15y + 9$

57. $x = -y^2 + 5y - 5$

58. $y = -x^2 + 9x - 20$

Rewrite in standard form and graph. Be sure to find the vertex and all intercepts.

59. $y = x^2 - 4x - 5$

60. $y = x^2 + 6x - 16$

61. $y = -x^2 + 12x - 32$

62. $y = -x^2 - 10x$

63. $y = 2x^2 + 4x + 9$

64. $y = 3x^2 - 6x + 4$

65. $y = -5x^2 + 30x - 45$

66. $y = -4x^2 - 16x - 16$

67. $x = y^2 - 2y - 8$

68. $x = y^2 + 4y + 8$

69. $x = y^2 - 2y - 3$

70. $x = y^2 + 6y - 7$

71. $x = -y^2 - 10y - 24$

72. $x = -y^2 - 12y - 40$

73. $x = 3y^2 + 12y + 12$

74. $x = -2y^2 + 12y - 18$

75. $x = y^2 - 4y - 3$

76. $x = y^2 + 6y + 1$

77. $x = -y^2 + 2y + 5$

78. $y = 2x^2 - 2x + 1$

79. $y = -3x^2 + 2x + 1$

80. $y = -x^2 + 3x + 10$

81. $x = -4y^2 - 4y - 5$

82. $x = y^2 - y + 2$

83. $y = x^2 + 5x - 1$

84. $y = 2x^2 + 6x + 3$

85. $x = 2y^2 + 10x + 12$

86. $x = y^2 + y - 1$

PART C: DISCUSSION BOARD

87. Research and discuss real-world applications that involve a parabola.

88. Do all parabolas have *x*-intercepts? Explain.

89. Do all parabolas have *y*-intercepts? Explain.

90. Make up your own parabola that opens left or right, write it in general form, and graph it.

ANSWERS

1. Distance: 10 units; midpoint: $(2, -7)$

3. Distance: $2\sqrt{13}$ units; midpoint: $(1, -4)$

5. Distance: $5\sqrt{2}$ units; midpoint: $\left(\frac{19}{2}, \frac{5}{2}\right)$

7. Distance: $\sqrt{5}$ units; midpoint: $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}\right)$

9. Distance: $\sqrt{5}$ units; midpoint: $\left(\frac{3\sqrt{5}}{2}, -\sqrt{3}\right)$

11. Distance: $\frac{5\sqrt{2}}{2}$ units; midpoint: $\left(-\frac{3}{4}, \frac{1}{4}\right)$

13. Distance: $\frac{\sqrt{2}}{2}$ units; midpoint: $\left(\frac{1}{4}, -\frac{43}{20}\right)$

15. Distance: $\sqrt{a^2 + b^2}$ units; midpoint: $\left(\frac{a}{2}, \frac{b}{2}\right)$

17. 5π square units

19. 2π square units

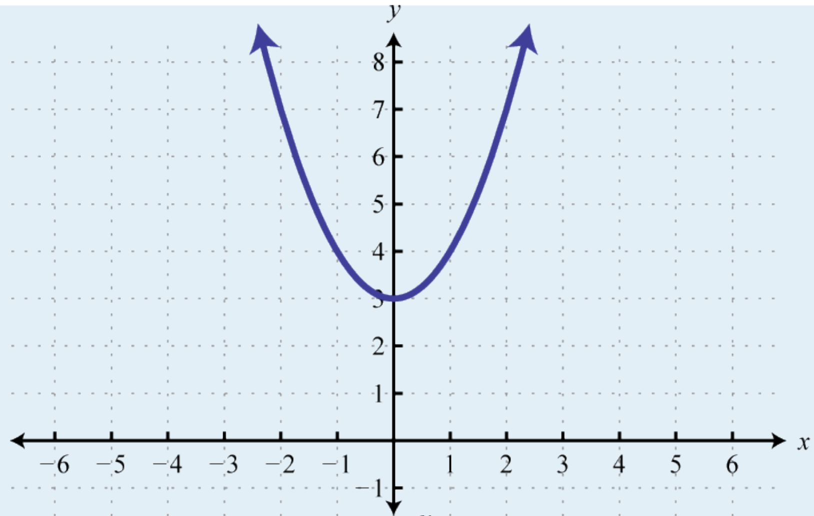
21. $\frac{9}{2}\pi$ square units

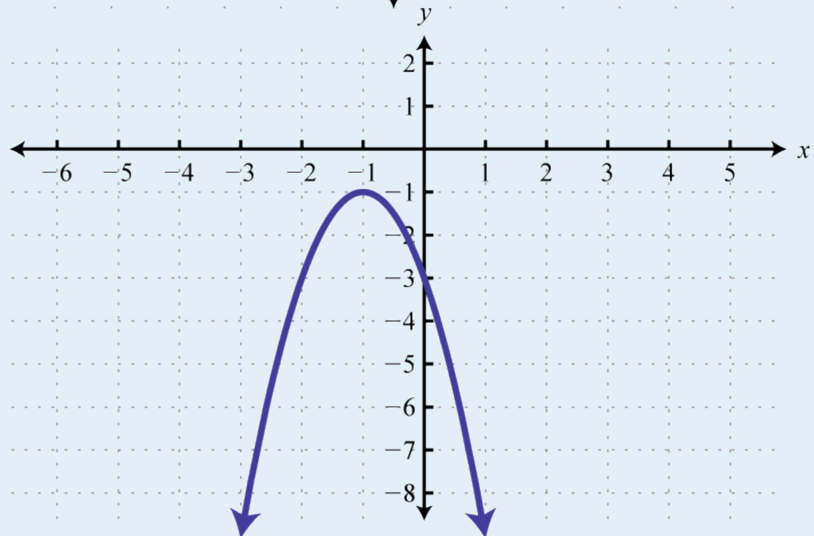
23. $6 + 6\sqrt{5}$ units

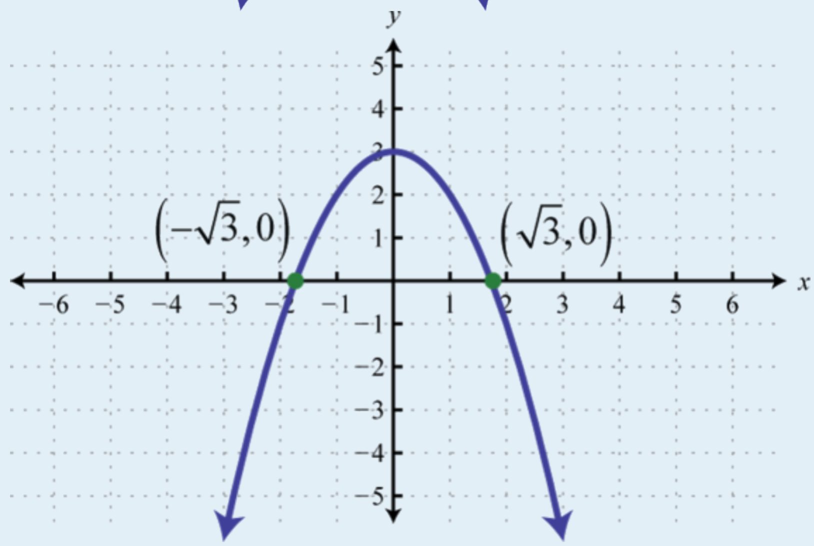
25. 12 units

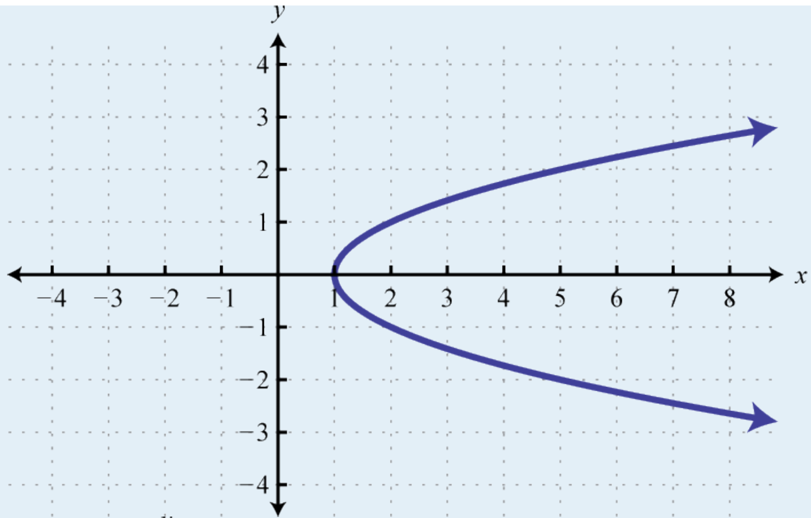
27. −2, 6

29. 2, 4

31. 

33. 

35. 



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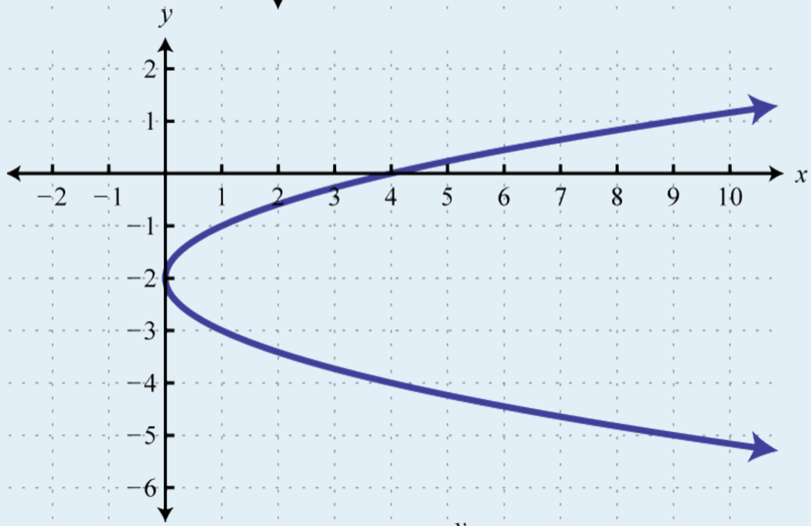

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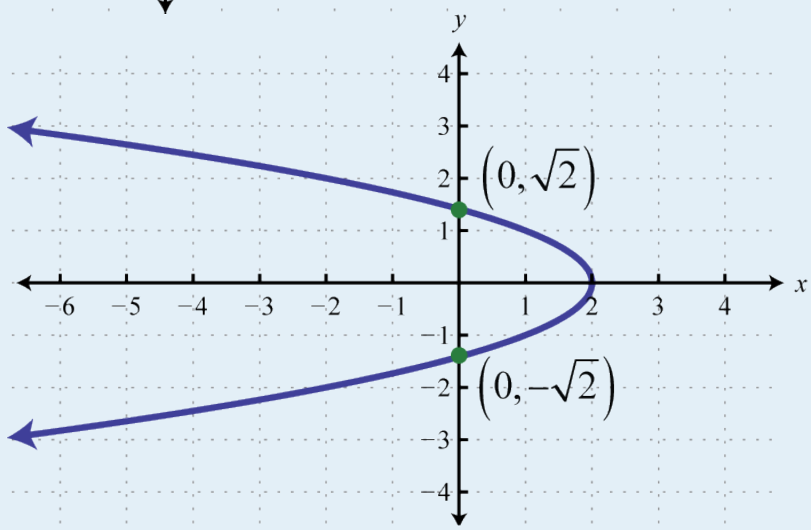

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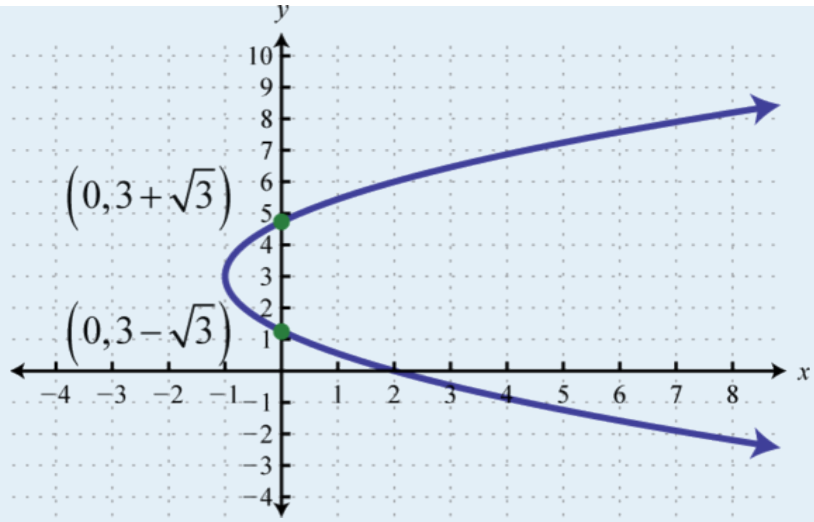
43. 

45. $y = (x - 3)^2 + 9$; vertex: $(3, 9)$

47. $x = \left(y + 10\right)^2 - 13$; vertex: $(-13, -10)$

49. $y = (x - 7)^2$; vertex: $(7, 0)$

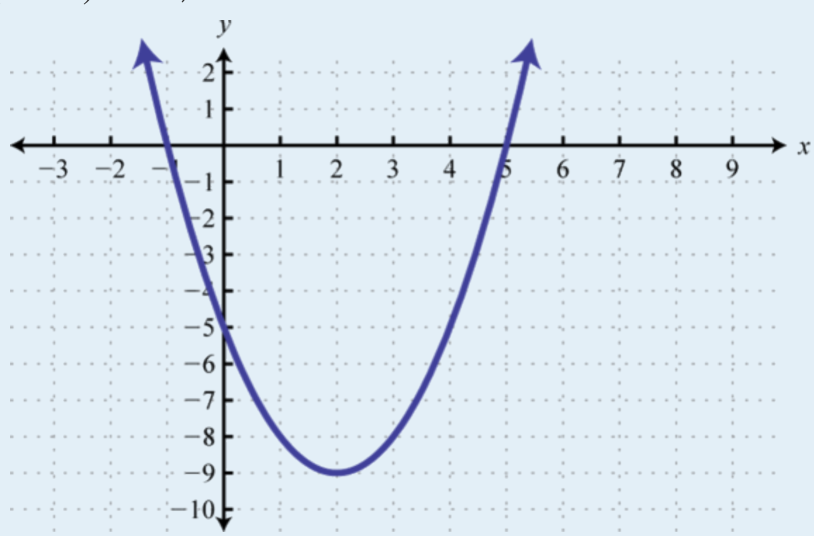
51. $x = 2\left(y - 1\right)^2 + 3$; vertex: $(3, 1)$

53. $y = 6(x + 3)^2$; vertex: $(-3, 0)$

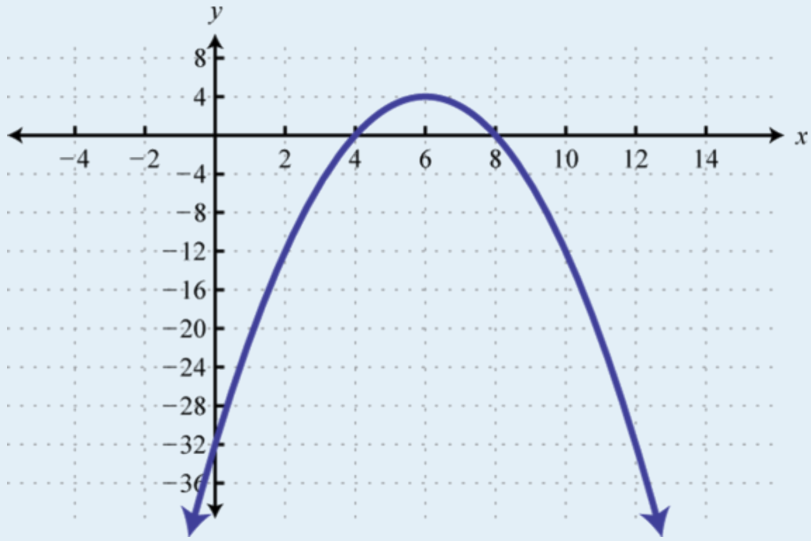
55. $y = 2\left(x - \frac{1}{2}\right)^2 - \frac{3}{2}$; vertex: $\left(\frac{1}{2}, -\frac{3}{2}\right)$

57. $x = -\left(y - \frac{5}{2}\right)^2 + \frac{5}{4}$; vertex: $\left(\frac{5}{4}, \frac{5}{2}\right)$

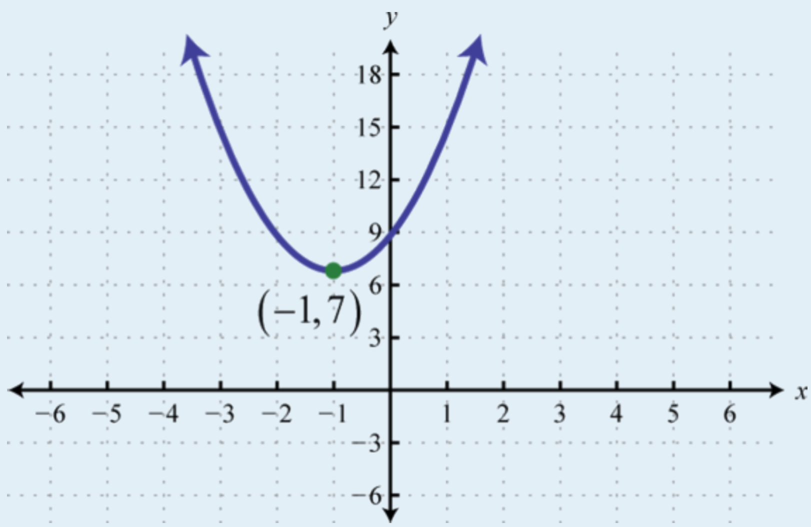
59. $y = (x - 2)^2 - 9$;



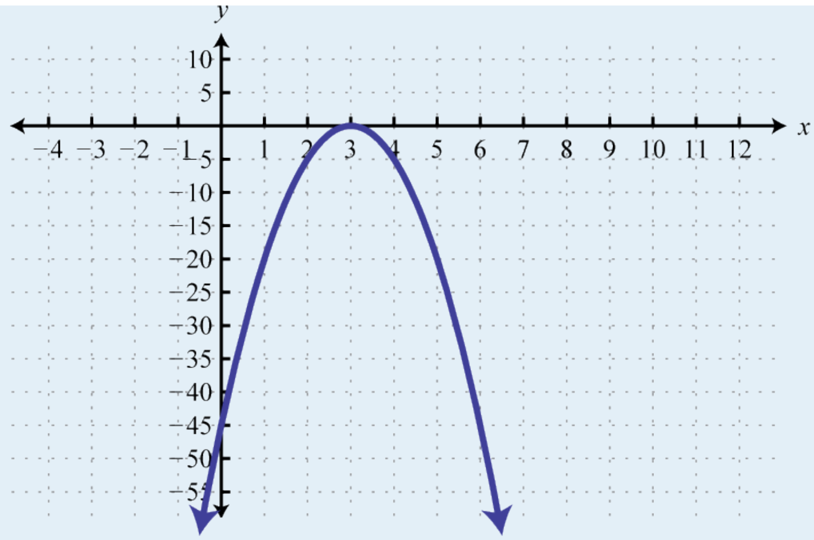
61. $y = -(x - 6)^2 + 4$;



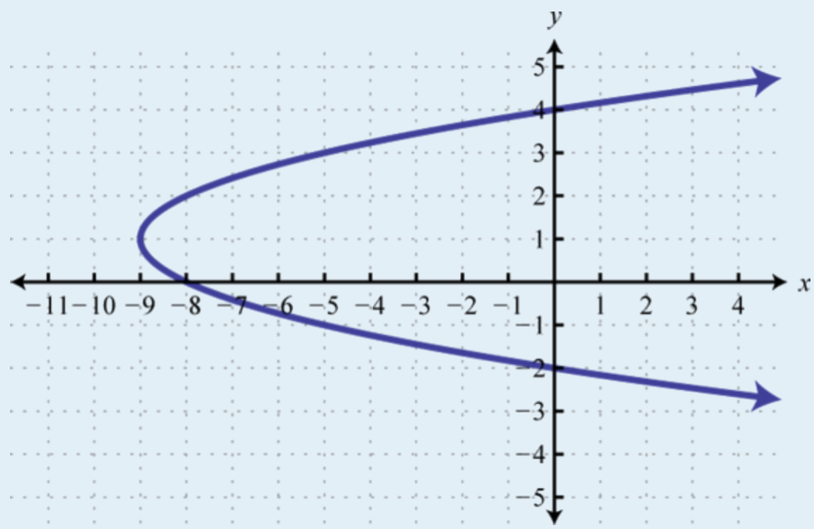
63. $y = 2(x + 1)^2 + 7$;



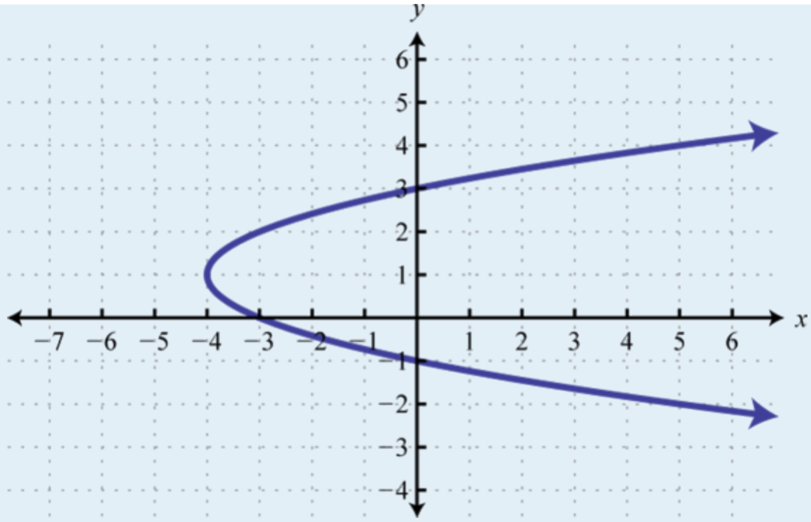
65. $y = -5(x - 3)^2$;



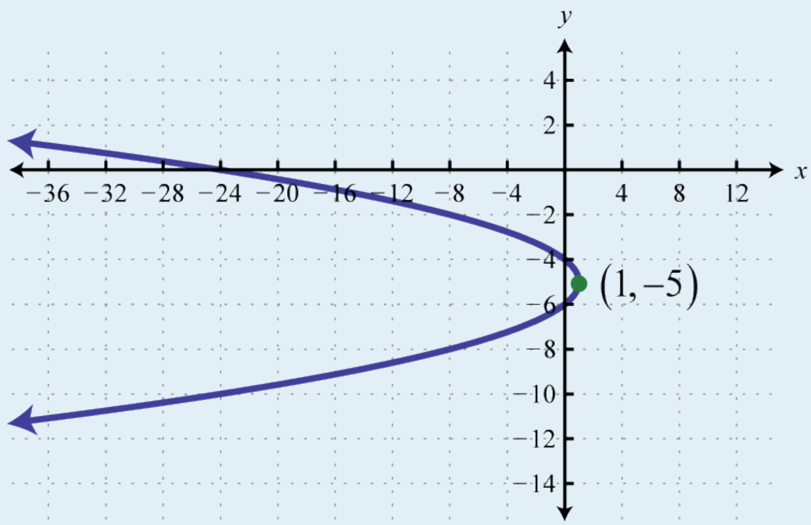
67. $x = (y - 1)^2 - 9$;



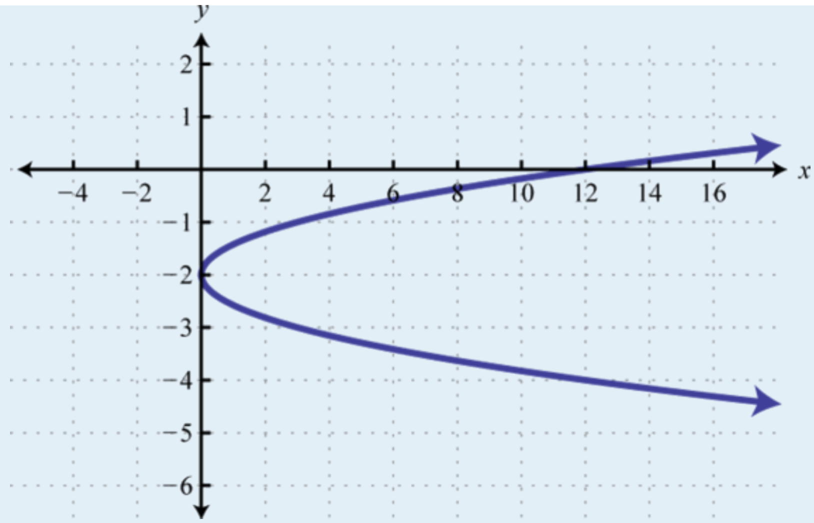
69. $x = (y - 1)^2 - 4$;



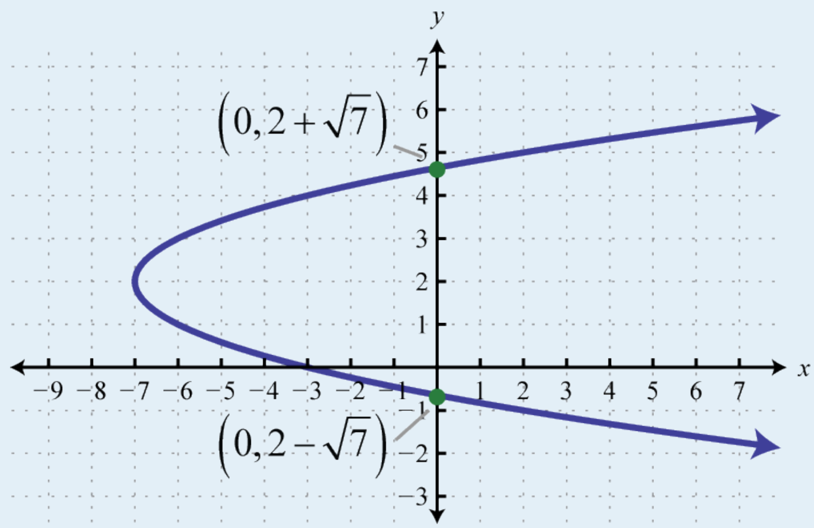
71. $x = -(y+5)^2 + 1$;



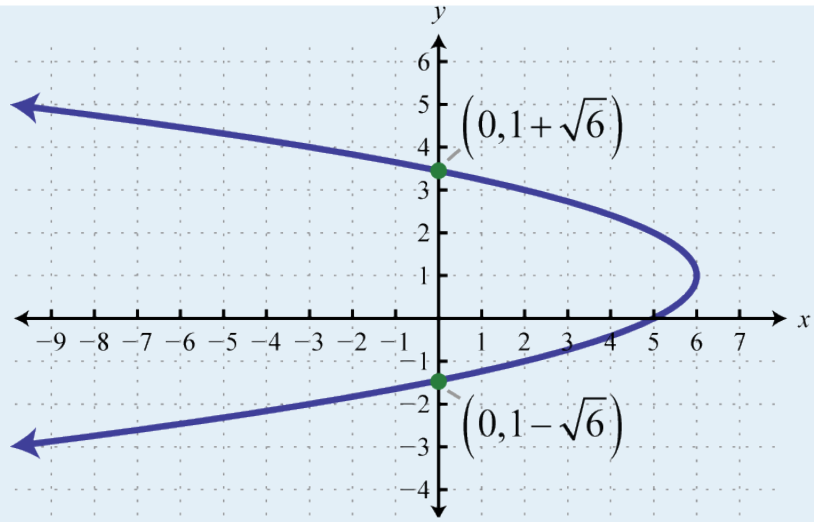
73. $x = 3(y+2)^2$;



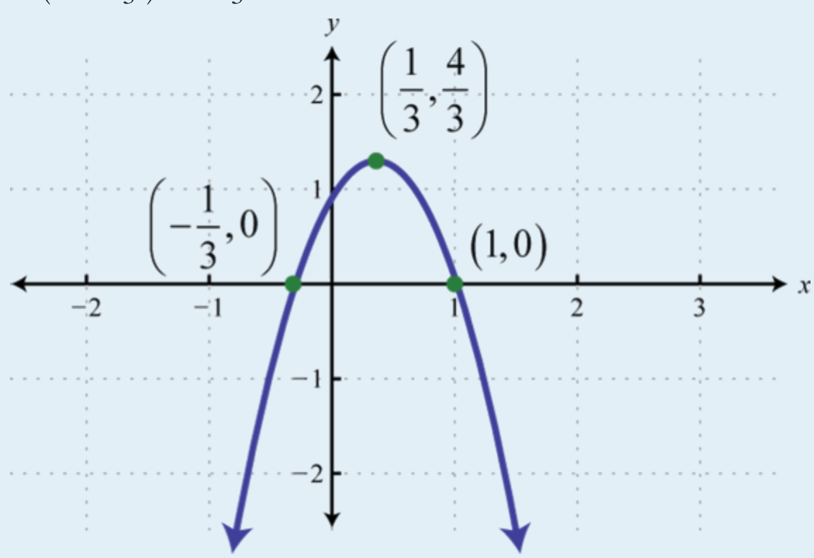
75. $x = (y-2)^2 - 7$;



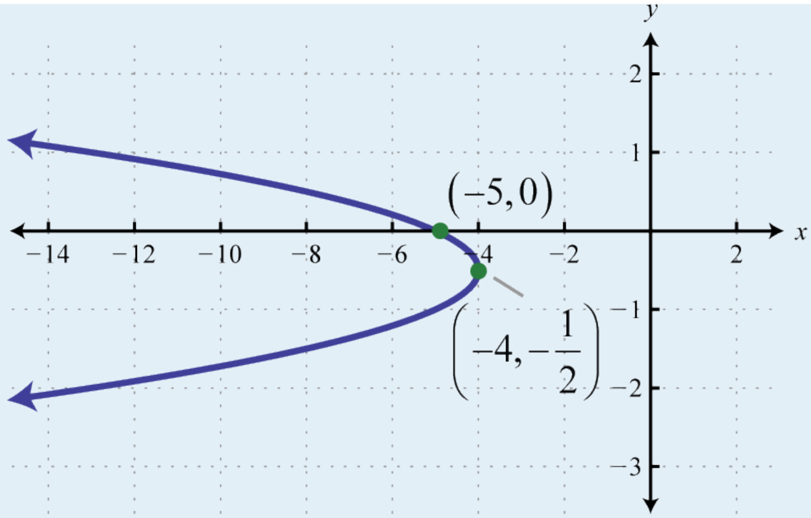
77. $x = -(y-1)^2 + 6$;



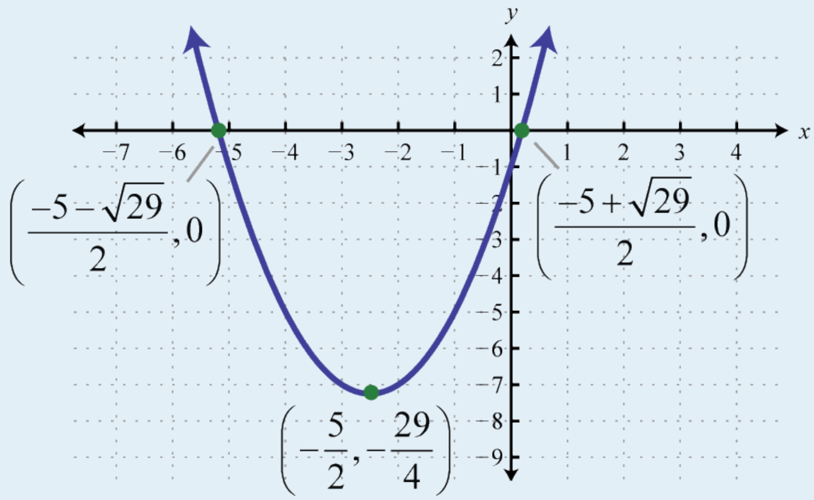
79. $y = -3\left(x - \frac{1}{3}\right)^2 + \frac{4}{3}$;



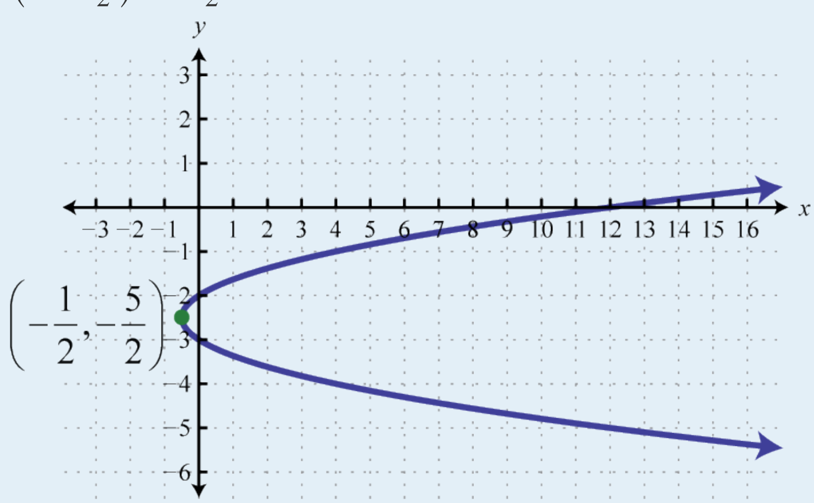
81. $x = -4\left(y + \frac{1}{2}\right)^2 - 4$;



83. $y = \left(x + \frac{5}{2}\right)^2 - \frac{29}{4}$;



85. $x = 2\left(y + \frac{5}{2}\right)^2 - \frac{1}{2}$;



87. Answer may vary

89. Answer may vary

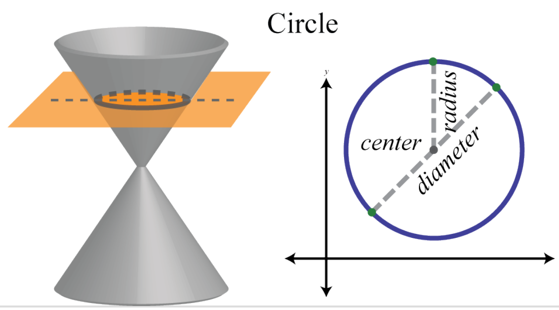
8.2 Circles

LEARNING OBJECTIVES

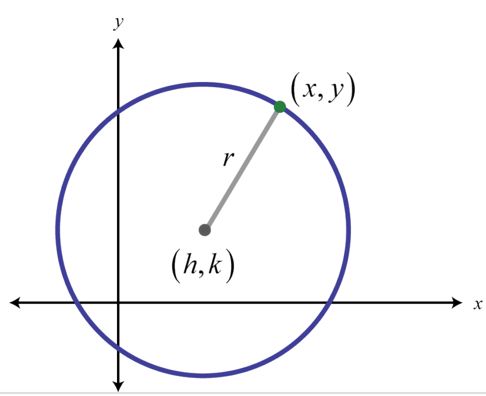
1. Graph a circle in standard form.
2. Determine the equation of a circle given its graph.
3. Rewrite the equation of a circle in standard form.

The Circle in Standard Form

A **circle**[8] is the set of points in a plane that lie a fixed distance, called the **radius**[9], from any point, called the center. The **diameter**[10] is the length of a line segment passing through the center whose endpoints are on the circle. In addition, a circle can be formed by the intersection of a cone and a plane that is perpendicular to the axis of the cone:



In a rectangular coordinate plane, where the center of a circle with radius r is (h, k), we have



8. A circle is the set of points in a plane that lie a fixed distance from a given point, called the center.

9. The fixed distance from the center of a circle to any point on the circle.

10. The length of a line segment passing through the center of a circle whose endpoints are on the circle.

Calculate the distance between (h, k) and (x, y) using the distance formula,

$$\sqrt{(x-h)^2 + (y-k)^2} = r$$

Squaring both sides leads us to the equation of a **circle in standard form**[11],

$$(x-h)^2 + (y-k)^2 = r^2$$

In this form, the center and radius are apparent. For example, given the equation $(x-2)^2 + (y+5)^2 = 16$ we have,

$$\begin{array}{ccccc} (x-h)^2 + & (x-k)^2 & = r^2 \\ \downarrow & \downarrow & \downarrow \\ (x-2)^2 + & [y-(-5)]^2 & = 4^2 \end{array}$$

In this case, the center is $(2, -5)$ and $r = 4$. More examples follow:

Equation	Center	Radius
$(x-3)^2 + (y-4)^2 = 25$	$(3, 4)$	$r = 5$

11. The equation of a circle written in the form $(x-h)^2 + (y-k)^2 = r^2$ where (h, k) is the center and r is the radius.

Equation	Center	Radius
$(x-1)^2 + (y+2)^2 = 7$	$(1, -2)$	$r = \sqrt{7}$
$(x+4)^2 + (y-3)^2 = 1$	$(-4, 3)$	$r = 1$
$x^2 + (y+6)^2 = 8$	$(0, -6)$	$r = 2\sqrt{2}$

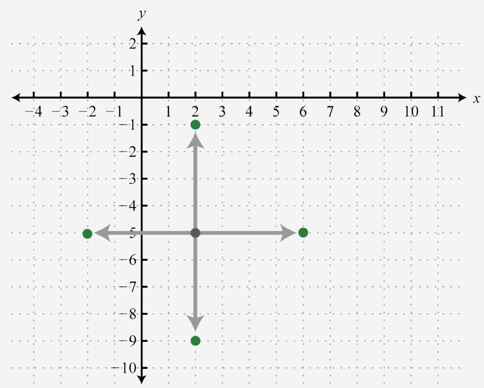
The graph of a circle is completely determined by its center and radius.

Example 1

Graph: $(x-2)^2 + (y+5)^2 = 16$.

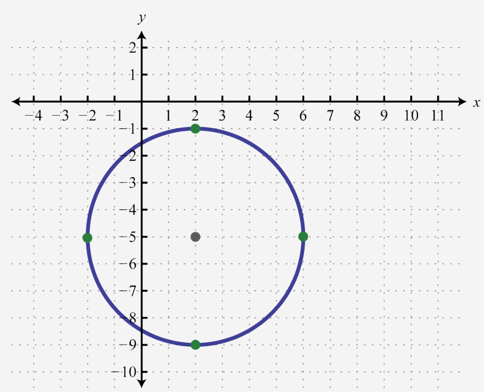
Solution:

Written in this form we can see that the center is $(2, -5)$ and that the radius $r = 4$ units. From the center mark points 4 units up and down as well as 4 units left and right.



Then draw in the circle through these four points.

Answer:



As with any graph, we are interested in finding the *x*- and *y*-intercepts.

Example 2

Find the intercepts: $(x-2)^2+(y+5)^2=16$.

Solution:

To find the y-intercepts set $x=0$:

$$\begin{aligned}(x-2)^2+(y+5)^2&=16\\(0-2)^2+(y+5)^2&=16\\4+(y+5)^2&=16\end{aligned}$$

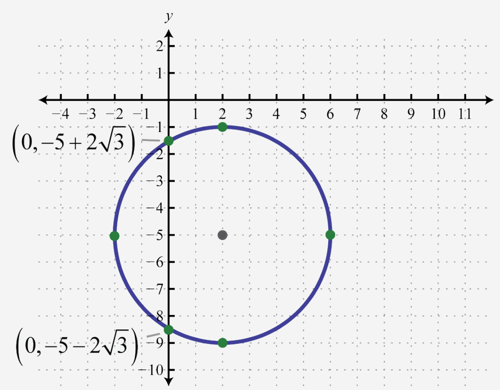
For this equation, we can solve by extracting square roots.

$$\begin{aligned}(y+5)^2&=12\\y+5&=\pm\sqrt{12}\\y+5&=\pm 2\sqrt{3}\\y&=-5\pm 2\sqrt{3}\end{aligned}$$

Therefore, the y-intercepts are $\left(0,-5-2\sqrt{3}\right)$ and $\left(0,-5+2\sqrt{3}\right)$. To find the x-intercepts set $y=0$:

$$
\begin{aligned}
(x-2)^2+(y+5)^2&=16\\
(x-2)^2+(0+5)^2&=16\\
(x-2)^2+25&=16\\
(x-2)^2&=-9\\
x-2&=\pm\sqrt{-9}\\
x&=2\pm 3i
\end{aligned}
$$

And because the solutions are complex we conclude that there are no real x-intercepts. Note that this does make sense given the graph.



Answer: x-intercepts: none; y-intercepts: $\left(0,-5-2\sqrt{3}\right)$ and $\left(0,-5+2\sqrt{3}\right)$

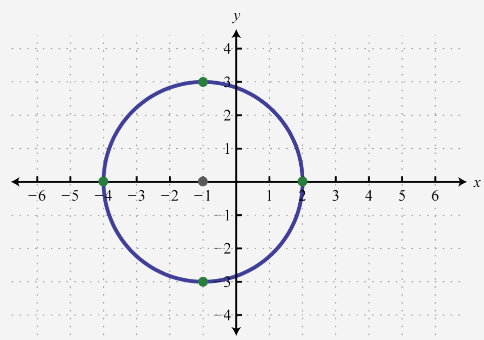
Given the center and radius of a circle, we can find its equation.

Example 3

Graph the circle with radius $r = 3$ units centered at $(-1, 0)$. Give its equation in standard form and determine the intercepts.

Solution:

Given that the center is $(-1, 0)$ and the radius is $r = 3$ we sketch the graph as follows:



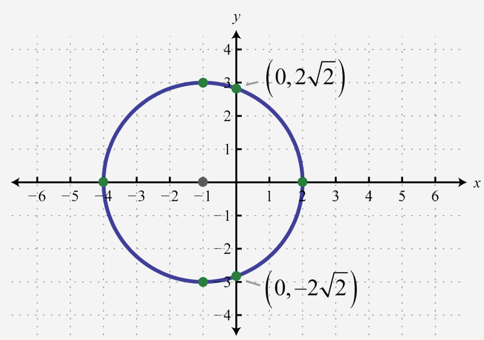
Substitute h, k, and r to find the equation in standard form. Since $(h, k) = (-1, 0)$ and $r = 3$ we have,

$$\begin{aligned}(x-h)^2+(y-k)^2&=r^2\\ [x-(-1)]^2+(y-0)^2&=3^2\\ (x+1)^2+y^2&=9\end{aligned}$$

The equation of the circle is $(x + 1)^2 + y^2 = 9$, use this to determine the y-intercepts.

$$\begin{aligned} (x+1)^2+y^2&=9 && \textit{Set } x=0 \textit{ to and solve for } y. \\ (0+1)^2+y^2&=9 \\ 1+y^2&=9 \\ y^2&=8 \\ y&=\pm\sqrt{8} \\ y&=\pm 2\sqrt{2} \end{aligned}$$

Therefore, the *y*-intercepts are $\left(0, -2\sqrt{2}\right)$ and $\left(0, 2\sqrt{2}\right)$. To find the *x*-intercepts algebraically, set $y = 0$ and solve for *x*; this is left for the reader as an exercise.



Answer: Equation: $(x+1)^2 + y^2 = 9$; *y*-intercepts: $\left(0, -2\sqrt{2}\right)$ and $\left(0, 2\sqrt{2}\right)$; *x*-intercepts: $(-4, 0)$ and $(2, 0)$

Of particular importance is the **unit circle**[12],

$$x^2 + y^2 = 1$$

Or,

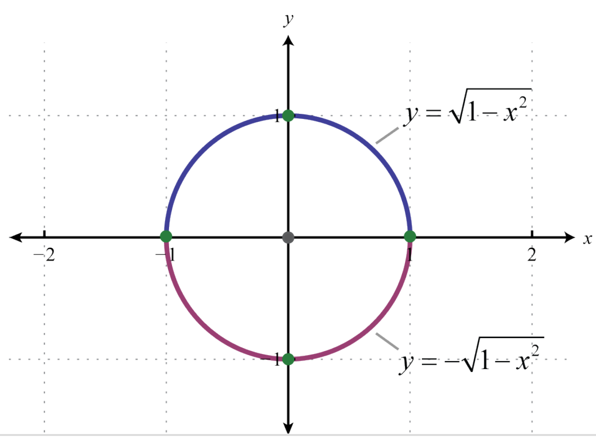
12. The circle centered at the origin with radius 1; its equation is $x^2 + y^2 = 1$.

$$(x-0)^2 + (y-0)^2 = 1^2$$

In this form, it should be clear that the center is $(0, 0)$ and that the radius is 1 unit. Furthermore, if we solve for y we obtain two functions:

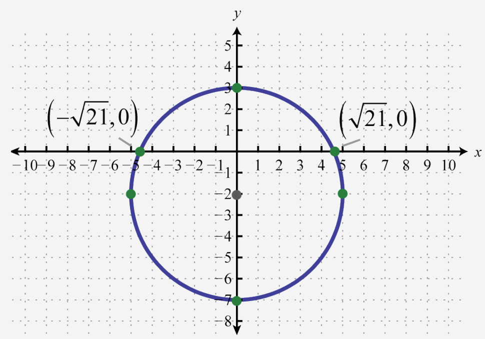
$$\begin{aligned} x^2 + y^2 &= 1 \\ y^2 &= 1 - x^2 \\ y &= \pm\sqrt{1-x^2} \end{aligned}$$

The function defined by $y = \sqrt{1-x^2}$ is the top half of the circle and the function defined by $y = -\sqrt{1-x^2}$ is the bottom half of the unit circle:



Try this! Graph and label the intercepts: $x^2 + (y + 2)^2 = 25$.

Answer:



(click to see video)

The Circle in General Form

We have seen that the graph of a circle is completely determined by the center and radius which can be read from its equation in standard form. However, the equation is not always given in standard form. The equation of a **circle in general form**[13] follows:

$$x^2 + y^2 + cx + dy + e = 0$$

Here c, d, and e are real numbers. The steps for graphing a circle given its equation in general form follow.

13. The equation of a circle written in the form $x^2 + y^2 + cx + dy + e = 0$.

Example 4

Graph: $x^2 + y^2 + 6x - 8y + 13 = 0.$

Solution:

Begin by rewriting the equation in standard form.

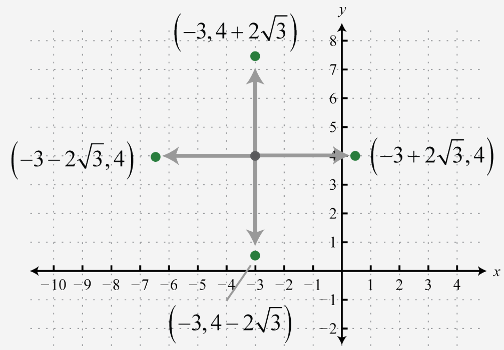
- **Step 1:** Group the terms with the same variables and move the constant to the right side. In this case, subtract 13 on both sides and group the terms involving x and the terms involving y as follows.

$$\begin{gathered} x^2 + y^2 + 6x - 8y + 13 = 0 \\ \left(x^2 + 6x + \_\_\_\right) + \left(y^2 - 8y + \_\_\_\right) = -13 \end{gathered}$$

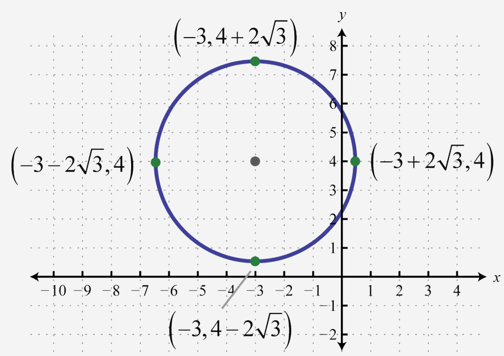
- **Step 2:** Complete the square for each grouping. The idea is to add the value that completes the square, $\left(\frac{b}{2}\right)^2$, to both sides for both groupings, and then factor. For the terms involving x use $\left(\frac{6}{2}\right)^2 = 3^2 = 9$ and for the terms involving y use $\left(\frac{-8}{2}\right)^2 = (-4)^2 = 16.$

$$\begin{gathered} \left(x^2 + 6x + 9\right) + \left(y^2 - 8y + 16\right) = -13 + 9 + 16 \\ (x+3)^2 + \left(y-4\right)^2 = 12 \end{gathered}$$

- **Step 3:** Determine the center and radius from the equation in standard form. In this case, the center is $(-3, 4)$ and the radius $r = \sqrt{12} = 2\sqrt{3}.$
- **Step 4:** From the center, mark the radius vertically and horizontally and then sketch the circle through these points.


$(-3, 4+2\sqrt{3})$
$(-3-2\sqrt{3}, 4)$
$(-3+2\sqrt{3}, 4)$
$(-3, 4-2\sqrt{3})$

Answer:


$(-3, 4+2\sqrt{3})$
$(-3-2\sqrt{3}, 4)$
$(-3+2\sqrt{3}, 4)$
$(-3, 4-2\sqrt{3})$

Example 5

Determine the center and radius: $4x^2 + 4y^2 - 8x + 12y - 3 = 0.$

Solution:

We can obtain the general form by first dividing both sides by 4.

$$\frac{4x^2 + 4y^2 - 8x + 12y - 3}{4} = \frac{0}{4}$$
$$x^2 + y^2 - 2x + 3y - \frac{3}{4} = 0$$

Now that we have the general form for a circle, where both terms of degree two have a leading coefficient of 1, we can use the steps for rewriting it in standard form. Begin by adding $\frac{3}{4}$ to both sides and group variables that are the same.

$$\left(x^2 - 2x + \_\_\_\right) + \left(y^2 + 3y + \_\_\_\right) = \frac{3}{4}$$

Next complete the square for both groupings. Use $\left(\frac{-2}{2}\right)^2 = (-1)^2 = 1$ for the first grouping and $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$ for the second grouping.

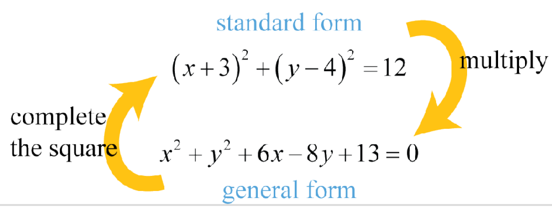
$$\left(x^2 - 2x + 1\right) + \left(y^2 + 3y + \frac{9}{4}\right) = \frac{3}{4} + 1 + \frac{9}{4}$$

$$(x-1)^2 + \left(y + \frac{3}{2}\right)^2 = \frac{16}{4}$$

$$(x-1)^2 + \left(y + \frac{3}{2}\right)^2 = 4$$

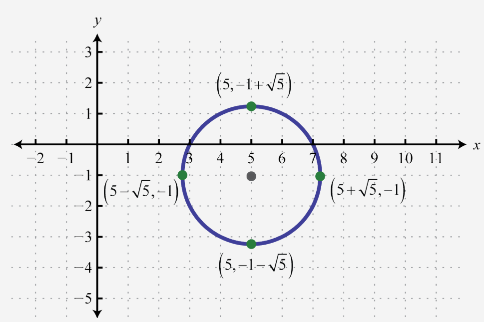
Answer: Center: $\left(1, -\frac{3}{2}\right)$; radius: $r = 2$

In summary, to convert from standard form to general form we multiply, and to convert from general form to standard form we complete the square.



Try this! Graph: $x^2 + y^2 - 10x + 2y + 21 = 0$.

Answer:



(click to see video)

KEY TAKEAWAYS

- The graph of a circle is completely determined by its center and radius.
- Standard form for the equation of a circle is $(x-h)^2+(y-k)^2=r^2$. The center is (h, k) and the radius measures *r* units.
- To graph a circle mark points *r* units up, down, left, and right from the center. Draw a circle through these four points.
- If the equation of a circle is given in general form $x^2+y^2+cx+dy+e=0$, group the terms with the same variables, and complete the square for both groupings. This will result in standard form, from which we can read the circle's center and radius.
- We recognize the equation of a circle if it is quadratic in both *x* and *y* where the coefficient of the squared terms are the same.

TOPIC EXERCISES

PART A: THE CIRCLE IN STANDARD FORM

Determine the center and radius given the equation of a circle in standard form.

1. $(x-5)^2+(y+4)^2=64$
2. $(x+9)^2+(y-7)^2=121$
3. $x^2+(y+6)^2=4$
4. $(x-1)^2+y^2=1$
5. $(x+1)^2+(y+1)^2=7$
6. $(x+2)^2+(y-7)^2=8$

Determine the standard form for the equation of the circle given its center and radius.

7. Center $(5, 7)$ with radius $r=7$.
8. Center $(-2, 8)$ with radius $r=5$.
9. Center $(6, -11)$ with radius $r=\sqrt{2}$.
10. Center $(-4, -5)$ with radius $r=\sqrt{6}$.
11. Center $(0, -1)$ with radius $r=2\sqrt{5}$.
12. Center $(0, 0)$ with radius $r=3\sqrt{10}$.

Graph.

13. $(x-1)^2+(y-2)^2=9$
14. $(x+3)^2+(y-3)^2=25$

15. $(x-2)^2+(y+6)^2=4$

16. $(x+6)^2+(y+4)^2=36$

17. $x^2+(y-4)^2=1$

18. $(x-3)^2+y^2=4$

19. $x^2+y^2=12$

20. $x^2+y^2=8$

21. $(x-7)^2+(y-6)^2=2$

22. $(x+2)^2+(y-5)^2=5$

23. $(x+3)^2+(y-1)^2=18$

24. $(x-3)^2+(y-2)^2=15$

Find the *x*- and *y*-intercepts.

25. $(x-1)^2+(y-2)^2=9$

26. $(x+5)^2+(y-3)^2=25$

27. $x^2+(y-4)^2=1$

28. $(x-3)^2+y^2=18$

29. $x^2+y^2=50$

30. $x^2+(y+9)^2=20$

31. $(x-4)^2+(y+5)^2=10$

32. $(x+10)^2+(y-20)^2=400$

Find the equation of the circle.

33. Circle with center $(1,-2)$ passing through $(3,-4)$.

34. Circle with center $(-4, -1)$ passing through $(0, -3)$.
35. Circle whose diameter is defined by $(5, 1)$ and $(-1, 7)$.
36. Circle whose diameter is defined by $(-5, 7)$ and $(-1, -5)$.
37. Circle with center $(5, -2)$ and area 9π square units.
38. Circle with center $(-8, -3)$ and circumference 12π square units.
39. Find the area of the circle with equation $(x + 12)^2 + (x - 5)^2 = 7$.
40. Find the circumference of the circle with equation $(x + 1)^2 + (y + 5)^2 = 8$.

PART B: THE CIRCLE IN GENERAL FORM

Rewrite in standard form and graph.

41. $x^2 + y^2 + 4x - 2y - 4 = 0$
42. $x^2 + y^2 - 10x + 2y + 10 = 0$
43. $x^2 + y^2 + 2x + 12y + 36 = 0$
44. $x^2 + y^2 - 14x - 8y + 40 = 0$
45. $x^2 + y^2 + 6y + 5 = 0$
46. $x^2 + y^2 - 12x + 20 = 0$
47. $x^2 + y^2 + 8x + 12y + 16 = 0$
48. $x^2 + y^2 - 20x - 18y + 172 = 0$
49. $4x^2 + 4y^2 - 4x + 8y + 1 = 0$
50. $9x^2 + 9y^2 + 18x + 6y + 1 = 0$
51. $x^2 + y^2 + 4x + 8y + 14 = 0$
52. $x^2 + y^2 - 2x - 4y - 15 = 0$
53. $x^2 + y^2 - x - 2y + 1 = 0$

54. $x^2 + y^2 - x + y - \frac{1}{2} = 0$

55. $4x^2 + 4y^2 + 8x - 12y + 5 = 0$

56. $9x^2 + 9y^2 + 12x - 36y + 4 = 0$

57. $2x^2 + 2y^2 + 6x + 10y + 9 = 0$

58. $9x^2 + 9y^2 - 6x + 12y + 4 = 0$

Given a circle in general form, determine the intercepts.

59. $x^2 + y^2 - 5x + 3y + 6 = 0$

60. $x^2 + y^2 + x - 2y - 7 = 0$

61. $x^2 + y^2 - 6y + 2 = 2$

62. $x^2 + y^2 - 6x - 8y + 5 = 0$

63. $2x^2 + 2y^2 - 3x - 9 = 0$

64. $3x^2 + 3y^2 + 8y - 16 = 0$

65. Determine the area of the circle whose equation is $x^2 + y^2 - 2x - 6y - 35 = 0.$

66. Determine the area of the circle whose equation is $4x^2 + 4y^2 - 12x - 8y - 59 = 0.$

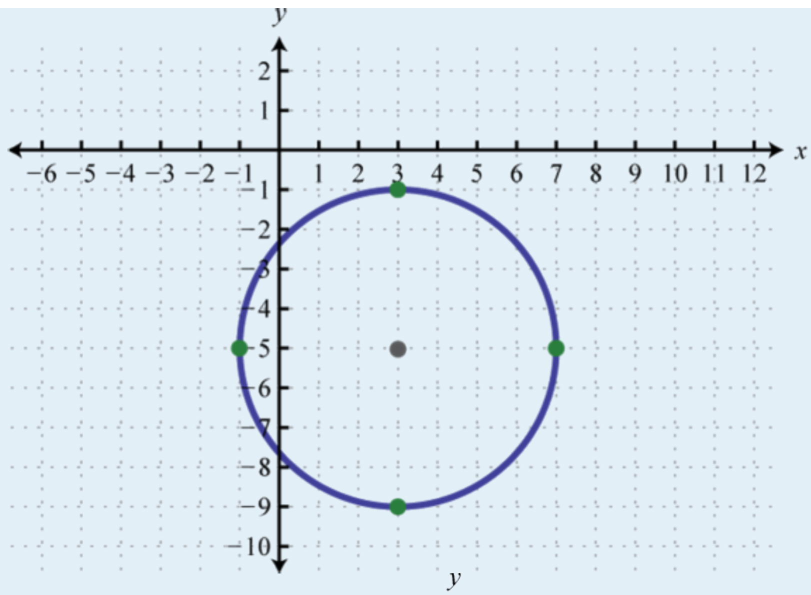
67. Determine the circumference of a circle whose equation is $x^2 + y^2 - 5x + 1 = 0.$

68. Determine the circumference of a circle whose equation is $x^2 + y^2 + 5x - 2y + 3 = 0.$

69. Find general form of the equation of a circle centered at $\left(-3, 5\right)$ passing through $(1, -2)$.

70. Find general form of the equation of a circle centered at $(-2, -3)$ passing through $(-1, 3)$.

Given the graph of a circle, determine its equation in general form.

71. 
y
x
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6 7 8 9 10 11 12
2
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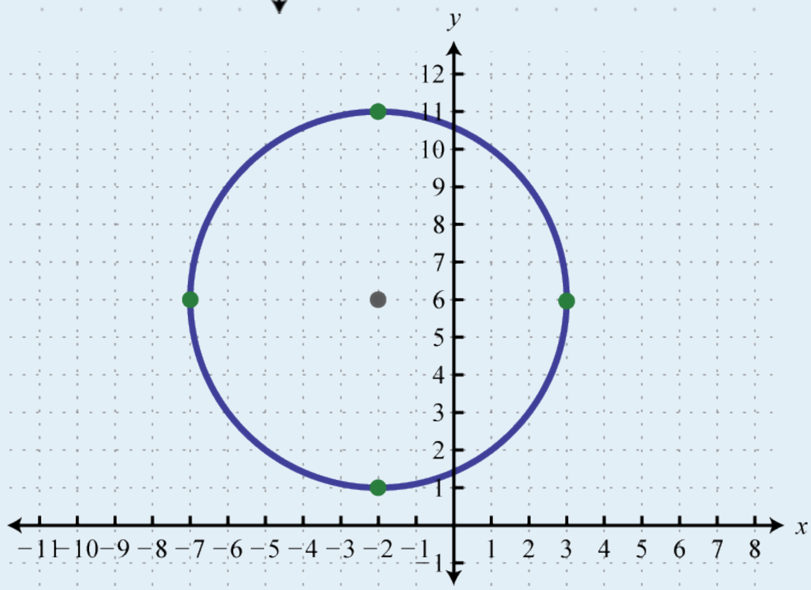
−9
−10

72. 
y
x
−11 −10 −9 −8 −7 −6 −5 −4 −3 −2 −1 1 2 3 4 5 6 7 8
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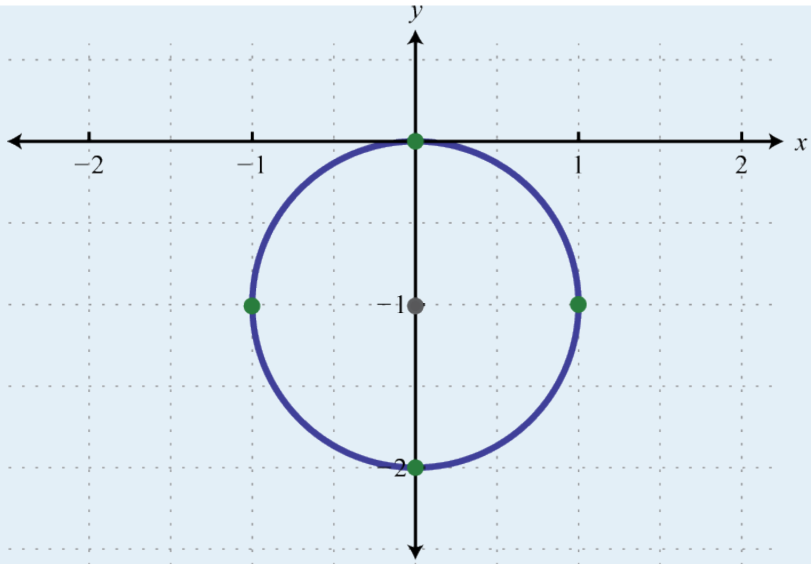
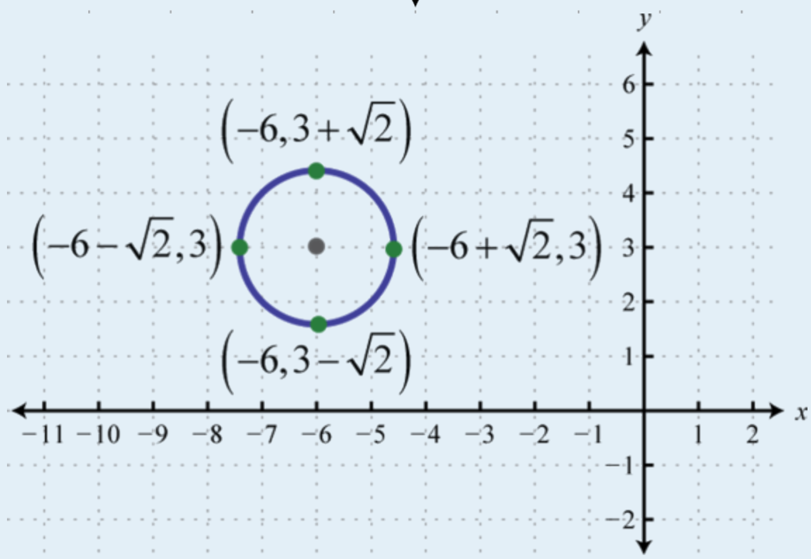
73. 

74. 

PART C: DISCUSSION BOARD

75. Is the center of a circle part of the graph? Explain.

76. Make up your own circle, write it in general form, and graph it.

77. Explain how we can tell the difference between the equation of a parabola in general form and the equation of a circle in general form. Give an example.

78. Do all circles have intercepts? What are the possible numbers of intercepts? Illustrate your explanation with graphs.

ANSWERS

1. Center: $(5, -4)$; radius: $r = 8$

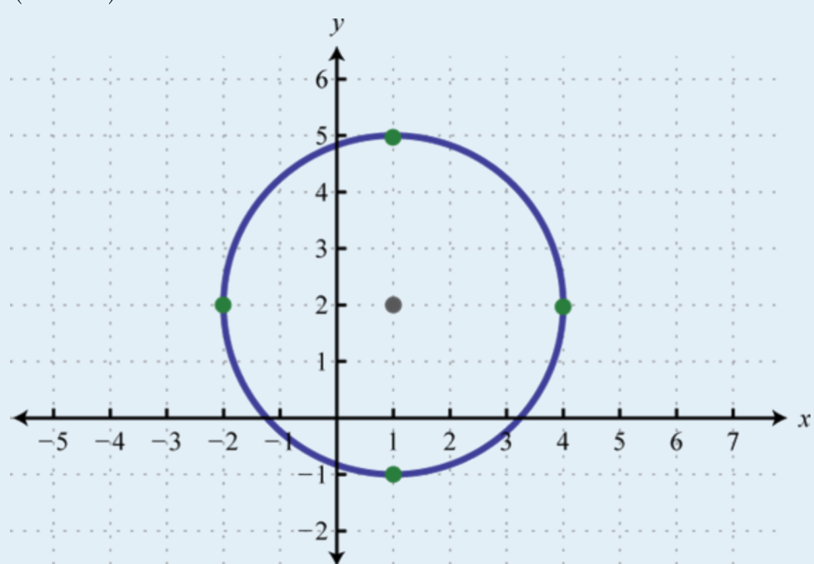
3. Center: $(0, -6)$; radius: $r = 2$

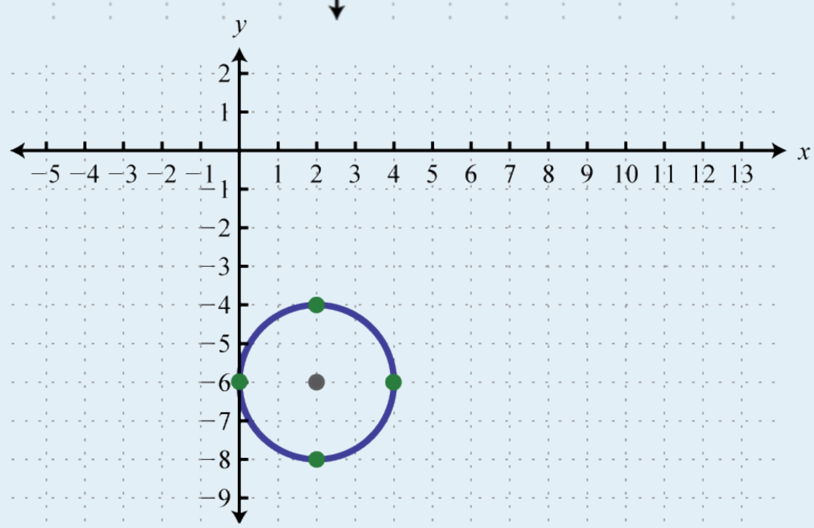
5. Center: $(-1, -1)$; radius: $r = \sqrt{7}$

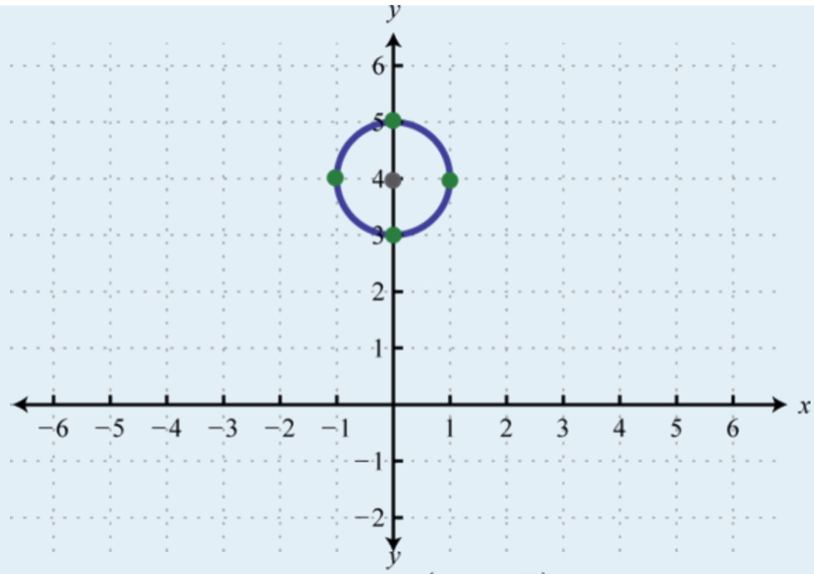
7. $(x - 5)^2 + (y - 7)^2 = 49$

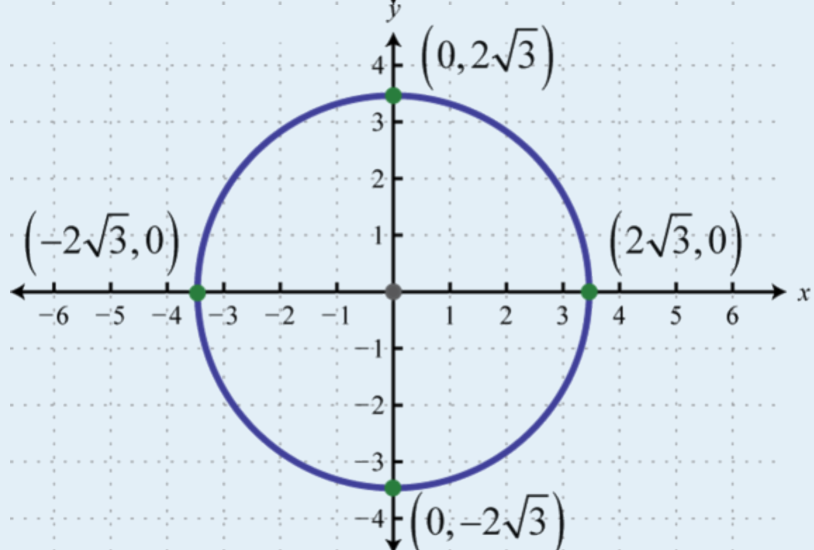
9. $(x - 6)^2 + (y + 11)^2 = 2$

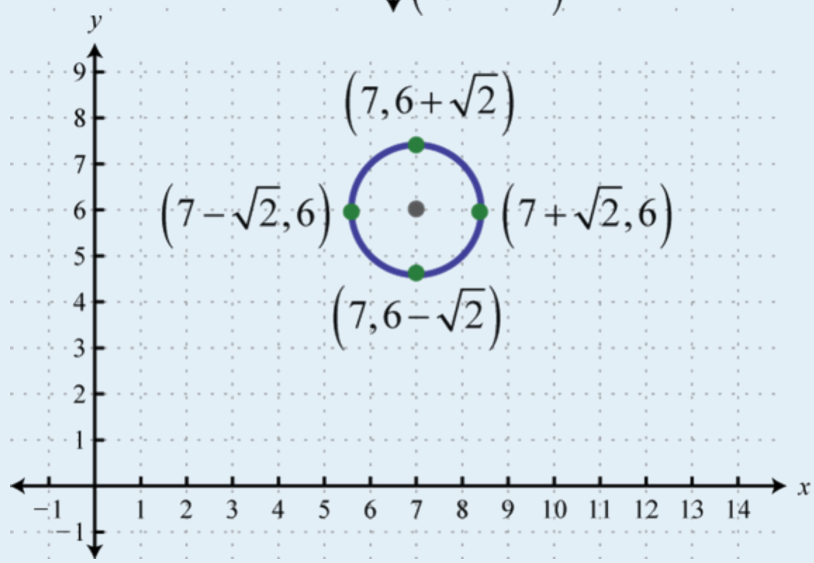
11. $x^2 + (y + 1)^2 = 20$

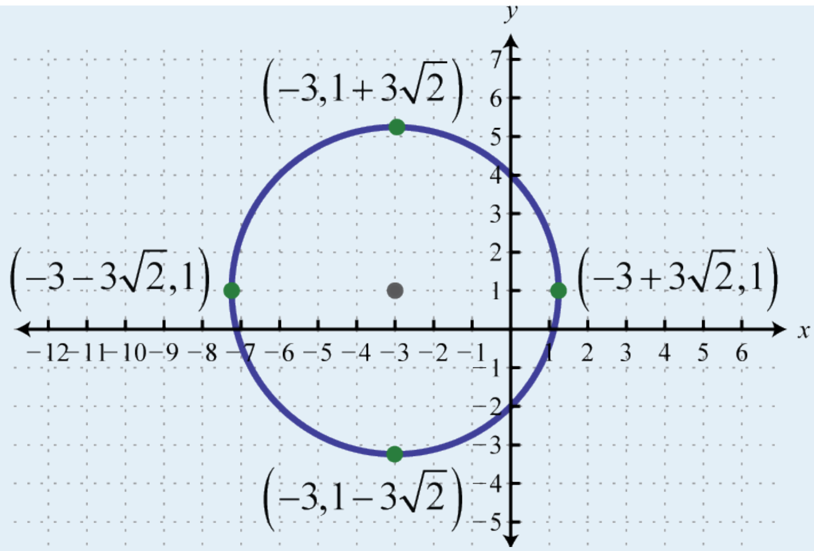
13. 

15. 

17. 
y
x

19. 
$(0, 2\sqrt{3})$
$(-2\sqrt{3}, 0)$
$(2\sqrt{3}, 0)$
$(0, -2\sqrt{3})$

21. 
$(7, 6+\sqrt{2})$
$(7-\sqrt{2}, 6)$
$(7+\sqrt{2}, 6)$
$(7, 6-\sqrt{2})$

23. 

25. x-intercepts: $\left(1 \pm \sqrt{5}, 0\right)$; y-intercepts: $\left(0, 2 \pm 2\sqrt{2}\right)$

27. x-intercepts: none; y-intercepts: $(0, 3)$, $(0, 5)$

29. x-intercepts: $\left(\pm 5\sqrt{2}, 0\right)$; y-intercepts: $\left(0, \pm 5\sqrt{2}\right)$

31. x-intercepts: none; y-intercepts: none

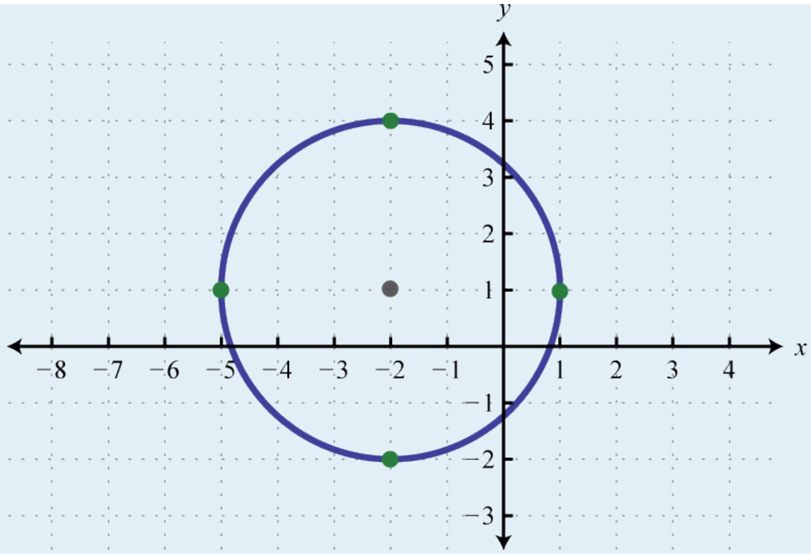
33. $(x-1)^2 + (y+2)^2 = 8$

35. $(x-2)^2 + (y-4)^2 = 18$

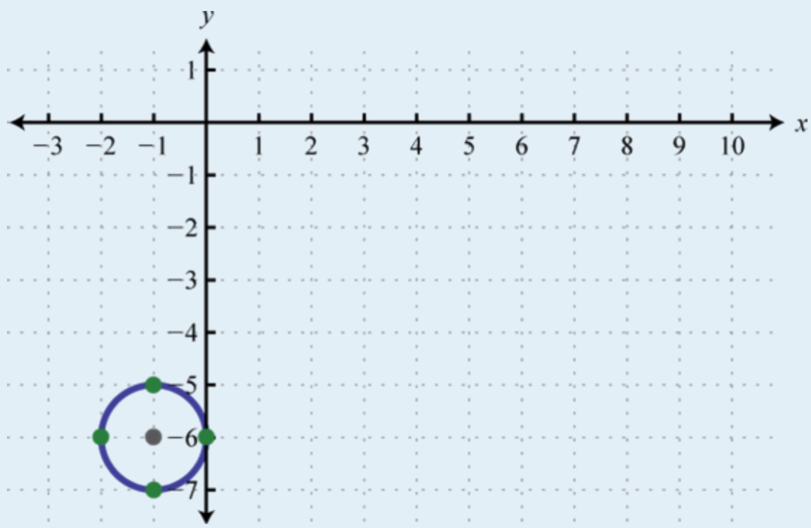
37. $(x-5)^2 + (y+2)^2 = 9$

39. 7π square units

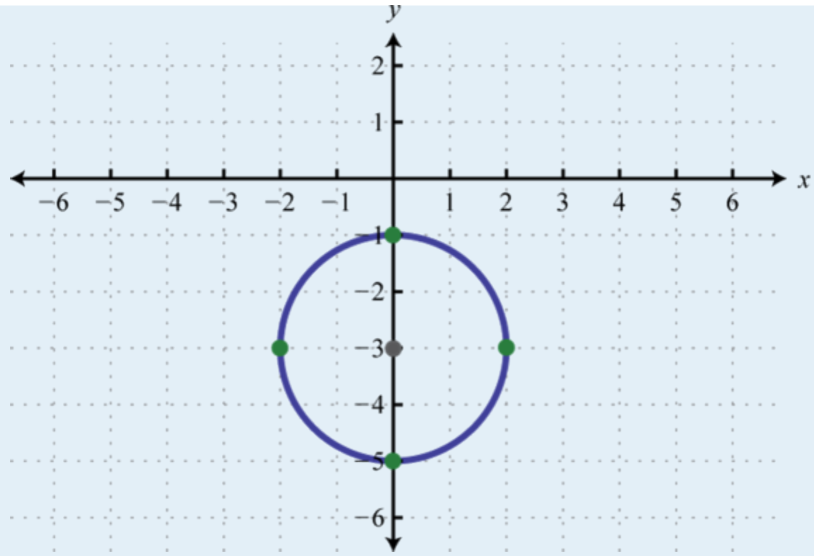
41. $(x+2)^2 + (y-1)^2 = 9$;



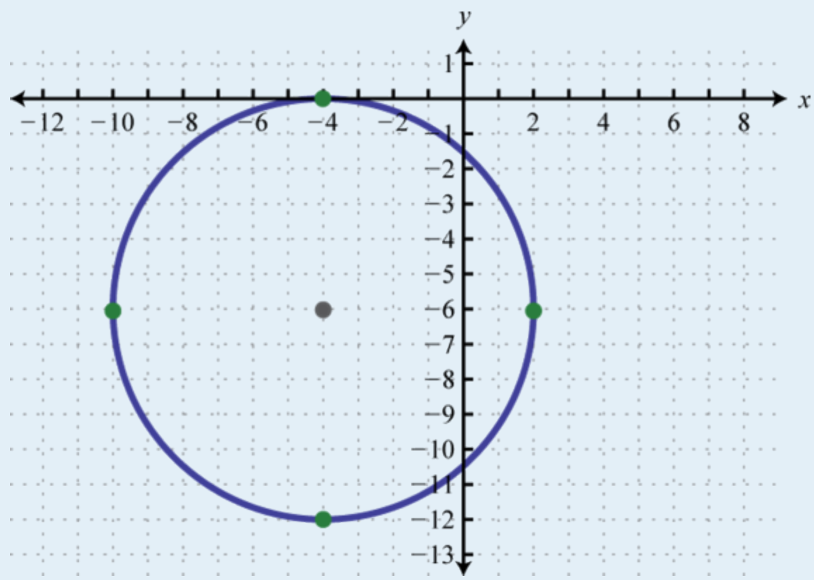
43. $(x+1)^2 + \left(y+6\right)^2 = 1$;



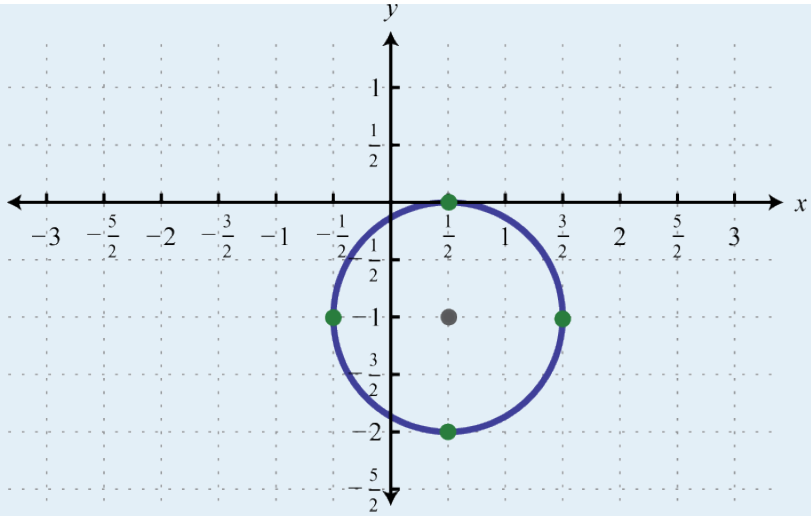
45. $x^2 + \left(y+3\right)^2 = 4$;



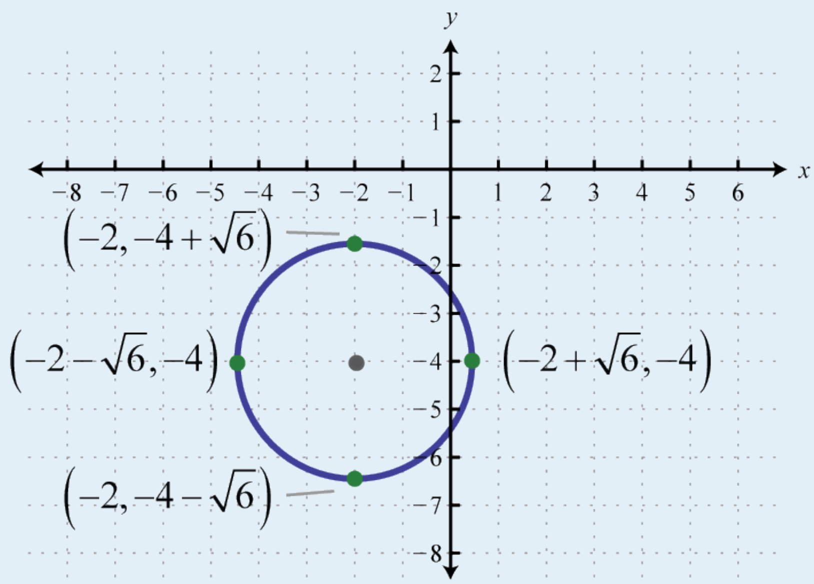
47. $(x+4)^2+(y+6)^2=36$;



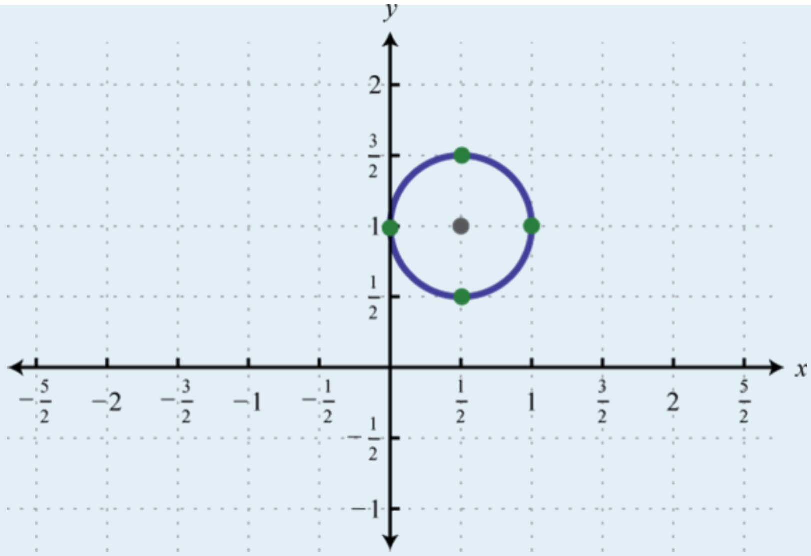
49. $\left(x-\frac{1}{2}\right)^2+(y+1)^2=1$;



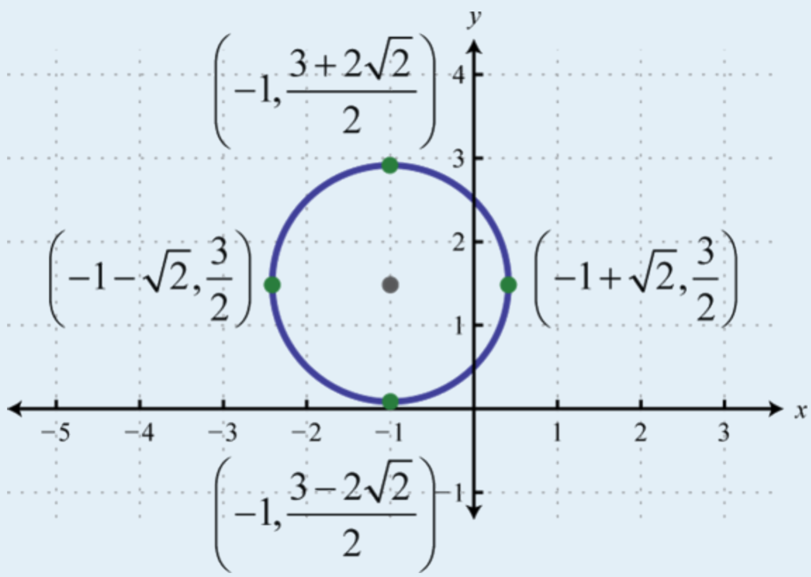
51. $(x+2)^2 + (y-4)^2 = 6$;



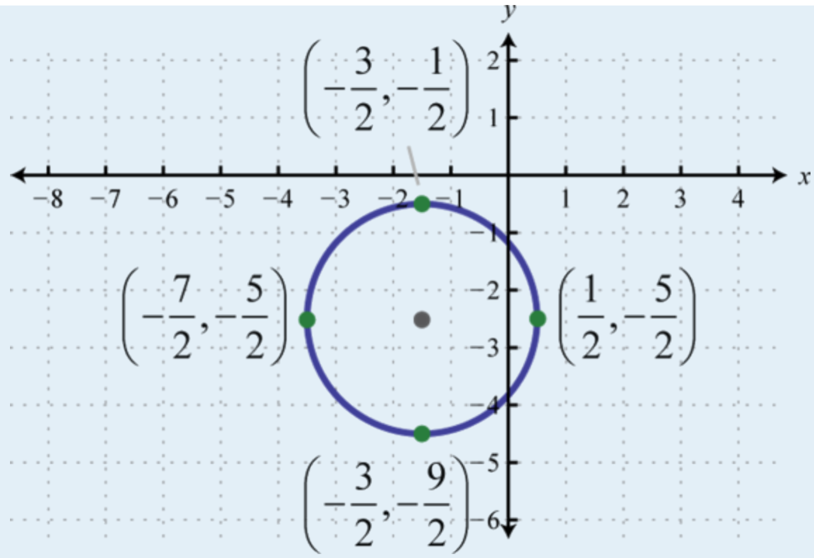
53. $\left(x - \frac{1}{2}\right)^2 + (y-1)^2 = \frac{1}{4}$;



55. $(x+1)^2+\left(y-\frac{3}{2}\right)^2=2$;



57. $\left(x+\frac{3}{2}\right)^2+\left(y+\frac{5}{2}\right)^2=4$;



59. x-intercepts: $(2, 0), (3, 0)$; y-intercepts: none

61. x-intercepts: $(0, 0)$; y-intercepts: $(0, 0), (0, 6)$

63. x-intercepts: $\left(-\frac{3}{2}, 0\right), (3, 0)$; y-intercepts: $\left(0, \pm\frac{3\sqrt{2}}{2}\right)$

65. 45π square units

67. $\pi\sqrt{21}$ units

69. $x^2 + y^2 + 6x - 10y - 31 = 0$

71. $x^2 + y^2 - 6x + 10y + 18 = 0$

73. $x^2 + y^2 + 2y = 0$

75. Answer may vary

77. Answer may vary

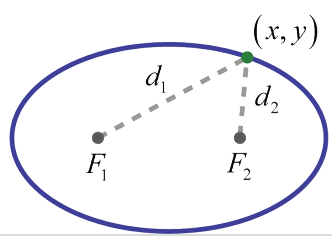
8.3 Ellipses

LEARNING OBJECTIVES

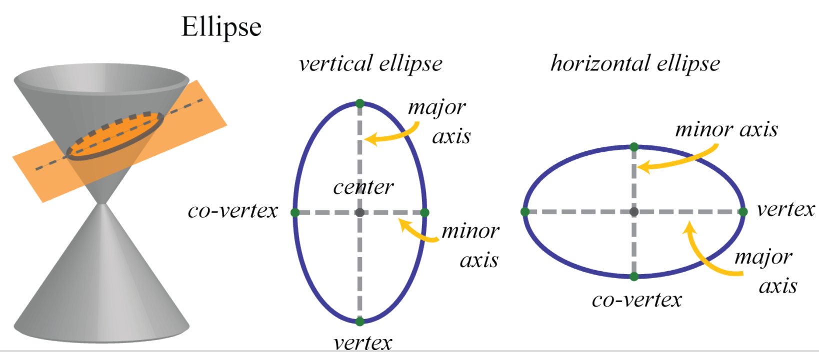
1. Graph an ellipse in standard form.
2. Determine the equation of an ellipse given its graph.
3. Rewrite the equation of an ellipse in standard form.

The Ellipse in Standard Form

An **ellipse**[14] is the set of points in a plane whose distances from two fixed points, called foci, have a sum that is equal to a positive constant. In other words, if points F_1 and F_2 are the foci (plural of focus) and d is some given positive constant then (x, y) is a point on the ellipse if $d = d_1 + d_2$ as pictured below:



In addition, an ellipse can be formed by the intersection of a cone with an oblique plane that is not parallel to the side of the cone and does not intersect the base of the cone. Points on this oval shape where the distance between them is at a maximum are called **vertices**[15] and define the **major axis**[16]. The center of an ellipse is the midpoint between the vertices. The **minor axis**[17] is the line segment through the center of an ellipse defined by two points on the ellipse where the distance between them is at a minimum. The endpoints of the minor axis are called **co-vertices**[18].



14. The set of points in a plane whose distances from two fixed points have a sum that is equal to a positive constant.

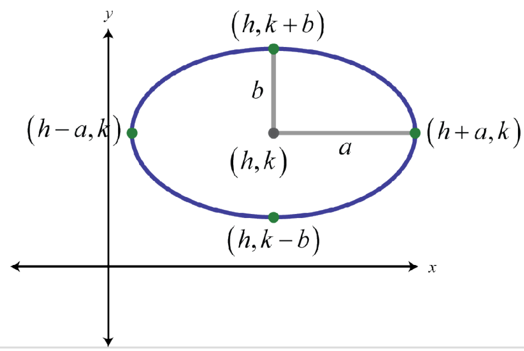
15. Points on the ellipse that mark the endpoints of the major axis.

16. The line segment through the center of an ellipse defined by two points on the ellipse where the distance between them is a maximum.

17. The line segment through the center of an ellipse defined by two points on the ellipse where the distance between them is a minimum.

18. Points on the ellipse that mark the endpoints of the minor axis.

If the major axis of an ellipse is parallel to the x-axis in a rectangular coordinate plane, we say that the ellipse is horizontal. If the major axis is parallel to the y-axis, we say that the ellipse is vertical. In this section, we are only concerned with sketching these two types of ellipses. However, the ellipse has many real-world applications and further research on this rich subject is encouraged. In a rectangular coordinate plane, where the center of a horizontal ellipse is (h, k), we have



As pictured $a > b$ where a, one-half of the length of the major axis, is called the **major radius**[19]. And b, one-half of the length of the minor axis, is called the **minor radius**[20]. The equation of an **ellipse in standard form**[21] follows:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

The vertices are $(h \pm a, k)$ and $(h, k \pm b)$ and the orientation depends on a and b. If $a > b$, then the ellipse is horizontal as shown above and if $a < b$, then the ellipse is vertical and b becomes the major radius. What do you think happens when $a = b$?

19. One-half of the length of the major axis.

20. One-half of the length of the minor axis.

21. The equation of an ellipse written in the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. The center is (h, k) and the larger of a and b is the major radius and the smaller is the minor radius.

Equation	Center	a	b	Orientation
$\frac{(x-1)^2}{4}+\frac{(y-8)^2}{9}=1$	$(1,8)$	$a=2$	$b=3$	Vertical
$\frac{(x-3)^2}{2}+\frac{(y+5)^2}{16}=1$	$(3,-5)$	$a=\sqrt{2}$	$b=4$	Vertical
$\frac{(x+1)^2}{1}+\frac{(y-7)^2}{8}=1$	$(-1,7)$	$a=1$	$b=2\sqrt{2}$	Vertical
$\frac{x^2}{25}+\frac{(y+6)^2}{10}=1$	$(0,-6)$	$a=5$	$b=\sqrt{10}$	Horizontal

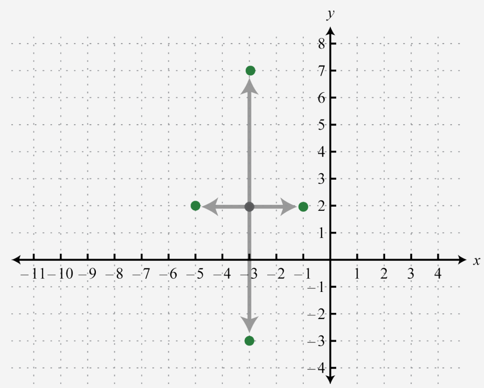
The graph of an ellipse is completely determined by its center, orientation, major radius, and minor radius, all of which can be determined from its equation written in standard from.

Example 1

Graph: $\frac{(x+3)^2}{4} + \frac{(y-2)^2}{25} = 1$.

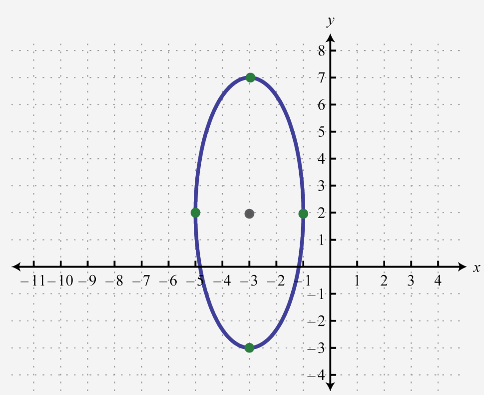
Solution:

Written in this form we can see that the center of the ellipse is $(-3, 2)$, $a = \sqrt{4} = 2$, and $b = \sqrt{25} = 5$. From the center mark points 2 units to the left and right and 5 units up and down.



Then draw an ellipse through these four points.

Answer:



As with any graph, we are interested in finding the *x*- and *y*-intercepts.

Example 2

Find the intercepts: $\frac{(x+3)^2}{4}+\frac{(y-2)^2}{25}=1$.

Solution:

To find the x-intercepts set $y=0$:

$$\begin{aligned}\frac{(x+3)^2}{4}+\frac{(0-2)^2}{25}&=1\\ \frac{(x+3)^2}{4}+\frac{4}{25}&=1\\ \frac{(x+3)^2}{4}&=1-\frac{4}{25}\\ \frac{(x+3)^2}{4}&=\frac{21}{25}\end{aligned}$$

At this point we extract the root by applying the square root property.

$$\begin{aligned}\frac{x+3}{2}&=\pm\sqrt{\frac{21}{25}}\\ x+3&=\pm\frac{2\sqrt{21}}{5}\\ x&=-3\pm\frac{2\sqrt{21}}{5}=\frac{-15\pm2\sqrt{21}}{5}\end{aligned}$$

Setting $x=0$ and solving for y leads to complex solutions, therefore, there are no y-intercepts. This is left as an exercise.

Answer: x-intercepts: $\left(\frac{-15 \pm 2\sqrt{21}}{5}, 0\right)$; y-intercepts: none.

Unlike a circle, standard form for an ellipse requires a 1 on one side of its equation.

Example 3

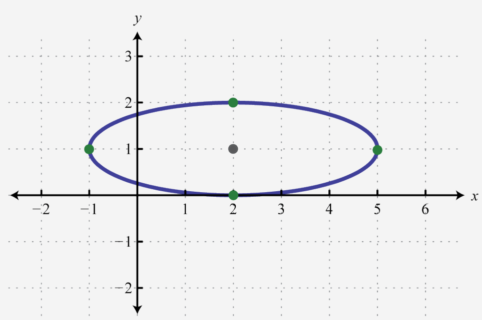
Graph and label the intercepts: $(x-2)^2 + 9(y-1)^2 = 9$.

Solution:

To obtain standard form, with 1 on the right side, divide both sides by 9.

$$\frac{(x-2)^2 + 9(y-1)^2}{9} = \frac{9}{9}$$
$$\frac{(x-2)^2}{9} + \frac{9(y-1)^2}{9} = \frac{9}{9}$$
$$\frac{(x-2)^2}{9} + \frac{(y-1)^2}{1} = 1$$

Therefore, the center of the ellipse is $(2, 1)$, $a = \sqrt{9} = 3$, and $b = \sqrt{1} = 1$. The graph follows:

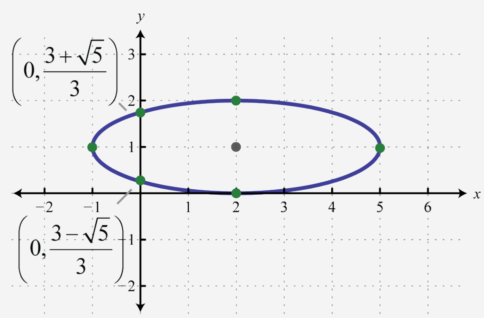


To find the intercepts we can use the standard form $\frac{(x-2)^2}{9} + (y-1)^2 = 1$:

x-intercepts set $y=0$	y-intercepts set $x=0$
$\frac{(x-2)^2}{9}+(0-1)^2=1$ $\frac{(x-2)^2}{9}+1=1$ $(x-2)^2=0$ $x-2=0$ $x=2$	$\frac{(0-2)^2}{9}+(y-1)^2=1$ $\frac{4}{9}+(y-1)^2=1$ $(y-1)^2=\frac{5}{9}$ $y-1=\pm\sqrt{\frac{5}{9}}$ $y=1\pm\frac{\sqrt{5}}{3}=\frac{3\pm\sqrt{5}}{3}$

Therefore the x-intercept is $(2, 0)$ and the y-intercepts are $\left(0, \frac{3+\sqrt{5}}{3}\right)$ and $\left(0, \frac{3-\sqrt{5}}{3}\right)$.

Answer:



Consider the ellipse centered at the origin,

$$x^2 + \frac{y^2}{4} = 1$$

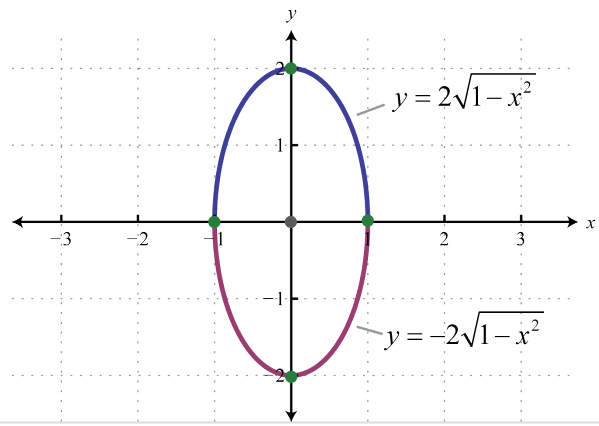
Given this equation we can write,

$$\frac{(x-0)^2}{1^2} + \frac{(y-0)^2}{2^2} = 1$$

In this form, it is clear that the center is $(0,0)$, $a = 1$, and $b = 2$. Furthermore, if we solve for y we obtain two functions:

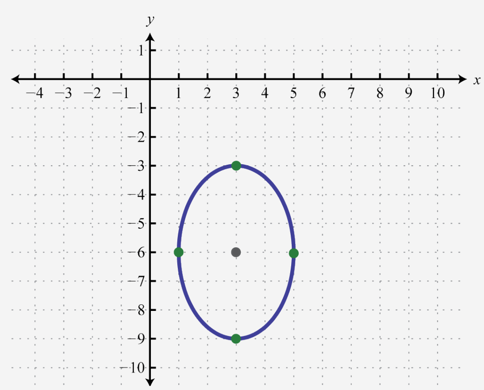
$$\begin{aligned} x^2 + \frac{y^2}{4} &= 1 \\ \frac{y^2}{4} &= 1 - x^2 \\ y^2 &= 4\left(1 - x^2\right) \\ y &= \pm\sqrt{4\left(1 - x^2\right)} \\ y &= \pm 2\sqrt{1 - x^2} \end{aligned}$$

The function defined by $y = 2\sqrt{1 - x^2}$ is the top half of the ellipse and the function defined by $y = -2\sqrt{1 - x^2}$ is the bottom half.



Try this! Graph: $9(x-3)^2 + 4(y+2)^2 = 36$.

Answer:



(click to see video)

The Ellipse in General Form

We have seen that the graph of an ellipse is completely determined by its center, orientation, major radius, and minor radius; which can be read from its equation in standard form. However, the equation is not always given in standard form. The equation of an **ellipse in general form**[22] follows,

$$px^2 + qy^2 + cx + dy + e = 0$$

22. The equation of an ellipse written in the form $px^2 + qy^2 + cx + dy + e = 0$ where $p, q > 0$.

where $p, q > 0$. The steps for graphing an ellipse given its equation in general form are outlined in the following example.

Example 4

Graph: $2x^2 + 9y^2 + 16x - 90y + 239 = 0$.

Solution:

Begin by rewriting the equation in standard form.

- **Step 1:** Group the terms with the same variables and move the constant to the right side. Factor so that the leading coefficient of each grouping is 1.

$$\begin{aligned} 2x^2 + 9y^2 + 16x - 90y + 239 &= 0 \\ \left(2x^2 + 16x + \_\_\_\right) + \left(9y^2 - 90y + \_\_\_\right) &= -239 \\ 2\left(x^2 + 8x + \_\_\_\right) + 9\left(y^2 - 10y + \_\_\_\right) &= -239 \end{aligned}$$

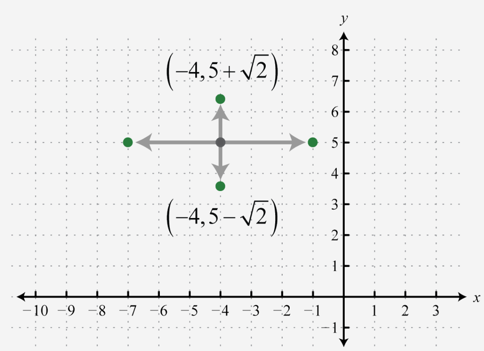
- **Step 2:** Complete the square for each grouping. In this case, for the terms involving x use $\left(\frac{8}{2}\right)^2 = 4^2 = 16$ and for the terms involving y use $\left(\frac{-10}{2}\right)^2 = (-5)^2 = 25$. The factor in front of the grouping affects the value used to balance the equation on the right side:

$$2\left(x^2 + 8x + 16\right) + 9\left(y^2 - 10y + 25\right) = -239 + 32 + 225$$

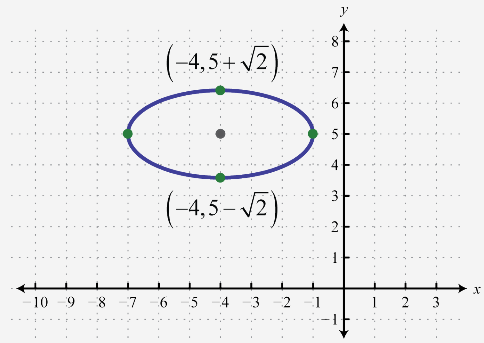
Because of the distributive property, adding 16 inside of the first grouping is equivalent to adding $2 \cdot 16 = 32$. Similarly, adding 25 inside of the second grouping is equivalent to adding $9 \cdot 25 = 225$. Now factor and then divide to obtain 1 on the right side.

$$2(x+4)^2+9(y-5)^2=18$$
$$\frac{2(x+4)^2+9(y-5)^2}{18}=\frac{18}{18}$$
$$\frac{2(x+4)^2}{18}+\frac{9(y-5)^2}{18}=\frac{18}{18}$$
$$\frac{(x+4)^2}{9}+\frac{(y-5)^2}{2}=1$$

- **Step 3:** Determine the center, a, and b. In this case, the center is $(-4, 5)$, $a=\sqrt{9}=3$, and $b=\sqrt{2}$.
- **Step 4:** Use a to mark the vertices left and right of the center, use b to mark the vertices up and down from the center, and then sketch the graph. In this case, the vertices along the minor axes $\left(-4, 5 \pm \sqrt{2}\right)$ are not apparent and should be labeled.



Answer:


$\left(-4, 5+\sqrt{2}\right)$
$\left(-4, 5-\sqrt{2}\right)$
y
x

Example 5

Determine the center of the ellipse as well as the lengths of the major and minor axes: $5x^2 + y^2 - 3x + 40 = 0$.

Solution:

In this example, we only need to complete the square for the terms involving x.

$$\begin{aligned} 5x^2 + y^2 - 30x + 40 &= 0 \\ \left(5x^2 - 30x + \_\_\_\right) + y^2 &= -40 \\ 5\left(x^2 - 6x + \_\_\_\right) + y^2 &= -40 \end{aligned}$$

Use $\left(\frac{-6}{2}\right)^2 = (-3)^2 = 9$ for the first grouping to be balanced by $5 \cdot 9 = 45$ on the right side.

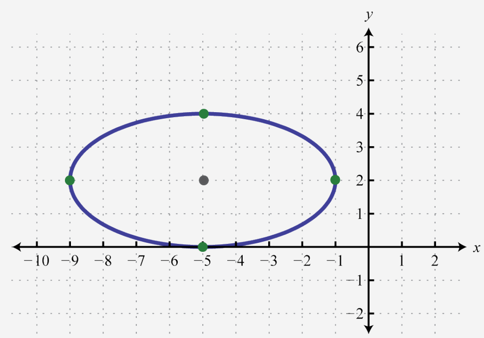
$$\begin{aligned} 5\left(x^2 - 6x \color{blue}{+ 9}\right) + y^2 &= -40 \color{blue}{+ 45} \\ 5(x-3)^2 + y^2 &= 5 \\ \frac{5(x-3)^2 + y^2}{\color{blue}{5}} &= \frac{5}{\color{blue}{5}} \\ \frac{(x-3)^2}{1} + \frac{y^2}{5} &= 1 \end{aligned}$$

Here, the center is $(3, 0)$, $a = \sqrt{1} = 1$, and $b = \sqrt{5}$. Because b is larger than a, the length of the major axis is $2b$ and the length of the minor axis is $2a$.

Answer: Center: $(3, 0)$; major axis: $2\sqrt{5}$ units; minor axis: 2 units.

Try this! Graph: $x^2 + 4y^2 + 10x - 16y + 25 = 0$.

Answer:



(click to see video)

KEY TAKEAWAYS

- The graph of an ellipse is completely determined by its center, orientation, major radius, and minor radius.
- The center, orientation, major radius, and minor radius are apparent if the equation of an ellipse is given in standard form: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.
- To graph an ellipse, mark points *a* units left and right from the center and points *b* units up and down from the center. Draw an ellipse through these points.
- The orientation of an ellipse is determined by *a* and *b*. If $a > b$ then the ellipse is wider than it is tall and is considered to be a horizontal ellipse. If $a < b$ then the ellipse is taller than it is wide and is considered to be a vertical ellipse.
- If the equation of an ellipse is given in general form $px^2 + qy^2 + cx + dy + e = 0$ where $p, q > 0$, group the terms with the same variables, and complete the square for both groupings.
- We recognize the equation of an ellipse if it is quadratic in both *x* and *y* and the coefficients of each square term have the same sign.

TOPIC EXERCISES

PART A: THE ELLIPSE IN STANDARD FORM

Given the equation of an ellipse in standard form, determine its center, orientation, major radius, and minor radius.

1. $\frac{(x-1)^2}{4} + \frac{(y+2)^2}{49} = 1$
2. $\frac{(x+3)^2}{64} + \frac{(y-2)^2}{9} = 1$
3. $\frac{x^2}{3} + (y+9)^2 = 1$
4. $\frac{(x-1)^2}{8} + y^2 = 1$
5. $4(x+5)^2 + 9(y+5)^2 = 36$
6. $16(x-1)^2 + 3(y+10)^2 = 48$

Determine the standard form for the equation of an ellipse given the following information.

7. Center $(3, 4)$ with $a = 5$ and $b = 2$.
8. Center $(-1, 9)$ with $a = 7$ and $b = 3$.
9. Center $(5, -1)$ with $a = \sqrt{6}$ and $b = 2\sqrt{3}$.
10. Center $(-7, -2)$ with $a = 5\sqrt{2}$ and $b = \sqrt{7}$.
11. Center $(0, -3)$ with $a = 1$ and $b = \sqrt{5}$.
12. Center $(0, 0)$ with $a = \sqrt{2}$ and $b = 4$.

Graph.

13. $\frac{(x-4)^2}{4} + \frac{(y+2)^2}{9} = 1$
14. $\frac{(x+1)^2}{25} + \frac{(y-2)^2}{4} = 1$

15. $\frac{(x-5)^2}{16} + \frac{(y+6)^2}{1} = 1$

16. $\frac{(x+4)^2}{4} + \frac{(y+3)^2}{36} = 1$

17. $\frac{(x-2)^2}{9} + \frac{(y-1)^2}{64} = 1$

18. $\frac{(x+1)^2}{49} + (y+3)^2 = 1$

19. $4(x+3)^2 + 9(y-3)^2 = 36$

20. $16x^2 + (y-1)^2 = 16$

21. $4(x-2)^2 + 25y^2 = 100$

22. $81x^2 + y^2 = 81$

23. $\frac{(x-2)^2}{8} + \frac{(y-4)^2}{9} = 1$

24. $\frac{(x+1)^2}{4} + \frac{(y-1)^2}{12} = 1$

25. $\frac{(x-6)^2}{2} + \frac{(y+2)^2}{5} = 1$

26. $\frac{(x+3)^2}{18} + \frac{(y-5)^2}{3} = 1$

27. $3x^2 + 2(y-3)^2 = 6$

28. $5(x+1)^2 + 3y^2 = 15$

29. $4x^2 + 6y^2 = 24$

30. $5x^2 + 10y^2 = 50$

Find the *x*- and *y*-intercepts.

31. $\frac{(x-3)^2}{4} + \frac{(y-2)^2}{9} = 1$

32. $\frac{(x+3)^2}{16} + \frac{(y-7)^2}{9} = 1$

33. $\frac{(x-2)^2}{4} + \frac{(y+6)^2}{36} = 1$

34. $\frac{(x+1)^2}{25} + \frac{(y-1)^2}{9} = 1$

35. $5x^2 + 2(y-4)^2 = 20$

36. $4(x-3)^2 + 9y^2 = 72$

37. $5x^2 + 2y^2 = 10$

38. $3x^2 + 4y^2 = 24$

Find the equation of the ellipse.

39. Ellipse with vertices $(\pm 5, 0)$ and $(0, \pm 6)$.

40. Ellipse whose major axis has vertices $(2, 9)$ and $(2, -1)$ and minor axis has vertices $(-2, 4)$ and $(6, 4)$.

41. Ellipse whose major axis has vertices $(-8, -2)$ and $(0, -2)$ and minor axis has a length of 4 units.

42. Ellipse whose major axis has vertices $(-2, 2)$ and $(-2, 8)$ and minor axis has a length of 2 units.

PART B: THE ELLIPSE IN GENERAL FORM

Rewrite in standard form and graph.

43. $4x^2 + 9y^2 + 8x - 36y + 4 = 0$

44. $9x^2 + 25y^2 - 18x + 100y - 116 = 0$

45. $4x^2 + 49y^2 + 24x + 98y - 111 = 0$

46. $9x^2 + 4y^2 - 72x + 24y + 144 = 0$

47. $x^2 + 64y^2 - 12x + 128y + 36 = 0$

48. $16x^2 + y^2 - 96x - 4y + 132 = 0$

49. $36x^2 + 4y^2 - 40y - 44 = 0$

50. $x^2 + 9y^2 - 2x - 8 = 0$

51. $x^2 + 9y^2 - 4x - 36y - 41 = 0$

52. $16x^2 + y^2 + 160x - 10y + 361 = 0$

53. $4x^2 + 5y^2 + 32x - 20y + 64 = 0$

54. $2x^2 + 3y^2 - 8x - 30y + 65 = 0$

55. $8x^2 + 5y^2 - 16x + 10y - 27 = 0$

56. $7x^2 + 2y^2 + 28x - 16y + 46 = 0$

57. $36x^2 + 16y^2 - 36x - 32y - 119 = 0$

58. $16x^2 + 100y^2 + 64x - 300y - 111 = 0$

59. $x^2 + 4y^2 - 20y + 21 = 0$

60. $9x^2 + y^2 + 12x - 2y - 4 = 0$

Given general form determine the intercepts.

61. $5x^2 + 4y^2 - 20x + 24y + 36 = 0$

62. $4x^2 + 3y^2 - 8x + 6y - 5 = 0$

63. $6x^2 + y^2 - 12x + 4y + 4 = 0$

64. $8x^2 + y^2 - 6y - 7 = 0$

65. $5x^2 + 2y^2 - 20x - 8y + 18 = 0$

66. $2x^2 + 3y^2 - 4x - 5y + 1 = 0$

Determine the area of the ellipse. (The area of an ellipse is given by the formula $A = \pi ab$, where *a* and *b* are the lengths of the major radius and the minor radius.)

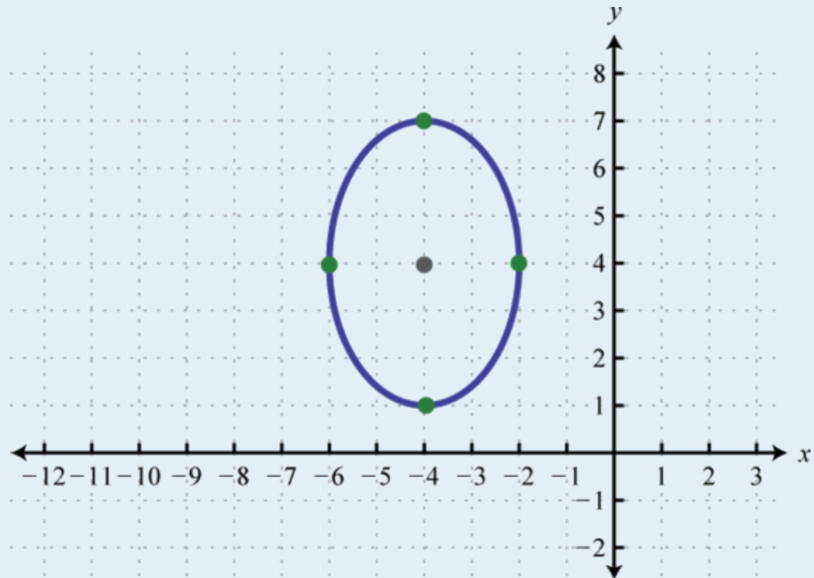
67. $\frac{(x-10)^2}{25} + \frac{(y+3)^2}{5} = 1$

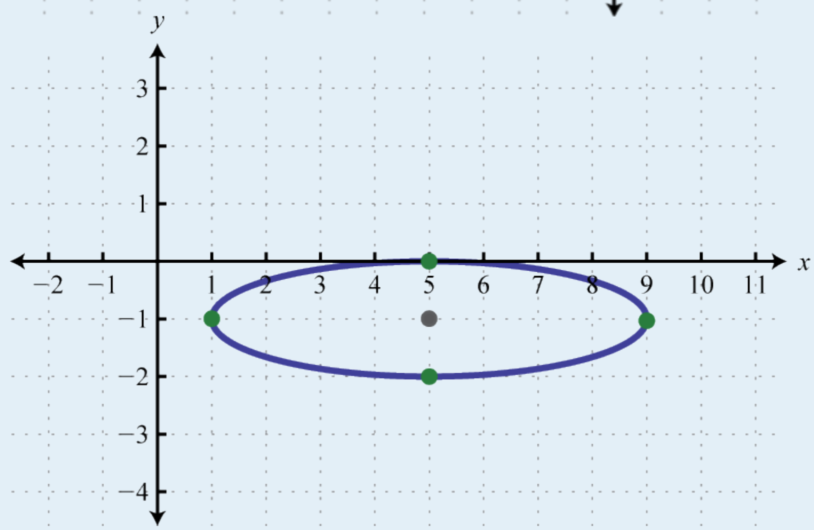
68. $\frac{(x+1)^2}{18} + \frac{y^2}{36} = 1$

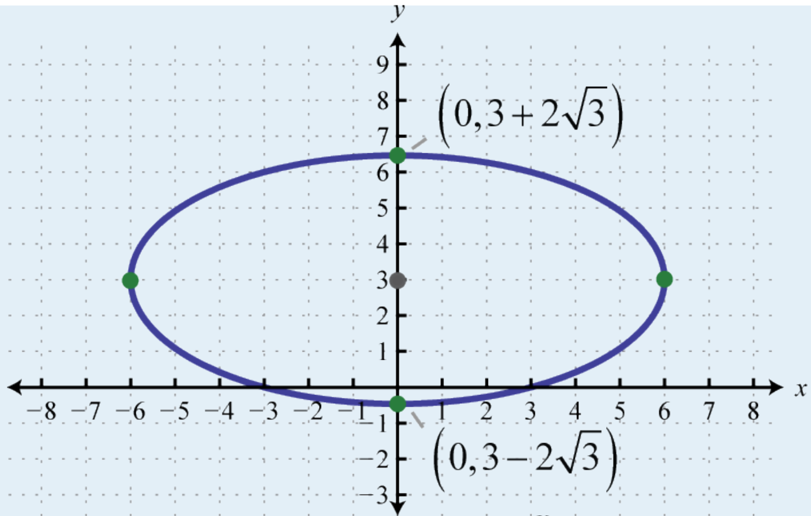
69. $7x^2 + 3y^2 - 14x + 36y + 94 = 0$

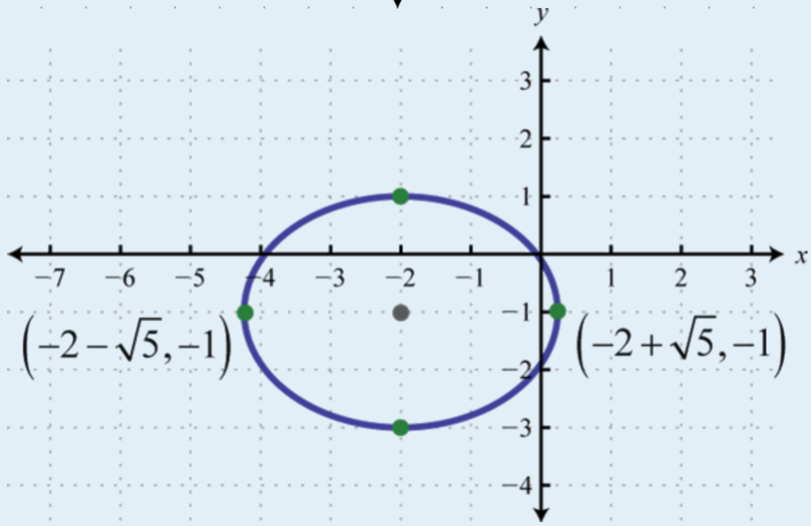
70. $4x^2 + 8y^2 + 20x - 8y + 11 = 0$

Given the graph of an ellipse, determine its equation in general form.

71. 

72. 

73. 

74. 

PART C: DISCUSSION BOARD

75. Explain why a circle can be thought of as a very special ellipse.

76. Make up your own equation of an ellipse, write it in general form and graph it.

77. Do all ellipses have intercepts? What are the possible numbers of intercepts for an ellipse? Explain.

78. Research and discuss real-world examples of ellipses.

ANSWERS

1. Center: $(1, -2)$; orientation: vertical; major radius: 7 units; minor radius: 2 units; $a = 2; b = 7$

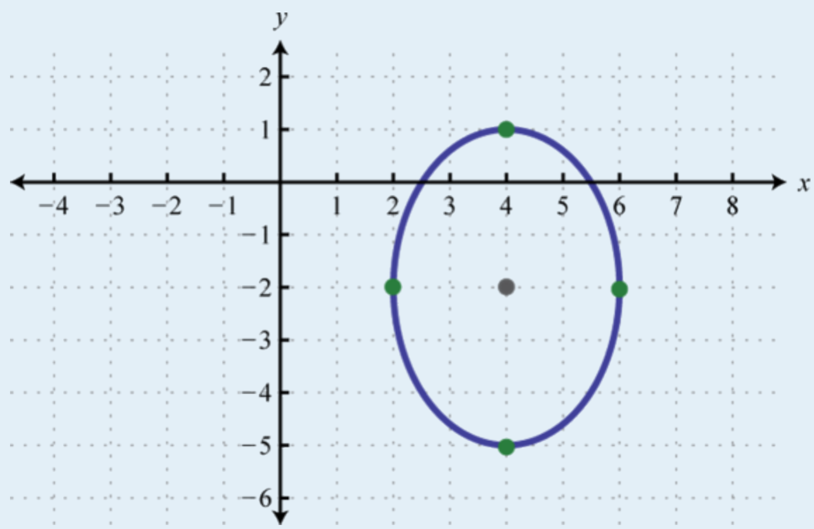
3. Center: $(0, -9)$; orientation: horizontal; major radius: $\sqrt{3}$ units; minor radius: 1 unit; $a = \sqrt{3}; b = 1$

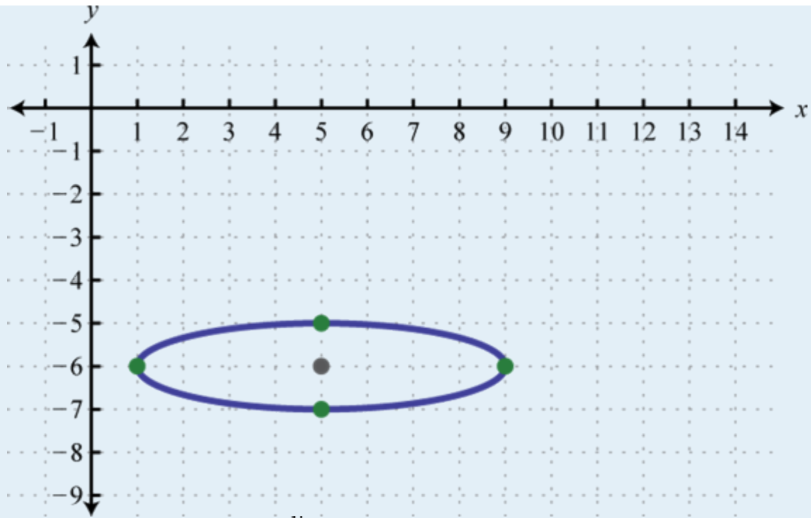
5. Center: $(-5, -5)$; orientation: horizontal; major radius: 3 units; minor radius: 2 units; $a = 3; b = 2$

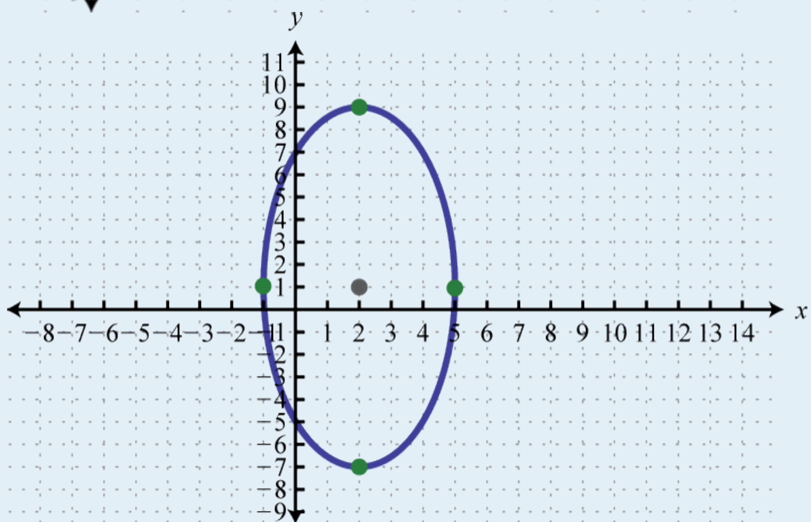
7. $\frac{(x-3)^2}{25} + \frac{(y-4)^2}{4} = 1$

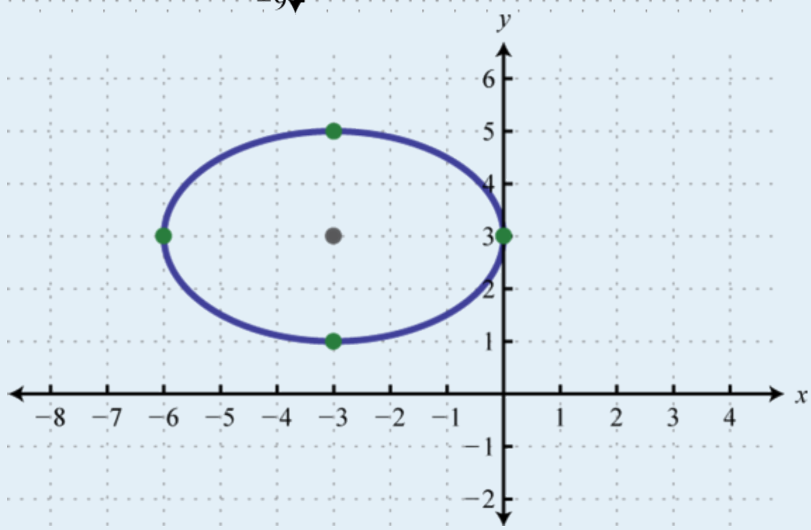
9. $\frac{(x-5)^2}{6} + \frac{(y+1)^2}{12} = 1$

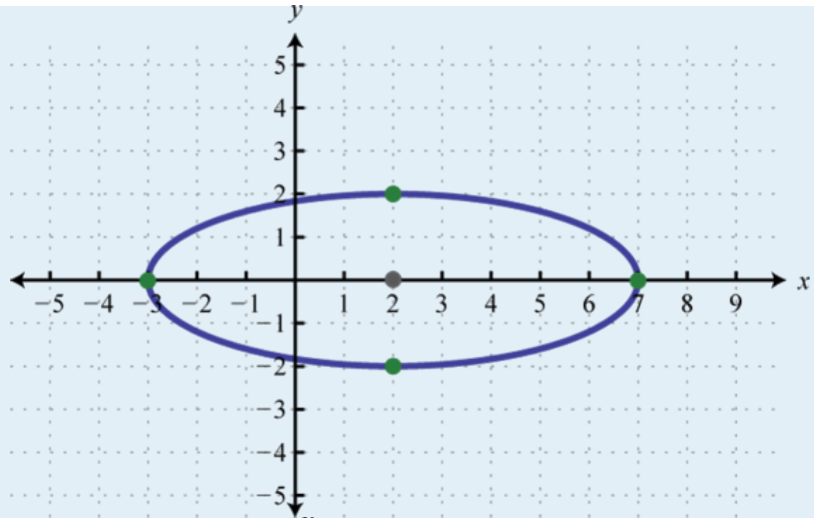
11. $x^2 + \frac{(y+3)^2}{5} = 1$

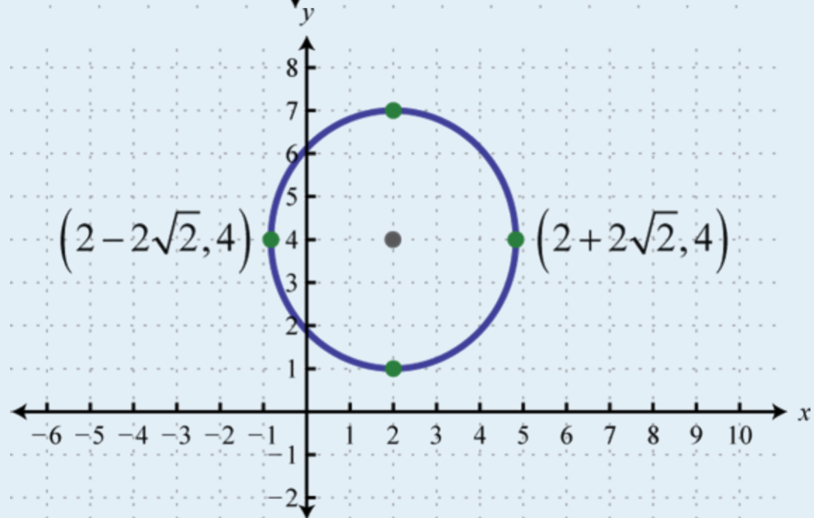
13. 

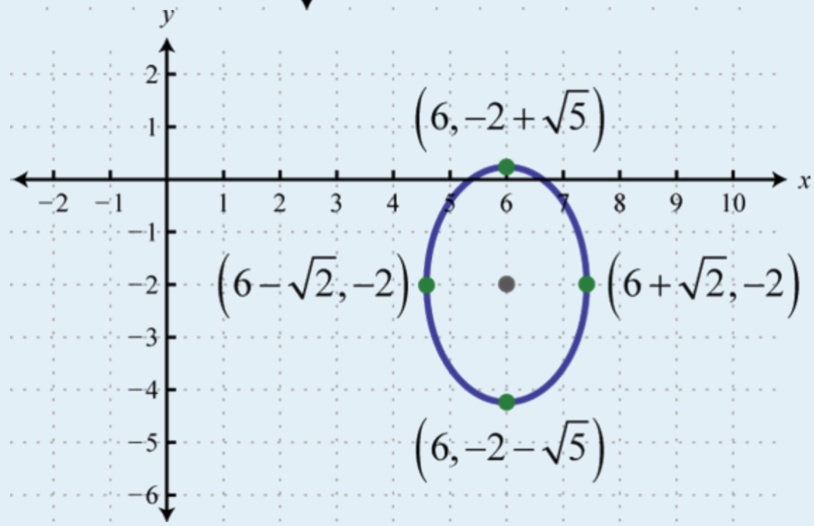
15. 

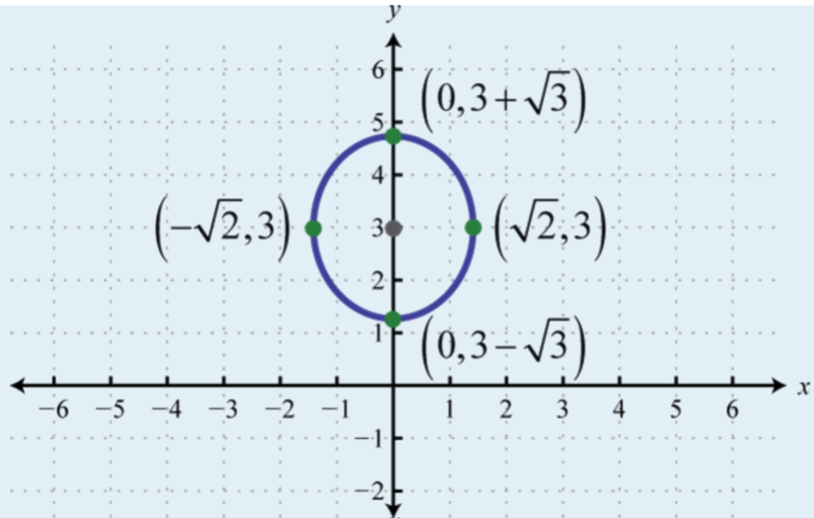
17. 

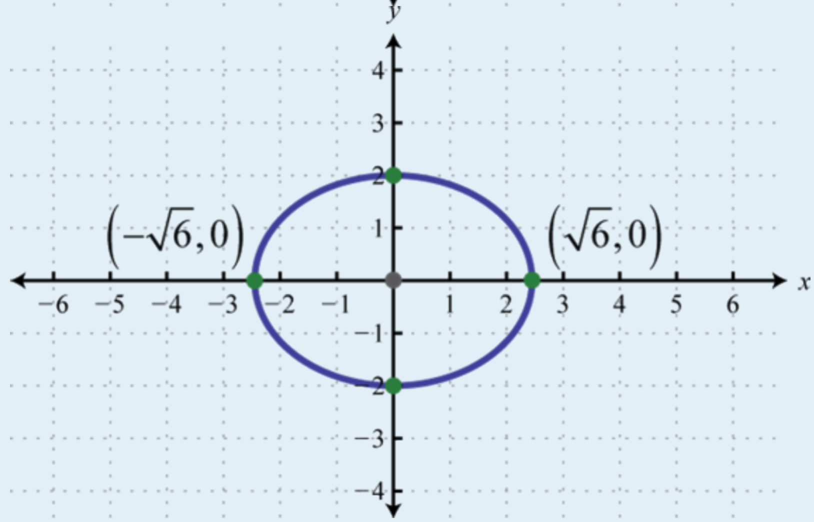
19. 

21. 

23. 

25. 

27. 

29. 

31. x-intercepts: $\left(\frac{9\pm2\sqrt{5}}{3}, 0\right)$; y-intercepts: none

33. x-intercepts: $(2, 0)$; y-intercepts: $(0, -6)$

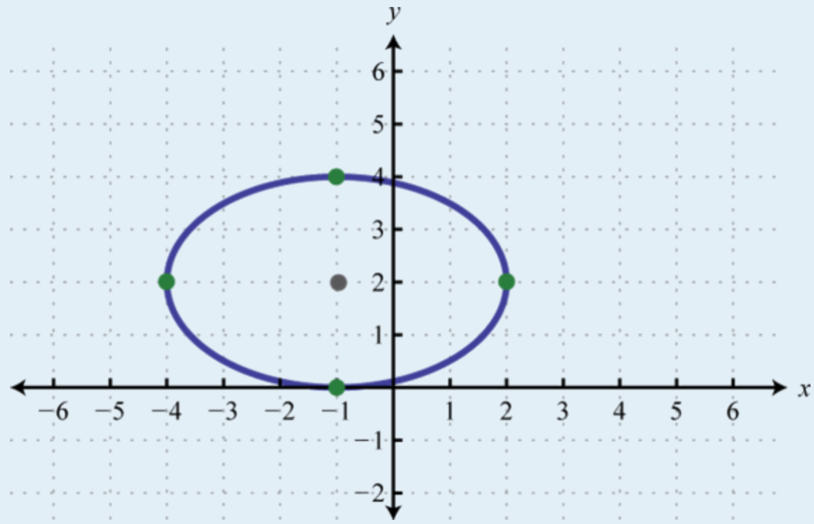
35. x-intercepts: none; y-intercepts: $\left(0, 4 \pm \sqrt{10}\right)$

37. x-intercepts: $\left(\pm\sqrt{2}, 0\right)$; y-intercepts: $\left(0, \pm\sqrt{5}\right)$

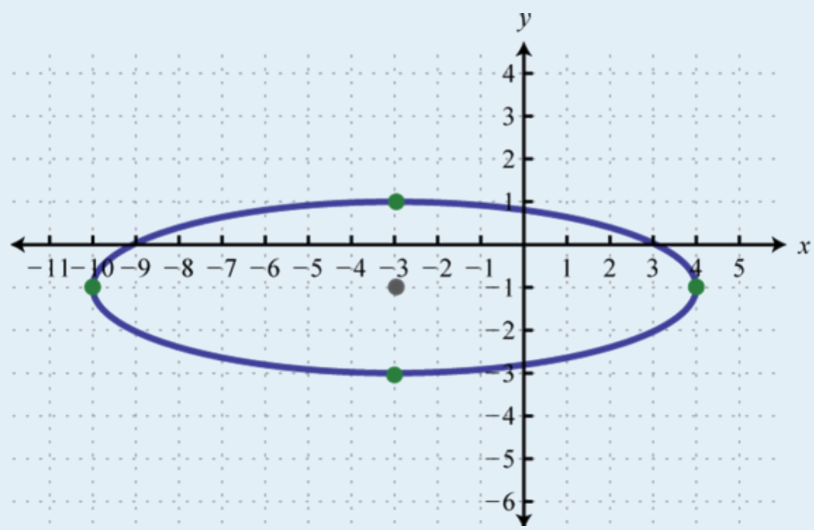
39. $\frac{x^2}{25} + \frac{y^2}{36} = 1$

41. $\frac{(x+4)^2}{16} + \frac{(y+2)^2}{4} = 1$

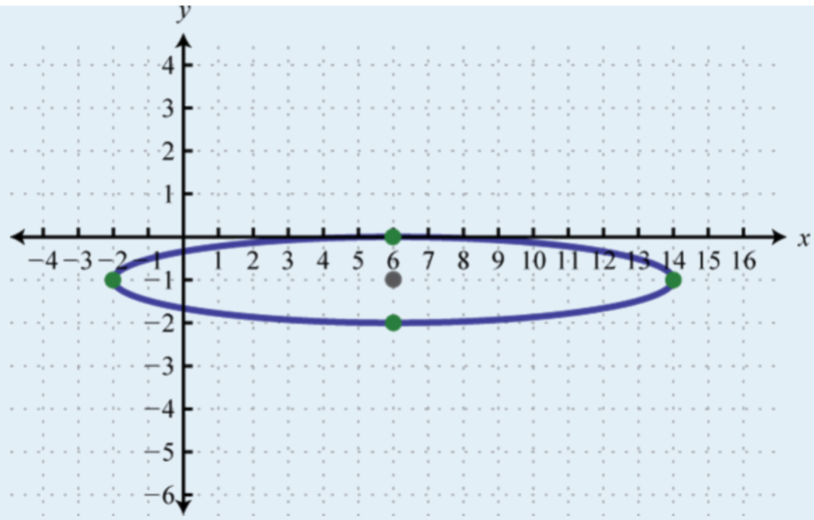
43. $\frac{(x+1)^2}{9} + \frac{(y-2)^2}{4} = 1$;



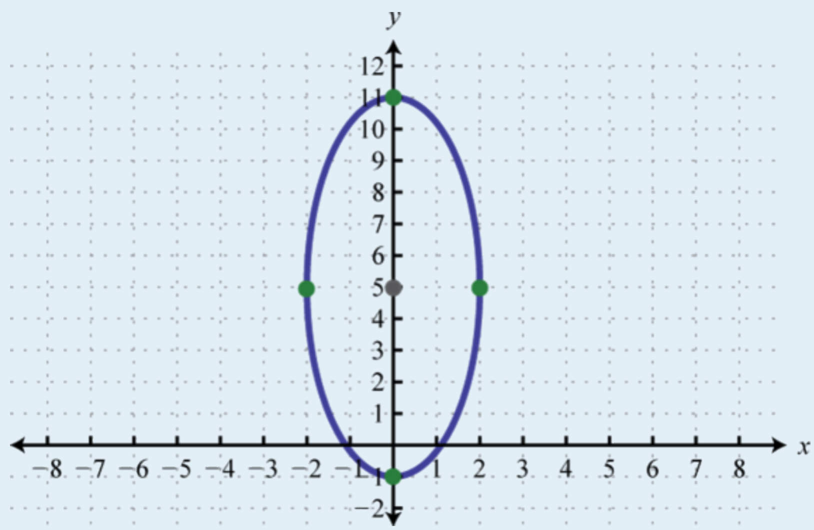
45. $\frac{(x+3)^2}{49} + \frac{(y+1)^2}{4} = 1$;



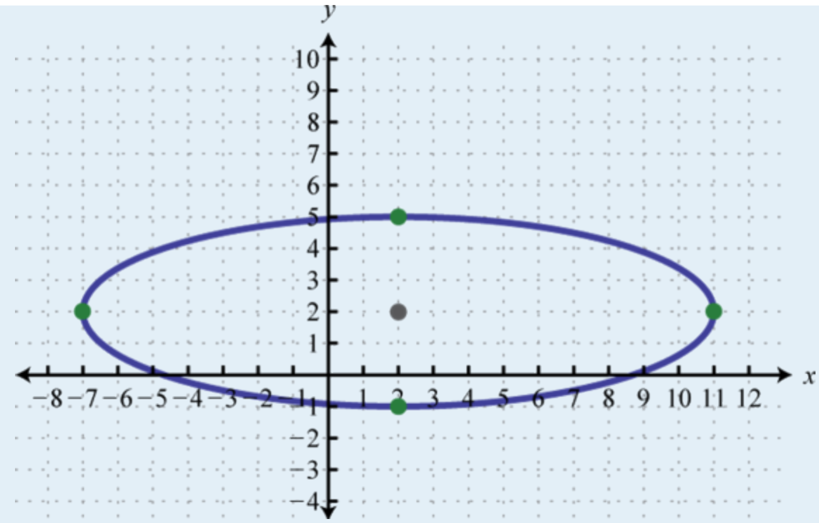
47. $\frac{(x-6)^2}{64} + (y+1)^2 = 1$;



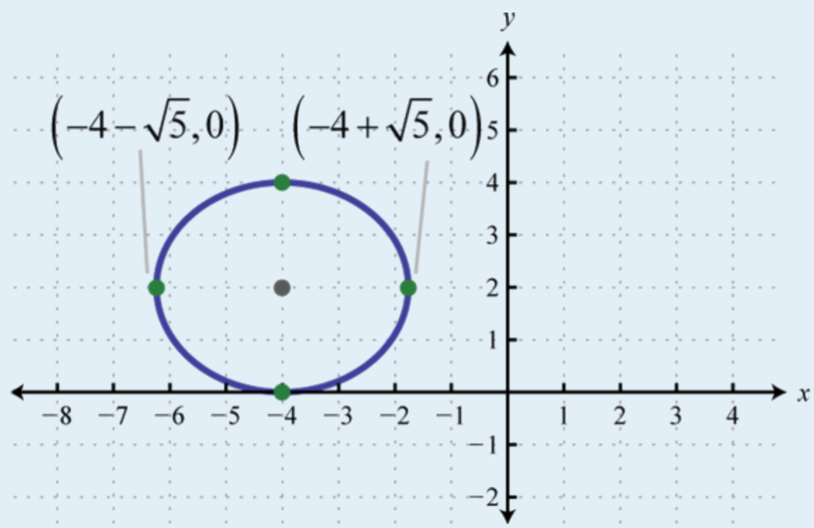
49. $\frac{x^2}{4} + \frac{(y-5)^2}{36} = 1;$



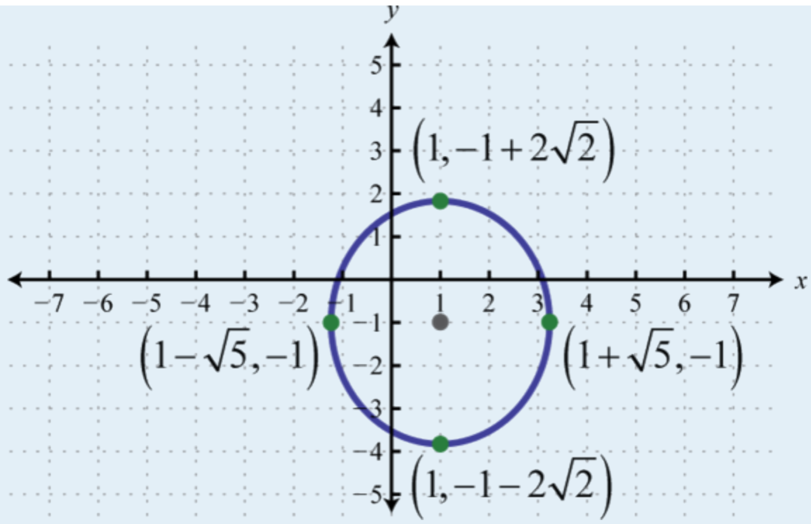
51. $\frac{(x-2)^2}{81} + \frac{(y-2)^2}{9} = 1;$



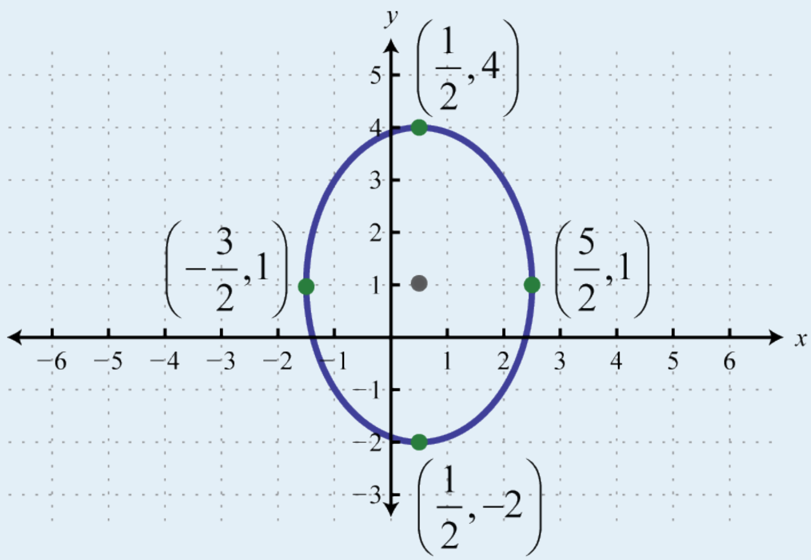
53. $\frac{(x+4)^2}{5} + \frac{(y-2)^2}{4} = 1$;



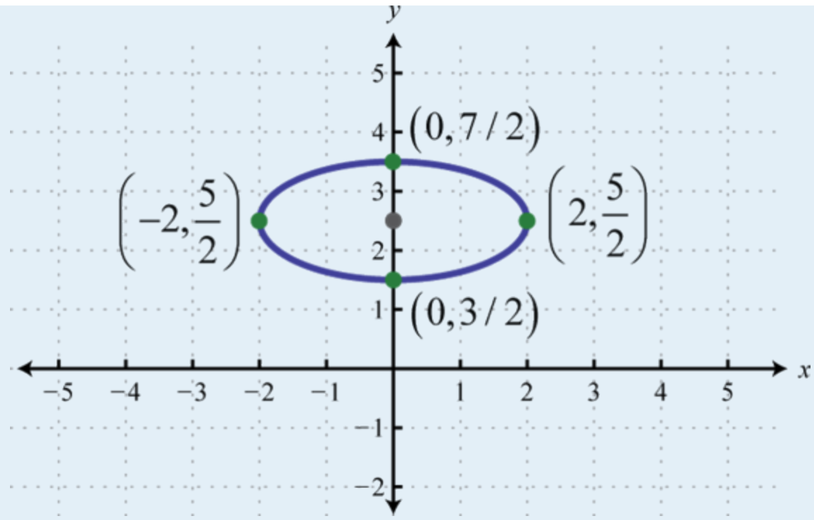
55. $\frac{(x-1)^2}{5} + \frac{(y+1)^2}{8} = 1$;



57. $\frac{\left(x-\frac{1}{2}\right)^2}{4}+\frac{(y-1)^2}{9}=1;$



59. $\frac{x^2}{4}+\left(y-\frac{5}{2}\right)^2=1;$



61. x-intercepts: none; y-intercepts: $(0, -3)$

63. x-intercepts: $\left(\frac{3\pm\sqrt{3}}{3}, 0\right)$; y-intercepts: $(0, -2)$

65. x-intercepts: $\left(\frac{10\pm\sqrt{10}}{5}, 0\right)$; y-intercepts: none

67. $5\pi\sqrt{5}$ square units

69. $\pi\sqrt{21}$ square units

71. $9x^2 + 4y^2 + 72x - 32y + 172 = 0$

73. $x^2 + 3y^2 - 18y - 9 = 0$

75. Answer may vary

77. Answer may vary

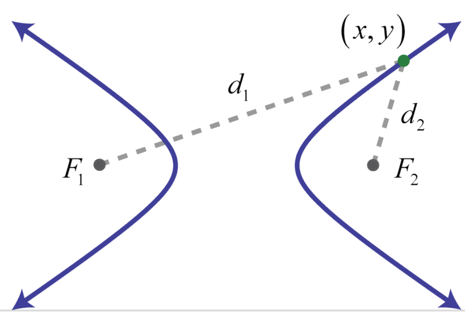
8.4 Hyperbolas

LEARNING OBJECTIVES

1. Graph a hyperbola in standard form.
2. Determine the equation of a hyperbola given its graph.
3. Rewrite the equation of a hyperbola in standard form.
4. Identify a conic section given its equation.

The Hyperbola in Standard Form

A **hyperbola**[23] is the set of points in a plane whose distances from two fixed points, called foci, has an absolute difference that is equal to a positive constant. In other words, if points F_1 and F_2 are the foci and d is some given positive constant then (x, y) is a point on the hyperbola if $d = |d_1 - d_2|$as pictured below:

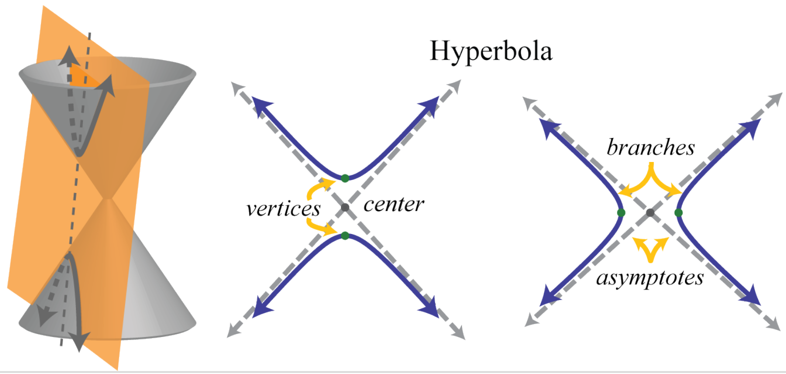


In addition, a hyperbola is formed by the intersection of a cone with an oblique plane that intersects the base. It consists of two separate curves, called **branches**[24]. Points on the separate branches of the graph where the distance is at a minimum are called **vertices.**[25] The midpoint between a hyperbola's vertices is its center. Unlike a parabola, a hyperbola is asymptotic to certain lines drawn through the center. In this section, we will focus on graphing hyperbolas that open left and right or upward and downward.

23. The set of points in a plane whose distances from two fixed points, called foci, has an absolute difference that is equal to a positive constant.

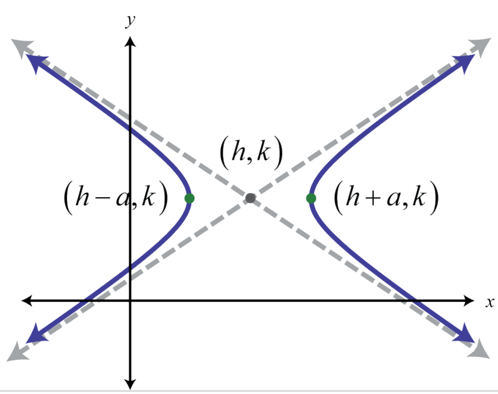
24. The two separate curves of a hyperbola.

25. Points on the separate branches of a hyperbola where the distance is a minimum.



The asymptotes are drawn dashed as they are not part of the graph; they simply indicate the end behavior of the graph. The equation of a **hyperbola opening left and right in standard form**[26] follows:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

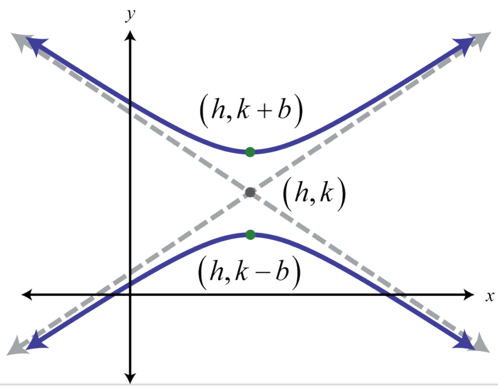


Here the center is (h, k) and the vertices are $(h \pm a, k)$ .The equation of a **hyperbola opening upward and downward in standard form**[27] follows:

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

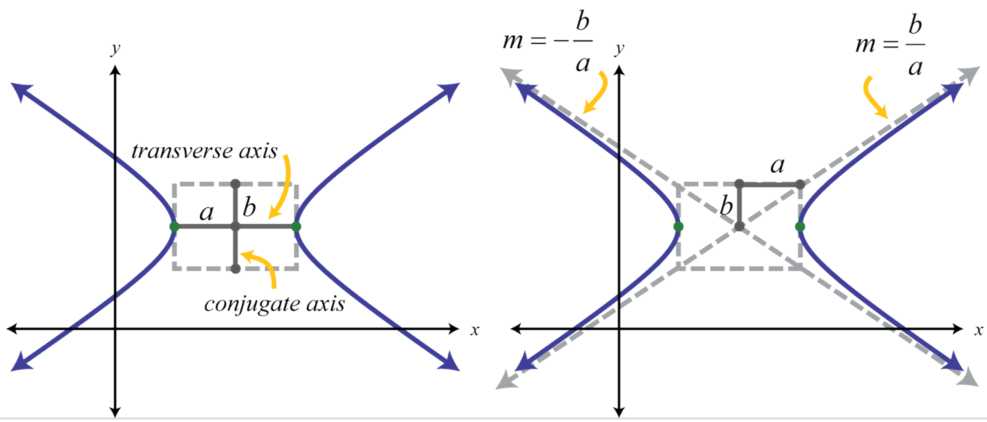
26. The equation of a hyperbola written in the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$.The center is (h, k), a defines the transverse axis, and b defines the conjugate axis.

27. The equation of a hyperbola written in the form $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$.The center is (h, k), b defines the transverse axis, and a defines the conjugate axis.



Here the center is (h, k) and the vertices are $(h, k \pm b)$.

The asymptotes are essential for determining the shape of any hyperbola. Given standard form, the asymptotes are lines passing through the center (h, k) with slope $m = \pm \frac{b}{a}$ .To easily sketch the asymptotes we make use of two special line segments through the center using *a* and *b*. Given any hyperbola, the **transverse axis**[28] is the line segment formed by its vertices. The **conjugate axis**[29] is the line segment through the center perpendicular to the transverse axis as pictured below:



The rectangle defined by the transverse and conjugate axes is called the **fundamental rectangle**[30]. The lines through the corners of this rectangle have slopes $m = \pm \frac{b}{a}$ .These lines are the asymptotes that define the shape of the hyperbola. Therefore, given standard form, many of the properties of a hyperbola are apparent.

28. The line segment formed by the vertices of a hyperbola.

29. A line segment through the center of a hyperbola that is perpendicular to the transverse axis.

30. The rectangle formed using the endpoints of a hyperbolas, transverse and conjugate axes.

Equation	Center	a	b	Opens
$\frac{(x-3)^2}{25} - \frac{(y-5)^2}{16} = 1$	$(3, 5)$	$a = 5$	$b = 4$	Left and right
$\frac{(y-2)^2}{36} - \frac{(x+1)^2}{9} = 1$	$(-1, 2)$	$a = 3$	$b = 6$	Upward and downward
$\frac{(y+2)^2}{3} - (x-5)^2 = 1$	$(5, -2)$	$a = 1$	$b = \sqrt{3}$	Upward and downward
$\frac{x^2}{49} - \frac{(y+4)^2}{8} = 1$	$(0, -4)$	$a = 7$	$b = 2\sqrt{2}$	Left and right

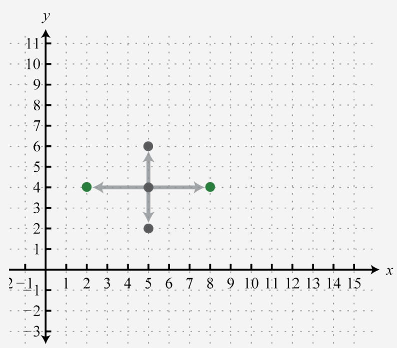
The graph of a hyperbola is completely determined by its center, vertices, and asymptotes.

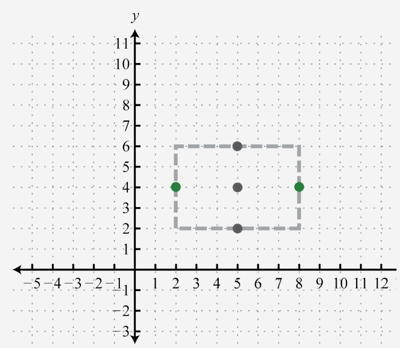
Example 1

Graph: $\frac{(x-5)^2}{9} - \frac{(y-4)^2}{4} = 1.$

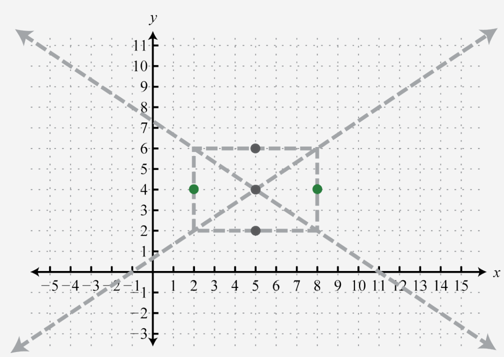
Solution:

In this case, the expression involving x has a positive leading coefficient; therefore, the hyperbola opens left and right. Here $a = \sqrt{9} = 3$ and $b = \sqrt{4} = 2$. From the center $(5, 4)$, mark points 3 units left and right as well as 2 units up and down. Connect these points with a rectangle as follows:



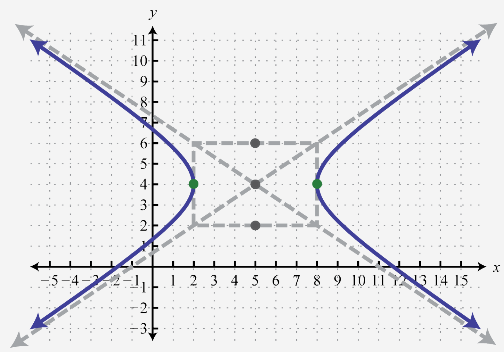


The lines through the corners of this rectangle define the asymptotes.



Use these dashed lines as a guide to graph the hyperbola opening left and right passing through the vertices.

Answer:

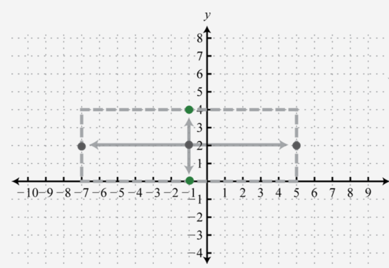


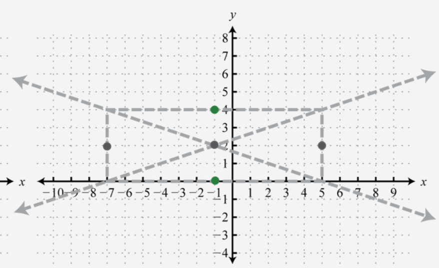
Example 2

Graph: $\frac{(y-2)^2}{4} - \frac{(x+1)^2}{36} = 1.$

Solution:

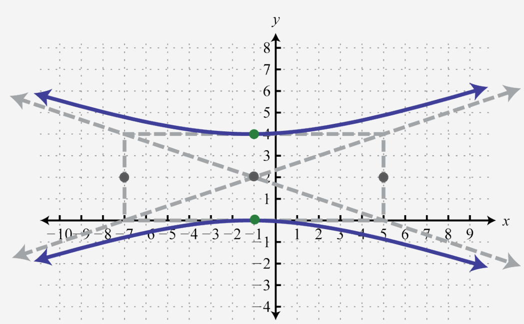
In this case, the expression involving *y* has a positive leading coefficient; therefore, the hyperbola opens upward and downward. Here $a = \sqrt{36} = 6$ and $b = \sqrt{4} = 2$. From the center $(-1, 2)$ mark points 6 units left and right as well as 2 units up and down. Connect these points with a rectangle. The lines through the corners of this rectangle define the asymptotes.





Use these dashed lines as a guide to graph the hyperbola opening upward and downward passing through the vertices.

Answer:



Note: When given a hyperbola opening upward and downward, as in the previous example, it is a common error to interchange the values for the center, *h* and *k*. This is the case because the quantity involving the variable y usually appears first in standard form. Take care to ensure that the y-value of the center comes from the quantity involving the variable y and that the x-value of the center is obtained from the quantity involving the variable x.

As with any graph, we are interested in finding the x- and y-intercepts.

Example 3

Find the intercepts: $\frac{(y-2)^2}{4} - \frac{(x+1)^2}{36} = 1$.

Solution:

To find the x-intercepts set $y = 0$ and solve for x.

$$
\begin{aligned}
\frac{(0-2)^2}{4} - \frac{(x+1)^2}{36} &= 1 \\
1 - \frac{(x+1)^2}{36} &= 1 \\
-\frac{(x+1)^2}{36} &= 0 \\
(x+1)^2 &= 0 \\
x+1 &= 0 \\
x &= -1
\end{aligned}
$$

Therefore there is only one x-intercept, $(-1, 0)$. To find the y-intercept set $x = 0$ and solve for y.

$$\frac{(y-2)^2}{4} - \frac{(0+1)^2}{36} = 1$$
$$\frac{(y-2)^2}{4} - \frac{1}{36} = 1$$
$$\frac{(y-2)^2}{4} = \frac{37}{36}$$
$$\frac{(y-2)}{2} = \pm\frac{\sqrt{37}}{6}$$
$$y-2 = \pm\frac{\sqrt{37}}{3}$$
$$y = 2 \pm \frac{\sqrt{37}}{3} = \frac{6 \pm \sqrt{37}}{3}$$

Therefore there are two y-intercepts, $\left(0, \frac{6-\sqrt{37}}{3}\right) \approx (0, -0.03)$ and $\left(0, \frac{6+\sqrt{37}}{3}\right) \approx (0, 4.03)$. Take a moment to compare these to the sketch of the graph in the previous example.

Answer: x-intercept: $(-1, 0)$; y-intercepts: $\left(0, \frac{6-\sqrt{37}}{3}\right)$ and $\left(0, \frac{6+\sqrt{37}}{3}\right)$.

Consider the hyperbola centered at the origin,

$$9x^2 - 5y^2 = 45$$

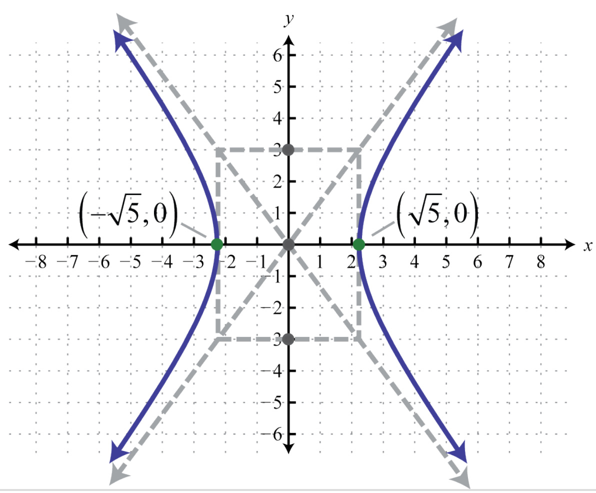
Standard form requires one side to be equal to 1. In this case, we can obtain standard form by dividing both sides by 45.

$$\frac{9x^2 - 5y^2}{45} = \frac{45}{45}$$
$$\frac{9x^2}{45} - \frac{5y^2}{45} = \frac{45}{45}$$
$$\frac{x^2}{5} - \frac{y^2}{9} = 1$$

This can be written as follows:

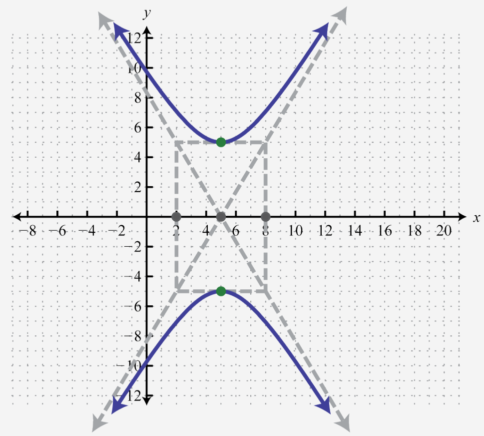
$$\frac{(x-0)^2}{5} - \frac{(y-0)^2}{9} = 1$$

In this form, it is clear that the center is $(0, 0)$, $a = \sqrt{5}$, and $b = 3$. The graph follows.



Try this! Graph: $\frac{y^2}{25} - \frac{(x-5)^2}{9} = 1$.

Answer:



(click to see video)

The Hyperbola in General Form

We have seen that the graph of a hyperbola is completely determined by its center, vertices, and asymptotes; which can be read from its equation in standard form. However, the equation is not always given in standard form. The equation of a **hyperbola in general form**[31] follows:

$$px^2 - qy^2 + cx + dy + e = 0 \quad \textit{Hyperbola opens left and right.}$$
$$qy^2 - px^2 + cx + dy + e = 0 \quad \textit{Hyperbola opens upward and downward.}$$

where $p, q > 0$. The steps for graphing a hyperbola given its equation in general form are outlined in the following example.

31. The equation of a hyperbola written in the form $px^2 - qy^2 + cx + dy + e = 0$ or $qy^2 - px^2 - cx + dy + e = 0$ where $p, q > 0$.

Example 4

Graph: $4x^2 - 9y^2 + 32x - 54y - 53 = 0$.

Solution:

Begin by rewriting the equation in standard form.

- **Step 1:** Group the terms with the same variables and move the constant to the right side. Factor so that the leading coefficient of each grouping is 1.

$$\begin{aligned} 4x^2 - 9y^2 + 32x - 54y - 53 &= 0 \\ \left(4x^2 + 32x + \_\_\_\right) + \left(-9y^2 - 54y + \_\_\_\right) &= 53 \\ 4\left(x^2 + 8x + \_\_\_\right) - 9\left(y^2 + 6y + \_\_\_\right) &= 53 \end{aligned}$$

- **Step 2:** Complete the square for each grouping. In this case, for the terms involving x use $\left(\frac{8}{2}\right)^2 = 4^2 = 16$ and for the terms involving y use $\left(\frac{6}{2}\right)^2 = (3)^2 = 9$. The factor in front of each grouping affects the value used to balance the equation on the right,

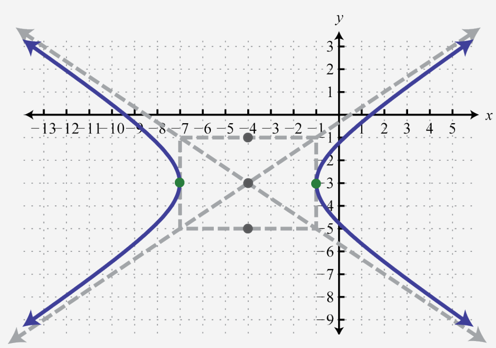
$$4\left(x^2 + 8x + 16\right) - 9\left(y^2 + 6y + 9\right) = 53 + 64 - 81$$

Because of the distributive property, adding 16 inside of the first grouping is equivalent to adding $4 \cdot 16 = 64$. Similarly, adding 9 inside of the second grouping is equivalent to adding $-9 \cdot 9 = -81$. Now factor and then divide to obtain 1 on the right side.

$$4(x+4)^2 - 9(y+3)^2 = 36$$
$$\frac{4(x+4)^2 - 9(y+3)^2}{36} = \frac{36}{36}$$
$$\frac{4(x+4)^2}{36} - \frac{9(y+3)^2}{36} = \frac{36}{36}$$
$$\frac{(x+4)^2}{9} - \frac{(y+3)^2}{4} = 1$$

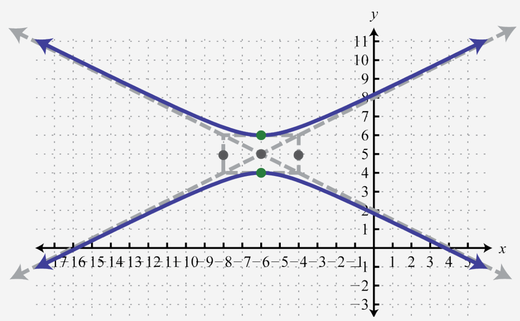
- **Step 3:** Determine the center, a, and b, and then use this information to sketch the graph. In this case, the center is $(-4, -3)$, $a = \sqrt{9} = 3$, and $b = \sqrt{4} = 2$. Because the leading coefficient of the expression involving x is positive and the coefficient of the expression involving y is negative, we graph a hyperbola opening left and right.

Answer:



Try this! Graph: $4y^2 - x^2 - 40y - 12x + 60 = 0$.

Answer:



(click to see video)

Identifying the Conic Sections

In this section, the challenge is to identify a conic section given its equation in general form. To distinguish between the conic sections, use the exponents and coefficients. If the equation is quadratic in only one variable and linear in the other, then its graph will be a parabola.

Parabola: $a > 0$	
$y = a(x-h)^2 + k$ $y = ax^2 + bx + c$	$x = a(y-k)^2 + h$ $x = ay^2 + by + c$

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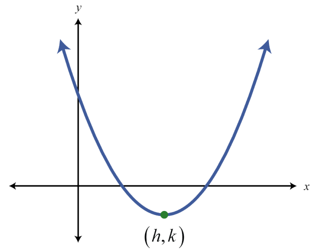
<td>
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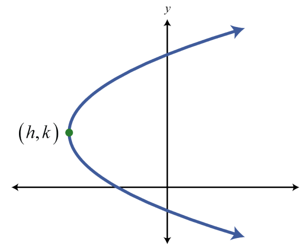
<td>
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<tr>
<td>Parabola: $a < 0$</td>
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<td>$y = a(x - h)^2 + k$
$y = ax^2 + bx + c$</td>
<td>$x = a(y - k)^2 + h$
$x = ay^2 + by + c$</td>
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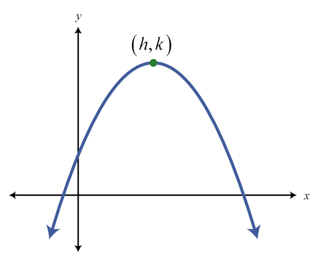
<td>
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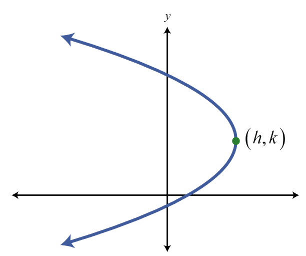
<td>
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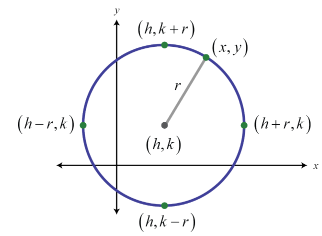
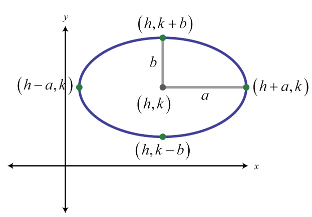
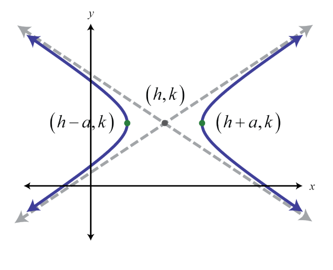
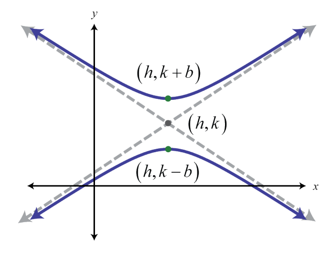
If the equation is quadratic in both variables, where the coefficients of the squared terms are the same, then its graph will be a circle.

Circle:	
$(x-h)^2+(y-k)^2=r^2$ $x^2+y^2+cx+dy+e=0$	

If the equation is quadratic in both variables where the coefficients of the squared terms are different but have the same sign, then its graph will be an ellipse.

Ellipse: $a,b>0$ and $p,q>0$	
$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ $px^2+qy^2+cx+dy+e=0$	

If the equation is quadratic in both variables where the coefficients of the squared terms have different signs, then its graph will be a hyperbola.

Hyperbola: $a, b > 0$ and $p, q > 0$	
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ $px^2 - qy^2 + cx + dy + e = 0$	$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$ $qy^2 - px^2 + cx + dy + e = 0$
	

Example 5

Identify the graph of each equation as a parabola, circle, ellipse, or hyperbola.

a. $4x^2 + 4y^2 - 1 = 0$
b. $3x^2 - 2y^2 - 12 = 0$
c. $x - y^2 - 6y + 11 = 0$

Solution:

a. The equation is quadratic in both *x* and *y* where the leading coefficients for both variables is the same, 4.

$$\begin{aligned} 4x^2 + 4y^2 - 1 &= 0 \\ 4x^2 + 4y^2 &= 1 \\ x^2 + y^2 &= \frac{1}{4} \end{aligned}$$

This is an equation of a circle centered at the origin with radius 1/2.

b. The equation is quadratic in both *x* and *y* where the leading coefficients for both variables have different signs.

$$\begin{aligned} 3x^2 - 2y^2 - 12 &= 0 \\ \frac{3x^2 - 2y^2}{12} &= \frac{12}{12} \\ \frac{x^2}{4} - \frac{y^2}{6} &= 1 \end{aligned}$$

This is an equation of a hyperbola opening left and right centered at the origin.

c. The equation is quadratic in y only.

$$\begin{aligned} x - y^2 + 6y - 11 &= 0 \\ x &= y^2 - 6y + \quad\; + 11 \\ x &= (y^2 - 6y + 9) + 11 - 9 \\ x &= (y - 3)^2 + 2 \end{aligned}$$

This is an equation of a parabola opening right with vertex $(2, 3)$.

Answer:

a. Circle
b. Hyperbola
c. Parabola

KEY TAKEAWAYS

- The graph of a hyperbola is completely determined by its center, vertices, and asymptotes.
- The center, vertices, and asymptotes are apparent if the equation of a hyperbola is given in standard form: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ or $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1.$
- To graph a hyperbola, mark points *a* units left and right from the center and points *b* units up and down from the center. Use these points to draw the fundamental rectangle; the lines through the corners of this rectangle are the asymptotes. If the coefficient of x^2 is positive, draw the branches of the hyperbola opening left and right through the points determined by *a*. If the coefficient of y^2 is positive, draw the branches of the hyperbola opening up and down through the points determined by *b*.
- The orientation of the transverse axis depends the coefficient of x^2 and y^2.
- If the equation of a hyperbola is given in general form $px^2 - qy^2 + cx + dy + e = 0$ or $qy^2 - px^2 + cx + dy + e = 0$ where $p, q > 0$, group the terms with the same variables, and complete the square for both groupings to obtain standard form.
- We recognize the equation of a hyperbola if it is quadratic in both *x* and *y* where the coefficients of the squared terms are opposite in sign.

TOPIC EXERCISES

PART A: THE HYPERBOLA IN STANDARD FORM

Given the equation of a hyperbola in standard form, determine its center, which way the graph opens, and the vertices.

1. $\frac{(x-6)^2}{16} - \frac{(y+4)^2}{9} = 1$
2. $\frac{(y-3)^2}{25} - \frac{(x+1)^2}{64} = 1$
3. $\frac{(y+9)^2}{5} - x^2 = 1$
4. $\frac{(x-5)^2}{12} - y^2 = 1$
5. $4(y+10)^2 - 25(x+1)^2 = 100$
6. $9(x-1)^2 - 5(y+10)^2 = 45$

Determine the standard form for the equation of a hyperbola given the following information.

7. Center $(2, 7)$, $a = 6$, $b = 3$, opens left and right.
8. Center $(-9, 1)$, $a = 7$, $b = 2$, opens up and down.
9. Center $(10, -3)$, $a = \sqrt{7}$, $b = 5\sqrt{2}$, opens up and down.
10. Center $(-7, -2)$, $a = 3\sqrt{3}$, $b = \sqrt{5}$, opens left and right.
11. Center $(0, -8)$, $a = \sqrt{2}$, $b = 1$,opens up and down.
12. Center $(0, 0)$, $a = 2\sqrt{6}$, $b = 4$, opens left and right.

Graph.

13. $\frac{(x-3)^2}{9} - \frac{(y+1)^2}{16} = 1$

14. $\frac{(x+3)^2}{4} - \frac{(y-1)^2}{25} = 1$

15. $\frac{(x-2)^2}{16} - \frac{(y+3)^2}{1} = 1$

16. $\frac{(y+2)^2}{9} - \frac{(x+2)^2}{36} = 1$

17. $\frac{(y-1)^2}{4} - \frac{(x-2)^2}{16} = 1$

18. $(y+2)^2 - \frac{(x+3)^2}{9} = 1$

19. $4(x+3)^2 - 9(y-3)^2 = 36$

20. $16x^2 - 4(y-1)^2 = 64$

21. $4(y-1)^2 - 25x^2 = 100$

22. $9y^2 - 16x^2 = 144$

23. $\frac{(x-2)^2}{12} - \frac{(y-4)^2}{9} = 1$

24. $\frac{(x+2)^2}{4} - \frac{(y-1)^2}{8} = 1$

25. $\frac{(y+1)^2}{5} - \frac{(x-3)^2}{2} = 1$

26. $\frac{(y-4)^2}{3} - \frac{(x+6)^2}{18} = 1$

27. $4x^2 - 3(y-3)^2 = 12$

28. $7(x+1)^2 - 2y^2 = 14$

29. $6y^2 - 3x^2 = 18$

30. $10x^2 - 3y^2 = 30$

Find the *x*- and *y*-intercepts.

31. $\frac{(x-1)^2}{9} - \frac{(y-4)^2}{4} = 1$

32. $\frac{(x+4)^2}{16} - \frac{(y-3)^2}{9} = 1$

33. $\frac{(y-1)^2}{4} - \frac{(x+1)^2}{36} = 1$

34. $\frac{(y+2)^2}{4} - \frac{(x-1)^2}{16} = 1$

35. $2x^2 - 3(y-1)^2 = 12$

36. $6(x-5)^2 - 2y^2 = 12$

37. $36x^2 - 2y^2 = 9$

38. $6y^2 - 4x^2 = 2$

39. Find the equation of the hyperbola with vertices $(\pm 2, 3)$ and a conjugate axis that measures 12 units.

40. Find the equation of the hyperbola with vertices $(4, 7)$ and $(4, 3)$ and a conjugate axis that measures 6 units.

PART B: THE HYPERBOLA IN GENERAL FORM

Rewrite in standard form and graph.

41. $4x^2 - 9y^2 + 16x + 54y - 101 = 0$

42. $9x^2 - 25y^2 - 18x - 100y - 316 = 0$

43. $4y^2 - 16x^2 - 64x + 8y - 124 = 0$

44. $9y^2 - 4x^2 - 24x - 72y + 72 = 0$

45. $y^2 - 36x^2 - 72x - 12y - 36 = 0$

46. $9y^2 - x^2 + 8x - 36y + 11 = 0$

47. $36x^2 - 4y^2 + 24y - 180 = 0$

48. $x^2 - 25y^2 - 2x - 24 = 0$

49. $25x^2 - 64y^2 + 200x + 640y - 2{,}800 = 0$

50. $49y^2 - 4x^2 + 40x + 490y + 929 = 0$

51. $3x^2 - 2y^2 + 24x + 8y + 34 = 0$

52. $4x^2 - 8y^2 - 24x + 80y - 196 = 0$

53. $3y^2 - x^2 - 2x - 6y - 16 = 0$

54. $12y^2 - 5x^2 + 40x + 48y - 92 = 0$

55. $4x^2 - 16y^2 + 12x + 16y - 11 = 0$

56. $4x^2 - y^2 - 4x - 2y - 16 = 0$

57. $4y^2 - 36x^2 + 108x - 117 = 0$

58. $4x^2 - 9y^2 + 8x + 6y - 33 = 0$

Given the general form, determine the intercepts.

59. $3x^2 - y^2 - 11x - 8y - 4 = 0$

60. $4y^2 - 8x^2 + 2x + 9y - 9 = 0$

61. $x^2 - y^2 + 2x + 2y - 4 = 0$

62. $y^2 - x^2 + 6y - 8x - 16 = 0$

63. $5x^2 - 2y^2 - 4x - 3y = 0$

64. $2x^2 - 3y^2 - 4x - 5y + 1 = 0$

Find the equations of the asymptotes to the given hyperbola.

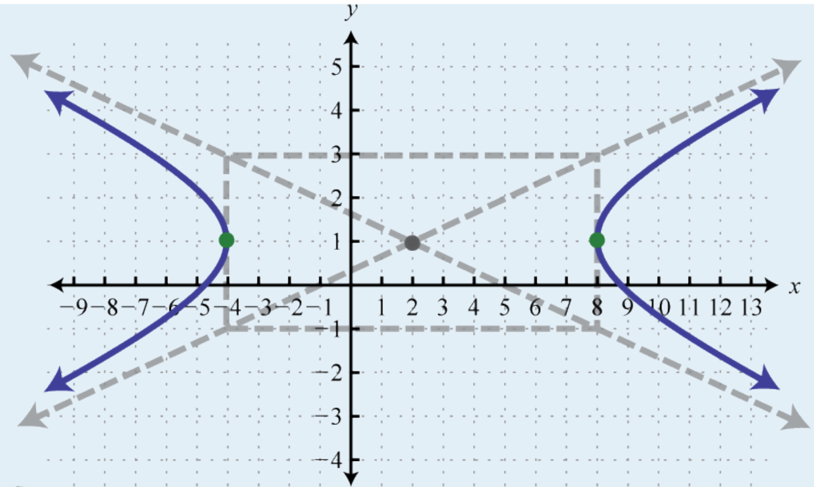
65. $\frac{(y-5)^2}{9} - \frac{(x+8)^2}{16} = 1.$

66. $\frac{(x+9)^2}{36} - \frac{(y-4)^2}{4} = 1.$

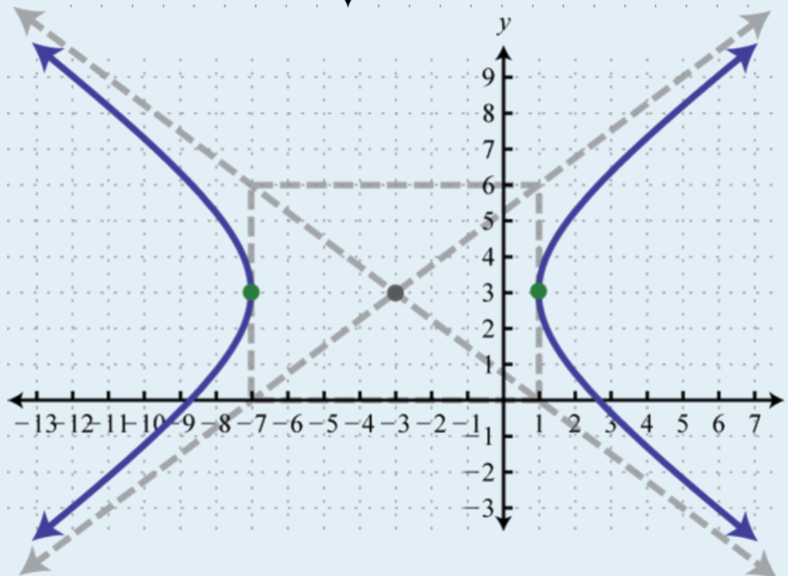
67. $16x^2 - 4y^2 - 24y - 96x + 44 = 0.$

68. $4y^2 - x^2 - 8y - 4x - 4 = 0.$

Given the graph of a hyperbola, determine its equation in general form.


y
x
−9 −8 −7 −6 −5 −4 −3 −2 −1 1 2 3 4 5 6 7 8 9 10 11 12 13
5 4 3 2 1 −1 −2 −3 −4

69.


y
x
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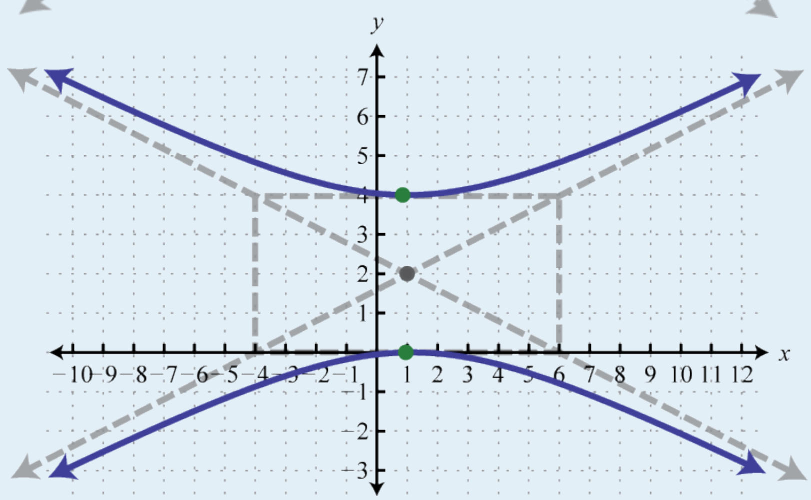
9 8 7 6 5 4 3 2 1 −1 −2 −3

70.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x
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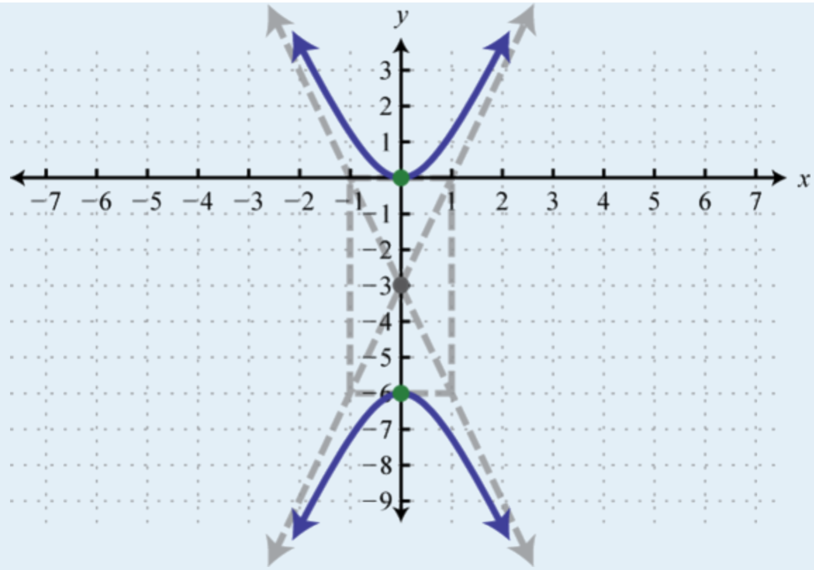
7 6 5 4 3 2 1 −1 −2 −3

71.

72. 

PART C: IDENTIFYING THE CONIC SECTIONS

Identify the following as the equation of a line, parabola, circle, ellipse, or hyperbola.

73. $x^2 + y^2 + 10x - 2y + 23 = 0$
74. $x^2 + y + 2x - 3 = 0$
75. $2x^2 + y^2 - 12x + 14 = 0$
76. $3x - 2y = 24$
77. $x^2 - y^2 + 36 = 0$
78. $4x^2 + 4y^2 - 32 = 0$
79. $x^2 - y^2 - 2x + 2y - 1 = 0$
80. $x - y^2 + 2y + 1 = 0$
81. $3x + 3y + 5 = 0$
82. $8x^2 + 4y^2 - 144x - 12y + 641 = 0$

Identify the conic sections and rewrite in standard form.

83. $x^2 - y - 6x + 11 = 0$

84. $x^2 + y^2 - 12x - 6y + 44 = 0$

85. $x^2 - 2y^2 - 4x - 12y - 18 = 0$

86. $25y^2 - 2x^2 + 36x - 50y - 187 = 0$

87. $7x^2 + 4y^2 - 84x + 16y + 240 = 0$

88. $4x^2 + 4y^2 - 80x + 399 = 0$

89. $4x^2 + 4y^2 + 4x - 32y + 29 = 0$

90. $16x^2 - 4y^2 - 32x + 20y - 25 = 0$

91. $9x - 18y^2 + 12y + 7 = 0$

92. $16x^2 + 12y^2 - 24x - 48y + 9 = 0$

PART D: DISCUSSION BOARD

93. Develop a formula for the equations of the asymptotes of a hyperbola. Share it along with an example on the discussion board.

94. Make up your own equation of a hyperbola, write it in general form, and graph it.

95. Do all hyperbolas have intercepts? What are the possible numbers of intercepts for a hyperbola? Explain.

96. Research and discuss real-world examples of hyperbolas.

ANSWERS

1. Center: $(6, -4)$; $a = 4$; $b = 3$; opens left and right; vertices: $(2, -4)$, $(10, -4)$

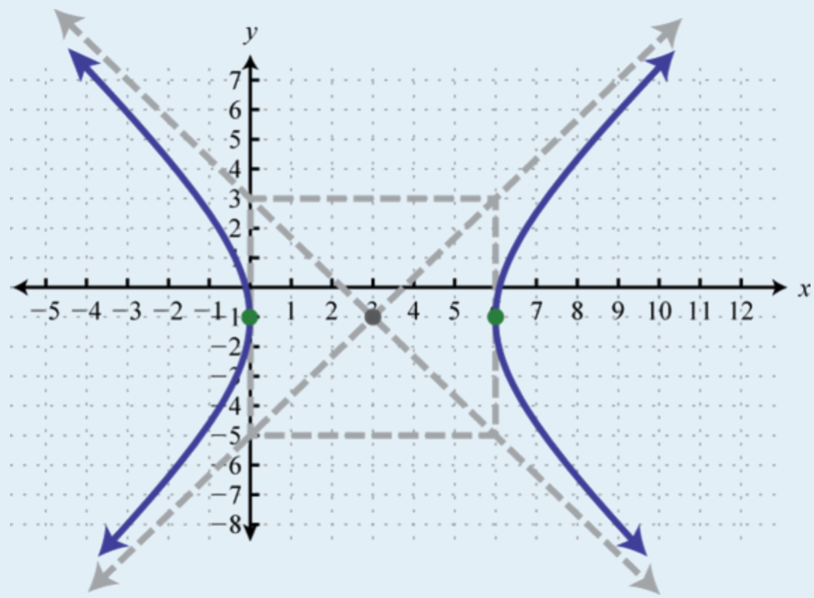
3. Center: $(0, -9)$; $a = 1, b = \sqrt{5}$; opens upward and downward; vertices: $\left(0, -9 - \sqrt{5}\right), \left(0, -9 + \sqrt{5}\right)$

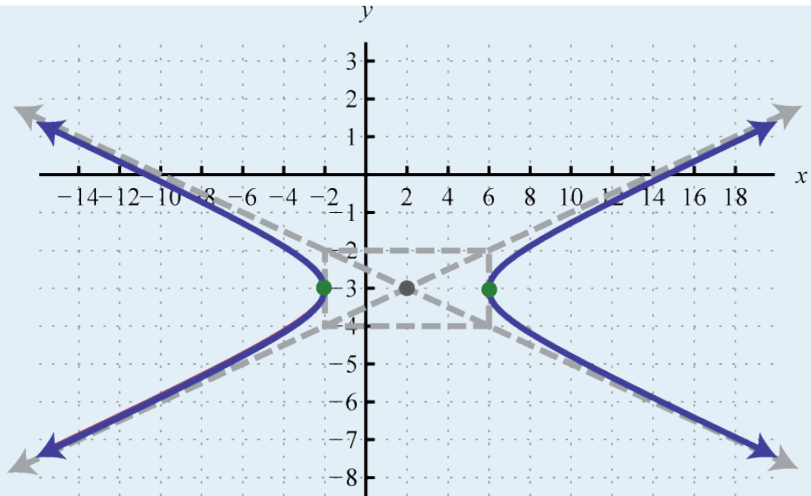
5. Center: $(-1, -10)$; $a = 2, b = 5$; opens upward and downward; vertices: $(-1, -15)$, $(-1, -5)$

7. $\frac{(x-2)^2}{36} - \frac{(y-7)^2}{9} = 1$

9. $\frac{(y+3)^2}{50} - \frac{(x-10)^2}{7} = 1$

11. $\frac{(y+8)^2}{1} - \frac{x^2}{2} = 1$

13. 



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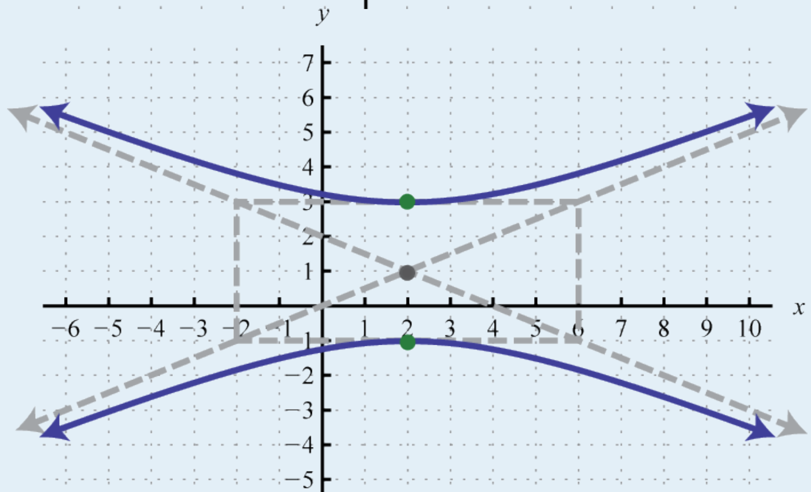

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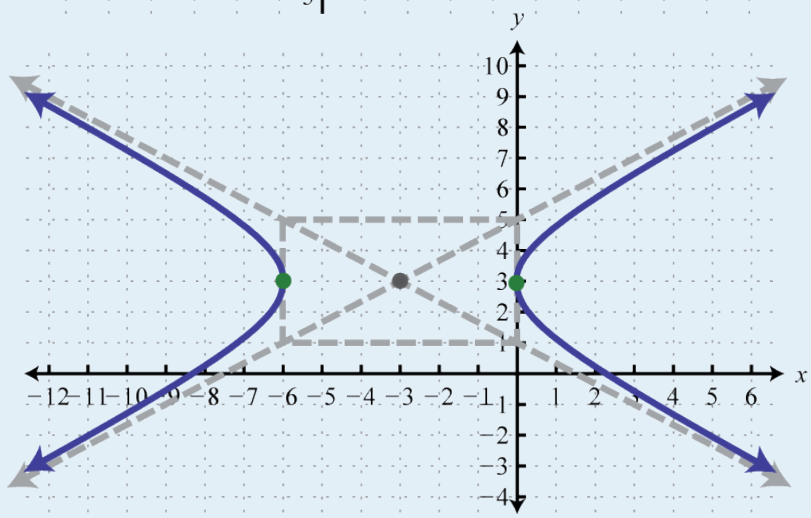

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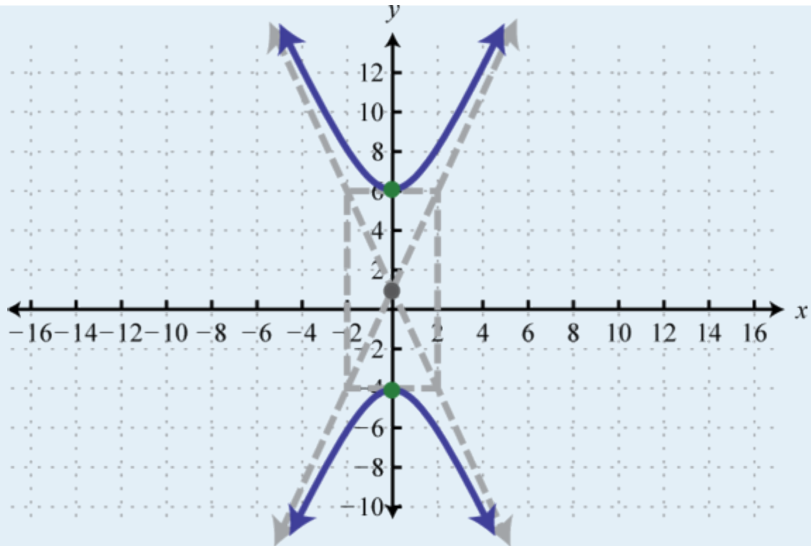
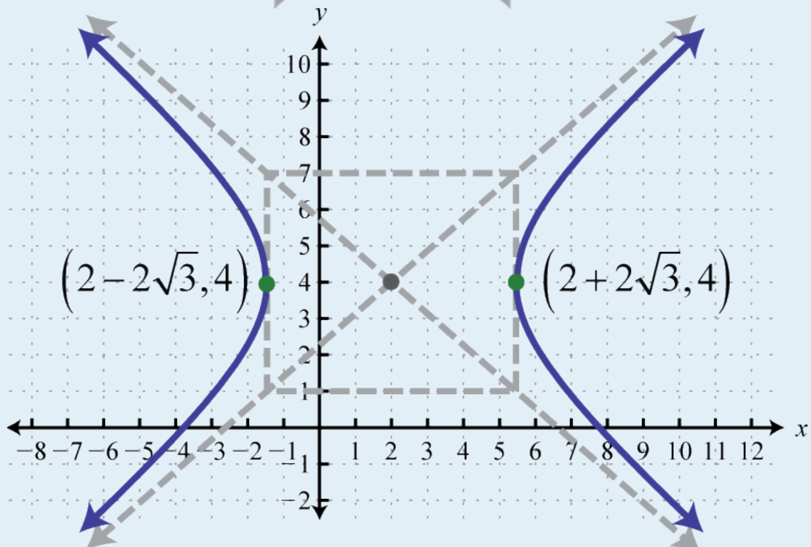
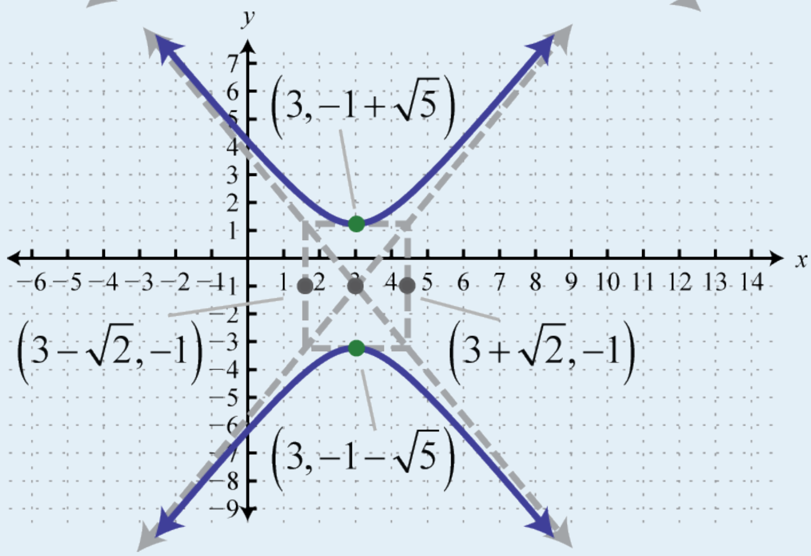

y
x

21.

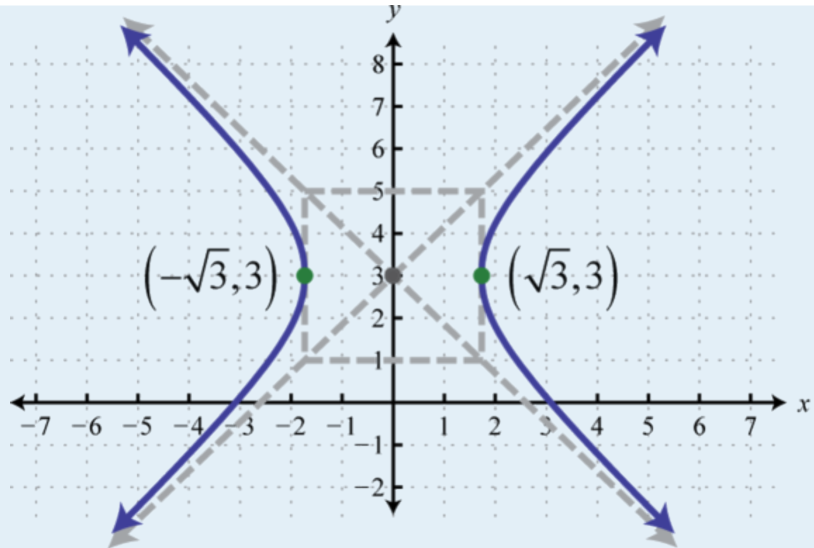

y
x
$(2-2\sqrt{3},4)$
$(2+2\sqrt{3},4)$

23.


y
x
$(3,-1+\sqrt{5})$
$(3-\sqrt{2},-1)$
$(3+\sqrt{2},-1)$
$(3,-1-\sqrt{5})$

25.

27.



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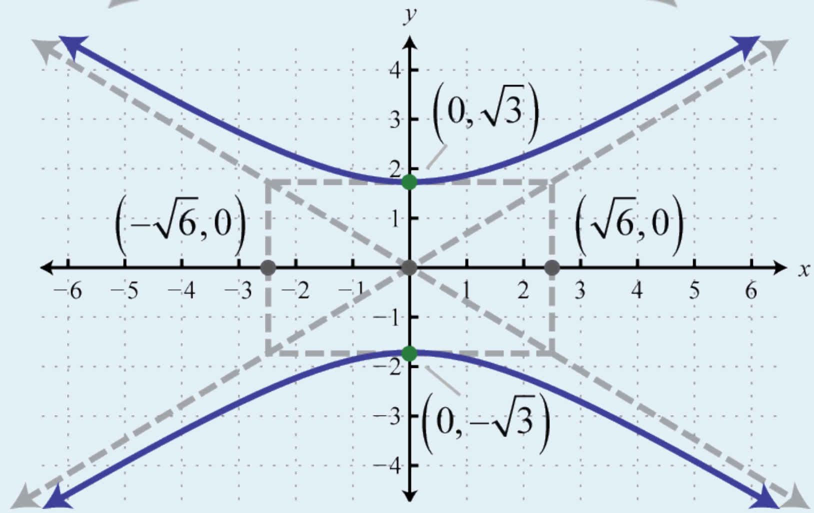


31. *x*-intercepts: $\left(1 \pm 3\sqrt{5}, 0\right)$; *y*-intercepts: none

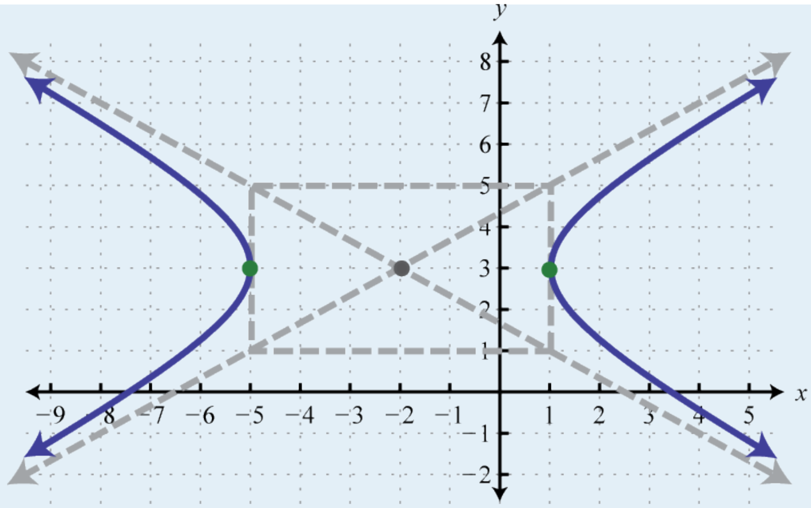
33. *x*-intercepts: none; *y*-intercepts: $\left(0, \frac{3\pm\sqrt{37}}{3}\right)$

35. *x*-intercepts: $\left(\pm \frac{\sqrt{30}}{2}, 0\right)$; *y*-intercepts: none

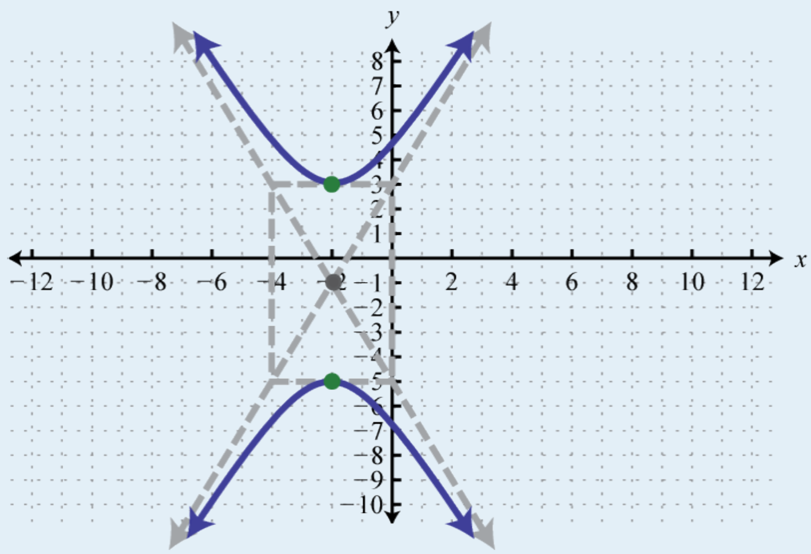
37. *x*-intercepts: $\left(\pm \frac{1}{2}, 0\right)$; *y*-intercepts: none

39. $\frac{x^2}{4} - \frac{(y-3)^2}{36} = 1$

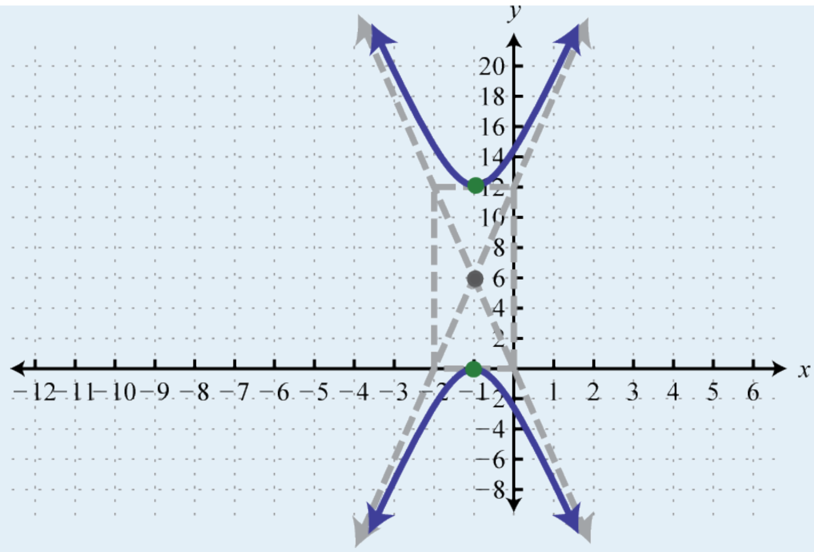
41. $\frac{(x+2)^2}{9} - \frac{(y-3)^2}{4} = 1$;



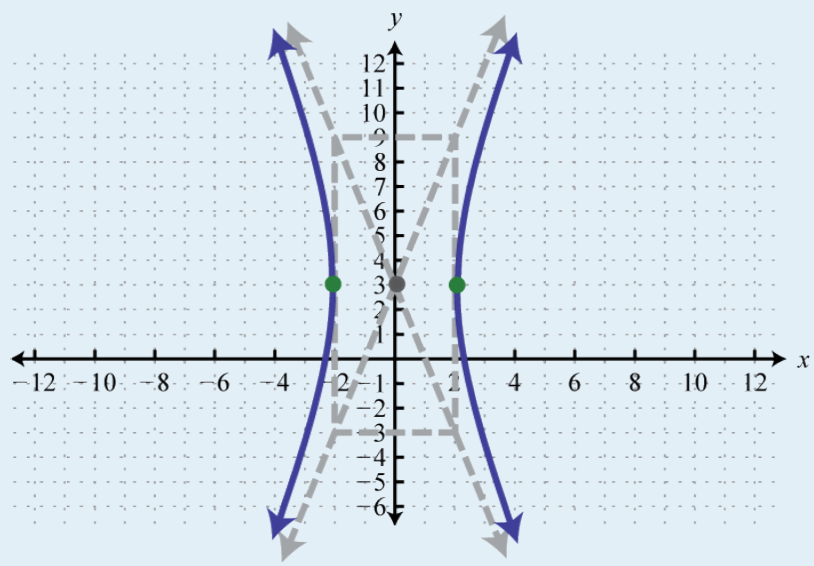
43. $\frac{(y+1)^2}{16} - \frac{(x+2)^2}{4} = 1$;



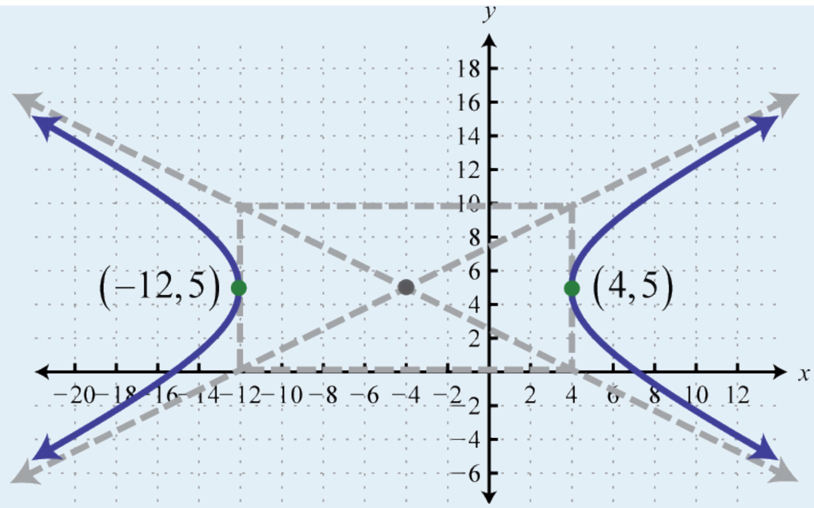
45. $\frac{(y-6)^2}{36} - (x+1)^2 = 1$;



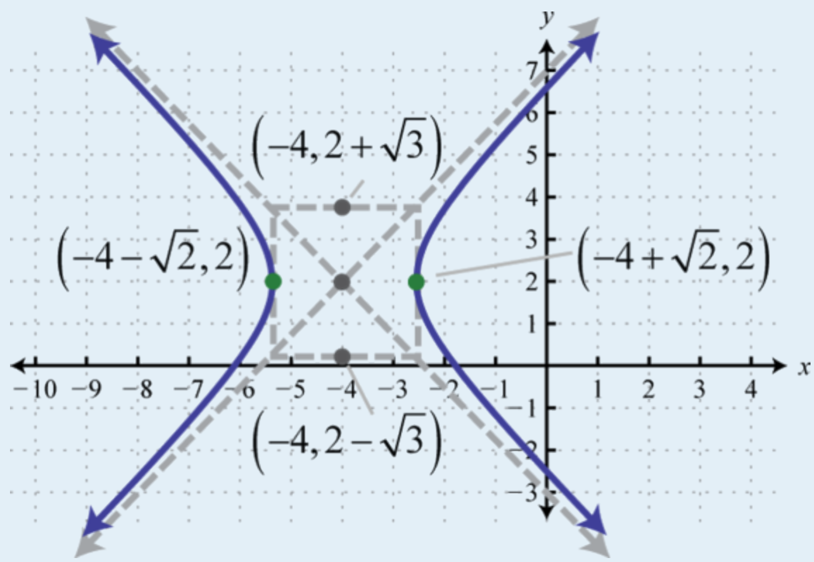
47. $\frac{x^2}{4} - \frac{(y-3)^2}{36} = 1;$



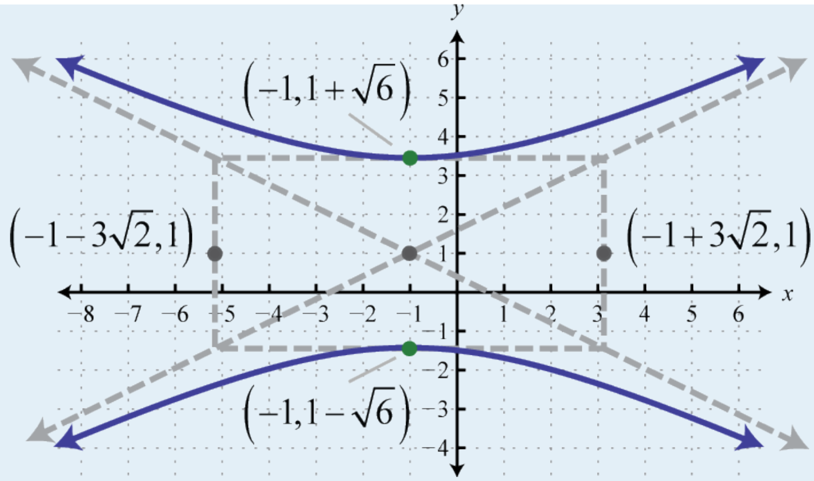
49. $\frac{(x+4)^2}{64} - \frac{(y-5)^2}{25} = 1;$



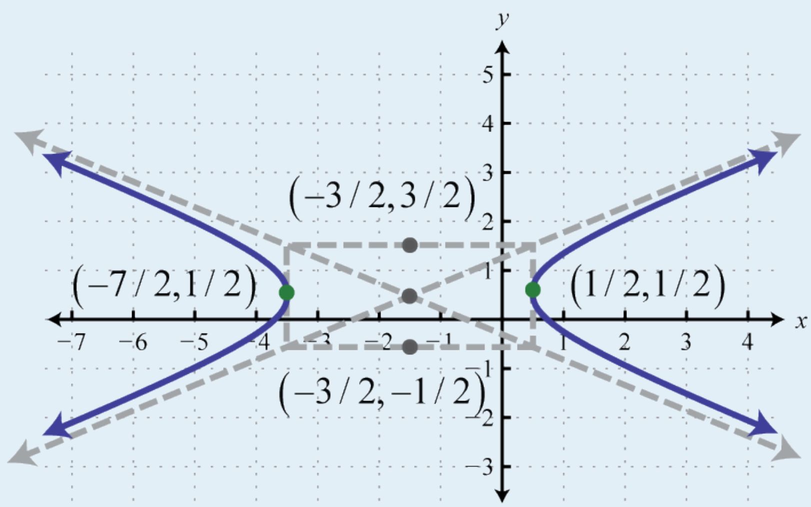
51. $\frac{(x+4)^2}{2} - \frac{(y-2)^2}{3} = 1$;



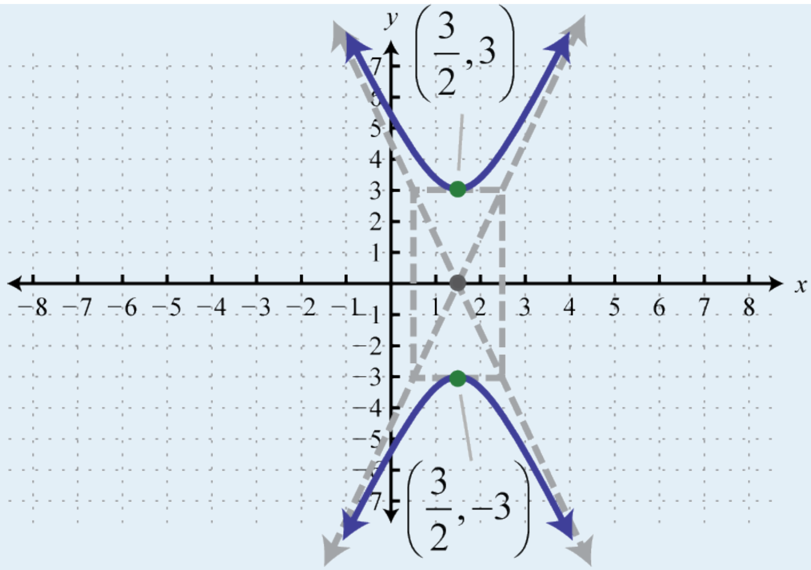
53. $\frac{(y-1)^2}{6} - \frac{(x+1)^2}{18} = 1$;



55. $\frac{\left(x+\frac{3}{2}\right)^2}{4} - \frac{\left(y-\frac{1}{2}\right)^2}{1} = 1;$



57. $\frac{y^2}{9} - \left(x - \frac{3}{2}\right)^2 = 1;$



59. x-intercepts: $\left(-\frac{1}{3}, 0\right), (4, 0)$; y-intercepts: $\left(0, -4 \pm 2\sqrt{3}\right)$

61. x-intercepts: $\left(-1 \pm \sqrt{5}, 0\right)$; y-intercepts: none

63. x-intercepts: $(0, 0), \left(\frac{4}{5}, 0\right)$; y-intercepts: $(0, 0), \left(0, -\frac{3}{2}\right)$

65. $y = -\frac{3}{4}x - 1, y = \frac{3}{4}x + 11$

67. $y = -2x + 3, y = 2x - 9$

69. $x^2 - 9y^2 - 4x + 18y - 41 = 0$

71. $25y^2 - 4x^2 - 100y + 8x - 4 = 0$

73. Circle

75. Ellipse

77. Hyperbola

79. Hyperbola

81. Line

83. Parabola; $y = (x - 3)^2 + 2$

85. Hyperbola; $\frac{(x-2)^2}{4} - \frac{(y+3)^2}{2} = 1$

87. Ellipse; $\frac{(x-6)^2}{4} + \frac{(y+2)^2}{7} = 1$

89. Circle; $\left(x + \frac{1}{2}\right)^2 + (y-4)^2 = 9$

91. Parabola; $x = 2\left(y - \frac{1}{3}\right)^2 - 1$

93. Answer may vary

95. Answer may vary

8.5 Solving Nonlinear Systems

LEARNING OBJECTIVES

1. Identify nonlinear systems.
2. Solve nonlinear systems using the substitution method.

Nonlinear Systems

A system of equations where at least one equation is not linear is called a **nonlinear system**[32]. In this section we will use the substitution method to solve nonlinear systems. Recall that solutions to a system with two variables are ordered pairs (x, y) that satisfy both equations.

32. A system of equations where at least one equation is not linear.

Example 1

Solve: $\begin{cases} x + 2y = 0 \\ x^2 + y^2 = 5 \end{cases}.$

Solution:

In this case we begin by solving for x in the first equation.

$$\begin{cases} x + 2y = 0 \Longrightarrow \quad x = -2y \\ x^2 + y^2 = 5 \end{cases}$$

Substitute $x = -2y$ into the second equation and then solve for y.

$$\begin{aligned} (-2y)^2 + y^2 &= 5 \\ 4y^2 + y^2 &= 5 \\ 5y^2 &= 5 \\ y^2 &= 1 \\ y &= \pm 1 \end{aligned}$$

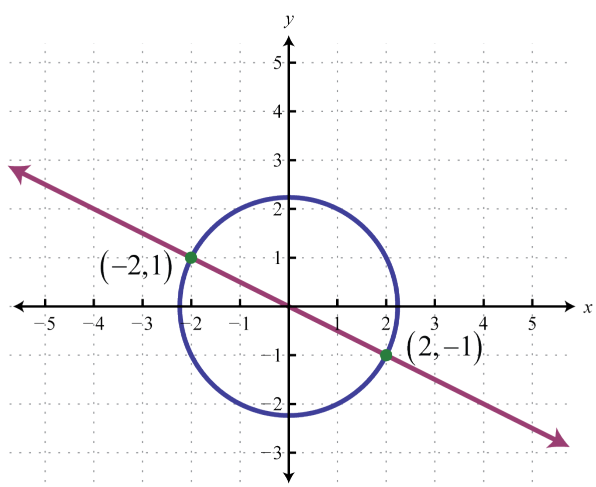
Here there are two answers for y; use $x = -2y$ to find the corresponding x-values.

Using $y = -1$	Using $y = 1$
$\begin{aligned} x &= -2y \\ &= -2(-1) \\ &= 2 \end{aligned}$	$\begin{aligned} x &= -2y \\ &= -2(1) \\ &= -2 \end{aligned}$

This gives us two ordered pair solutions, $(2, -1)$ and $(-2, 1)$.

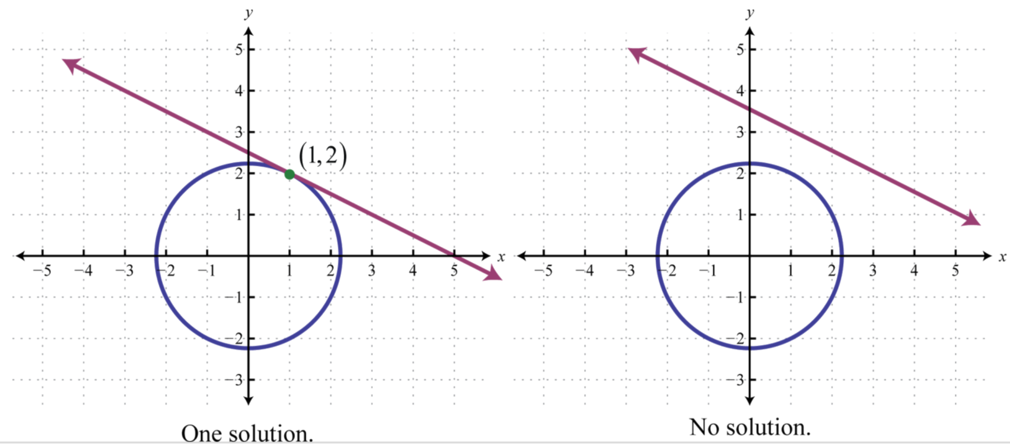
Answer: $(2, -1)$, $(-2, 1)$

In the previous example, the given system consisted of a line and a circle. Graphing these equations on the same set of axes, we can see that the two ordered pair solutions correspond to the two points of intersection.



Two solutions.

If we are given a system consisting of a circle and a line, then there are 3 possibilities for real solutions—two solutions as pictured above, one solution, or no solution.


(1, 2)
One solution.
No solution.

Example 2

Solve: $\begin{cases} x+y=3 \\ x^2+y^2=2 \end{cases}$.

Solution:

Solve for y in the first equation.

$$\begin{cases} x+y=3 \Longrightarrow \quad y=3-x \\ x^2+y^2=2 \end{cases}$$

Next, substitute $y=3-x$ into the second equation and then solve for x.

$$\begin{aligned} x^2+(3-x)^2 &= 2 \\ x^2+9-6x+x^2 &= 2 \\ 2x^2-6x+9 &= 2 \\ 2x^2-6x+7 &= 0 \end{aligned}$$

The resulting equation does not factor. Furthermore, using $a=2$, $b=-6$, and $c=7$ we can see that the discriminant is negative:

$$\begin{aligned} b^2-4ac &= (-6)^2-4(2)(7) \\ &= 36-56 \\ &= -20 \end{aligned}$$

We conclude that there are no real solutions to this equation and thus no solution to the system.

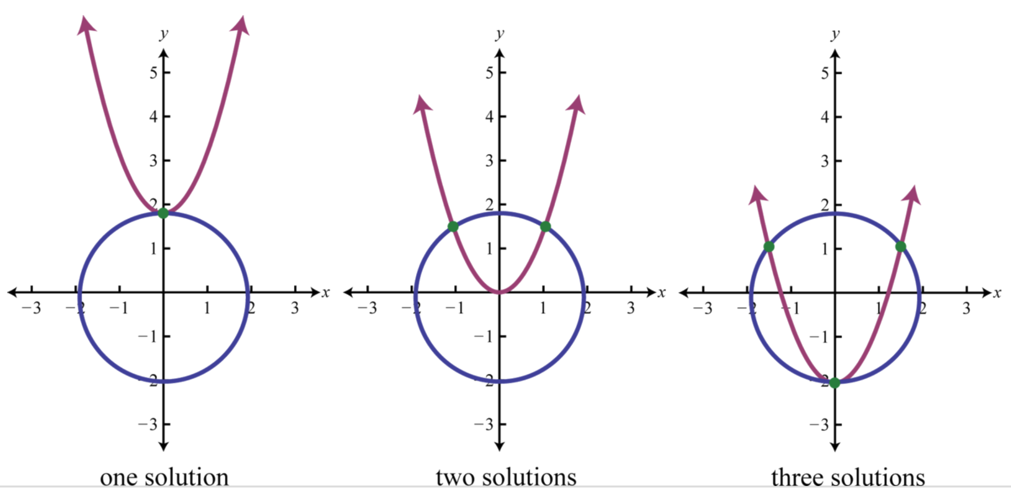
Answer: Ø

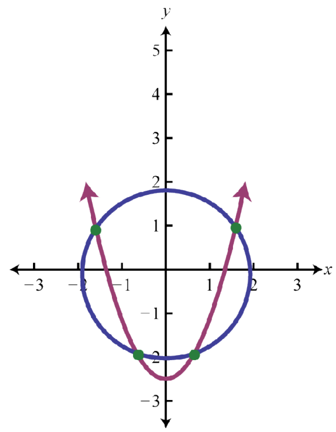
Try this! Solve: $\begin{cases} x - y = 5 \\ x^2 + (y+1)^2 = 8 \end{cases}$.

Answer: $(2, -3)$

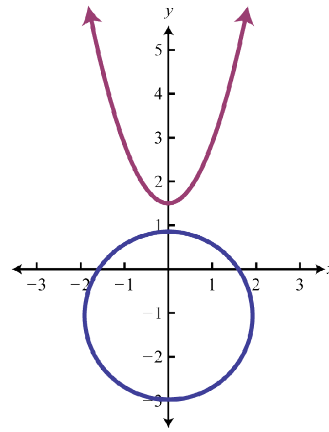
(click to see video)

If given a circle and a parabola, then there are 5 possibilities for solutions.





four solutions



no solution

When using the substitution method, we can perform the substitution step using entire algebraic expressions. The goal is to produce a single equation in one variable that can be solved using the techniques learned up to this point in our study of algebra.

Example 3

Solve: $\begin{cases} x^2 + y^2 = 2 \\ y - x^2 = -2 \end{cases}.$

Solution:

We can solve for x^2 in the second equation.

$$\begin{cases} x^2 + y^2 = 2 \\ y - x^2 = -2 \end{cases} \quad \Rightarrow \quad y + 2 = x^2$$

Substitute $x^2 = y + 2$ into the first equation and then solve for y.

$$\begin{aligned} y + 2 + y^2 &= 2 \\ y^2 + y &= 0 \\ y\left(y + 1\right) &= 0 \\ y &= 0 \quad or \quad y = -1 \end{aligned}$$

Back substitute into $x^2 = y + 2$ to find the corresponding x-values.

Using $y = -1$	Using $y = 0$
$x^2 = y + 2$ $x^2 = -1 + 2$ $x^2 = 1$ $x = \pm 1$	$x^2 = y + 2$ $x^2 = 0 + 2$ $x^2 = 2$ $x = \pm\sqrt{2}$

This leads us to four solutions, $(\pm 1, -1)$ and $\left(\pm\sqrt{2}, 0\right)$.

Answer: $(\pm 1, -1)$, $\left(\pm\sqrt{2}, 0\right)$

Example 4

Solve: $\begin{cases} (x-1)^2 - 2y^2 = 4 \\ x^2 + y^2 = 9 \end{cases}.$

Solution:

We can solve for y^2 in the second equation,

$$\begin{cases} (x-1)^2 - 2y^2 = 4 \\ x^2 + y^2 = 9 \Longrightarrow \quad y^2 = 9 - x^2 \end{cases}$$

Substitute $y^2 = 9 - x^2$ into the first equation and then solve for x.

$$\begin{aligned} (x-1)^2 - 2(9 - x^2) &= 4 \\ x^2 - 2x + 1 - 18 + 2x^2 &= 0 \\ 3x^2 - 2x - 21 &= 0 \\ (3x+7)(x-3) &= 0 \\ 3x + 7 &= 0 \;\; or \;\; x - 3 = 0 \\ x &= -\frac{7}{3} \qquad\quad x = 3 \end{aligned}$$

Back substitute into $y^2 = 9 - x^2$ to find the corresponding y-values.

Using $x = -\frac{7}{3}$	Using $x = 3$
$y^2 = 9 - \left(-\frac{7}{3}\right)^2$ $y^2 = \frac{9}{1} - \frac{49}{9}$ $y^2 = \frac{32}{9}$ $y = \pm\frac{\sqrt{32}}{3} = \pm\frac{4\sqrt{2}}{3}$	$y^2 = 9 - (3)^2$ $y^2 = 0$ $y = 0$

This leads to three solutions, $\left(-\frac{7}{3}, \pm\frac{4\sqrt{2}}{3}\right)$ and $(3, 0)$.

Answer: $(3, 0)$, $\left(-\frac{7}{3}, \pm\frac{4\sqrt{2}}{3}\right)$

Example 5

Solve: $\begin{cases} x^2 + y^2 = 2 \\ xy = 1 \end{cases}$.

Solution:

Solve for y in the second equation.

$$\begin{cases} x^2 + y^2 = 2 \\ xy = 1 \Longrightarrow \quad y = \dfrac{1}{x} \end{cases}$$

Substitute $y = \frac{1}{x}$ into the first equation and then solve for x.

$$\begin{aligned} x^2 + \left(\frac{1}{x}\right)^2 &= 2 \\ x^2 + \frac{1}{x^2} &= 2 \end{aligned}$$

This leaves us with a rational equation. Make a note that $x \neq 0$ and multiply both sides by x^2.

$$\begin{aligned} x^2\left(x^2+\frac{1}{x^2}\right) &= 2\cdot x^2 \\ x^4+1 &= 2x^2 \\ x^4-2x^2+1 &= 0 \\ (x^2-1)(x^2-1) &= 0 \end{aligned}$$

At this point we can see that both factors are the same. Apply the zero product property.

$$\begin{aligned} x^2-1 &= 0 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

Back substitute into $y = \frac{1}{x}$ to find the corresponding y-values.

Using $x = -1$	Using $x = 1$
$\begin{aligned} y &= \frac{1}{x} \\ &= \frac{1}{-1} \\ &= -1 \end{aligned}$	$\begin{aligned} y &= \frac{1}{x} \\ &= \frac{1}{1} \\ &= 1 \end{aligned}$

This leads to two solutions.

Answer: $(1, 1), (-1, -1)$

Try this! Solve: $\begin{cases} \frac{1}{x} + \frac{1}{y} = 4 \\ \frac{1}{x^2} + \frac{1}{y^2} = 40 \end{cases}$.

Answer: $\left(-\frac{1}{2}, \frac{1}{6}\right), \left(\frac{1}{6}, -\frac{1}{2}\right)$

(click to see video)

KEY TAKEAWAYS

- Use the substitution method to solve nonlinear systems.
- Streamline the solving process by using entire algebraic expressions in the substitution step to obtain a single equation with one variable.
- Understanding the geometric interpretation of the system can help in finding real solutions.

TOPIC EXERCISES

PART A: NONLINEAR SYSTEMS

Solve.

1. $\begin{cases} x^2 + y^2 = 10 \\ x + y = 4 \end{cases}$
2. $\begin{cases} x^2 + y^2 = 5 \\ x - y = -3 \end{cases}$
3. $\begin{cases} x^2 + y^2 = 30 \\ x - 3y = 0 \end{cases}$
4. $\begin{cases} x^2 + y^2 = 10 \\ 2x - y = 0 \end{cases}$
5. $\begin{cases} x^2 + y^2 = 18 \\ 2x - 2y = -12 \end{cases}$
6. $\begin{cases} (x-4)^2 + y^2 = 25 \\ 4x - 3y = 16 \end{cases}$
7. $\begin{cases} 3x^2 + 2y^2 = 21 \\ 3x - y = 0 \end{cases}$
8. $\begin{cases} x^2 + 5y^2 = 36 \\ x - 2y = 0 \end{cases}$
9. $\begin{cases} 4x^2 + 9y^2 = 36 \\ 2x + 3y = 6 \end{cases}$
10. $\begin{cases} 4x^2 + y^2 = 4 \\ 2x + y = -2 \end{cases}$
11. $\begin{cases} 2x^2 + y^2 = 1 \\ x + y = 1 \end{cases}$

12. $\begin{cases} 4x^2 + 3y^2 = 12 \\ 2x - y = 2 \end{cases}$

13. $\begin{cases} x^2 - 2y^2 = 35 \\ x - 3y = 0 \end{cases}$

14. $\begin{cases} 5x^2 - 7y^2 = 39 \\ 2x + 4y = 0 \end{cases}$

15. $\begin{cases} 9x^2 - 4y^2 = 36 \\ 3x + 2y = 0 \end{cases}$

16. $\begin{cases} x^2 + y^2 = 25 \\ x - 2y = -12 \end{cases}$

17. $\begin{cases} 2x^2 + 3y = 9 \\ 8x - 4y = 12 \end{cases}$

18. $\begin{cases} 2x - 4y^2 = 3 \\ 3x - 12y = 6 \end{cases}$

19. $\begin{cases} 4x^2 + 3y^2 = 12 \\ x - \frac{3}{2} = 0 \end{cases}$

20. $\begin{cases} 5x^2 + 4y^2 = 40 \\ y - 3 = 0 \end{cases}$

21. The sum of the squares of two positive integers is 10. If the first integer is added to twice the second integer, the sum is 7. Find the integers.

22. The diagonal of a rectangle measures $\sqrt{5}$ units and has a perimeter equal to 6 units. Find the dimensions of the rectangle.

23. For what values of b will the following system have real solutions?

$$\begin{cases} x^2 + y^2 = 1 \\ y = x + b \end{cases}$$

24. For what values of m will the following system have real solutions?

$$\begin{cases} x^2 - y^2 = 1 \\ y = mx \end{cases}$$

Solve.

25. $\begin{cases} x^2 + y^2 = 4 \\ y - x^2 = 2 \end{cases}$

26. $\begin{cases} x^2 + y^2 = 4 \\ y - x^2 = -2 \end{cases}$

27. $\begin{cases} x^2 + y^2 = 4 \\ y - x^2 = 3 \end{cases}$

28. $\begin{cases} x^2 + y^2 = 4 \\ 4y - x^2 = -4 \end{cases}$

29. $\begin{cases} x^2 + 3y^2 = 9 \\ y^2 - x = 3 \end{cases}$

30. $\begin{cases} x^2 + 3y^2 = 9 \\ x + y^2 = -4 \end{cases}$

31. $\begin{cases} 4x^2 - 3y^2 = 12 \\ x^2 + y^2 = 1 \end{cases}$

32. $\begin{cases} x^2 + y^2 = 1 \\ x^2 - y^2 = 1 \end{cases}$

33. $\begin{cases} x^2 + y^2 = 1 \\ 4y^2 - x^2 - 4y = 0 \end{cases}$

34. $\begin{cases} x^2 + y^2 = 4 \\ 2x^2 - y^2 + 4x = 0 \end{cases}$

35. $\begin{cases} 2(x-2)^2 + y^2 = 6 \\ (x-3)^2 + y^2 = 4 \end{cases}$

36. $\begin{cases} x^2 + y^2 - 6y = 0 \\ 4x^2 + 5y^2 + 20y = 0 \end{cases}$

37. $\begin{cases} x^2 + 4y^2 = 25 \\ 4x^2 + y^2 = 40 \end{cases}$

38. $\begin{cases} x^2 - 2y^2 = -10 \\ 4x^2 + y^2 = 10 \end{cases}$

39. $\begin{cases} 2x^2 + y^2 = 14 \\ x^2 - (y-1)^2 = 6 \end{cases}$

40. $\begin{cases} 3x^2 - (y-2)^2 = 12 \\ x^2 + (y-2)^2 = 1 \end{cases}$

41. The difference of the squares of two positive integers is 12. The sum of the larger integer and the square of the smaller is equal to 8. Find the integers.

42. The difference between the length and width of a rectangle is 4 units and the diagonal measures 8 units. Find the dimensions of the rectangle. Round off to the nearest tenth.

43. The diagonal of a rectangle measures p units and has a perimeter equal to $2q$ units. Find the dimensions of the rectangle in terms of p and q.

44. The area of a rectangle is p square units and its perimeter is $2q$ units. Find the dimensions of the rectangle in terms of p and q.

Solve.

45. $\begin{cases} x^2 + y^2 = 26 \\ xy = 5 \end{cases}$

46. $\begin{cases} x^2 + y^2 = 10 \\ xy = 3 \end{cases}$

47. $\begin{cases} 2x^2 - 3y^2 = 5 \\ xy = 1 \end{cases}$

48. $\begin{cases} 3x^2 - 4y^2 = -11 \\ xy = 1 \end{cases}$

49. $\begin{cases} x^2 + y^2 = 2 \\ xy - 2 = 0 \end{cases}$

50. $\begin{cases} x^2 + y^2 = 1 \\ 2xy - 1 = 0 \end{cases}$

51. $\begin{cases} 4x - y^2 = 0 \\ xy = 2 \end{cases}$

52. $\begin{cases} 3y - x^2 = 0 \\ xy - 9 = 0 \end{cases}$

53. $\begin{cases} 2y - x^2 = 0 \\ xy - 1 = 0 \end{cases}$

54. $\begin{cases} x - y^2 = 0 \\ xy = 3 \end{cases}$

55. The diagonal of a rectangle measures $2\sqrt{10}$ units. If the area of the rectangle is 12 square units, find its dimensions.

56. The area of a rectangle is 48 square meters and the perimeter measures 32 meters. Find the dimensions of the rectangle.

57. The product of two positive integers is 72 and their sum is 18. Find the integers.

58. The sum of the squares of two positive integers is 52 and their product is 24. Find the integers.

Solve.

59. $\begin{cases} \frac{1}{x} + \frac{1}{y} = 4 \\ \frac{1}{x} - \frac{1}{y} = 2 \end{cases}$

60. $\begin{cases} \frac{2}{x} - \frac{1}{y} = 5 \\ \frac{1}{x} + \frac{1}{y} = 2 \end{cases}$

61. $\begin{cases} \dfrac{1}{x} + \dfrac{2}{y} = 1 \\ \dfrac{3}{x} - \dfrac{1}{y} = 2 \end{cases}$

62. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 6 \\ \dfrac{1}{x^2} + \dfrac{1}{y^2} = 20 \end{cases}$

63. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 2 \\ \dfrac{1}{x^2} + \dfrac{1}{y^2} = 34 \end{cases}$

64. $\begin{cases} xy - 16 = 0 \\ 2x^2 - y = 0 \end{cases}$

65. $\begin{cases} x + y^2 = 4 \\ y = \sqrt{x} \end{cases}$

66. $\begin{cases} y^2 - (x-1)^2 = 1 \\ y = \sqrt{x} \end{cases}$

67. $\begin{cases} y = 2^x \\ y = 2^{2x} - 56 \end{cases}$

68. $\begin{cases} y = 3^{2x} - 72 \\ y - 3^x = 0 \end{cases}$

69. $\begin{cases} y = e^{4x} \\ y = e^{2x} + 6 \end{cases}$

70. $\begin{cases} y - e^{2x} = 0 \\ y - e^x = 0 \end{cases}$

PART B: DISCUSSION BOARD

71. How many real solutions can be obtained from a system that consists of a circle and a hyperbola? Explain.

72. Make up your own nonlinear system, solve it, and provide the answer. Also, provide a graph and discuss the geometric interpretation of the solutions.

ANSWERS

1. $(1, 3), (3, 1)$
3. $\left(-3\sqrt{3}, -\sqrt{3}\right), \left(3\sqrt{3}, \sqrt{3}\right)$
5. $(-3, 3)$
7. $(-1, -3), (1, 3)$
9. $(0, 2), (3, 0)$
11. $(0, 1), \left(\frac{2}{3}, \frac{1}{3}\right)$
13. $\left(-3\sqrt{5}, -\sqrt{5}\right), \left(3\sqrt{5}, \sqrt{5}\right)$
15. Ø
17. $\left(\frac{-3+3\sqrt{5}}{2}, -6 + 3\sqrt{5}\right), \left(\frac{-3-3\sqrt{5}}{2}, -6 - 3\sqrt{5}\right)$
19. $\left(\frac{3}{2}, -1\right), \left(\frac{3}{2}, 1\right)$
21. 1, 3
23. $b \in \left[-\sqrt{2}, \sqrt{2}\right]$
25. $(0, 2)$
27. Ø
29. $(-3, 0), \left(0, -\sqrt{3}\right), \left(0, \sqrt{3}\right)$
31. Ø
33. $(0, 1), \left(-\frac{2\sqrt{5}}{5}, -\frac{1}{5}\right), \left(\frac{2\sqrt{5}}{5}, -\frac{1}{5}\right)$
35. $(3, -2), (3, 2)$
37. $(-3, -2), (-3, 2), (3, -2), (3, 2)$

39. $\left(-\sqrt{7}, 0\right), \left(\sqrt{7}, 0\right), \left(-\frac{\sqrt{55}}{3}, \frac{4}{3}\right), \left(\frac{\sqrt{55}}{3}, \frac{4}{3}\right)$

41. 2, 4

43. $\frac{q+\sqrt{2p^2-q^2}}{2}$ units by $\frac{q-\sqrt{2p^2-q^2}}{2}$ units

45. $(-5, -1), (5, 1), (-1, -5), (1, 5)$

47. $\left(-\sqrt{3}, -\frac{\sqrt{3}}{3}\right), \left(\sqrt{3}, \frac{\sqrt{3}}{3}\right)$

49. Ø

51. $(1, 2)$

53. $\left(\sqrt[3]{2}, \frac{\sqrt[3]{4}}{2}\right)$

55. 2 units by 6 units

57. 6, 12

59. $\left(\frac{1}{3}, 1\right)$

61. $\left(\frac{7}{5}, 7\right)$

63. $\left(-\frac{1}{3}, \frac{1}{5}\right), \left(\frac{1}{5}, -\frac{1}{3}\right)$

65. $\left(2, \sqrt{2}\right)$

67. $(3, 8)$

69. $\left(\frac{\ln 3}{2}, 9\right)$

71. Answer may vary

8.6 Review Exercises and Sample Exam

REVIEW EXERCISES

DISTANCE, MIDPOINT, AND THE PARABOLA

Calculate the distance and midpoint between the given two points.

1. $(0, 2)$ and $(-4, -1)$
2. $(6, 0)$ and $(-2, -6)$
3. $(-2, 4)$ and $(-6, -8)$
4. $\left(\frac{1}{2}, -1\right)$ and $\left(\frac{5}{2}, -\frac{1}{2}\right)$
5. $\left(0, -3\sqrt{2}\right)$ and $\left(\sqrt{5}, -4\sqrt{2}\right)$
6. $\left(-5\sqrt{3}, \sqrt{6}\right)$ and $\left(-3\sqrt{3}, \sqrt{6}\right)$

Determine the area of a circle whose diameter is defined by the given two points.

7. $(-3, 3)$ and $(3, -3)$
8. $(-2, -9)$ and $(-10, -15)$
9. $\left(\frac{2}{3}, -\frac{1}{2}\right)$ and $\left(-\frac{1}{3}, \frac{3}{2}\right)$
10. $\left(2\sqrt{5}, -2\sqrt{2}\right)$ and $\left(0, -4\sqrt{2}\right)$

Rewrite in standard form and give the vertex.

11. $y = x^2 - 10x + 33$
12. $y = 2x^2 - 4x - 1$
13. $y = x^2 - 3x - 1$
14. $y = -x^2 - x - 2$
15. $x = y^2 + 10y + 10$

16. $x = 3y^2 + 12y + 7$
17. $x = -y^2 + 8y - 3$
18. $x = 5y^2 - 5y + 2$

Rewrite in standard form and graph. Be sure to find the vertex and all intercepts.

19. $y = x^2 - 20x + 75$
20. $y = -x^2 - 10x + 75$
21. $y = -2x^2 - 12x - 24$
22. $y = 4x^2 + 4x + 6$
23. $x = y^2 - 10y + 16$
24. $x = -y^2 + 4y + 12$
25. $x = -4y^2 + 12y$
26. $x = 9y^2 + 18y + 12$
27. $x = -4y^2 + 4y + 2$
28. $x = -y^2 - 5y + 2$

CIRCLES

Determine the center and radius given the equation of a circle in standard form.

29. $\left(x - 6\right)^2 + y^2 = 9$
30. $(x + 8)^2 + \left(y - 10\right)^2 = 1$
31. $x^2 + y^2 = 5$
32. $\left(x - \frac{3}{8}\right)^2 + \left(y + \frac{5}{2}\right)^2 = \frac{1}{2}$

Determine standard form for the equation of the circle:

33. Center $(-7, 2)$ with radius $r = 10$.
34. Center $\left(\frac{1}{3}, -1\right)$ with radius $r = \frac{2}{3}$.
35. Center $(0, -5)$ with radius $r = 2\sqrt{7}$.
36. Center $(1, 0)$ with radius $r = \frac{5\sqrt{3}}{2}$.
37. Circle whose diameter is defined by $(-4, 10)$ and $(-2, 8)$.
38. Circle whose diameter is defined by $(3, -6)$ and $(0, -4)$.

Find the *x*- and *y*-intercepts.

39. $(x-3)^2 + (y+5)^2 = 16$
40. $(x+5)^2 + (y-1)^2 = 4$
41. $x^2 + (y-2)^2 = 20$
42. $(x-3)^2 + (y+3)^2 = 8$
43. $x^2 + y^2 - 12y + 27 = 0$
44. $x^2 + y^2 - 4x + 2y + 1 = 0$

Graph.

45. $(x+8)^2 + (y-6)^2 = 4$
46. $(x-20)^2 + \left(y + \frac{15}{2}\right)^2 = \frac{225}{4}$
47. $x^2 + y^2 = 24$
48. $(x-1)^2 + y^2 = \frac{1}{4}$
49. $x^2 + (y-7)^2 = 27$
50. $(x+1)^2 + (y-1)^2 = 2$

Rewrite in standard form and graph.

51. $x^2 + y^2 - 6x + 4y - 3 = 0$

52. $x^2 + y^2 + 8x - 10y + 16 = 0$

53. $2x^2 + 2y^2 - 2x - 6y - 3 = 0$

54. $4x^2 + 4y^2 + 8y + 1 = 0$

55. $x^2 + y^2 - 5x + y - \frac{1}{2} = 0$

56. $x^2 + y^2 + 12x - 8y = 0$

ELLIPSES

Given the equation of an ellipse in standard form, determine its center, orientation, major radius, and minor radius.

57. $\frac{(x+12)^2}{16} + \frac{(y-10)^2}{4} = 1$

58. $\frac{(x+3)^2}{3} + \frac{y^2}{25} = 1$

59. $x^2 + \frac{(y-5)^2}{12} = 1$

60. $\frac{(x-8)^2}{5} + \frac{(y+8)}{18} = 1$

Determine the standard form for the equation of the ellipse given the following information.

61. Center $(0, -4)$ with $a = 3$ and $b = 4$.

62. Center $(3, 8)$ with $a = 1$ and $b = \sqrt{7}$.

63. Center $(0, 0)$ with $a = 5$ and $b = \sqrt{2}$.

64. Center $(-10, -30)$ with $a = 10$ and $b = 1$.

Find the *x*- and *y*-intercepts.

65. $\frac{(x+2)^2}{4} + \frac{y^2}{9} = 1$

66. $\frac{(x-1)^2}{2} + \frac{(y+1)^2}{3} = 1$

67. $5x^2 + 2y^2 = 20$

68. $5(x-3)^2 + 6y^2 = 120$

Graph.

69. $\frac{(x-10)^2}{25} + \frac{(y+5)^2}{4} = 1$

70. $\frac{(x+6)^2}{9} + \frac{(y-8)^2}{36} = 1$

71. $\frac{\left(x-\frac{3}{2}\right)^2}{4} + \left(y - \frac{7}{2}\right)^2 = 1$

72. $\left(x - \frac{2}{3}\right)^2 + \frac{y^2}{4} = 1$

73. $\frac{x^2}{2} + \frac{y^2}{5} = 1$

74. $\frac{(x+2)^2}{8} + \frac{(y-3)^2}{12} = 1$

Rewrite in standard form and graph.

75. $4x^2 + 9y^2 - 8x + 90y + 193 = 0$

76. $9x^2 + 4y^2 + 108x - 80y + 580 = 0$

77. $x^2 + 9y^2 + 6x + 108y + 324 = 0$

78. $25x^2 + y^2 - 350x - 8y + 1{,}216 = 0$

79. $8x^2 + 12y^2 - 16x - 36y - 13 = 0$

80. $10x^2 + 2y^2 - 50x + 14y + 7 = 0$

HYPERBOLAS

Given the equation of a hyperbola in standard form, determine its center, which way the graph opens, and the vertices.

81. $\frac{(x-10)^2}{4} - \frac{(y+5)^2}{16} = 1$

82. $\frac{(x+7)^2}{2} - \frac{(y-8)^2}{8} = 1$

83. $\frac{(y-20)^2}{3} - (x-15)^2 = 1$

84. $3y^2 - 12(x-1)^2 = 36$

Determine the standard form for the equation of the hyperbola.

85. Center $(-25, 10)$, $a = 3, b = \sqrt{5}$, opens up and down.

86. Center $(9, -12)$, $a = 5\sqrt{3}, b = 7$, opens left and right.

87. Center $(-4, 0)$, $a = 1, b = 6$, opens left and right.

88. Center $(-2, -3)$, $a = 10\sqrt{2}, b = 2\sqrt{3}$, opens up and down.

Find the *x*- and *y*-intercepts.

89. $\frac{(x-1)^2}{4} - \frac{(y+3)^2}{9} = 1$

90. $\frac{(x+4)^2}{8} - \frac{(y-2)^2}{12} = 1$

91. $4(y-2)^2 - x^2 = 16$

92. $6(y+1)^2 - 3(x-1)^2 = 18$

Graph.

93. $\frac{(x-10)^2}{25} - \frac{(y+5)^2}{100} = 1$

94. $\frac{(x-4)^2}{4} - \frac{(y-8)^2}{16} = 1$

95. $\frac{(y-3)^2}{9} - \frac{(x-6)^2}{81} = 1$

96. $\frac{(y+1)^2}{4} - \frac{(x+1)^2}{25} = 1$

97. $\frac{y^2}{27} - \frac{(x-3)^2}{9} = 1$

98. $\frac{x^2}{2} - \frac{y^2}{3} = 1$

Rewrite in standard form and graph.

99. $4x^2 - 9y^2 - 8x - 90y - 257 = 0$

100. $9x^2 - y^2 - 108x + 16y + 224 = 0$

101. $25y^2 - 2x^2 - 100y + 50 = 0$

102. $3y^2 - x^2 - 2x - 10 = 0$

103. $8y^2 - 12x^2 + 24y - 12x - 33 = 0$

104. $4y^2 - 4x^2 - 16y - 28x - 37 = 0$

Identify the conic sections and rewrite in standard form.

105. $x^2 + y^2 - 2x - 8y + 16 = 0$

106. $x^2 + 2y^2 + 4x - 24y + 74 = 0$

107. $x^2 - y^2 - 6x - 4y + 3 = 0$

108. $x^2 + y - 10x + 22 = 0$

109. $x^2 + 12y^2 - 12x + 24 = 0$

110. $x^2 + y^2 + 10y + 22 = 0$

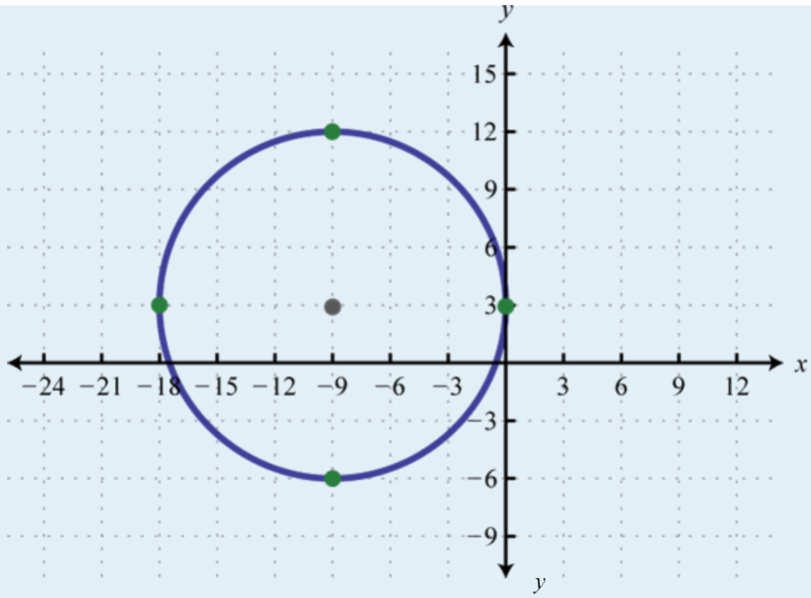
111. $4y^2 - 20x^2 + 16y + 20x - 9 = 0$

112. $16x - 16y^2 + 24y - 25 = 0$

113. $9x^2 - 9y^2 - 6x - 18y - 17 = 0$

114. $4x^2 + 4y^2 + 4x - 8y + 1 = 0$

Given the graph, write the equation in general form.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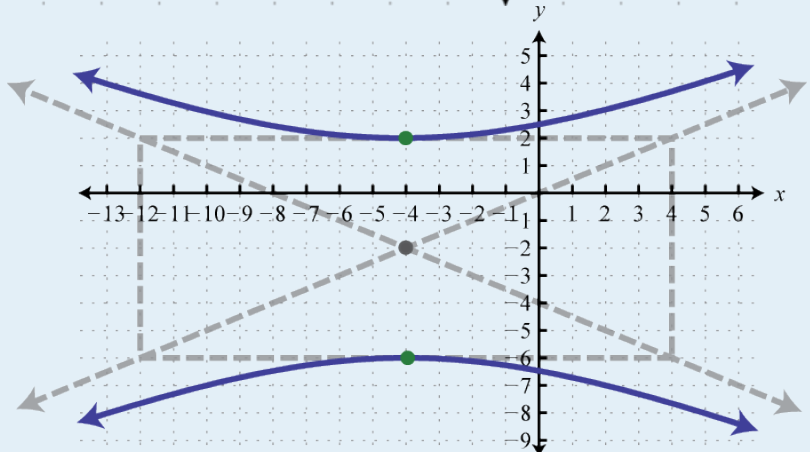
115.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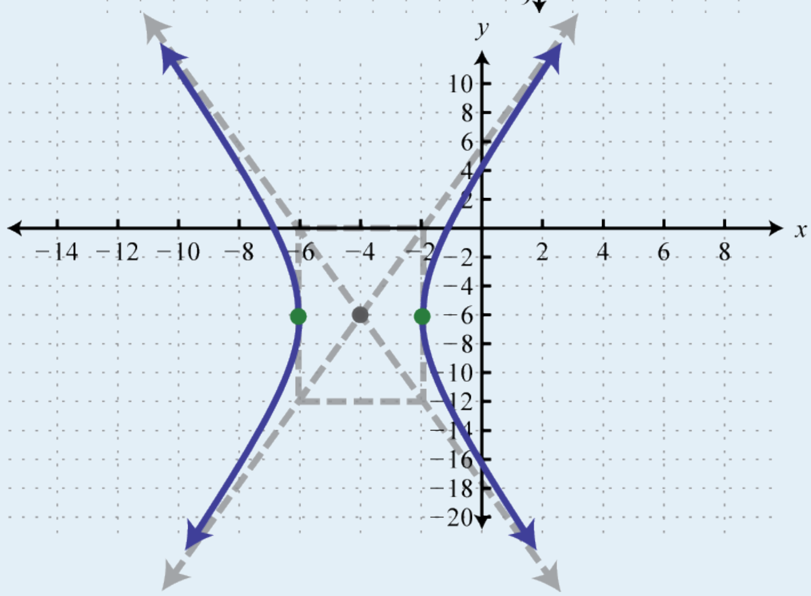
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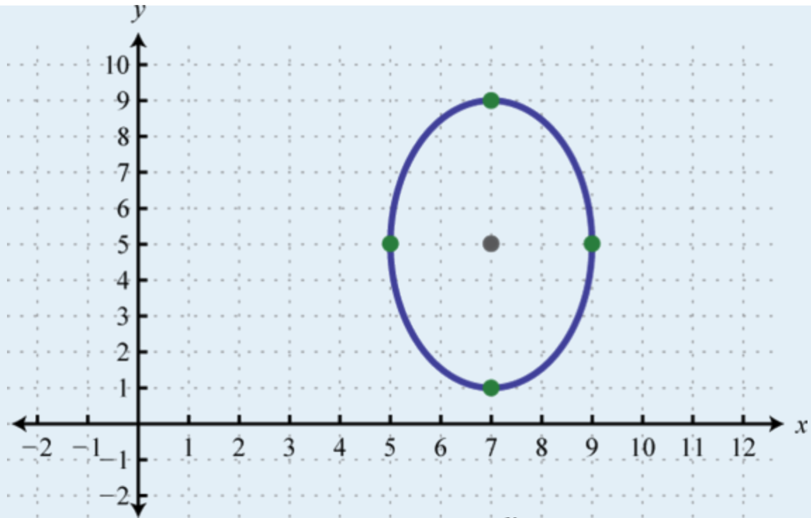
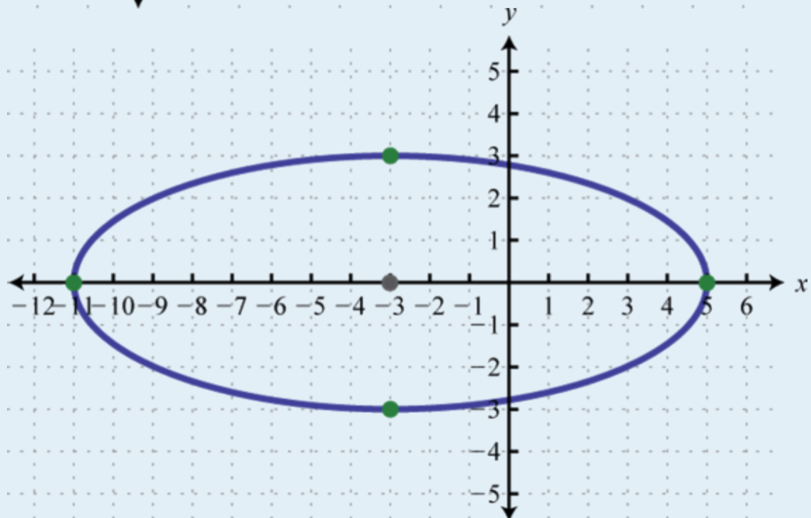
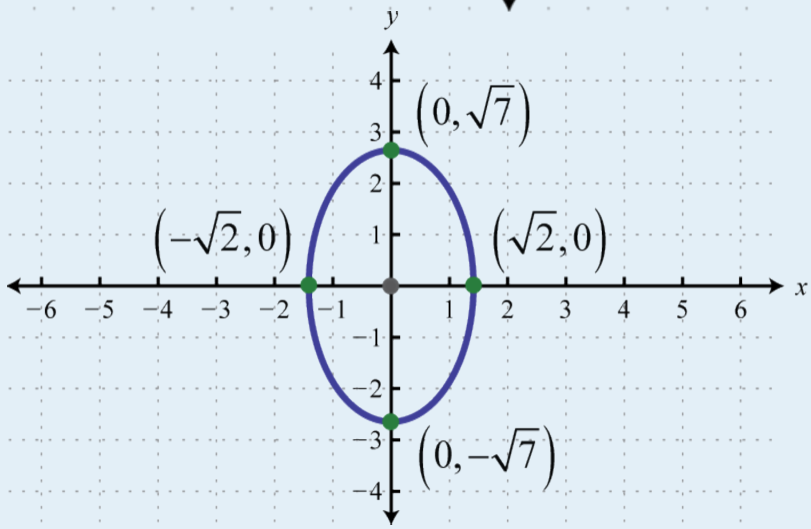
118. 

119. 

120. 

SOLVING NONLINEAR SYSTEMS

Solve.

121. $\begin{cases} x^2 + y^2 = 8 \\ x - y = 4 \end{cases}$

122. $\begin{cases} x^2 + y^2 = 1 \\ x + 2y = 1 \end{cases}$

123. $\begin{cases} x^2 + 3y^2 = 4 \\ 2x - y = 1 \end{cases}$

124. $\begin{cases} 2x^2 + y^2 = 5 \\ x + y = 3 \end{cases}$

125. $\begin{cases} 3x^2 - 2y^2 = 1 \\ x - y = 2 \end{cases}$

126. $\begin{cases} x^2 - 3y^2 = 10 \\ x - 2y = 1 \end{cases}$

127. $\begin{cases} 2x^2 + y^2 = 11 \\ 4x + y^2 = 5 \end{cases}$

128. $\begin{cases} x^2 + 4y^2 = 1 \\ 2x^2 + 4y = 5 \end{cases}$

129. $\begin{cases} 5x^2 - y^2 = 10 \\ x^2 + y = 2 \end{cases}$

130. $\begin{cases} 2x^2 + y^2 = 1 \\ 2x - 4y^2 = -3 \end{cases}$

131. $\begin{cases} x^2 + 4y^2 = 10 \\ xy = 2 \end{cases}$

132. $\begin{cases} y + x^2 = 0 \\ xy - 8 = 0 \end{cases}$

133. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 10 \\ \dfrac{1}{x} - \dfrac{1}{y} = 6 \end{cases}$

134. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 1 \\ y - x = 2 \end{cases}$

135. $\begin{cases} x - 2y^2 = 3 \\ y = \sqrt{x - 4} \end{cases}$

136. $\begin{cases} (x - 1)^2 + y^2 = 1 \\ y - \sqrt{x} = 0 \end{cases}$

ANSWERS

1. Distance: 5 units; midpoint: $\left(-2, \frac{1}{2}\right)$

3. Distance: $4\sqrt{10}$ units; midpoint: $(-4, -2)$

5. Distance: $\sqrt{7}$ units; midpoint: $\left(\frac{\sqrt{5}}{2}, -\frac{7\sqrt{2}}{2}\right)$

7. 18π square units

9. $\frac{5\pi}{4}$ square units

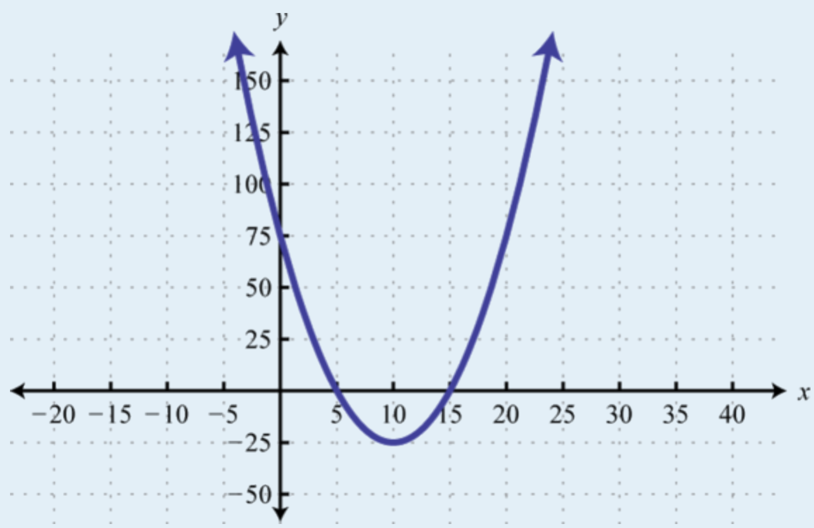
11. $y = (x-5)^2 + 8$; vertex: $(5, 8)$

13. $y = \left(x - \frac{3}{2}\right)^2 - \frac{13}{4}$; vertex: $\left(\frac{3}{2}, -\frac{13}{4}\right)$

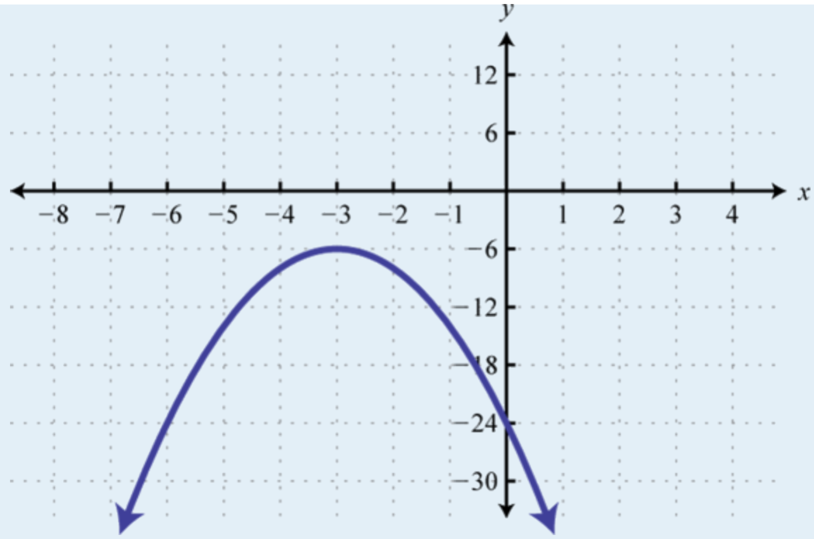
15. $x = (y+5)^2 - 15$; vertex: $(-15, -5)$

17. $x = -(y-4)^2 + 13$; vertex: $(13, 4)$

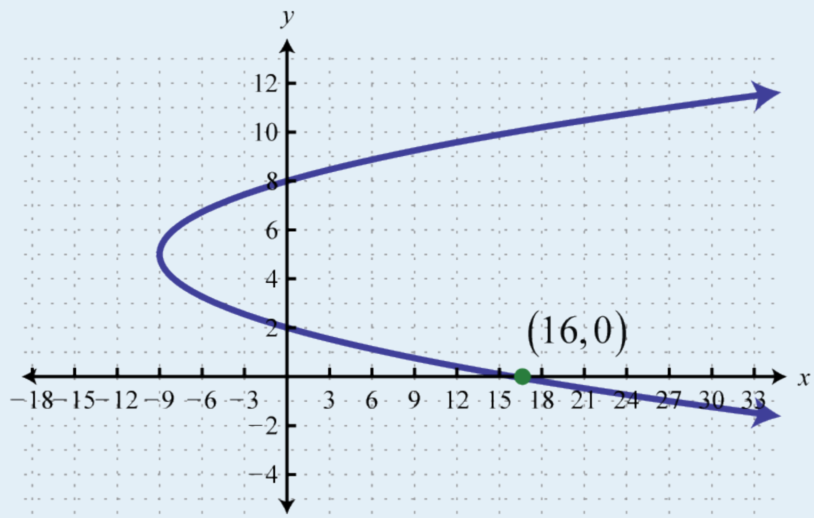
19. $y = (x-10)^2 - 25$;



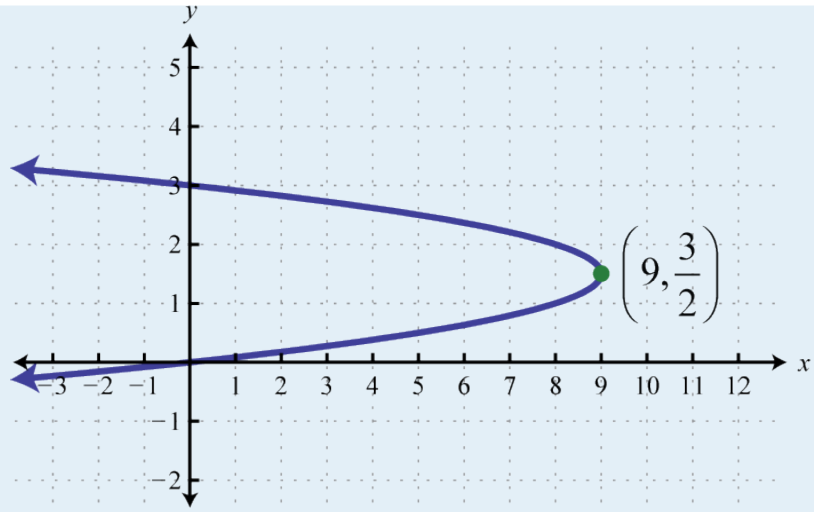
21. $y = -2(x+3)^2 - 6$;



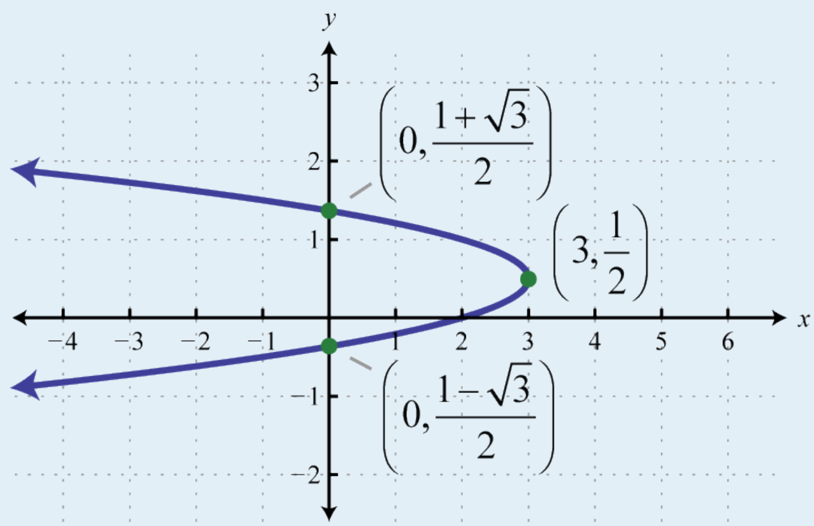
23. $x = (y-5)^2 - 9$;



25. $x = -4\left(y - \frac{3}{2}\right)^2 + 9$;



27. $x = -4\left(y - \frac{1}{2}\right)^2 + 3$;



29. Center: $(6, 0)$; radius: $r = 3$

31. Center: $(0, 0)$; radius: $r = \sqrt{5}$

33. $(x + 7)^2 + (y - 2)^2 = 100$

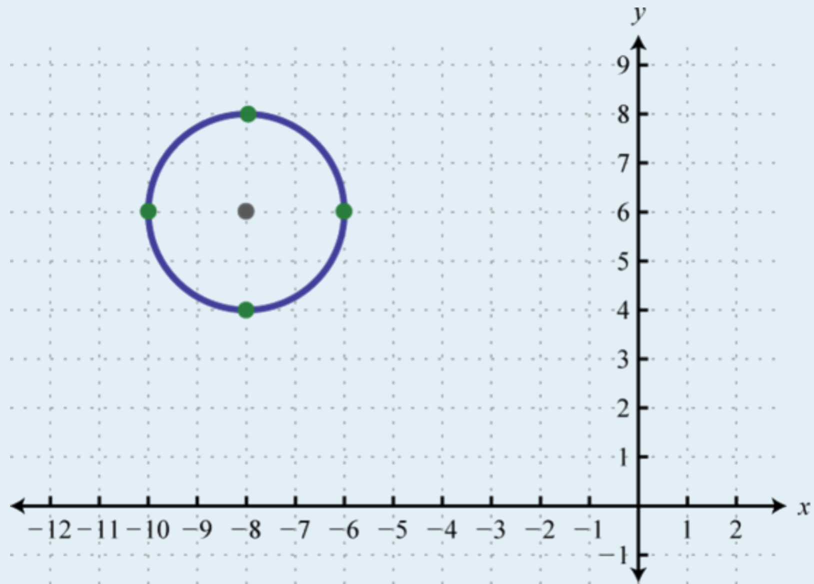
35. $x^2 + (y + 5)^2 = 28$

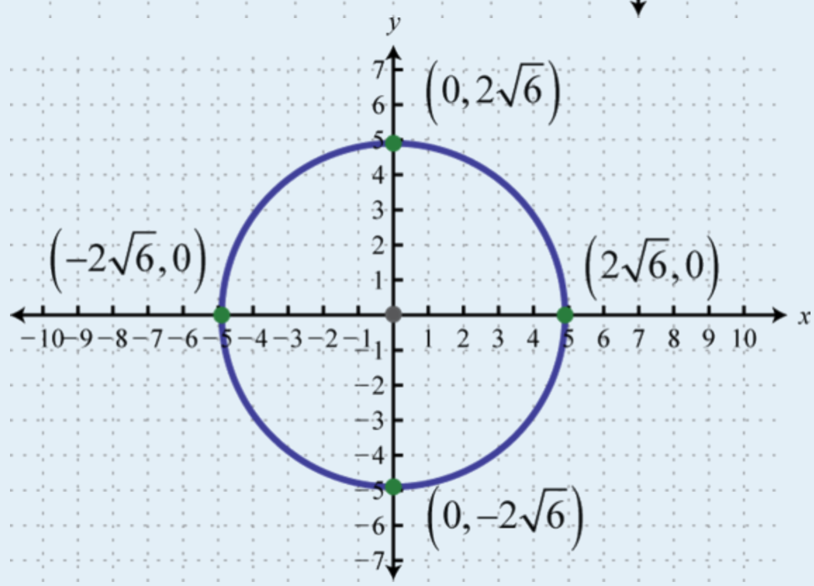
37. $(x + 3)^2 + (y - 9)^2 = 2$

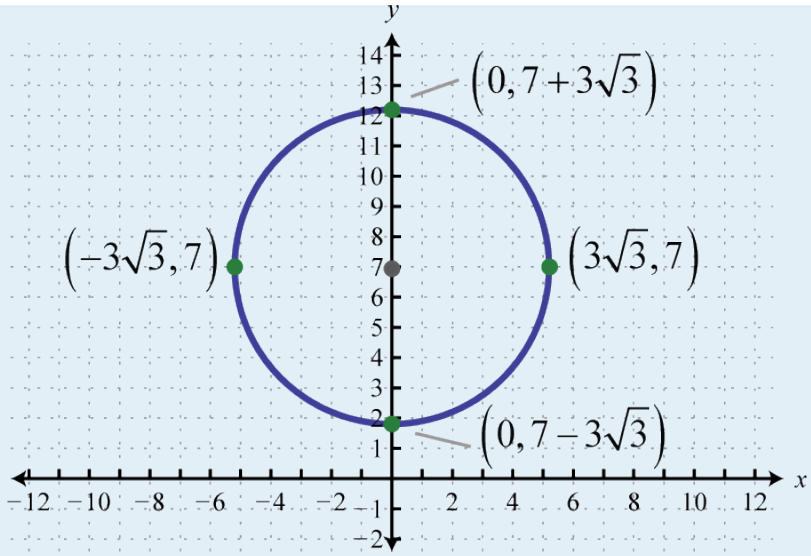
39. x-intercepts: none; y-intercepts: $\left(0, -5 \pm \sqrt{7}\right)$

41. x-intercepts: $(\pm 4, 0)$; y-intercepts: $\left(0, 2 \pm 2\sqrt{5}\right)$

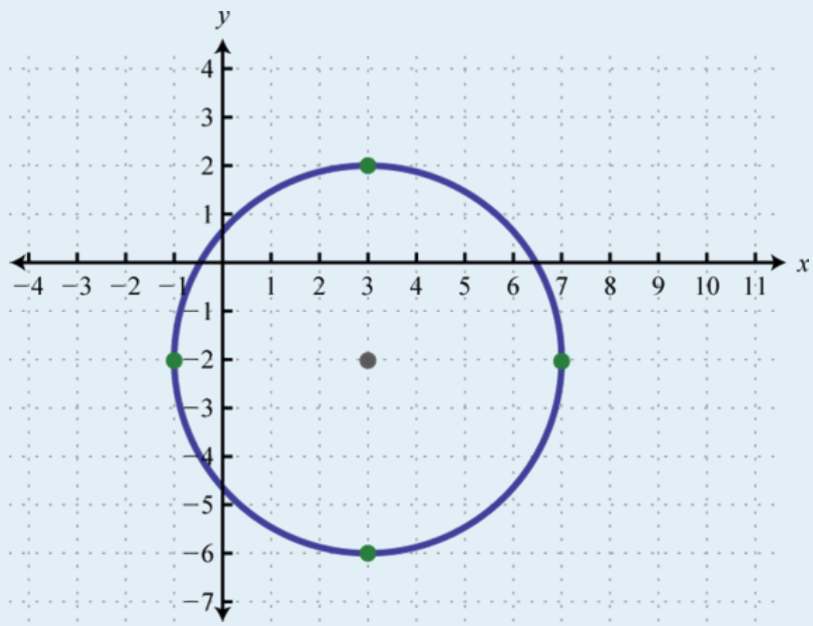
43. x-intercepts: none; y-intercepts: $(0, 3)$, $(0, 9)$

45. 

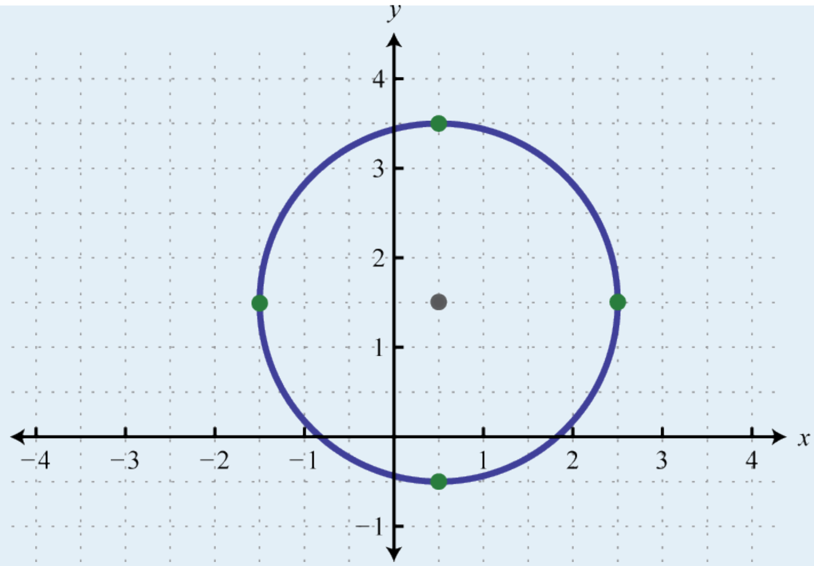
47. 

49. 

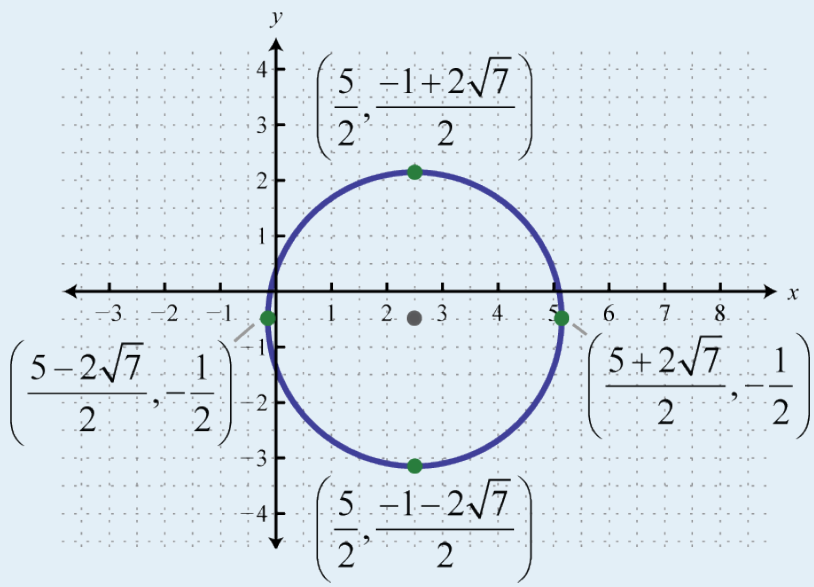
51. $(x-3)^2 + \left(y+2\right)^2 = 16;$



53. $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 = 4;$



55. $\left(x - \frac{5}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = 7$;



57. Center: $(-12, 10)$; orientation: horizontal; major radius: 4 units; minor radius: 2 units

59. Center: $(0, 5)$; orientation: vertical; major radius: $2\sqrt{3}$ units; minor radius: 1 unit

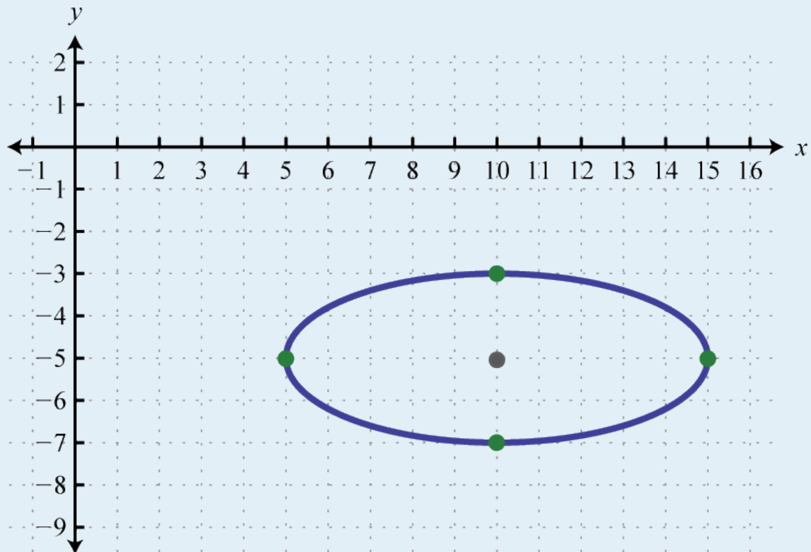
61. $\frac{x^2}{9} + \frac{(y+4)^2}{16} = 1$

63. $\frac{x^2}{25} + \frac{y^2}{2} = 1$

65. x-intercepts: $(-4, 0), (0, 0)$; y-intercepts: $(0, 0)$

67. x-intercepts: $(\pm 2, 0)$; y-intercepts: $\left(0, \pm\sqrt{10}\right)$

69.



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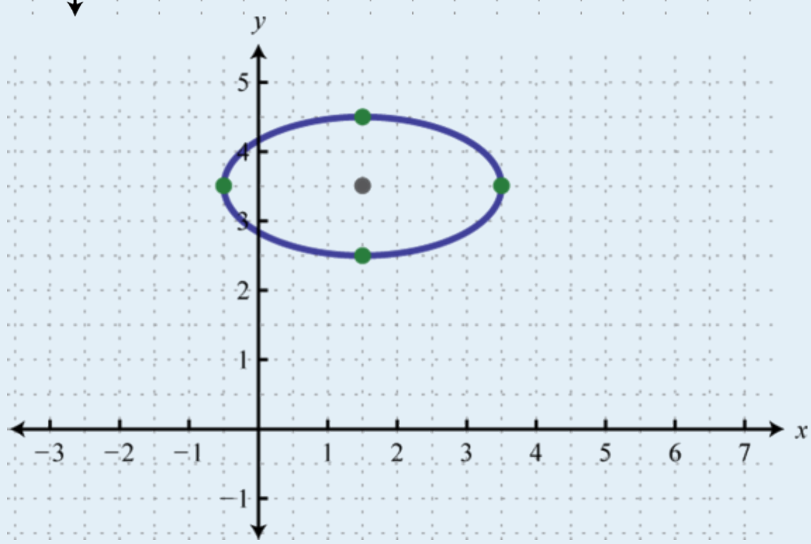
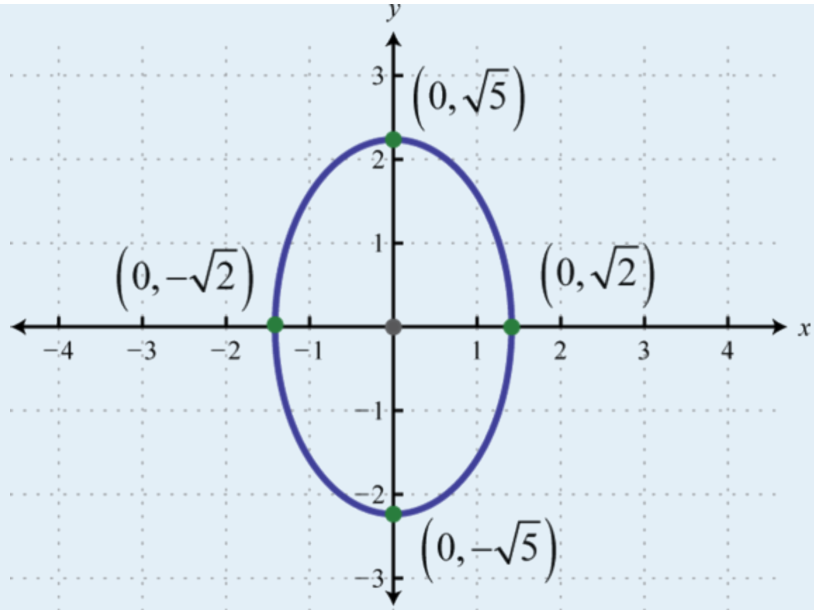
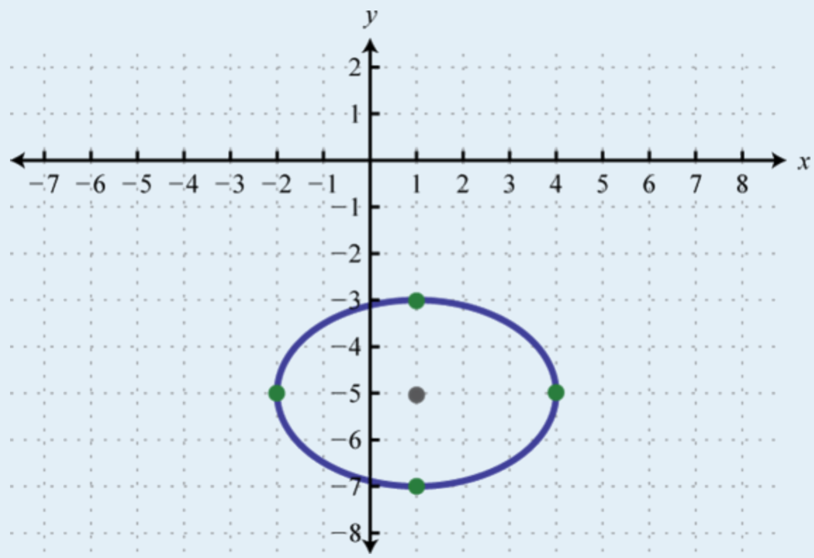


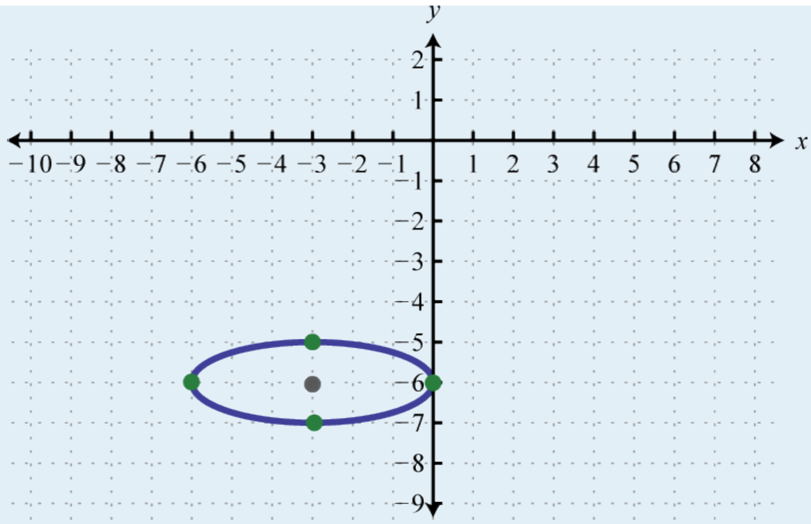


73.

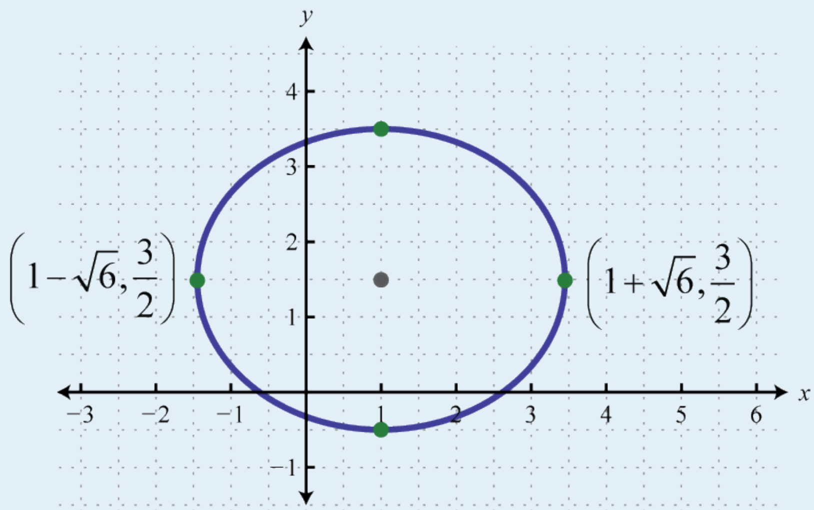
75. $\frac{(x-1)^2}{9} + \frac{(y+5)^2}{4} = 1;$



77. $\frac{(x+3)^2}{9} + (y+6)^2 = 1;$



79. $\frac{(x-1)^2}{6} + \frac{\left(y-\frac{3}{2}\right)^2}{4} = 1$;



81. Center: $(10, -5)$; opens left and right; vertices: $(8, -5)$, $(12, -5)$

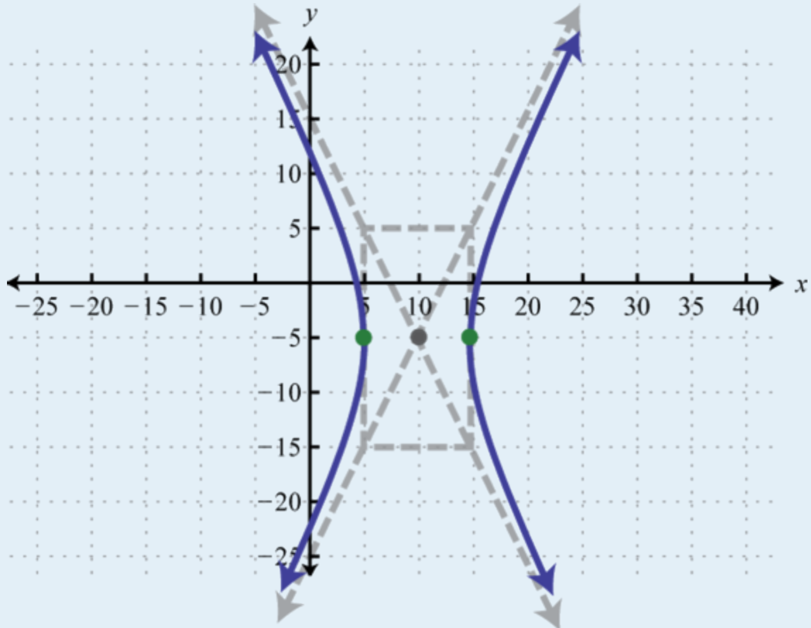
83. Center: $(15, 20)$; opens upward and downward; vertices: $\left(15, 20 - \sqrt{3}\right)$, $\left(15, 20 + \sqrt{3}\right)$

85. $\frac{(y-10)^2}{5} - \frac{(x+25)^2}{9} = 1$

87. $(x+4)^2 - \frac{y^2}{36} = 1$

89. x-intercepts: $\left(1 \pm 2\sqrt{2}, 0\right)$; y-intercepts: none

91. x-intercepts: $(0, 0)$; y-intercepts: $(0, 0), (0, 4)$



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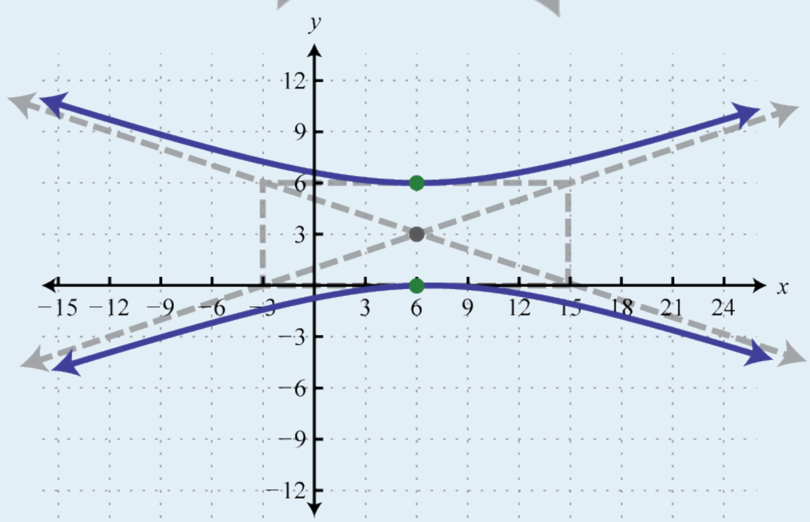

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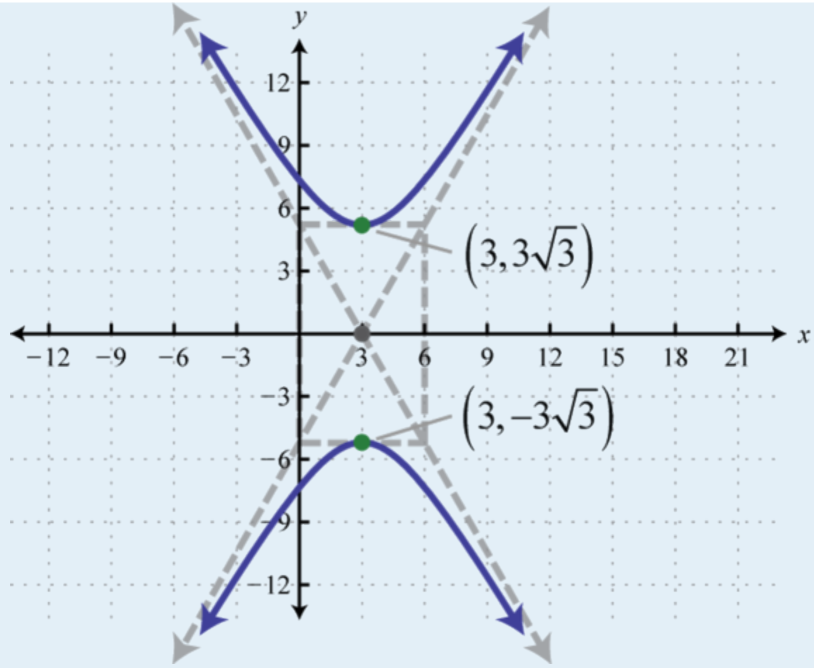
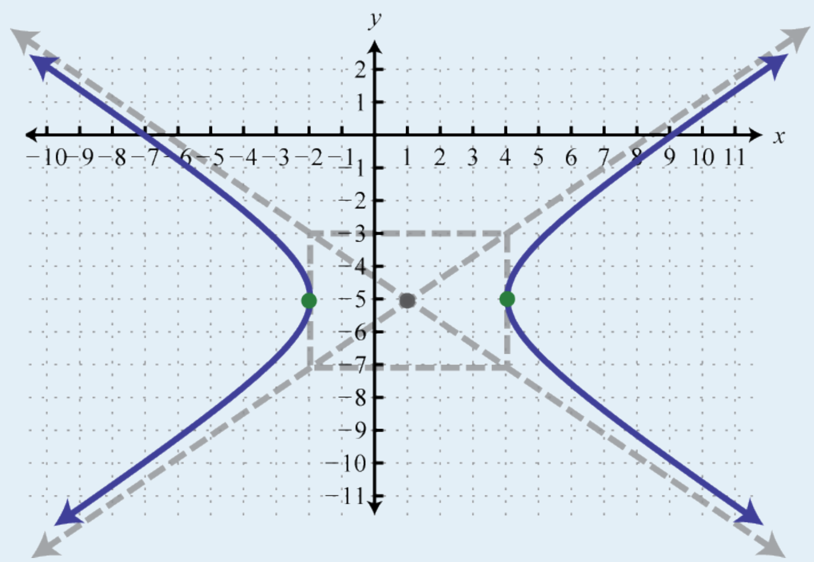
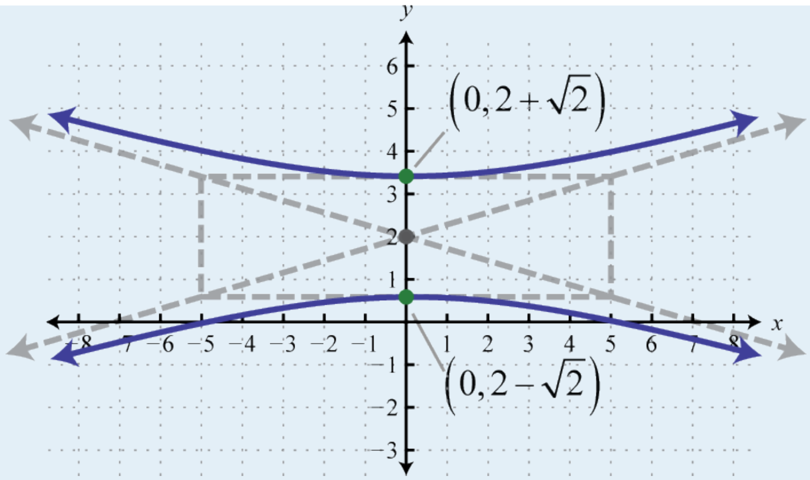


97.

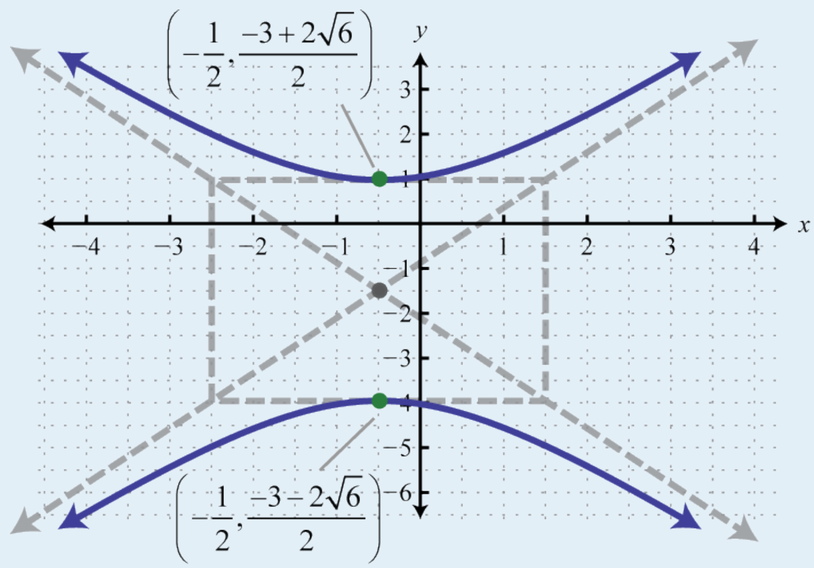
99. $\frac{(x-1)^2}{9} - \frac{(y+5)^2}{4} = 1;$



101. $\frac{(y-2)^2}{2} - \frac{x^2}{25} = 1;$



103. $\frac{\left(y+\frac{3}{2}\right)^2}{6} - \frac{\left(x+\frac{1}{2}\right)^2}{4} = 1$;



105. Circle; $(x-1)^2 + (y-4)^2 = 1$

107. Hyperbola; $\frac{(x-3)^2}{2} - \frac{(y+2)^2}{2} = 1$

109. Ellipse; $\frac{(x-6)^2}{12} + y^2 = 1$

111. Hyperbola; $\frac{(y+2)^2}{5} - \left(x - \frac{1}{2}\right)^2 = 1$

113. Hyperbola; $\left(x - \frac{1}{3}\right)^2 - (y+1)^2 = 1$

115. $x^2 + y^2 + 18x - 6y + 9 = 0$

117. $9x^2 - y^2 + 72x - 12y + 72 = 0$

119. $9x^2 + 64y^2 + 54x - 495 = 0$

121. $(2, -2)$

123. $\left(-\frac{1}{13}, -\frac{15}{13}\right), (1, 1)$

125. $(-9, -11), (1, -1)$

127. $(-1, -3), (-1, 3)$

129. $\left(-\sqrt{2}, 0\right), \left(\sqrt{2}, 0\right), \left(-\sqrt{7}, -5\right), \left(\sqrt{7}, -5\right)$

131. $\left(\sqrt{2}, \sqrt{2}\right), \left(-\sqrt{2}, -\sqrt{2}\right), \left(2\sqrt{2}, \frac{\sqrt{2}}{2}\right), \left(-2\sqrt{2}, -\frac{\sqrt{2}}{2}\right)$

133. $\left(\frac{1}{8}, \frac{1}{2}\right)$

135. $(5, 1)$

SAMPLE EXAM

1. Given two points $\left(-4, -6\right)$ and $(2, -8)$:

   a. Calculate the distance between them.
   b. Find the midpoint between them.

2. Determine the area of a circle whose diameter is defined by the points $(4, -3)$ and $(-1, 2)$.

Rewrite in standard form and graph. Find the vertex and all intercepts if any.

3. $y = -x^2 + 6x - 5$
4. $x = 2y^2 + 4y - 6$
5. $x = -3y^2 + 3y + 1$
6. Find the equation of a circle in standard form with center $\left(-6, 3\right)$ and radius $2\sqrt{5}$ units.

Sketch the graph of the conic section given its equation in standard form.

7. $(x - 4)^2 + \left(y + 1\right)^2 = 45$
8. $\frac{(x+3)^2}{4} + \frac{y^2}{9} = 1$
9. $\frac{y^2}{3} - \frac{x^2}{9} = 1$
10. $\frac{x^2}{16} - \left(y - 2\right)^2 = 1$

Rewrite in standard form and graph.

11. $9x^2 + 4y^2 - 144x + 16y + 556 = 0$
12. $x - y^2 + 6y + 7 = 0$
13. $x^2 + y^2 + 20x - 20y + 100 = 0$
14. $4y^2 - x^2 + 40y - 30x - 225 = 0$

Find the *x*- and *y*-intercepts.

15. $x = -2(y-4)^2 + 9$

16. $\frac{(y-1)^2}{12} - (x+1)^2 = 1$

Solve.

17. $\begin{cases} x + y = 2 \\ y = -x^2 + 4 \end{cases}$

18. $\begin{cases} y - x^2 = -3 \\ x^2 + y^2 = 9 \end{cases}$

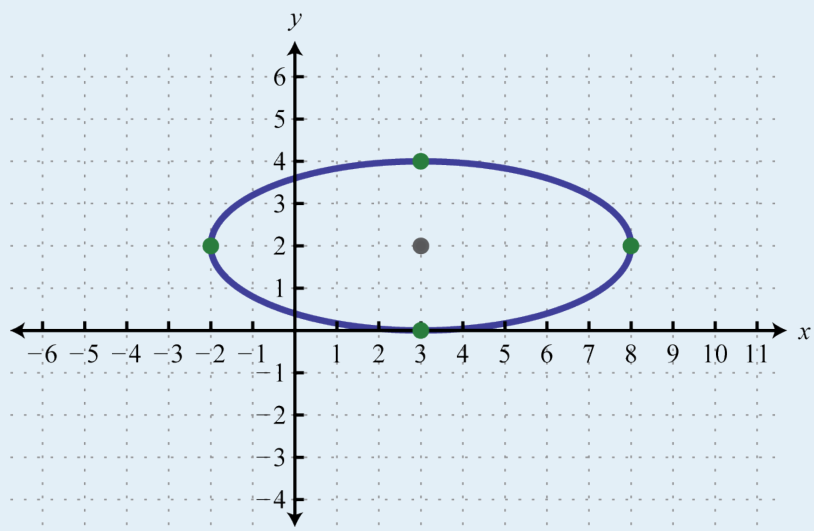
19. $\begin{cases} 2x - y = 1 \\ (x+1)^2 + 2y^2 = 1 \end{cases}$

20. $\begin{cases} x^2 + y^2 = 6 \\ xy = 3 \end{cases}$

21. Find the equation of an ellipse in standard form with vertices $(-3, -5)$ and $(5, -5)$ and a minor radius 2 units in length.

22. Find the equation of a hyperbola in standard form opening left and right with vertices $\left(\pm\sqrt{5}, 0\right)$ and a conjugate axis that measures 10 units.

23. Given the graph of the ellipse, determine its equation in general form.



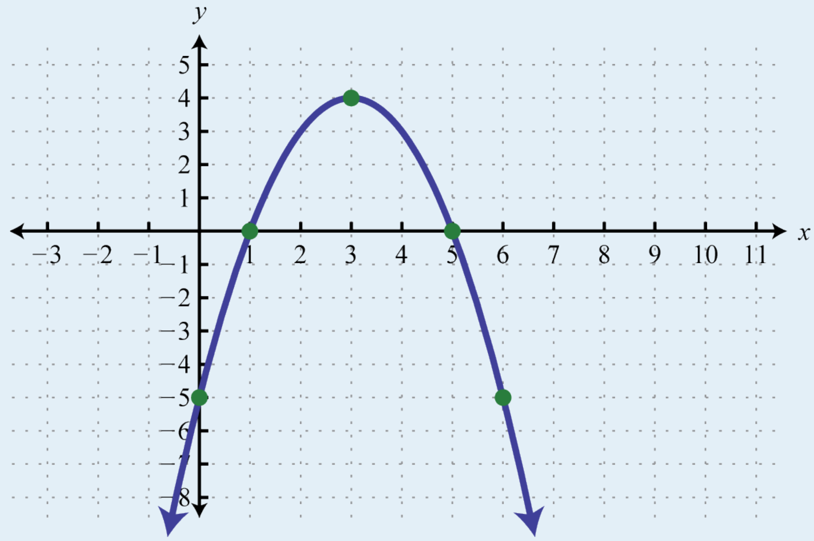
24. A rectangular deck has an area of 80 square feet and a perimeter that measures 36 feet. Find the dimensions of the deck.

25. The diagonal of a rectangle measures $2\sqrt{13}$ centimeters and the perimeter measures 20 centimeters. Find the dimensions of the rectangle.

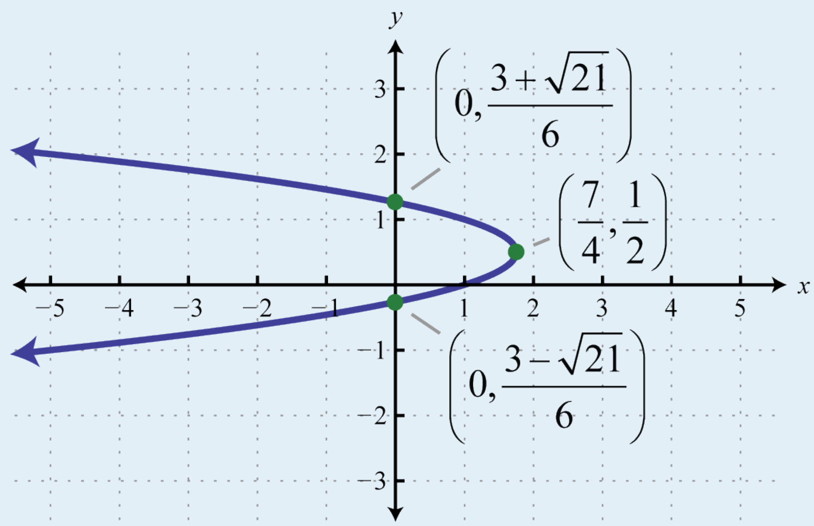
ANSWERS

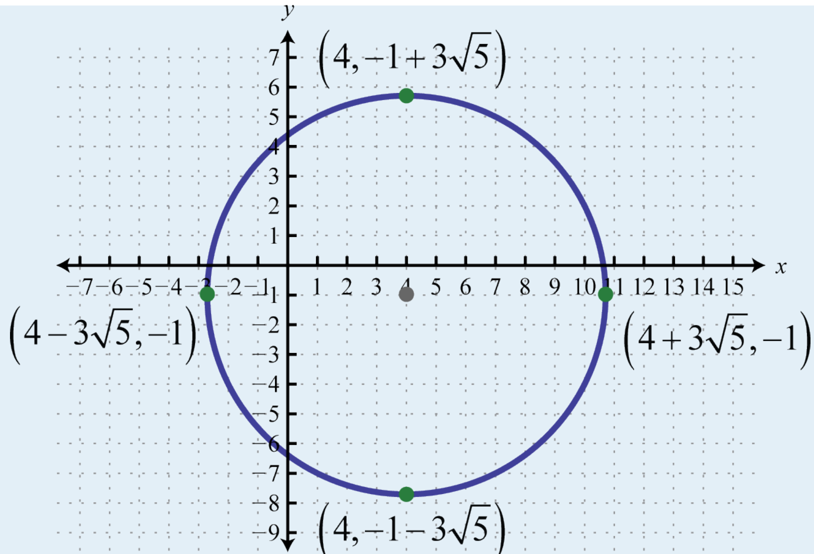
1. a. $2\sqrt{10}$ units;
   b. $(-1, -7)$

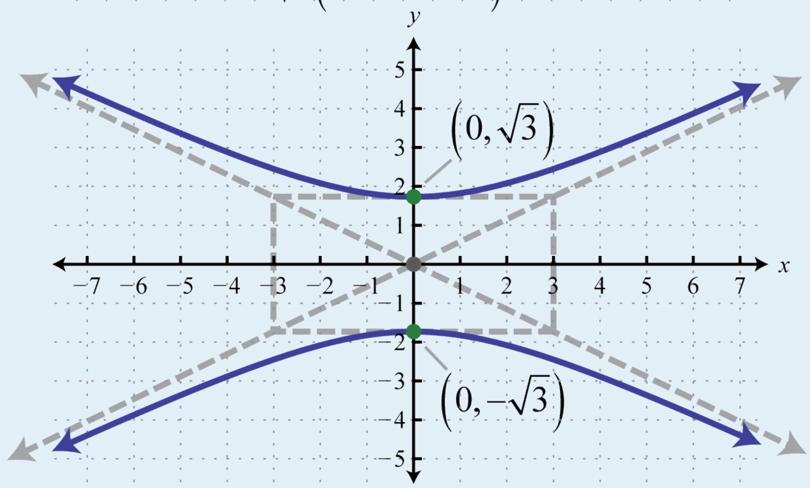
3. $y = -(x - 3)^2 + 4$;



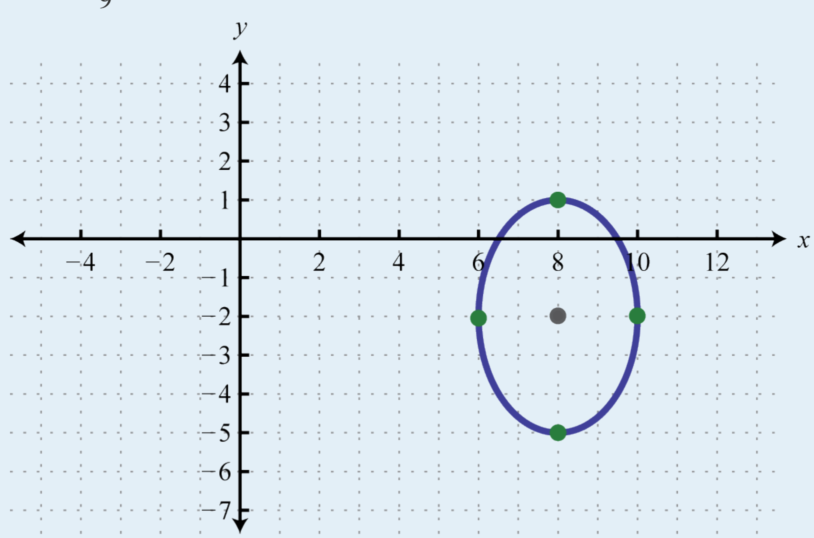
5. $x = -3\left(y - \frac{1}{2}\right)^2 + \frac{7}{4}$;



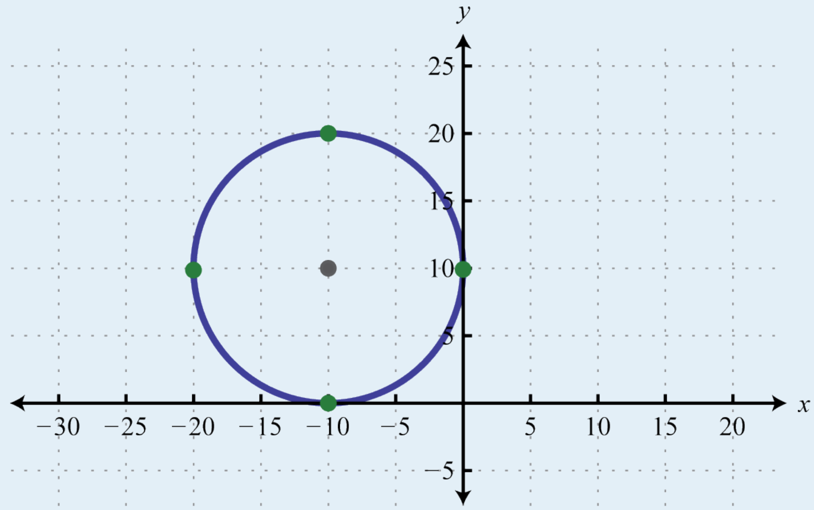
7. 

9. 

11. $\frac{(x-8)^2}{4} + \frac{(y+2)^2}{9} = 1;$



13. $(x + 10)^2 + \left(y - 10\right)^2 = 100$;



15. x-intercept: $(-23, 0)$; y-intercepts: $\left(0, \frac{8 \pm 3\sqrt{2}}{2}\right)$

17. $(-1, 3), (2, 0)$

19. Ø

21. $\frac{(x-1)^2}{16} + \frac{\left(y+5\right)^2}{4} = 1$

23. $4x^2 + 25y^2 - 24x - 100y + 36 = 0$

25. 6 centimeters by 4 centimeters

Chapter 9

Sequences, Series, and the Binomial Theorem

9.1 Introduction to Sequences and Series

LEARNING OBJECTIVES

1. Find any element of a sequence given a formula for its general term.
2. Use sigma notation and expand corresponding series.
3. Distinguish between a sequence and a series.
4. Calculate the *n*th partial sum of sequence.

Sequences

A **sequence**[1] is a function whose domain is a set of consecutive natural numbers beginning with 1. For example, the following equation with domain $\{1, 2, 3, \ldots\}$ defines an **infinite sequence**[2]:

$$a(n) = 5n - 3 \quad or \quad a_n = 5n - 3$$

The elements in the range of this function are called terms of the sequence. It is common to define the *n*th term, or the **general term of a sequence**[3], using the subscritped notation a_n, which reads "*a* sub *n*." Terms can be found using substitution as follows:

$$\begin{aligned} &\text{General term}: & a_n &= 5n - 3 \\ \\ &\text{First term } (n = 1): & a_1 &= 5(1) - 3 = 2 \\ &\text{Second term } (n = 2): & a_2 &= 5(2) - 3 = 7 \\ &\text{Third term } (n = 3): & a_3 &= 5(3) - 3 = 12 \\ &\text{Fourth term } (n = 4): & a_3 &= 5(4) - 3 = 17 \\ &\text{Fifth term } (n = 5): & a_3 &= 5\left(5\right) - 3 = 22 \\ & & &\vdots \end{aligned}$$

1. A function whose domain is a set of consecutive natural numbers starting with 1.
2. A sequence whose domain is the set of natural numbers $\{1, 2, 3, \ldots\}$.
3. An equation that defines the *n*th term of a sequence commonly denoted using subscripts a_n.

This produces an ordered list,

$$2, 7, 12, 17, 22, \ldots$$

The ellipsis (...) indicates that this sequence continues forever. Unlike a set, order matters. If the domain of a sequence consists of natural numbers that end, such as $\{1, 2, 3, \ldots, k\}$, then it is called a **finite sequence**[4].

4. A sequence whose domain is $\{1, 2, 3, \ldots, k\}$ where k is a natural number.

Example 1

Given the general term of a sequence, find the first 5 terms as well as the 100th term: $a_n = \frac{n(n-1)}{2}$.

Solution:

To find the first 5 terms, substitute 1, 2, 3, 4, and 5 for n and then simplify.

$$\begin{aligned} a_1 &= \frac{1(1-1)}{2} = \frac{1(0)}{2} = \frac{0}{2} = 0 \\ a_2 &= \frac{2(2-1)}{2} = \frac{2(1)}{2} = \frac{2}{2} = 1 \\ a_3 &= \frac{3(3-1)}{2} = \frac{3(2)}{2} = \frac{6}{2} = 3 \\ a_4 &= \frac{4(4-1)}{2} = \frac{4(3)}{2} = \frac{12}{2} = 6 \\ a_5 &= \frac{5\left(5-1\right)}{2} = \frac{5(4)}{2} = \frac{20}{2} = 10 \end{aligned}$$

Use $n = 100$ to determine the 100th term in the sequence.

$$a_{100} = \frac{100(100-1)}{2} = \frac{100(99)}{2} = \frac{9{,}900}{2} = 4{,}950$$

Answer: First five terms: 0, 1, 3, 6, 10; $a_{100} = 4{,}950$

Sometimes the general term of a sequence will alternate in sign and have a variable other than n.

Example 2

Find the first 5 terms of the sequence: $a_n = (-1)^n x^{n+1}$.

Solution:

Here we take care to replace *n* with the first 5 natural numbers and not *x*.

$$\begin{aligned} a_1 &= (-1)^1 x^{1+1} &&= -x^2 \\ a_2 &= (-1)^2 x^{2+1} &&= x^3 \\ a_3 &= (-1)^3 x^{3+1} &&= -x^4 \\ a_4 &= (-1)^4 x^{4+1} &&= x^5 \\ a_5 &= (-1)^5 x^{5+1} &&= -x^6 \end{aligned}$$

Answer: $-x^2, x^3, -x^4, x^5, -x^6$

Try this! Find the first 5 terms of the sequence: $a_n = (-1)^{n+1} 2^n$.

Answer: 2, -4, 8, -16, 32.

(click to see video)

One interesting example is the Fibonacci sequence. The first two numbers in the Fibonacci sequence are 1, and each successive term is the sum of the previous two. Therefore, the general term is expressed in terms of the previous two as follows:

$$F_n = F_{n-2} + F_{n-1}$$

Here $F_1 = 1, F_2 = 1$, and $n > 2$. A formula that describes a sequence in terms of its previous terms is called a **recurrence relation**[5].

Example 3

Find the first 7 Fibonacci numbers.

Solution:

Given that $F_1 = 1$ and $F_2 = 1$, use the recurrence relation $F_n = F_{n-2} + F_{n-1}$ where n is an integer starting with $n = 3$ to find the next 5 terms:

$$\begin{aligned}
F_3 &= F_{3-2} + F_{3-1} = F_1 + F_2 = 1 + 1 = 2 \\
F_4 &= F_{4-2} + F_{4-1} = F_2 + F_3 = 1 + 2 = 3 \\
F_5 &= F_{5-2} + F_{5-1} = F_3 + F_4 = 2 + 3 = 5 \\
F_6 &= F_{6-2} + F_{6-1} = F_4 + F_5 = 3 + 5 = 8 \\
F_7 &= F_{7-2} + F_{7-1} = F_5 + F_6 = 5 + 8 = 13
\end{aligned}$$

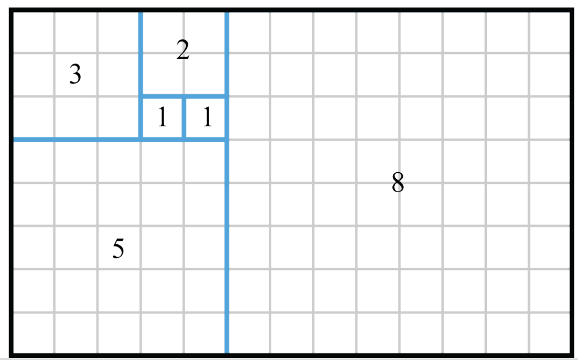
Answer: 1, 1, 2, 3, 5, 8, 13

Figure 9.1

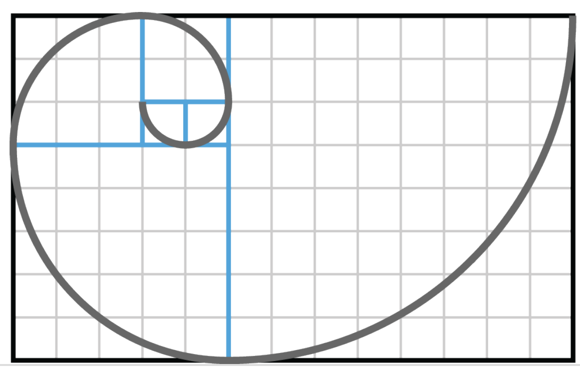
5. A formula that uses previous terms of a sequence to describe subsequent terms.

Leonardo Fibonacci (1170–1250) Wikipedia

Fibonacci numbers appear in applications ranging from art to computer science and biology. The beauty of this sequence can be visualized by constructing a Fibonacci spiral. Consider a tiling of squares where each side has a length that matches each Fibonacci number:



Connecting the opposite corners of the squares with an arc produces a special spiral shape.



This shape is called the Fibonacci spiral and approximates many spiral shapes found in nature.

Series

A **series**[6] is the sum of the terms of a sequence. The sum of the terms of an infinite sequence results in an **infinite series**[7], denoted S_∞. The sum of the first n terms in a sequence is called a **partial sum**[8], denoted S_n. For example, given the sequence of positive odd integers 1, 3, 5,... we can write:

6. The sum of the terms of a sequence.
7. The sum of the terms of an infinite sequence denoted S_∞.
8. The sum of the first n terms in a sequence denoted S_n.

$$S_\infty = 1 + 3 + 5 + 7 + 9 + \cdots \quad \textit{Infinite series}$$
$$S_5 = 1 + 3 + 5 + 7 + 9 = 25 \quad \textit{5th partial sum}$$

Example 4

Determine the 3rd and 5th partial sums of the sequence: 3,−6, 12,−24, 48,...

Solution:

$$S_3 = 3 + (-6) + 12 = 9$$
$$S_5 = 3 + (-6) + 12 + (-24) + 48 = 33$$

Answer: $S_3 = 9$; $S_5 = 33$

If the general term is known, then we can express a series using **sigma**[9] (or **summation**[10]) **notation**:

$$S_\infty = \sum_{n=1}^{\infty} n^2 = 1^2 + 2^2 + 3^2 + \ldots \quad \textit{Infinite series}$$
$$S_3 = \sum_{n=1}^{3} n^2 = 1^2 + 2^2 + 3^2 \quad \textit{3rd partial sum}$$

The symbol Σ (upper case Greek letter sigma) is used to indicate a series. The expressions above and below indicate the range of the **index of summation**[11], in this case represented by n. The lower number indicates the starting integer and the

9. A sum denoted using the symbol Σ (upper case Greek letter sigma).

10. Used when referring to sigma notation.

11. The variable used in sigma notation to indicate the lower and upper bounds of the summation.

upper value indicates the ending integer. The nth partial sum S_n can be expressed using sigma notation as follows:

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

This is read, "the sum of a_k as k goes from 1 to n." Replace n with ∞ to indicate an infinite sum.

Example 5

Evaluate: $\sum_{k=1}^{5} (-3)^{n-1}$.

$$\begin{aligned}\sum_{k=1}^{5} (-3)^{k-1} &= (-3)^{1-1} + (-3)^{2-1} + (-3)^{3-1} + (-3)^{4-1} + (-3)^{5-1} \\ &= (-3)^0 + (-3)^1 + (-3)^2 + (-3)^3 + (-3)^4 \\ &= 1 - 3 + 9 - 27 + 81 \\ &= 61\end{aligned}$$

Answer: 61

When working with sigma notation, the index does not always start at 1.

Example 6

Evaluate: $\sum_{k=2}^{5}(-1)^k\left(3k\right)$.

Solution:

Here the index is expressed using the variable k, which ranges from 2 to 5.

$$\begin{aligned}\sum_{k=2}^{5}(-1)^k(3k) &= (-1)^2(3\cdot 2)+(-1)^3(3\cdot 3)+(-1)^4(3\cdot 4)+(-1)^5(3\cdot 5)\\ &= 6-9+12-15\\ &= -6\end{aligned}$$

Answer: -6

Try this! Evaluate: $\sum_{n=1}^{5}\left(15-9n\right)$.

Answer: -60

(click to see video)

Infinity is used as the upper bound of a sum to indicate an infinite series.

Example 7

Write in expanded form: $\sum_{n=0}^{\infty} \frac{n}{n+1}$.

Solution:

In this case we begin with $n = 0$ and add three dots to indicate that this series continues forever.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{n}{n+1} &= \frac{0}{0+1} + \frac{1}{1+1} + \frac{2}{2+1} + \frac{3}{3+1} + \cdots \\ &= \frac{0}{1} + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots \\ &= 0 + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots\end{aligned}$$

Answer: $0 + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$

When expanding a series, take care to replace only the variable indicated by the index.

Example 8

Write in expanded form: $\sum_{i=1}^{\infty}(-1)^{i-1}x^{2i}$.

Solution:

$$\begin{aligned}\sum_{i=1}^{\infty}(-1)^{i-1}x^{2i} &= (-1)^{1-1}x^{2(1)} + (-1)^{2-1}x^{2(2)} + (-1)^{3-1}x^{2(3)} + \cdots \\ &= (-1)^{0}x^{2(1)} + (-1)^{1}x^{2(2)} + (-1)^{2}x^{2(3)} + \cdots \\ &= x^2 - x^4 + x^6 - \cdots\end{aligned}$$

Answer: $x^2 - x^4 + x^6 - \cdots$

KEY TAKEAWAYS

- A sequence is a function whose domain consists of a set of natural numbers beginning with 1. In addition, a sequence can be thought of as an ordered list.
- Formulas are often used to describe the *n*th term, or general term, of a sequence using the subscripted notation a_n.
- A series is the sum of the terms in a sequence. The sum of the first n terms is called the *n*th partial sum and is denoted S_n.
- Use sigma notation to denote summations in a compact manner. The *n*th partial sum, using sigma notation, can be written $S_n = \sum_{k=1}^{n} a_k$. The symbol Σ denotes a summation where the expression below indicates that the index k starts at 1 and iterates through the natural numbers ending with the value n above.

TOPIC EXERCISES

PART A: SEQUENCES

Find the first 5 terms of the sequence as well as the 30th term.

1. $a_n = 2n$
2. $a_n = 2n + 1$
3. $a_n = \frac{n^2-1}{2}$
4. $a_n = \frac{n}{2n-1}$
5. $a_n = (-1)^n (n+1)^2$
6. $a_n = (-1)^{n+1} n^2$
7. $a_n = 3^{n-1}$
8. $a_n = 2^{n-2}$
9. $a_n = \left(\frac{1}{2}\right)^n$
10. $a_n = \left(-\frac{1}{3}\right)^n$
11. $a_n = \frac{(-1)^{n-1}}{3n-1}$
12. $a_n = \frac{2(-1)^n}{n+5}$
13. $a_n = 1 + \frac{1}{n}$
14. $a_n = \frac{n^2+1}{n}$

Find the first 5 terms of the sequence.

15. $a_n = 2x^{2n-1}$
16. $a_n = (2x)^{n-1}$
17. $a_n = \frac{x^n}{n+4}$

18. $a_n = \frac{x^{2n}}{x-2}$

19. $a_n = \frac{nx^{2n}}{n+1}$

20. $a_n = \frac{(n+1)x^n}{n^2}$

21. $a_n = (-1)^n x^{3n}$

22. $a_n = (-1)^{n-1} x^{n+1}$

Find the first 5 terms of the sequence defined by the given recurrence relation.

23. $a_n = a_{n-1} + 5$ where $a_1 = 3$

24. $a_n = a_{n-1} - 3$ where $a_1 = 4$

25. $a_n = 3a_{n-1}$ where $a_1 = -2$

26. $a_n = -2a_{n-1}$ where $a_1 = -1$

27. $a_n = na_{n-1}$ where $a_1 = 1$

28. $a_n = (n-1)a_{n-1}$ where $a_1 = 1$

29. $a_n = 2a_{n-1} - 1$ where $a_1 = 0$

30. $a_n = 3a_{n-1} + 1$ where $a_1 = -1$

31. $a_n = a_{n-2} + 2a_{n-1}$ where $a_1 = -1$ and $a_2 = 0$

32. $a_n = 3a_{n-1} - a_{n-2}$ where $a_1 = 0$ and $a_2 = 2$

33. $a_n = a_{n-1} - a_{n-2}$ where $a_1 = 1$ and $a_2 = 3$

34. $a_n = a_{n-2} + a_{n-1} + 2$ where $a_1 = -4$ and $a_2 = -1$

Find the indicated term.

35. $a_n = 2 - 7n; a_{12}$

36. $a_n = 3n - 8; a_{20}$

37. $a_n = -4(5)^{n-4}; a_7$

38. $a_n = 6\left(\frac{1}{3}\right)^{n-6}; a_9$

39. $a_n = 1 + \frac{1}{n}; a_{10}$

40. $a_n = (n+1)\,5^{n-3}; a_5$

41. $a_n = (-1)^n 2^{2n-3}; a_4$

42. $a_n = n(n-1)(n-2); a_6$

43. An investment of $4,500 is made in an account earning 2% interest compounded quarterly. The balance in the account after n quarters is given by $a_n = 4500\left(1 + \frac{0.02}{4}\right)^n$. Find the amount in the account after each quarter for the first two years. Round to the nearest cent.

44. The value of a new car after n years is given by the formula $a_n = 18{,}000\left(\frac{3}{4}\right)^n$. Find and interpret a_7. Round to the nearest whole dollar.

45. The number of comparisons a computer algorithm makes to sort n names in a list is given by the formula $a_n = n\log_2 n$. Determine the number of comparisons it takes this algorithm to sort 2×10^6 (2 million) names.

46. The number of comparisons a computer algorithm makes to search n names in a list is given by the formula $a_n = n^2$. Determine the number of comparisons it takes this algorithm to search 2×10^6 (2 million) names.

PART B: SERIES

Find the indicated partial sum.

47. 3, 5, 9, 17, 33,…; S_4

48. −5, 7, −29, 79, −245,…; S_4

49. 4, 1, −4, −11, −20,…; S_5

50. 0, 2, 6, 12, 20,…; S_3

51. $a_n = 2 - 7n; S_5$

52. $a_n = 3n - 8; S_5$

53. $a_n = -4(5)^{n-4}; S_3$

54. $a_n = 6\left(\frac{1}{3}\right)^{n-6}; S_3$

55. $a_n = 1 + \frac{1}{n}; S_4$

56. $a_n = (n+1)\, 5^{n-3}; S_3$

57. $a_n = (-1)^n 2^{2n-3}; S_5$

58. $a_n = n\,(n-1)\ (n-2); S_4$

Evaluate.

59. $\sum_{k=1}^{5} 3k$

60. $\sum_{k=1}^{6} 2k$

61. $\sum_{i=2}^{6} i^2$

62. $\sum_{i=0}^{4} (i+1)^2$

63. $\sum_{n=1}^{5} (-1)^{n+1} 2^n$

64. $\sum_{n=5}^{10} (-1)^n n^2$

65. $\sum_{k=-2}^{2} \left(\frac{1}{2}\right)^k$

66. $\sum_{k=-4}^{0} \left(\frac{1}{3}\right)^k$

67. $\sum_{k=0}^{4} (-2)^{k+1}$

68. $\sum_{k=-1}^{3} (-3)^{k-1}$

69. $\sum_{n=1}^{5} 3$

70. $\sum_{n=1}^{7} -5$

71. $\sum_{k=-2}^{3} k\left(k+1\right)$

72. $\sum_{k=-2}^{2} \left(k-2\right)\left(k+2\right)$

Write in expanded form.

73. $\sum_{n=1}^{\infty} \frac{n-1}{n}$

74. $\sum_{n=1}^{\infty} \frac{n}{2n-1}$

75. $\sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^{n-1}$

76. $\sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)^{n+1}$

77. $\sum_{n=1}^{\infty} 3\left(\frac{1}{5}\right)^{n}$

78. $\sum_{n=0}^{\infty} 2\left(\frac{1}{3}\right)^{n}$

79. $\sum_{k=0}^{\infty} (-1)^{k} x^{k+1}$

80. $\sum_{k=1}^{\infty} (-1)^{k+1} x^{k-1}$

81. $\sum_{i=0}^{\infty} (-2)^{i+1} x^{i}$

82. $\sum_{i=1}^{\infty} (-3)^{i-1} x^{3i}$

83. $\sum_{k=1}^{\infty} (2k-1) x^{2k}$

84. $\sum_{k=1}^{\infty} \frac{kx^{k-1}}{k+1}$

Express the following series using sigma notation.

85. $x + 2x^2 + 3x^3 + 4x^4 + 5x^5$

86. $\frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4 + \frac{4}{5}x^5 + \frac{5}{6}x^6$

87. $2 + 2^2x + 2^3x^2 + 2^4x^3 + 2^5x^4$

88. $3x + 3^2x^2 + 3^3x^3 + 3^4x^4 + 3^5x^5$

89. $2x + 4x^2 + 8x^3 + \cdots + 2^n x^n$

90. $x + 3x^2 + 9x^3 + \cdots + 3^n x^{n+1}$

91. $5 + (5+d) + (5+2d) + \cdots + (5+nd)$

92. $2 + 2r^1 + 2r^2 + \cdots + 2r^{n-1}$

93. $\frac{3}{4} + \frac{3}{8} + \frac{3}{16} + \cdots + 3\left(\frac{1}{2}\right)^n$

94. $\frac{8}{3} + \frac{16}{4} + \frac{32}{5} + \cdots + \frac{2^n}{n}$

95. A structured settlement yields an amount in dollars each year, represented by n, according to the formula $p_n = 10{,}000(0.70)^{n-1}$. What is the total amount gained from the settlement after 5 years?

96. The first row of seating in a small theater consists of 14 seats. Each row thereafter consists of 2 more seats than the previous row. If there are 7 rows, how many total seats are in the theater?

PART C: DISCUSSION BOARD

97. Research and discuss Fibonacci numbers as they are found in nature.

98. Research and discuss the life and contributions of Leonardo Fibonacci.

99. Explain the difference between a sequence and a series. Provide an example of each.

ANSWERS

1. 2, 4, 6, 8, 10; $a_{30} = 60$

3. 0, $\frac{3}{2}$, 4, $\frac{15}{2}$, 12; $a_{30} = \frac{899}{2}$

5. −4, 9, −16, 25, −36; $a_{30} = 961$

7. 1, 3, 9, 27, 81; $a_{30} = 3^{29}$

9. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}; a_{30} = \frac{1}{2^{30}}$

11. $\frac{1}{2}, -\frac{1}{5}, \frac{1}{8}, -\frac{1}{11}, \frac{1}{14}; a_{30} = -\frac{1}{89}$

13. $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}; a_{30} = \frac{31}{30}$

15. $2x, 2x^3, 2x^5, 2x^7, 2x^9$

17. $\frac{x}{5}, \frac{x^2}{6}, \frac{x^3}{7}, \frac{x^4}{8}, \frac{x^5}{9}$

19. $\frac{x^2}{2}, \frac{2x^4}{3}, \frac{3x^6}{4}, \frac{4x^8}{5}, \frac{5x^{10}}{6}$

21. $-x^3, x^6, -x^9, x^{12}, -x^{15}$

23. 3, 8, 13, 18, 23

25. −2, −6, −18, −54, −162

27. 1, 2, 6, 24, 120

29. 0, −1, −3, −7, −15

31. −1, 0, −1, −2, −5

33. 1, 3, 2, −1, −3

35. −82

37. −500

39. $\frac{11}{10}$

41. 32

43. Year 1: QI: \$4,522.50; QII: \$4,545.11; QIII: \$4,567.84; QIV: \$4,590.68; Year 2: QI: \$4,613.63; QII: \$4,636.70; QIII: \$4,659.88; QIV: \$4,683.18

45. Approximately 4×10^7 comparisons

47. 34

49. –30

51. –95

53. $-\frac{124}{125}$

55. $\frac{73}{12}$

57. $-\frac{205}{2}$

59. 45

61. 90

63. 22

65. $\frac{31}{4}$

67. –22

69. 15

71. 22

73. $0 + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$

75. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$

77. $\frac{3}{5} + \frac{3}{25} + \frac{3}{125} + \frac{3}{625} + \cdots$

79. $x - x^2 + x^3 - x^4 + \cdots$

81. $-2 + 4x - 8x^2 + 16x^3 - \cdots$

83. $x^2 + 3x^4 + 5x^6 + 7x^8 + \cdots$

85. $\sum_{k=1}^{5} k x^k$

87. $\sum_{k=1}^{5} 2^k x^{k-1}$

89. $\sum_{k=1}^{n} 2^k x^k$

91. $\sum_{k=0}^{n} (5 + kd)$

93. $\sum_{k=2}^{n} 3\left(\frac{1}{2}\right)^k$

95. $27,731

97. Answer may vary

99. Answer may vary

9.2 Arithmetic Sequences and Series

LEARNING OBJECTIVES

1. Identify the common difference of an arithmetic sequence.
2. Find a formula for the general term of an arithmetic sequence.
3. Calculate the *n*th partial sum of an arithmetic sequence.

Arithmetic Sequences

An **arithmetic sequence**[12], or **arithmetic progression**[13], is a sequence of numbers where each successive number is the sum of the previous number and some constant *d*.

$$a_n = a_{n-1} + d \quad \textit{Arithmetic Sequence}$$

And because $a_n - a_{n-1} = d$, the constant *d* is called the **common difference**[14]. For example, the sequence of positive odd integers is an arithmetic sequence,

$$1, 3, 5, 7, 9, \ldots$$

Here $a_1 = 1$ and the difference between any two successive terms is 2. We can construct the general term $a_n = a_{n-1} + 2$ where,

12. A sequence of numbers where each successive number is the sum of the previous number and some constant *d*.

13. Used when referring to an arithmetic sequence.

14. The constant *d* that is obtained from subtracting any two successive terms of an arithmetic sequence; $a_n - a_{n-1} = d$.

$$\begin{aligned} a_1 &= 1 \\ a_2 &= a_1 + 2 = 1 + 2 = 3 \\ a_3 &= a_2 + 2 = 3 + 2 = 5 \\ a_4 &= a_3 + 2 = 5 + 2 = 7 \\ a_5 &= a_4 + 2 = 7 + 2 = 9 \\ &\vdots \end{aligned}$$

In general, given the first term a_1 of an arithmetic sequence and its common difference d, we can write the following:

$$\begin{aligned} a_2 &= a_1 + d \\ a_3 &= a_2 + d = (a_1 + d) + d = a_1 + 2d \\ a_4 &= a_3 + d = (a_1 + 2d) + d = a_1 + 3d \\ a_5 &= a_4 + d = (a_1 + 3d) + d = a_1 + 4d \\ &\vdots \end{aligned}$$

From this we see that any arithmetic sequence can be written in terms of its first element, common difference, and index as follows:

$$a_n = a_1 + (n - 1)\,d \quad \textit{Arithmetic Sequence}$$

In fact, any general term that is linear in n defines an arithmetic sequence.

Example 1

Find an equation for the general term of the given arithmetic sequence and use it to calculate its 100th term: $7, 10, 13, 16, 19, \ldots$

Solution:

Begin by finding the common difference,

$$d = 10 - 7 = 3$$

Note that the difference between any two successive terms is 3. The sequence is indeed an arithmetic progression where $a_1 = 7$ and $d = 3$.

$$\begin{aligned} a_n &= a_1 + (n-1)\,d \\ &= 7 + (n-1)\cdot 3 \\ &= 7 + 3n - 3 \\ &= 3n + 4 \end{aligned}$$

Therefore, we can write the general term $a_n = 3n + 4$. Take a minute to verify that this equation describes the given sequence. Use this equation to find the 100th term:

$$a_{100} = 3\,(100) + 4 = 304$$

Answer: $a_n = 3n + 4$; $a_{100} = 304$

The common difference of an arithmetic sequence may be negative.

Example 2

Find an equation for the general term of the given arithmetic sequence and use it to calculate its 75th term: $6, 4, 2, 0, -2, \ldots$

Solution:

Begin by finding the common difference,

$$d = 4 - 6 = -2$$

Next find the formula for the general term, here $a_1 = 6$ and $d = -2$.

$$\begin{aligned} a_n &= a_1 + (n-1)\,d \\ &= 6 + (n-1)\cdot(-2) \\ &= 6 - 2n + 2 \\ &= 8 - 2n \end{aligned}$$

Therefore, $a_n = 8 - 2n$ and the 75th term can be calculated as follows:

$$\begin{aligned} a_{75} &= 8 - 2\,(75) \\ &= 8 - 150 \\ &= -142 \end{aligned}$$

Answer: $a_n = 8 - 2n$; $a_{100} = -142$

The terms between given terms of an arithmetic sequence are called **arithmetic means**[15].

15. The terms between given terms of an arithmetic sequence.

Example 3

Find all terms in between $a_1 = -8$ and $a_7 = 10$ of an arithmetic sequence. In other words, find all arithmetic means between the 1^{st} and 7^{th} terms.

Solution:

Begin by finding the common difference d. In this case, we are given the first and seventh term:

$$a_n = a_1 + (n-1)\,d \quad \textit{Use } n = 7.$$

$$\begin{aligned} a_7 &= a_1 + (7-1)\,d \\ a_7 &= a_1 + 6d \end{aligned}$$

Substitute $a_1 = -8$ and $a_7 = 10$ into the above equation and then solve for the common difference d.

$$\begin{aligned} 10 &= -8 + 6d \\ 18 &= 6d \\ 3 &= d \end{aligned}$$

Next, use the first term $a_1 = -8$ and the common difference $d = 3$ to find an equation for the nth term of the sequence.

$$
\begin{aligned}
a_n &= -8 + (n-1)\cdot 3 \\
&= -8 + 3n - 3 \\
&= -11 + 3n
\end{aligned}
$$

With $a_n = 3n - 11$, where n is a positive integer, find the missing terms.

$$
\begin{aligned}
a_1 &= 3(1) - 11 = 3 - 11 = -8 \\
&\left.\begin{aligned}
a_2 &= 3(2) - 11 = 6 - 11 = -5 \\
a_3 &= 3(3) - 11 = 9 - 11 = -2 \\
a_4 &= 3(4) - 11 = 12 - 11 = 1 \\
a_5 &= 3\left(5\right) - 11 = 15 - 11 = 4 \\
a_6 &= 3\left(6\right) - 11 = 18 - 11 = 7
\end{aligned}\right\} \textit{arithmetic means} \\
a_7 &= 3(7) - 11 = 21 - 11 = 10
\end{aligned}
$$

Answer: -5, -2, 1, 4, 7

In some cases, the first term of an arithmetic sequence may not be given.

Example 4

Find the general term of an arithmetic sequence where $a_3 = -1$ and $a_{10} = 48$.

Solution:

To determine a formula for the general term we need a_1 and d. A linear system with these as variables can be formed using the given information and $a_n = a_1 + (n-1)d$:

$$\begin{cases} a_3 = a_1 + (3-1)d \\ a_{10} = a_1 + (10-1)d \end{cases} \Longrightarrow \begin{cases} -1 = a_1 + 2d & \color{blue}{\textit{Use } a_3 = -1.} \\ 48 = a_1 + 9d & \color{blue}{\textit{Use } a_{10} = 48.} \end{cases}$$

Eliminate a_1 by multiplying the first equation by -1 and add the result to the second equation.

$$\begin{cases} -1 = a_1 + 2d \\ 48 = a_1 + 9d \end{cases} \xrightarrow{\times(-1)} + \begin{cases} 1 = -a_1 - 2d \\ 48 = a_1 + 9d \end{cases}$$

$$\begin{aligned} 49 &= 7d \\ 7 &= d \end{aligned}$$

Substitute $d = 7$ into $-1 = a_1 + 2d$ to find a_1.

$$\begin{aligned} -1 &= a_1 + 2(7) \\ -1 &= a_1 + 14 \\ -15 &= a_1 \end{aligned}$$

Next, use the first term $a_1 = -15$ and the common difference $d = 7$ to find a formula for the general term.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= -15 + (n-1)\cdot 7 \\ &= -15 + 7n - 7 \\ &= -22 + 7n \end{aligned}$$

Answer: $a_n = 7n - 22$

Try this! Find an equation for the general term of the given arithmetic sequence and use it to calculate its 100$^{\text{th}}$ term: $\frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, \ldots$

Answer: $a_n = \frac{1}{2}n + 1$; $a_{100} = 51$

(click to see video)

Arithmetic Series

An **arithmetic series**[16] is the sum of the terms of an arithmetic sequence. For example, the sum of the first 5 terms of the sequence defined by $a_n = 2n - 1$ follows:

16. The sum of the terms of an arithmetic sequence.

$$\begin{aligned} S_5 &= \sum_{n=1}^{5} (2n-1) \\ &= [2(1)-1] + [2(2)-1] + [2(3)-1] + [2(4)-1] + [2(5)-1] \\ &= 1+3+5+7+9 \\ &= 25 \end{aligned}$$

Adding 5 positive odd integers, as we have done above, is managable. However, consider adding the first 100 positive odd integers. This would be very tedious. Therefore, we next develop a formula that can be used to calculate the sum of the first n terms, denoted S_n, of any arithmetic sequence. In general,

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + a_n$$

Writing this series in reverse we have,

$$S_n = a_n + (a_n - d) + (a_n - 2d) + \ldots + a_1$$

And adding these two equations together, the terms involving d add to zero and we obtain n factors of $a_1 + a_n$:

$$\begin{aligned} 2S_n &= (a_1 + a_n) + (a_1 + a_n) + \ldots + (a_n + a_1) \\ 2S_n &= n(a_1 + a_n) \end{aligned}$$

Dividing both sides by 2 leads us the formula for the ***n*th partial sum of an arithmetic sequence**[17]:

17. The sum of the first n terms of an arithmetic sequence given by the formula: $S_n = \frac{n(a_1+a_n)}{2}$.

$$S_n = \frac{n\left(a_1 + a_n\right)}{2}$$

Use this formula to calculate the sum of the first 100 terms of the sequence defined by $a_n = 2n - 1$. Here $a_1 = 1$ and $a_{100} = 199$.

$$\begin{aligned} S_{100} &= \frac{100\left(a_1 + a_{100}\right)}{2} \\ &= \frac{100\,(1 + 199)}{2} \\ &= 10{,}000 \end{aligned}$$

Example 5

Find the sum of the first 50 terms of the given sequence: 4, 9, 14, 19, 24, ...

Solution:

Determine whether or not there is a common difference between the given terms.

$$d = 9 - 4 = 5$$

Note that the difference between any two successive terms is 5. The sequence is indeed an arithmetic progression and we can write

$$\begin{aligned} a_n &= a_1 + (n-1)\,d \\ &= 4 + (n-1)\cdot 5 \\ &= 4 + 5n - 5 \\ &= 5n - 1 \end{aligned}$$

Therefore, the general term is $a_n = 5n - 1$. To calculate the 50th partial sum of this sequence we need the 1st and the 50th terms:

$$\begin{aligned} a_1 &= 4 \\ a_{50} &= 5\,(50) - 1 = 249 \end{aligned}$$

Next use the formula to determine the 50$^{\text{th}}$ partial sum of the given arithmetic sequence.

$$
\begin{aligned}
S_n &= \frac{n(a_1 + a_n)}{2} \\
S_{50} &= \frac{50.(a_1 + a_{50})}{2} \\
&= \frac{50(4 + 249)}{2} \\
&= 25(253) \\
&= 6{,}325
\end{aligned}
$$

Answer: $S_{50} = 6{,}325$

Example 6

Evaluate: $\sum_{n=1}^{35} (10 - 4n)$.

Solution:

In this case, we are asked to find the sum of the first 35 terms of an arithmetic sequence with general term $a_n = 10 - 4n$. Use this to determine the 1st and the 35th term.

$$\begin{aligned} a_1 &= 10 - 4\,(1) = 6 \\ a_{35} &= 10 - 4\left(35\right) = -130 \end{aligned}$$

Next use the formula to determine the 35th partial sum.

$$\begin{aligned} S_n &= \frac{n\,(a_1 + a_n)}{2} \\ S_{35} &= \frac{35 \cdot \ (a_1 + a_{35})}{2} \\ &= \frac{35\left[6 + \ (-130)\right]}{2} \\ &= \frac{35\,(-124)}{2} \\ &= -2{,}170 \end{aligned}$$

Answer: −2,170

Example 7

The first row of seating in an outdoor amphitheater contains 26 seats, the second row contains 28 seats, the third row contains 30 seats, and so on. If there are 18 rows, what is the total seating capacity of the theater?

Figure 9.2

Roman Theater (Wikipedia)

Solution:

Begin by finding a formula that gives the number of seats in any row. Here the number of seats in each row forms a sequence:

$$26, 28, 30, \ldots$$

Note that the difference between any two successive terms is 2. The sequence is an arithmetic progression where $a_1 = 26$ and $d = 2$.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= 26 + (n-1)\cdot 2 \\ &= 26 + 2n - 2 \\ &= 2n + 24 \end{aligned}$$

Therefore, the number of seats in each row is given by $a_n = 2n + 24$. To calculate the total seating capacity of the 18 rows we need to calculate the 18th partial sum. To do this we need the 1st and the 18th terms:

$$\begin{aligned} a_1 &= 26 \\ a_{18} &= 2(18) + 24 = 60 \end{aligned}$$

Use this to calculate the 18th partial sum as follows:

$$\begin{aligned} S_n &= \frac{n(a_1 + a_n)}{2} \\ S_{18} &= \frac{18\cdot(a_1 + a_{18})}{2} \\ &= \frac{18\left(26+60\right)}{2} \\ &= 9\left(86\right) \\ &= 774 \end{aligned}$$

Answer: There are 774 seats total.

Try this! Find the sum of the first 60 terms of the given sequence: 5, 0, −5, −10, −15, ...

Answer: $S_{60} = -8{,}550$

(click to see video)

KEY TAKEAWAYS

- An arithmetic sequence is a sequence where the difference *d* between successive terms is constant.
- The general term of an arithmetic sequence can be written in terms of its first term a_1, common difference *d*, and index *n* as follows: $a_n = a_1 + (n-1)\,d$.
- An arithmetic series is the sum of the terms of an arithmetic sequence.
- The *n*th partial sum of an arithmetic sequence can be calculated using the first and last terms as follows: $S_n = \frac{n(a_1 + a_n)}{2}$.

TOPIC EXERCISES

PART A: ARITHMETIC SEQUENCES

Write the first 5 terms of the arithmetic sequence given its first term and common difference. Find a formula for its general term.

1. $a_1 = 5; d = 3$
2. $a_1 = 12; d = 2$
3. $a_1 = 15; d = -5$
4. $a_1 = 7; d = -4$
5. $a_1 = \frac{1}{2}; d = 1$
6. $a_1 = \frac{2}{3}; d = \frac{1}{3}$
7. $a_1 = 1; d = -\frac{1}{2}$
8. $a_1 = -\frac{5}{4}; d = \frac{1}{4}$
9. $a_1 = 1.8; d = 0.6$
10. $a_1 = -4.3; d = 2.1$

Given the arithmetic sequence, find a formula for the general term and use it to determine the 100th term.

11. 3, 9, 15, 21, 27,...
12. 3, 8, 13, 18, 23,...
13. −3, −7, −11, −15, −19,...
14. −6, −14, −22, −30, −38,...
15. −5, −10, −15, −20, −25,...
16. 2, 4, 6, 8, 10,...
17. $\frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \frac{13}{2}, \frac{17}{2}, \ldots$
18. $-\frac{1}{3}, \frac{2}{3}, \frac{5}{3}, \frac{8}{3}, \frac{11}{3}, \ldots$

19. $\frac{1}{3}, 0, -\frac{1}{3}, -\frac{2}{3}, -1, \ldots$

20. $\frac{1}{4}, -\frac{1}{2}, -\frac{5}{4}, -2, -\frac{11}{4}, \ldots$

21. 0.8, 2, 3.2, 4.4, 5.6,...

22. 4.4, 7.5, 10.6, 13.7, 16.8,...

23. Find the 50th positive odd integer.

24. Find the 50th positive even integer.

25. Find the 40th term in the sequence that consists of every other positive odd integer: 1, 5, 9, 13,...

26. Find the 40th term in the sequence that consists of every other positive even integer: 2, 6, 10, 14,...

27. What number is the term 355 in the arithmetic sequence −15, −5, 5, 15, 25,...?

28. What number is the term −172 in the arithmetic sequence 4, −4, −12, −20, −28,...?

29. Given the arithmetic sequence defined by the recurrence relation $a_n = a_{n-1} + 5$ where $a_1 = 2$ and $n > 1$, find an equation that gives the general term in terms of a_1 and the common difference *d*.

30. Given the arithmetic sequence defined by the recurrence relation $a_n = a_{n-1} - 9$ where $a_1 = 4$ and $n > 1$, find an equation that gives the general term in terms of a_1 and the common difference *d*.

Given the terms of an arithmetic sequence, find a formula for the general term.

31. $a_1 = 6$ and $a_7 = 42$

32. $a_1 = -\frac{1}{2}$ and $a_{12} = -6$

33. $a_1 = -19$ and $a_{26} = 56$

34. $a_1 = -9$ and $a_{31} = 141$

35. $a_1 = \frac{1}{6}$ and $a_{10} = \frac{37}{6}$

36. $a_1 = \frac{5}{4}$ and $a_{11} = \frac{65}{4}$

37. $a_3 = 6$ and $a_{26} = -40$

38. $a_3 = 16$ and $a_{15} = 76$

39. $a_4 = -8$ and $a_{23} = 30$
40. $a_5 = -7$ and $a_{37} = -135$
41. $a_4 = -\frac{23}{10}$ and $a_{21} = -\frac{25}{2}$
42. $a_3 = \frac{1}{8}$ and $a_{12} = -\frac{11}{2}$
43. $a_5 = 13.2$ and $a_{26} = 61.5$
44. $a_4 = -1.2$ and $a_{13} = 12.3$

Find all arithmetic means between the given terms.

45. $a_1 = -3$ and $a_6 = 17$
46. $a_1 = 5$ and $a_5 = -7$
47. $a_2 = 4$ and $a_8 = 7$
48. $a_5 = \frac{1}{2}$ and $a_9 = -\frac{7}{2}$
49. $a_5 = 15$ and $a_7 = 21$
50. $a_6 = 4$ and $a_{11} = -1$

PART B: ARITHMETIC SERIES

Calculate the indicated sum given the formula for the general term.

51. $a_n = 3n + 5; S_{100}$
52. $a_n = 5n - 11; S_{100}$
53. $a_n = \frac{1}{2} - n; S_{70}$
54. $a_n = 1 - \frac{3}{2}n; S_{120}$
55. $a_n = \frac{1}{2}n - \frac{3}{4}; S_{20}$
56. $a_n = n - \frac{3}{5}; S_{150}$
57. $a_n = 45 - 5n; S_{65}$
58. $a_n = 2n - 48; S_{95}$

59. $a_n = 4.4 - 1.6n; S_{75}$

60. $a_n = 6.5n - 3.3; S_{67}$

Evaluate.

61. $\sum_{n=1}^{160} (3n)$

62. $\sum_{n=1}^{121} (-2n)$

63. $\sum_{n=1}^{250} (4n - 3)$

64. $\sum_{n=1}^{120} (2n + 12)$

65. $\sum_{n=1}^{70} (19 - 8n)$

66. $\sum_{n=1}^{220} (5 - n)$

67. $\sum_{n=1}^{60} \left(\frac{5}{2} - \frac{1}{2}n\right)$

68. $\sum_{n=1}^{51} \left(\frac{3}{8}n + \frac{1}{4}\right)$

69. $\sum_{n=1}^{120} (1.5n - 2.6)$

70. $\sum_{n=1}^{175} (-0.2n - 1.6)$

71. Find the sum of the first 200 positive integers.

72. Find the sum of the first 400 positive integers.

The general term for the sequence of positive odd integers is given by $a_n = 2n - 1$ and the general term for the sequence of positive even integers is given by $a_n = 2n$. Find the following.

73. The sum of the first 50 positive odd integers.

74. The sum of the first 200 positive odd integers.

75. The sum of the first 50 positive even integers.

76. The sum of the first 200 positive even integers.

77. The sum of the first *k* positive odd integers.

78. The sum of the first *k* positive even integers.

79. The first row of seating in a small theater consists of 8 seats. Each row thereafter consists of 3 more seats than the previous row. If there are 12 rows, how many total seats are in the theater?

80. The first row of seating in an outdoor amphitheater contains 42 seats, the second row contains 44 seats, the third row contains 46 seats, and so on. If there are 22 rows, what is the total seating capacity of the theater?

81. If a triangular stack of bricks has 37 bricks on the bottom row, 34 bricks on the second row and so on with one brick on top. How many bricks are in the stack?

82. Each successive row of a triangular stack of bricks has one less brick until there is only one brick on top. How many rows does the stack have if there are 210 total bricks?

83. A 10-year salary contract offers $65,000 for the first year with a $3,200 increase each additional year. Determine the total salary obligation over the 10 year period.

84. A clock tower strikes its bell the number of times indicated by the hour. At one o'clock it strikes once, at two o'clock it strikes twice and so on. How many times does the clock tower strike its bell in a day?

PART C: DISCUSSION BOARD

85. Is the Fibonacci sequence an arithmetic sequence? Explain.

86. Use the formula for the *n*th partial sum of an arithmetic sequence $S_n = \frac{n(a_1 + a_n)}{2}$ and the formula for the general term $a_n = a_1 + (n-1)d$ to derive a new formula for the *n*th partial sum

$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$.Under what circumstances would this formula be useful? Explain using an example of your own making.

87. Discuss methods for calculating sums where the index does not start at 1. For example, $\sum_{n=15}^{35}(3n+4) = 1{,}659.$

88. A famous story involves Carl Friedrich Gauss misbehaving at school. As punishment, his teacher assigned him the task of adding the first 100 integers. The legend is that young Gauss answered correctly within seconds. What is the answer and how do you think he was able to find the sum so quickly?

ANSWERS

1. 5, 8, 11, 14, 17; $a_n = 3n + 2$
3. 15, 10, 5, 0, -5; $a_n = 20 - 5n$
5. $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}; a_n = n - \frac{1}{2}$
7. 1, $\frac{1}{2}$, 0, $-\frac{1}{2}$, -1; $a_n = \frac{3}{2} - \frac{1}{2}n$
9. 1.8, 2.4, 3, 3.6, 4.2; $a_n = 0.6n + 1.2$
11. $a_n = 6n - 3; a_{100} = 597$
13. $a_n = 1 - 4n; a_{100} = -399$
15. $a_n = -5n; a_{100} = -500$
17. $a_n = 2n - \frac{3}{2}; a_{100} = \frac{397}{2}$
19. $a_n = \frac{2}{3} - \frac{1}{3}n; a_{100} = -\frac{98}{3}$
21. $a_n = 1.2n - 0.4; a_{100} = 119.6$
23. 99
25. 157
27. 38
29. $a_n = 5n - 3$
31. $a_n = 6n$
33. $a_n = 3n - 22$
35. $a_n = \frac{2}{3}n - \frac{1}{2}$
37. $a_n = 12 - 2n$
39. $a_n = 2n - 16$
41. $a_n = \frac{1}{10} - \frac{3}{5}n$
43. $a_n = 2.3n + 1.7$
45. 1, 5, 9, 13

47. $\frac{9}{2}, 5, \frac{11}{2}, 6, \frac{13}{2}$

49. 18

51. 15,650

53. −2,450

55. 90

57. −7,800

59. −4,230

61. 38,640

63. 124,750

65. −18,550

67. −765

69. 10,578

71. 20,100

73. 2,500

75. 2,550

77. k^2

79. 294 seats

81. 247 bricks

83. $794,000

85. Answer may vary

87. Answer may vary

9.3 Geometric Sequences and Series

LEARNING OBJECTIVES

1. Identify the common ratio of a geometric sequence.
2. Find a formula for the general term of a geometric sequence.
3. Calculate the *n*th partial sum of a geometric sequence.
4. Calculate the sum of an infinite geometric series when it exists.

Geometric Sequences

A **geometric sequence**[18], or **geometric progression**[19], is a sequence of numbers where each successive number is the product of the previous number and some constant *r*.

$$a_n = r a_{n-1} \quad \textit{Geometic Sequence}$$

And because $\frac{a_n}{a_{n-1}} = r$, the constant factor *r* is called the **common ratio**[20]. For example, the following is a geometric sequence,

$$9, 27, 81, 243, 729 \ldots$$

Here $a_1 = 9$ and the ratio between any two successive terms is 3. We can construct the general term $a_n = 3a_{n-1}$ where,

18. A sequence of numbers where each successive number is the product of the previous number and some constant *r*.

19. Used when referring to a geometric sequence.

20. The constant *r* that is obtained from dividing any two successive terms of a geometric sequence; $\frac{a_n}{a_{n-1}} = r$.

$$
\begin{aligned}
a_1 &= 9 \\
a_2 &= 3a_1 = 3(9) = 27 \\
a_3 &= 3a_2 = 3(27) = 81 \\
a_4 &= 3a_3 = 3(81) = 243 \\
a_5 &= 3a_4 = 3(243) = 729 \\
&\vdots
\end{aligned}
$$

In general, given the first term a_1 and the common ratio r of a geometric sequence we can write the following:

$$
\begin{aligned}
a_2 &= ra_1 \\
a_3 &= ra_2 = r(a_1 r) = a_1 r^2 \\
a_4 &= ra_3 = r\left(a_1 r^2\right) = a_1 r^3 \\
a_5 &= ra_3 = r\left(a_1 r^3\right) = a_1 r^4 \\
&\vdots
\end{aligned}
$$

From this we see that any geometric sequence can be written in terms of its first element, its common ratio, and the index as follows:

$$a_n = a_1 r^{n-1} \qquad \textit{Geometric Sequence}$$

In fact, any general term that is exponential in n is a geometric sequence.

Example 1

Find an equation for the general term of the given geometric sequence and use it to calculate its 10th term: $3, 6, 12, 24, 48\ldots$

Solution:

Begin by finding the common ratio,

$$r = \frac{6}{3} = 2$$

Note that the ratio between any two successive terms is 2. The sequence is indeed a geometric progression where $a_1 = 3$ and $r = 2$.

$$\begin{aligned} a_n &= a_1 r^{n-1} \\ &= 3(2)^{n-1} \end{aligned}$$

Therefore, we can write the general term $a_n = 3(2)^{n-1}$ and the 10th term can be calculated as follows:

$$\begin{aligned} a_{10} &= 3(2)^{10-1} \\ &= 3(2)^9 \\ &= 1{,}536 \end{aligned}$$

Answer: $a_n = 3(2)^{n-1}$; $a_{10} = 1{,}536$

The terms between given terms of a geometric sequence are called **geometric means**[21].

21. The terms between given terms of a geometric sequence.

Example 2

Find all terms between $a_1 = -5$ and $a_4 = -135$ of a geometric sequence. In other words, find all geometric means between the 1^{st} and 4^{th} terms.

Solution:

Begin by finding the common ratio r. In this case, we are given the first and fourth terms:

$$a_n = a_1 r^{n-1} \quad \textit{Use } n = 4.$$

$$\begin{aligned} a_4 &= a_1 r^{4-1} \\ a_4 &= a_1 r^3 \end{aligned}$$

Substitute $a_1 = -5$ and $a_4 = -135$ into the above equation and then solve for r.

$$\begin{aligned} -135 &= -5r^3 \\ 27 &= r^3 \\ 3 &= r \end{aligned}$$

Next use the first term $a_1 = -5$ and the common ratio $r = 3$ to find an equation for the nth term of the sequence.

$$\begin{aligned} a_n &= a_1 r^{n-1} \\ a_n &= -5(3)^{n-1} \end{aligned}$$

Now we can use $a_n = -5(3)^{n-1}$ where n is a positive integer to determine the missing terms.

$$
\begin{aligned}
a_1 &= -5(3)^{1-1} = -5 \cdot 3^0 = -5 \\
\left.\begin{aligned}
a_2 &= -5(3)^{2-1} = -5 \cdot 3^1 = -15 \\
a_3 &= -5(3)^{3-1} = -5 \cdot 3^2 = -45
\end{aligned}\right\} &\ \textit{geometic means} \\
a_4 &= -5(3)^{4-1} = -5 \cdot 3^3 = -135
\end{aligned}
$$

Answer: -15, -45,

The first term of a geometric sequence may not be given.

Example 3

Find the general term of a geometric sequence where $a_2 = -2$ and $a_5 = \frac{2}{125}$.

Solution:

To determine a formula for the general term we need a_1 and r. A nonlinear system with these as variables can be formed using the given information and $a_n = a_1 r^{n-1}$:

$$\begin{cases} a_2 = a_1 r^{2-1} \\ a_5 = a_1 r^{5-1} \end{cases} \Longrightarrow \begin{cases} -2 = a_1 r & \text{Use } a_2 = -2. \\ \dfrac{2}{125} = a_1 r^4 & \text{Use } a_5 = \dfrac{2}{125}. \end{cases}$$

Solve for a_1 in the first equation,

$$\begin{cases} -2 = a_1 r & \Rightarrow \quad \dfrac{-2}{r} = a_1 \\ \dfrac{2}{125} = a_1 r^4 \end{cases}$$

Substitute $a_1 = \frac{-2}{r}$ into the second equation and solve for r.

$$\begin{aligned}\frac{2}{125} &= a_1 r^4 \\ \frac{2}{125} &= \left(\frac{-2}{r}\right) r^4 \\ \frac{2}{125} &= -2r^3 \\ -\frac{1}{125} &= r^3 \\ -\frac{1}{5} &= r\end{aligned}$$

Back substitute to find a_1:

$$\begin{aligned}a_1 &= \frac{-2}{r} \\ &= \frac{-2}{\left(-\frac{1}{5}\right)} \\ &= 10\end{aligned}$$

Therefore, $a_1 = 10$ and $r = -\frac{1}{5}$.

Answer: $a_n = 10\left(-\frac{1}{5}\right)^{n-1}$

Try this! Find an equation for the general term of the given geometric sequence and use it to calculate its 6th term: $2, \frac{4}{3}, \frac{8}{9}, \ldots$

Answer: $a_n = 2\left(\frac{2}{3}\right)^{n-1}$; $a_6 = \frac{64}{243}$

(click to see video)

Geometric Series

A **geometric series**[22] is the sum of the terms of a geometric sequence. For example, the sum of the first 5 terms of the geometric sequence defined by $a_n = 3^{n+1}$ follows:

$$\begin{aligned} S_5 &= \sum_{n=1}^{5} 3^{n+1} \\ &= 3^{1+1} + 3^{2+1} + 3^{3+1} + 3^{4+1} + 3^{5+1} \\ &= 3^2 + 3^3 + 3^4 + 3^5 + 3^6 \\ &= 9 + 27 + 81 + 243 + 729 \\ &= 1{,}089 \end{aligned}$$

Adding 5 positive integers is managable. However, the task of adding a large number of terms is not. Therefore, we next develop a formula that can be used to calculate the sum of the first *n* terms of any geometric sequence. In general,

$$S_n = a_1 + a_1 r + a_1 r^2 + \ldots + a_1 r^{n-1}$$

Multiplying both sides by *r* we can write,

$$rS_n = a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^n$$

Subtracting these two equations we then obtain,

$$\begin{aligned} S_n - rS_n &= a_1 - a_1 r^n \\ S_n (1 - r) &= a_1 \left(1 - r^n\right) \end{aligned}$$

22. The sum of the terms of a geometric sequence.

Assuming $r \neq 1$ dividing both sides by $(1 - r)$leads us to the formula for the ***n*th partial sum of a geometric sequence**[23]:

$$S_n = \frac{a_1 (1 - r^n)}{1 - r} (r \neq 1)$$

In other words, the *n*th partial sum of any geometric sequence can be calculated using the first term and the common ratio. For example, to calculate the sum of the first 15 terms of the geometric sequence defined by $a_n = 3^{n+1}$, use the formula with $a_1 = 9$ and $r = 3$.

$$\begin{aligned} S_{15} &= \frac{a_1 \left(1 - r^{15}\right)}{1 - r} \\ &= \frac{9 \cdot \left(1 - 3^{15}\right)}{1 - 3} \\ &= \frac{9 \left(-14{,}348{,}906\right)}{-2} \\ &= 64{,}570{,}077 \end{aligned}$$

23. The sum of the first *n* terms of a geometric sequence, given by the formula: $S_n = \frac{a_1(1-r^n)}{1-r}$, $r \neq 1$.

Example 4

Find the sum of the first 10 terms of the given sequence: 4, −8, 16, −32, 64,...

Solution:

Determine whether or not there is a common ratio between the given terms.

$$r = \frac{-8}{4} = -2$$

Note that the ratio between any two successive terms is −2; hence, the given sequence is a geometric sequence. Use $r = -2$ and the fact that $a_1 = 4$ to calculate the sum of the first 10 terms,

$$\begin{aligned} S_n &= \frac{a_1\left(1 - r^n\right)}{1 - r} \\ S_{10} &= \frac{4\left[1 - (-2)^{10}\right]}{1 - (-2)} \\ &= \frac{4\left(1 - 1{,}024\right)}{1 + 2} \\ &= \frac{4\left(-1{,}023\right)}{3} \\ &= -1{,}364 \end{aligned}$$

Answer: $S_{10} = -1{,}364$

Example 5

Evaluate: $\sum_{n=1}^{6} 2(-5)^n$.

Solution:

In this case, we are asked to find the sum of the first 6 terms of a geometric sequence with general term $a_n = 2(-5)^n$. Use this to determine the 1st term and the common ratio r:

$$a_1 = 2(-5)^1 = -10$$

To show that there is a common ratio we can use successive terms in general as follows:

$$\begin{aligned} r &= \frac{a_n}{a_{n-1}} \\ &= \frac{2(-5)^n}{2(-5)^{n-1}} \\ &= (-5)^{n-(n-1)} \\ &= (-5)^1 \\ &= -5 \end{aligned}$$

Use $a_1 = -10$ and $r = -5$ to calculate the 6th partial sum.

$$
\begin{aligned}
S_n &= \frac{a_1\left(1-r^n\right)}{1-r}\\
S_6 &= \frac{-10\left[1-(-5)^6\right]}{1-\ (-5)}\\
&= \frac{-10\left(1-15{,}625\right)}{1+5}\\
&= \frac{-10\left(-15{,}624\right)}{6}\\
&= 26{,}040
\end{aligned}
$$

Answer: 26,040

Try this! Find the sum of the first 9 terms of the given sequence: -2, 1, -1/2,...

Answer: $S_9 = -\frac{171}{128}$

(click to see video)

If the common ratio r of an infinite geometric sequence is a fraction where $|r| < 1$ (that is $-1 < r < 1$), then the factor $(1 - r^n)$ found in the formula for the nth partial sum tends toward 1 as n increases. For example, if $r = \frac{1}{10}$ and $n = 2, 4, 6$ we have,

$$
\begin{aligned}
1-\left(\frac{1}{10}\right)^2 &= 1-0.01 = 0.99\\
1-\left(\frac{1}{10}\right)^4 &= 1-0.0001 = 0.9999\\
1-\left(\frac{1}{10}\right)^6 &= 1-0.000001 = 0.999999
\end{aligned}
$$

Here we can see that this factor gets closer and closer to 1 for increasingly larger values of *n*. This illustrates the idea of a limit, an important concept used extensively in higher-level mathematics, which is expressed using the following notation:

$$\lim_{n\to\infty}\left(1-r^n\right)=1 \text{ where } |r|<1$$

This is read, "the limit of $(1-r^n)$ as *n* approaches infinity equals 1." While this gives a preview of what is to come in your continuing study of mathematics, at this point we are concerned with developing a formula for special infinite geometric series. Consider the *n*th partial sum of any geometric sequence,

$$S_n=\frac{a_1\left(1-r^n\right)}{1-r}=\frac{a_1}{1-r}\left(1-r^n\right)$$

If $|r|<1$ then the limit of the partial sums as *n* approaches infinity exists and we can write,

$$S_n=\frac{a_1}{1-r}\left(1-r^n\right)\xrightarrow[n\to\infty]{} S_\infty=\frac{a_1}{1-r}\cdot 1$$

Therefore, a **convergent geometric series**[24] is an infinite geometric series where $|r|<1$; its sum can be calculated using the formula:

$$S_\infty=\frac{a_1}{1-r}$$

24. An infinite geometric series where $|r|<1$ whose sum is given by the formula: $S_\infty=\frac{a_1}{1-r}$.

Example 6

Find the sum of the infinite geometric series: $\frac{3}{2} + \frac{1}{2} + \frac{1}{6} + \frac{1}{18} + \frac{1}{54} + \cdots$

Solution:

Determine the common ratio,

$$r = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$$

Since the common ratio $r = \frac{1}{3}$is a fraction between −1 and 1, this is a convergent geometric series. Use the first term $a_1 = \frac{3}{2}$and the common ratio to calculate its sum.

$$\begin{aligned} S_\infty &= \frac{a_1}{1 - r} \\ &= \frac{\frac{3}{2}}{1 - \left(\frac{1}{3}\right)} \\ &= \frac{\frac{3}{2}}{\frac{2}{3}} \\ &= \frac{3}{2} \cdot \frac{3}{2} \\ &= \frac{9}{4} \end{aligned}$$

Answer: $S_\infty = \frac{9}{4}$

Note: In the case of an infinite geometric series where $|r| \geq 1$, the series diverges and we say that there is no sum. For example, if $a_n = (5)^{n-1}$ then $r = 5$ and we have

$$S_\infty = \sum_{n=1}^{\infty} (5)^{n-1} = 1 + 5 + 25 + \cdots$$

We can see that this sum grows without bound and has no sum.

Try this! Find the sum of the infinite geometric series: $\sum_{n=1}^{\infty} -2\left(\frac{5}{9}\right)^{n-1}$.

Answer: -9/2

(click to see video)

A repeating decimal can be written as an infinite geometric series whose common ratio is a power of 1/10. Therefore, the formula for a convergent geometric series can be used to convert a repeating decimal into a fraction.

Example 7

Write as a fraction: $1.181818\ldots$

Solution:

Begin by identifying the repeating digits to the right of the decimal and rewrite it as a geometric progression.

$$\begin{aligned} 0.181818\ldots &= 0.18 + 0.0018 + 0.000018 + \ldots \\ &= \frac{18}{100} + \frac{18}{10{,}000} + \frac{18}{1{,}000{,}000} + \ldots \end{aligned}$$

In this form we can determine the common ratio,

$$\begin{aligned} r &= \frac{\frac{18}{10{,}000}}{\frac{18}{100}} \\ &= \frac{18}{10{,}000} \times \frac{100}{18} \\ &= \frac{1}{100} \end{aligned}$$

Note that the ratio between any two successive terms is $\frac{1}{100}$. Use this and the fact that $a_1 = \frac{18}{100}$ to calculate the infinite sum:

$$\begin{aligned} S_\infty &= \frac{a_1}{1-r} \\ &= \frac{\frac{18}{100}}{1-\left(\frac{1}{100}\right)} \\ &= \frac{\frac{18}{100}}{\frac{99}{100}} \\ &= \frac{18}{100} \cdot \frac{100}{99} \\ &= \frac{2}{11} \end{aligned}$$

Therefore, $0.181818\ldots = \frac{2}{11}$and we have,

$$1.181818\ldots = 1 + \frac{2}{11} = 1\,\frac{2}{11}$$

Answer: $1\,\frac{2}{11}$

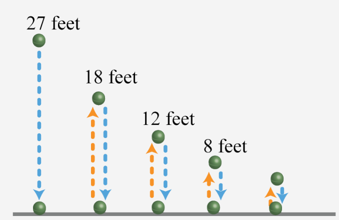
Example 8

A certain ball bounces back to two-thirds of the height it fell from. If this ball is initially dropped from 27 feet, approximate the total distance the ball travels.

Solution:

We can calculate the height of each successive bounce:

$$\begin{aligned} 27 \cdot \frac{2}{3} &= 18 \text{ feet} \quad \textit{Height of the first bounce} \\ 18 \cdot \frac{2}{3} &= 12 \text{ feet} \quad \textit{Height of the second bounce} \\ 12 \cdot \frac{2}{3} &= 8 \text{ feet} \quad \textit{Height of the third bounce} \end{aligned}$$



The total distance that the ball travels is the sum of the distances the ball is falling and the distances the ball is rising. The distances the ball falls forms a geometric series,

$$27 + 18 + 12 + \cdots \qquad \textit{Distance the ball is falling}$$

where $a_1 = 27$ and $r = \frac{2}{3}$.Because r is a fraction between −1 and 1, this sum can be calculated as follows:

$$\begin{aligned} S_\infty &= \frac{a_1}{1-r} \\ &= \frac{27}{1-\frac{2}{3}} \\ &= \frac{27}{\frac{1}{3}} \\ &= 81 \end{aligned}$$

Therefore, the ball is falling a total distance of 81 feet. The distances the ball rises forms a geometric series,

$$18 + 12 + 8 + \cdots \qquad \textit{Distance the ball is rising}$$

where $a_1 = 18$ and $r = \frac{2}{3}$. Calculate this sum in a similar manner:

$$\begin{aligned} S_\infty &= \frac{a_1}{1-r} \\ &= \frac{18}{1-\frac{2}{3}} \\ &= \frac{18}{\frac{1}{3}} \\ &= 54 \end{aligned}$$

Therefore, the ball is rising a total distance of 54 feet. Approximate the total distance traveled by adding the total rising and falling distances:

$$81 + 54 = 135 \quad \text{feet}$$

Answer: 135 feet

KEY TAKEAWAYS

- A geometric sequence is a sequence where the ratio *r* between successive terms is constant.
- The general term of a geometric sequence can be written in terms of its first term a_1, common ratio *r*, and index *n* as follows: $a_n = a_1 r^{n-1}$.
- A geometric series is the sum of the terms of a geometric sequence.
- The *n*th partial sum of a geometric sequence can be calculated using the first term a_1 and common ratio *r* as follows: $S_n = \frac{a_1(1-r^n)}{1-r}$.
- The infinite sum of a geometric sequence can be calculated if the common ratio is a fraction between −1 and 1 (that is $|r| < 1$) as follows: $S_\infty = \frac{a_1}{1-r}$. If $|r| \geq 1$, then no sum exists.

TOPIC EXERCISES

PART A: GEOMETRIC SEQUENCES

Write the first 5 terms of the geometric sequence given its first term and common ratio. Find a formula for its general term.

1. $a_1 = 1; r = 5$
2. $a_1 = 1; r = 3$
3. $a_1 = 2; r = 3$
4. $a_1 = 5; r = 4$
5. $a_1 = 2; r = -3$
6. $a_1 = 6; r = -2$
7. $a_1 = 3; r = \frac{2}{3}$
8. $a_1 = 6; r = \frac{1}{2}$
9. $a_1 = 1.2; r = 0.6$
10. $a_1 = -0.6; r = -3$

Given the geometric sequence, find a formula for the general term and use it to determine the 5^{th} term in the sequence.

11. 7, 28, 112,…
12. −2, −10, −50,…
13. 2, $\frac{1}{2}$, $\frac{1}{8}$,…
14. 1, $\frac{2}{5}$, $\frac{4}{25}$,…
15. 8, 4, 2,…
16. 6, 2, $\frac{2}{3}$,…
17. −1, $\frac{2}{3}$, $-\frac{4}{9}$,…
18. 2, $-\frac{3}{2}$, $\frac{9}{8}$,…

19. $\frac{1}{3}, -2, 12,...$

20. $\frac{2}{5}, -2, 10,...$

21. −3.6, −4.32, −5.184,...

22. 0.8, −2.08, 5.408,...

23. Find the general term and use it to determine the 20th term in the sequence: $1, \frac{x}{2}, \frac{x^2}{4},...$

24. Find the general term and use it to determine the 20th term in the sequence: $2, -6x, 18x^2,...$

25. The number of cells in a culture of a certain bacteria doubles every 4 hours. If 200 cells are initially present, write a sequence that shows the population of cells after every *n*th 4-hour period for one day. Write a formula that gives the number of cells after any 4-hour period.

26. A certain ball bounces back at one-half of the height it fell from. If this ball is initially dropped from 12 feet, find a formula that gives the height of the ball on the *n*th bounce and use it to find the height of the ball on the 6$^{\text{th}}$ bounce.

27. Given a geometric sequence defined by the recurrence relation $a_n = 4a_{n-1}$ where $a_1 = 2$ and $n > 1$, find an equation that gives the general term in terms of a_1 and the common ratio *r*.

28. Given the geometric sequence defined by the recurrence relation $a_n = 6a_{n-1}$ where $a_1 = \frac{1}{2}$ and $n > 1$, find an equation that gives the general term in terms of a_1 and the common ratio *r*.

Given the terms of a geometric sequence, find a formula for the general term.

29. $a_1 = -3$ and $a_6 = -96$

30. $a_1 = 5$ and $a_4 = -40$

31. $a_1 = -2$ and $a_8 = -\frac{1}{64}$

32. $a_1 = \frac{3}{4}$ and $a_4 = -\frac{1}{36}$

33. $a_2 = 18$ and $a_5 = 486$

34. $a_2 = 10$ and $a_7 = 320$

35. $a_4 = -2$ and $a_9 = 64$

36. $a_3 = -\frac{4}{3}$ and $a_6 = \frac{32}{81}$

37. $a_5 = 153.6$ and $a_8 = 9{,}830.4$

38. $a_4 = -2.4 \times 10^{-3}$ and $a_9 = -7.68 \times 10^{-7}$

Find all geometric means between the given terms.

39. $a_1 = 2$ and $a_4 = 250$

40. $a_1 = \frac{1}{3}$ and $a_6 = -\frac{1}{96}$

41. $a_2 = -20$ and $a_5 = -20{,}000$

42. $a_3 = 49$ and $a_6 = -16{,}807$

PART B: GEOMETRIC SERIES

Calculate the indicated sum.

43. $a_n = 2^{n+1}; S_{12}$

44. $a_n = (-2)^{n+1}; S_{12}$

45. $a_n = \left(\frac{1}{2}\right)^n; S_7$

46. $a_n = \left(\frac{2}{3}\right)^{n-1}; S_6$

47. $a_n = 5(-3)^{n-1}; S_5$

48. $a_n = -7(-4)^n; S_5$

49. $a_n = 2\left(-\frac{1}{4}\right)^n; S_5$

50. $a_n = \frac{1}{3}(2)^{n+1}; S_{10}$

51. $\sum_{n=1}^{5} 5^n$

52. $\sum_{n=1}^{6} (-4)^n$

53. $\sum_{k=1}^{10} 2^{k+1}$

54. $\sum_{k=1}^{14} 2^{k-1}$

55. $\sum_{k=1}^{10} -2(3)^k$

56. $\sum_{k=1}^{8} 5(-2)^k$

57. $\sum_{n=1}^{5} 2\left(\frac{1}{2}\right)^{n+2}$

58. $\sum_{n=1}^{4} -3\left(\frac{2}{3}\right)^{n}$

59. $a_n = \left(\frac{1}{5}\right)^n; S_\infty$

60. $a_n = \left(\frac{2}{3}\right)^{n-1}; S_\infty$

61. $a_n = 2\left(-\frac{3}{4}\right)^{n-1}; S_\infty$

62. $a_n = 3\left(-\frac{1}{6}\right)^n; S_\infty$

63. $a_n = -2\left(\frac{1}{2}\right)^{n+1}; S_\infty$

64. $a_n = -\frac{1}{3}\left(-\frac{1}{2}\right)^n; S_\infty$

65. $\sum_{n=1}^{\infty} 2\left(\frac{1}{3}\right)^{n-1}$

66. $\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^{n}$

67. $\sum_{n=1}^{\infty} 3(2)^{n-2}$

68. $\sum_{n=1}^{\infty} -\frac{1}{4}(3)^{n-2}$

69. $\sum_{n=1}^{\infty} \frac{1}{2}\left(-\frac{1}{6}\right)^{n}$

70. $\sum_{n=1}^{\infty} \frac{1}{3}\left(-\frac{2}{5}\right)^{n}$

Write as a mixed number.

71. 1.222...

72. 5.777 ...

73. 2.252525...

74. 3.272727...

75. 1.999...

76. 1.090909...

77. Suppose you agreed to work for pennies a day for 30 days. You will earn 1 penny on the first day, 2 pennies the second day, 4 pennies the third day, and so on. How many total pennies will you have earned at the end of the 30 day period? What is the dollar amount?

78. An initial roulette wager of $100 is placed (on red) and lost. To make up the difference, the player doubles the bet and places a $200 wager and loses. Again, to make up the difference, the player doubles the wager to $400 and loses. If the player continues doubling his bet in this manner and loses 7 times in a row, how much will he have lost in total?

79. A certain ball bounces back to one-half of the height it fell from. If this ball is initially dropped from 12 feet, approximate the total distance the ball travels.

80. A golf ball bounces back off of a cement sidewalk three-quarters of the height it fell from. If the ball is initially dropped from 8 meters, approximate the total distance the ball travels.

81. A structured settlement yields an amount in dollars each year, represented by *n*, according to the formula $p_n = 6{,}000(0.80)^{n-1}$. What is the total amount gained from the settlement after 10 years?

82. Beginning with a square, where each side measures 1 unit, inscribe another square by connecting the midpoints of each side. Continue inscribing squares in this manner indefinitely, as pictured:

Find the sum of the area of all squares in the figure. (Hint: Begin by finding the sequence formed using the areas of each square.)

PART C: SEQUENCES AND SERIES

Categorize the sequence as arithmetic, geometric, or neither. Give the common difference or ratio, if it exists.

83. $-12, 24, -48, \ldots$
84. $-7, -5, -3, \ldots$
85. $-3, -11, -19, \ldots$
86. $4, 9, 16, \ldots$
87. $2, \frac{3}{2}, \frac{4}{3}, \ldots$
88. $\frac{4}{3}, \frac{8}{9}, \frac{16}{27}, \ldots$
89. $\frac{1}{6}, -\frac{1}{6}, -\frac{1}{2}, \ldots$
90. $\frac{1}{3}, \frac{1}{4}, \frac{3}{16}, \ldots$
91. $\frac{1}{2}, \frac{1}{4}, \frac{1}{6} \ldots$
92. $-\frac{1}{10}, -\frac{1}{5}, -\frac{3}{10}, \ldots$
93. $1.26, 0.252, 0.0504, \ldots$

94. $0.02, 0.08, 0.18, \ldots$

95. 1, −1, 1, −1,...

96. 0, 0, 0,...

Categorize the sequence as arithmetic or geometric, and then calculate the indicated sum.

97. $a_n = 3(5)^{n-1}; S_8$

98. $a_n = 5 - 6n; S_{22}$

99. $a_n = 2n; S_{14}$

100. $a_n = 2^n; S_{10}$

101. $a_n = -2\left(\frac{1}{7}\right)^{n-1}; S_\infty$

102. $a_n = -2 + \frac{1}{7}n; S_8$

Calculate the indicated sum.

103. $\sum_{n=1}^{50} (3n - 5)$

104. $\sum_{n=1}^{25} (4 - 8n)$

105. $\sum_{n=1}^{12} (-2)^{n-1}$

106. $\sum_{n=1}^{\infty} 5\left(-\frac{1}{2}\right)^{n-1}$

107. $\sum_{n=1}^{40} 5$

108. $\sum_{n=1}^{\infty} 0.6^n$

PART D: DISCUSSION BOARD

109. Use the techniques found in this section to explain why $0.999 \ldots = 1.$

110. Construct a geometric sequence where $r = 1.$ Explore the *n*th partial sum of such a sequence. What conclusions can we make?

ANSWERS

1. 1, 5, 25, 125, 625; $a_n = 5^{n-1}$
3. 2, 6, 18, 54, 162; $a_n = 2(3)^{n-1}$
5. 2, −6, 18, −54, 162; $a_n = 2(-3)^{n-1}$
7. 3, 2, $\frac{4}{3}$, $\frac{8}{9}$, $\frac{16}{27}$; $a_n = 3\left(\frac{2}{3}\right)^{n-1}$
9. 1.2, 0.72, 0.432, 0.2592, 0.15552; $a_n = 1.2(0.6)^{n-1}$
11. $a_n = 7(4)^{n-1}, a_5 = 1{,}792$
13. $a_n = 2\left(\frac{1}{4}\right)^{n-1}, a_5 = \frac{1}{128}$
15. $a_n = 8\left(\frac{1}{2}\right)^{n-1}, a_5 = \frac{1}{2}$
17. $a_n = -\left(-\frac{2}{3}\right)^{n-1}, a_5 = -\frac{16}{81}$
19. $a_n = \frac{1}{3}(-6)^{n-1}, a_5 = 432$
21. $a_n = -3.6(1.2)^{n-1}, a_5 = -7.46496$
23. $a_n = \left(\frac{x}{2}\right)^{n-1}; a_{20} = \frac{x^{19}}{2^{19}}$
25. 400 cells; 800 cells; 1,600 cells; 3,200 cells; 6,400 cells; 12,800 cells; $p_n = 400(2)^{n-1}$ cells
27. $a_n = 2(4)^{n-1}$
29. $a_n = -3(2)^{n-1}$
31. $a_n = -2\left(\frac{1}{2}\right)^{n-1}$
33. $a_n = 6(3)^{n-1}$
35. $a_n = \frac{1}{4}(-2)^{n-1}$
37. $a_n = 0.6(4)^{n-1}$
39. 10, 50

41. −200; −2,000

43. 16,380

45. $\frac{127}{128}$

47. 305

49. $-\frac{205}{512}$

51. 3,905

53. 4,092

55. −177,144

57. $\frac{31}{64}$

59. $\frac{1}{4}$

61. $\frac{8}{7}$

63. −1

65. 3

67. No sum

69. $-\frac{1}{14}$

71. $1\frac{2}{9}$

73. $2\frac{25}{99}$

75. 2

77. 1,073,741,823 pennies; $10,737,418.23

79. 36 feet

81. $26,778.77

83. Geometric; $r = -2$

85. Arithmetic; $d = -8$

87. Neither

89. Arithmetic; $d = -\frac{1}{3}$

91. Neither

93. Geometric; $r = 0.2$

95. Geometric; $r = -1$

97. Geometric; 292,968

99. Arithmetic; 210

101. Geometric; $-\frac{7}{3}$

103. 3,575

105. −1,365

107. 200

109. Answer may vary

9.4 Binomial Theorem

LEARNING OBJECTIVES

1. Evaluate expressions involving factorials.
2. Calculate binomial coefficients.
3. Expand powers of binomials using the binomial theorem.

Factorials and the Binomial Coefficient

We begin by defining the **factorial**[25] of a natural number n, denoted $n!$, as the product of all natural numbers less than or equal to n.

$$n! = n\,(n-1)\,(n-2)\cdots 3 \cdot 2 \cdot 1$$

For example,

$$\begin{aligned} 7! &= 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5{,}040 && \textit{Seven factorial} \\ 5! &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 && \textit{Five factorial} \\ 3! &= 3 \cdot 2 \cdot 1 = 6 && \textit{Three factorial} \\ 1! &= 1 = 1 && \textit{One factorial} \end{aligned}$$

We define **zero factorial**[26] to be equal to 1,

$$0! = 1 \qquad \textit{Zero factorial}$$

The factorial of a negative number is not defined.

25. The product of all natural numbers less than or equal to a given natural number, denoted $n!$.

26. The factorial of zero is defined to be equal to 1; $0! = 1$.

Note: On most modern calculators you will find a factorial function. Some calculators do not provide a button dedicated to it. However, it usually can be found in the menu system if one is provided.

The factorial can also be expressed using the following recurrence relation,

$$n! = n\,(n-1)!$$

For example, the factorial of 8 can be expressed as the product of 8 and 7!:

$$\begin{aligned} 8! &= 8 \cdot 7! \\ &= 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\ &= 40{,}320 \end{aligned}$$

When working with ratios involving factorials, it is often the case that many of the factors cancel.

Example 1

Evaluate: $\frac{12!}{6!}$.

Solution:

$$\begin{aligned}\frac{12!}{6!} &= \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\ &= \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot \cancel{6!}}{\cancel{6!}} \\ &= 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \\ &= 665{,}280\end{aligned}$$

Answer: 665,280

The **binomial coefficient**[27], denoted ${}_nC_k = \binom{n}{k}$, is read "n choose k" and is given by the following formula:

$${}_nC_k = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

This formula is very important in a branch of mathematics called combinatorics. It gives the number of ways k elements can be chosen from a set of n elements where order does not matter. In this section, we are concerned with the ability to calculate this quantity.

27. An integer that is calculated using the formula: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Example 2

Calculate: $\binom{7}{3}$.

Solution:

Use the formula for the binomial coefficent,

$$\binom{n}{k} = \frac{n!}{k!\left(n-k\right)!}$$

where $n = 7$ and $k = 3$. After substituting, look for factors to cancel.

$$\begin{aligned}\binom{7}{3} &= \frac{7!}{3!\,(7-3)!}\\ &= \frac{7!}{3!\ 4!}\\ &= \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4!}}{3!\ \cancel{4!}}\\ &= \frac{210}{6}\\ &= 35\end{aligned}$$

Answer: 35

Note: Check the menu system of your calculator for a function that calculates this quantity. Look for the notation ${}_nC_k$ in the probability subsection.

Try this! Calculate: $\binom{8}{5}$.

Answer: 56

(click to see video)

Consider the following binomial raised to the 3rd power in its expanded form:

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

Compare it to the following calculations,

$$\begin{aligned}\binom{3}{0} &= \frac{3!}{0!\,(3-0)!} = \frac{3!}{1\cdot 3!} = 1\\ \binom{3}{1} &= \frac{3!}{1!\,(3-1)!} = \frac{3\cdot 2!}{1\cdot 2!} = 3\\ \binom{3}{2} &= \frac{3!}{2!\,(3-2)!} = \frac{3\cdot 2!}{2!\;1!} = 3\\ \binom{3}{3} &= \frac{3!}{3!\,(3-3)!} = \frac{3!}{3!\;0!} = 1\end{aligned}$$

Notice that there appears to be a connection between these calculations and the coefficients of the expanded binomial. This observation is generalized in the next section.

Binomial Theorem

Consider expanding $(x+2)^5$:

$$(x+2)^5 = (x+2)\,(x+2)\,(x+2)\,(x+2)\,(x+2)$$

One quickly realizes that this is a very tedious calculation involving multiple applications of the distributive property. The **binomial theorem**[28] provides a method of expanding binomials raised to powers without directly multiplying each factor:

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}x^1 y^{n-1}$$

More compactly we can write,

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k \qquad \textit{Binomial theorem}$$

28. Describes the algebraic expansion of binomials raised to powers: $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k.$

Example 3

Expand using the binomial theorem: $(x + 2)^5$.

Solution:

Use the binomial theorem where $n = 5$ and $y = 2$.

$$(x+2)^5 = \binom{5}{0} x^5 2^0 + \binom{5}{1} x^4 2^1 + \binom{5}{2} x^3 2^2 + \binom{5}{3} x^2 2^3 + \binom{5}{4} x^1 2^4$$

Sometimes it is helpful to identify the pattern that results from applying the binomial theorem. Notice that powers of the variable *x* start at 5 and decrease to zero. The powers of the constant term start at 0 and increase to 5. The binomial coefficents can be calculated off to the side and are left to the reader as an exercise.

$$\begin{aligned}(x+2)^5 &= \binom{5}{0} x^5 2^0 + \binom{5}{1} x^4 2^1 + \binom{5}{2} x^3 2^2 + \binom{5}{3} x^2 2^3 + \binom{5}{4} x^1 2^4 \\ &= 1x^5 \times 1 + 5x^4 \times 2 + 10x^3 \times 4 + 10x^2 \times 8 + 5x^1 \times 16 + 1 \times 1 \times \\ &= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32\end{aligned}$$

Answer: $x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$

The binomial may have negative terms, in which case we will obtain an alternating series.

Example 4

Expand using the binomial theorem: $(u - 2v)^4$.

Solution:

Use the binomial theorem where $n = 4, x = u$, and $y = -2v$ and then simplify each term.

$$
\begin{aligned}
(u - 2v)^4 &= \binom{4}{0} u^4(-2v)^0 + \binom{4}{1} u^3(-2v)^1 + \binom{4}{2} u^2(-2v)^2 + \binom{4}{3} u^1(-2 \\
&= \quad 1 \times u^4 \times 1 \quad + \quad 4u^3(-2v) \quad + \quad 6u^2\left(4v^2\right) \quad + \quad 4u\left(-8v^3\right) \\
&= u^4 - 8u^3v + 24u^2v^2 - 32uv^3 + 16v^4
\end{aligned}
$$

Answer: $u^4 - 8u^3v + 24u^2v^2 - 32uv^3 + 16v^4$

Try this! Expand using the binomial theorem: $\left(a^2 - 3\right)^4$.

Answer: $a^8 - 12a^6 + 54a^4 - 108a^2 + 81$

(click to see video)

Next we study the coefficients of the expansions of $\left(x + y\right)^n$ starting with $n = 0$:

$$
\begin{aligned}
(x+y)^0 &= 1\\
(x+y)^1 &= x+y\\
(x+y)^2 &= x^2+2xy+y^2\\
(x+y)^3 &= x^3+3x^2y+3xy^2+y^3\\
(x+y)^4 &= x^4+4x^3y+6x^2y^2+4xy^3+y^4
\end{aligned}
$$

Write the coefficients in a triangular array and note that each number below is the sum of the two numbers above it, always leaving a 1 on either end.

$(x+y)^0$	1
$(x+y)^1$	1 1
$(x+y)^2$	1 2 1
$(x+y)^3$	1 3 3 1
$(x+y)^4$	1 4 6 4 1

This is **Pascal's triangle**[29]; it provides a quick method for calculating the binomial coefficients. Use this in conjunction with the binomial theorem to streamline the process of expanding binomials raised to powers. For example, to expand $(x-1)^6$ we would need two more rows of Pascal's triangle,

$(x+y)^0$	1
$(x+y)^1$	1 1
$(x+y)^2$	1 2 1
$(x+y)^3$	1 3 3 1
$(x+y)^4$	1 4 6 4 1
$(x+y)^5$	1 5 10 10 5 1
$(x+y)^6$	1 6 15 20 15 6 1

The binomial coefficients that we need are in blue. Use these numbers and the binomial theorem to quickly expand $(x-1)^6$ as follows:

$$
\begin{aligned}
(x-1)^6 &= 1x^6(-1)^0 + 6x^5(-1)^1 + 15x^4(-1)^2 + 20x^3(-1)^3 + 15x^2(-1)^4 + 6x(-1)^5 \\
&= x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1
\end{aligned}
$$

29. A triangular array of numbers that correspond to the binomial coefficients.

Example 5

Expand using the binomial theorem and Pascal's triangle: $(2x-5)^4$.

Solution:

From Pascal's triangle we can see that when $n = 4$ the binomial coefficients are 1, 4, 6, 4, and 1. Use these numbers and the binomial theorem as follows:

$$\begin{aligned}(2x-5)^4 &= 1(2x)^4(-5)^0 + 4(2x)^3(-5)^1 + 6(2x)^2(-5)^2 + 4(2x)^1(-5)^3 + 1(2x)^0(- \\ &= 16x^4 \cdot 1 + 4 \cdot 8x^3(-5) + 6 \cdot 4x^2 \cdot 25 + 4 \cdot 2x(-125) + 1 \cdot 625 \\ &= 16x^4 - 160x^3 + 600x^2 - 1{,}000x + 625\end{aligned}$$

Answer: $16x^4 - 160x^3 + 600x^2 - 1{,}000x + 625$

KEY TAKEAWAYS

- To calculate the factorial of a natural number, multiply that number by all natural numbers less than it: $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. Remember that we have defined $0! = 1$.
- The binomial coefficients are the integers calculated using the formula: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
- The binomial theorem provides a method for expanding binomials raised to powers without directly multiplying each factor: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.
- Use Pascal's triangle to quickly determine the binomial coefficients.

TOPIC EXERCISES

PART A: FACTORIALS AND THE BINOMIAL COEFFICIENT

Evaluate.

1. $6!$
2. $4!$
3. $10!$
4. $9!$
5. $\frac{6!}{3!}$
6. $\frac{8!}{4!}$
7. $\frac{13!}{9!}$
8. $\frac{15!}{10!}$
9. $\frac{12!}{3!\,7!}$
10. $\frac{10!}{2!\,5!}$
11. $\frac{n!}{(n-2)!}$
12. $\frac{(n+1)!}{(n-1)!}$
13. a. $4! + 3!$
    b. $(4+3)!$
14. a. $4! - 3!$
    b. $(4-3)!$

Rewrite using factorial notation.

15. $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7$
16. $1 \times 2 \times 3 \times 4 \times 5$

17. $15 \times 14 \times 13$

18. $10 \times 9 \times 8 \times 7$

19. 13

20. 8×7

21. $n\,(n-1)\ (n-2)$

22. $1 \times 2 \times 3 \times \cdots \times n \times\ (n+1)$

Calculate the indicated binomial coefficient.

23. $\binom{6}{4}$

24. $\binom{8}{4}$

25. $\binom{7}{2}$

26. $\binom{9}{5}$

27. $\binom{9}{0}$

28. $\binom{13}{12}$

29. $\binom{n}{0}$

30. $\binom{n}{n}$

31. $\binom{n}{1}$

32. $\binom{n}{n-1}$

33. ${}_{10}C_8$

34. ${}_{5}C_1$

35. ${}_{12}C_{12}$

36. ${}_{10}C_5$

37. ${}_{n}C_{n-2}$

38. ${}_{n}C_{n-3}$

PART B: BINOMIAL THEOREM

Expand using the binomial theorem.

39. $(4x-3)^3$

40. $(2x-5)^3$

41. $\left(\frac{x}{2}+y\right)^3$

42. $\left(x+\frac{1}{y}\right)^3$

43. $(x+3)^4$

44. $(x+5)^4$

45. $(x-4)^4$

46. $(x-2)^4$

47. $\left(x+\frac{2}{y}\right)^4$

48. $\left(\frac{x}{3}-y\right)^4$

49. $(x+1)^5$

50. $(x-3)^5$

51. $(x-2)^6$

52. $(x+1)^6$

53. $(x-1)^7$
54. $(x+1)^7$
55. $(5x-1)^4$
56. $(3x-2)^4$
57. $(4u+v)^4$
58. $(3u-v)^4$
59. $(u-5v)^5$
60. $(2u+3v)^5$
61. $(a-b^2)^5$
62. $(a^2+b^2)^4$
63. $(a^2+b^4)^6$
64. $(a^5+b^2)^5$
65. $\left(x+\sqrt{2}\right)^3$
66. $\left(x-\sqrt{2}\right)^4$
67. $\left(\sqrt{x}-\sqrt{y}\right)^4, x, y \geq 0$
68. $\left(\sqrt{x}+2\sqrt{y}\right)^5, x, y \geq 0$
69. $(x+y)^7$
70. $(x+y)^8$
71. $(x+y)^9$

72. $(x - y)^7$

73. $(x - y)^8$

74. $(x - y)^9$

PART C: DISCUSSION BOARD

75. Determine the factorials of the integers 5, 10, 15, 20, and 25. Which grows faster, the common exponential function $a_n = 10^n$ or the factorial function $a_n = n!$? Explain.

76. Research and discuss the history of the binomial theorem.

ANSWERS

1. 720

3. 3,628,800

5. 120

7. 17,160

9. 15,840

11. $n^2 - n$

13. a. 30
    b. 5,040

15. $7!$

17. $\frac{15!}{12!}$

19. $\frac{13!}{12!}$

21. $\frac{n!}{(n-3)!}$

23. 15

25. 21

27. 1

29. 1

31. n

33. 45

35. 1

37. $\frac{n^2-n}{2}$

39. $64x^3 - 144x^2 + 108x - 27$

41. $\frac{x^3}{8} + \frac{3x^2y}{4} + \frac{3xy^2}{2} + y^3$

43. $x^4 + 12x^3 + 54x^2 + 108x + 81$

45. $x^4 - 16x^3 + 96x^2 - 256x + 256$

47. $x^4 + \frac{8x^3}{y} + \frac{24x^2}{y^2} + \frac{32x}{y^3} + \frac{16}{y^4}$

49. $x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$

51. $x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x + 64$

53. $x^7 - 7x^6 + 21x^5 - 35x^4 + 35x^3 - 21x^2 + 7x - 1$

55. $625x^4 - 500x^3 + 150x^2 - 20x + 1$

57. $256u^4 + 256u^3v + 96u^2v^2 + 16uv^3 + v^4$

59. $u^5 - 25u^4v + 250u^3v^2 - 1{,}250u^2v^3 + 3{,}125uv^4 - 3{,}125v^5$

61. $a^5 - 5a^4b^2 + 10a^3b^4 - 10a^2b^6 + 5ab^8 - b^{10}$

63. $a^{12} + 6a^{10}b^4 + 15a^8b^8 + 20a^6b^{12} + 15a^4b^{16} + 6a^2b^{20} + b^{24}$

65. $x^3 + 3\sqrt{2}x^2 + 6x + 2\sqrt{2}$

67. $x^2 - 4x\sqrt{xy} + 6xy - 4y\sqrt{xy} + y^2$

69. $x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$

71. $x^9 + 9x^8y + 36x^7y^2 + 84x^6y^3 + 126x^5y^4 + 126x^4y^5 + 84x^3y^6 + 36x^2y^7 + 9xy^8 + y^9$

73. $x^8 - 8x^7y + 28x^6y^2 - 56x^5y^3 + 70x^4y^4 - 56x^3y^5 + 28x^2y^6 - 8xy^7 + y^8$

75. Answer may vary

9.5 Review Exercises and Sample Exam

REVIEW EXERCISES

INTRODUCTION TO SEQUENCES AND SERIES

Find the first 5 terms of the sequence as well as the 30th term.

1. $a_n = 5n - 3$
2. $a_n = -4n + 3$
3. $a_n = -10n$
4. $a_n = 3n$
5. $a_n = (-1)^n(n - 2)^2$
6. $a_n = \frac{(-1)^n}{2n-1}$
7. $a_n = \frac{2n+1}{n}$
8. $a_n = (-1)^{n+1}(n - 1)$

Find the first 5 terms of the sequence.

9. $a_n = \frac{nx^n}{2n+1}$
10. $a_n = \frac{(-1)^{n-1}x^{n+2}}{n}$
11. $a_n = 2^n x^{2n}$
12. $a_n = (-3x)^{n-1}$
13. $a_n = a_{n-1} + 5$ where $a_1 = 0$
14. $a_n = 4a_{n-1} + 1$ where $a_1 = -2$
15. $a_n = a_{n-2} - 3a_{n-1}$ where $a_1 = 0$ and $a_2 = -3$
16. $a_n = 5a_{n-2} - a_{n-1}$ where $a_1 = -1$ and $a_2 = 0$

Find the indicated partial sum.

17. 1, 4, 7, 10, 13,...; S_5
18. 3, 1, -1, -3, -5,...; S_5

19. −1, 3, −5, 7, −9,...; S_4

20. $a_n = (-1)^n n^2; S_4$

21. $a_n = -3(n-2)^2; S_4$

22. $a_n = \left(-\frac{1}{5}\right)^{n-2}; S_4$

Evaluate.

23. $\sum_{k=1}^{6} (1-2k)$

24. $\sum_{k=1}^{4} (-1)^k 3k^2$

25. $\sum_{n=1}^{3} \frac{n+1}{n}$

26. $\sum_{n=1}^{7} 5(-1)^{n-1}$

27. $\sum_{k=4}^{8} (1-k)^2$

28. $\sum_{k=-2}^{2} \left(\frac{2}{3}\right)^k$

ARITHMETIC SEQUENCES AND SERIES

Write the first 5 terms of the arithmetic sequence given its first term and common difference. Find a formula for its general term.

29. $a_1 = 6; d = 5$

30. $a_1 = 5; d = 7$

31. $a_1 = 5; d = -3$

32. $a_1 = -\frac{3}{2}; d = -\frac{1}{2}$

33. $a_1 = -\frac{3}{4}; d = -\frac{3}{4}$

34. $a_1 = -3.6; d = 1.2$

35. $a_1 = 7; d = 0$

36. $a_1 = 1; d = 1$

Given the terms of an arithmetic sequence, find a formula for the general term.

37. 10, 20, 30, 40, 50,…

38. −7, −5, −3, −1, 1,…

39. −2, −5, −8, −11, −14,…

40. $-\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}, 1, \ldots$

41. $a_4 = 11$ and $a_9 = 26$

42. $a_5 = -5$ and $a_{10} = -15$

43. $a_6 = 6$ and $a_{24} = 15$

44. $a_3 = -1.4$ and $a_7 = 1$

Calculate the indicated sum given the formula for the general term of an arithmetic sequence.

45. $a_n = 4n - 3; S_{60}$

46. $a_n = -2n + 9; S_{35}$

47. $a_n = \frac{1}{5}n - \frac{1}{2}; S_{15}$

48. $a_n = -n + \frac{1}{4}; S_{20}$

49. $a_n = 1.8n - 4.2; S_{45}$

50. $a_n = -6.5n + 3; S_{35}$

Evaluate.

51. $\sum_{n=1}^{22} (7n - 5)$

52. $\sum_{n=1}^{100} (1 - 4n)$

53. $\sum_{n=1}^{35} \left(\frac{2}{3}n\right)$

54. $\sum_{n=1}^{30} \left(-\frac{1}{4}n + 1\right)$

55. $\sum_{n=1}^{40} (2.3n - 1.1)$

56. $\sum_{n=1}^{300} n$

57. Find the sum of the first 175 positive odd integers.

58. Find the sum of the first 175 positive even integers.

59. Find all arithmetic means between $a_1 = \frac{2}{3}$ and $a_5 = -\frac{2}{3}$

60. Find all arithmetic means between $a_3 = -7$ and $a_7 = 13$.

61. A 5-year salary contract offers $58,200 for the first year with a $4,200 increase each additional year. Determine the total salary obligation over the 5-year period.

62. The first row of seating in a theater consists of 10 seats. Each successive row consists of four more seats than the previous row. If there are 14 rows, how many total seats are there in the theater?

GEOMETRIC SEQUENCES AND SERIES

Write the first 5 terms of the geometric sequence given its first term and common ratio. Find a formula for its general term.

63. $a_1 = 5; r = 2$

64. $a_1 = 3; r = -2$

65. $a_1 = 1; r = -\frac{3}{2}$

66. $a_1 = -4; r = \frac{1}{3}$

67. $a_1 = 1.2; r = 0.2$

68. $a_1 = -5.4; r = -0.1$

Given the terms of a geometric sequence, find a formula for the general term.

69. 4, 40, 400,...

70. −6, −30, −150,...

71. $6, \frac{9}{2}, \frac{27}{8},\ldots$

72. $1, \frac{3}{5}, \frac{9}{25},\ldots$

73. $a_4 = -4$ and $a_9 = 128$

74. $a_2 = -1$ and $a_5 = -64$

75. $a_2 = -\frac{5}{2}$ and $a_5 = -\frac{625}{16}$

76. $a_3 = 50$ and $a_6 = -6{,}250$

77. Find all geometric means between $a_1 = -1$ and $a_4 = 64.$

78. Find all geometric means between $a_3 = 6$ and $a_6 = 162.$

Calculate the indicated sum given the formula for the general term of a geometric sequence.

79. $a_n = 3(4)^{n-1}; S_6$

80. $a_n = -5(3)^{n-1}; S_{10}$

81. $a_n = \frac{3}{2}(-2)^n; S_{14}$

82. $a_n = \frac{1}{5}(-3)^{n+1}; S_{12}$

83. $a_n = 8\left(\frac{1}{2}\right)^{n+2}; S_8$

84. $a_n = \frac{1}{8}(-2)^{n+2}; S_{10}$

Evaluate.

85. $$\sum_{n=1}^{10} 3(-4)^n$$

86. $$\sum_{n=1}^{9} -\frac{3}{5}(-2)^{n-1}$$

87. $\sum_{n=1}^{\infty} -3\left(\frac{2}{3}\right)^n$

88. $\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{4}{5}\right)^{n+1}$

89. $\sum_{n=1}^{\infty} \frac{1}{2}\left(-\frac{3}{2}\right)^n$

90. $\sum_{n=1}^{\infty} \frac{3}{2}\left(-\frac{1}{2}\right)^n$

91. After the first year of operation, the value of a company van was reported to be \$40,000. Because of depreciation, after the second year of operation the van was reported to have a value of \$32,000 and then \$25,600 after the third year of operation. Write a formula that gives the value of the van after the *n*th year of operation. Use it to determine the value of the van after 10 years of operation.

92. The number of cells in a culture of bacteria doubles every 6 hours. If 250 cells are initially present, write a sequence that shows the number of cells present after every 6-hour period for one day. Write a formula that gives the number of cells after the *n*th 6-hour period.

93. A ball bounces back to one-half of the height that it fell from. If dropped from 32 feet, approximate the total distance the ball travels.

94. A structured settlement yields an amount in dollars each year *n* according to the formula $p_n = 12{,}500(0.75)^{n-1}$. What is the total value of a 10-year settlement?

Classify the sequence as arithmetic, geometric, or neither.

95. 4, 9, 14,...

96. 6, 18, 54,...

97. $-1, -\frac{1}{2}, 0, \ldots$

98. 10, 30, 60,...

99. 0, 1, 8,...

100. $-1, \frac{2}{3}, -\frac{4}{9}, \ldots$

Evaluate.

101. $\sum_{n=1}^{4} n^2$

102. $\sum_{n=1}^{4} n^3$

103. $\sum_{n=1}^{32} \left(-4n + 5\right)$

104. $\sum_{n=1}^{\infty} -2\left(\frac{1}{5}\right)^{n-1}$

105. $\sum_{n=1}^{8} \frac{1}{3}(-3)^n$

106. $\sum_{n=1}^{46} \left(\frac{1}{4}n - \frac{1}{2}\right)$

107. $\sum_{n=1}^{22} (3 - n)$

108. $\sum_{n=1}^{31} 2n$

109. $\sum_{n=1}^{28} 3$

110. $\sum_{n=1}^{30} 3(-1)^{n-1}$

111. $\sum_{n=1}^{31} 3(-1)^{n-1}$

BINOMIAL THEOREM

Evaluate.

112. $8!$

113. $11!$

114. $\frac{10!}{2!6!}$

115. $\frac{9!3!}{8!}$

116. $\frac{(n+3)!}{n!}$

117. $\frac{(n-2)!}{(n+1)!}$

Calculate the indicated binomial coefficient.

118. $\binom{7}{4}$

119. $\binom{8}{3}$

120. $\binom{10}{5}$

121. $\binom{11}{10}$

122. $\binom{12}{0}$

123. $\binom{n+1}{n-1}$

124. $\binom{n}{n-2}$

Expand using the binomial theorem.

125. $(x+7)^3$

126. $(x-9)^3$

127. $(2y-3)^4$

128. $(y+4)^4$

129. $(x+2y)^5$

130. $(3x-y)^5$

131. $(u-v)^6$

132. $(u+v)^6$

133. $(5x^2+2y^2)^4$

134. $(x^3-2y^2)^4$

ANSWERS

1. 2, 7, 12, 17, 22; $a_{30} = 147$
3. −10, −20, −30, −40, −50; $a_{30} = -300$
5. −1, 0, −1, 4, −9; $a_{30} = 784$
7. 3, $\frac{5}{2}$, $\frac{7}{3}$, $\frac{9}{4}$, $\frac{11}{5}$; $a_{30} = \frac{61}{30}$
9. $\frac{x}{3}, \frac{2x^2}{5}, \frac{3x^3}{7}, \frac{4x^4}{9}, \frac{5x^5}{11}$
11. $2x^2, 4x^4, 8x^6, 16x^8, 32x^{10}$
13. 0, 5, 10, 15, 20
15. 0, −3, 9, −30, 99
17. 35
19. −5
21. −18
23. −36
25. $\frac{29}{6}$
27. 135
29. 6, 11, 16, 21, 26; $a_n = 5n + 1$
31. 5, 2, −1, −4, −7; $a_n = 8 - 3n$
33. $-\frac{3}{4}, -\frac{3}{2}, -\frac{9}{4}$, -3, $-\frac{15}{4}$; $a_n = -\frac{3}{4}n$
35. 7, 7, 7, 7, 7; $a_n = 7$
37. $a_n = 10n$
39. $a_n = 1 - 3n$
41. $a_n = 3n - 1$
43. $a_n = \frac{1}{2}n + 3$
45. 7,140

47. $\frac{33}{2}$

49. 1,674

51. 1,661

53. 420

55. 1,842

57. 30,625

59. $\frac{1}{3}, 0, -\frac{1}{3}$

61. $333,000

63. 5, 10, 20, 40, 80; $a_n = 5(2)^{n-1}$

65. $1, -\frac{3}{2}, \frac{9}{4}, -\frac{27}{8}, \frac{81}{16}; a_n = \left(-\frac{3}{2}\right)^{n-1}$

67. 1.2, 0.24, 0.048, 0.0096, 0.00192; $a_n = 1.2(0.2)^{n-1}$

69. $a_n = 4(10)^{n-1}$

71. $a_n = 6\left(\frac{3}{4}\right)^{n-1}$

73. $a_1 = \frac{1}{2}(-2)^{n-1}$

75. $a_n = -\left(\frac{5}{2}\right)^{n-1}$

77. 4, −16

79. 4,095

81. 16,383

83. $\frac{255}{128}$

85. 2,516,580

87. −6

89. No sum

91. $v_n = 40{,}000(0.8)^{n-1}; v_{10} = \$5{,}368.71$

93. 96 feet

95. Arithmetic; $d = 5$

97. Arithmetic; $d = \frac{1}{2}$

99. Neither

101. 30

103. −1,952

105. 1,640

107. −187

109. 84

111. 3

113. 39,916,800

115. 54

117. $\frac{1}{n(n+1)(n-1)}$

119. 56

121. 11

123. $\frac{n(n+1)}{2}$

125. $x^3 + 21x^2 + 147x + 343$

127. $16y^4 - 96y^3 + 216y^2 - 216y + 81$

129. $x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5$

131. $u^6 - 6u^5v + 15u^4v^2 - 20u^3v^3 + 15u^2v^4 - 6uv^5 + v^6$

133. $625x^8 + 1{,}000x^6y^2 + 600x^4y^4 + 160x^2y^6 + 16y^8$

SAMPLE EXAM

Find the first 5 terms of the sequence.

1. $a_n = 6n - 15$
2. $a_n = 5(-4)^{n-2}$
3. $a_n = \frac{n-1}{2n-1}$
4. $a_n = (-1)^{n-1} x^{2n}$

Find the indicated partial sum.

5. $a_n = (n-1)\, n^2; S_4$
6. $$\sum_{k=1}^{5} (-1)^k 2^{k-2}$$

Classify the sequence as arithmetic, geometric, or neither.

7. $-1, -\frac{3}{2}, -2, \ldots$
8. $1, -6, 36, \ldots$
9. $\frac{3}{8}, -\frac{3}{4}, \frac{3}{2}, \ldots$
10. $\frac{1}{2}, \frac{1}{4}, \frac{2}{9}, \ldots$

Given the terms of an arithmetic sequence, find a formula for the general term.

11. $10, 5, 0, -5, -10, \ldots$
12. $a_4 = -\frac{1}{2}$ and $a_9 = 2$

Given the terms of a geometric sequence, find a formula for the general term.

13. $-\frac{1}{8}, -\frac{1}{2}, -2, -8, -32, \ldots$
14. $a_3 = 1$ and $a_8 = -32$

Calculate the indicated sum.

15. $a_n = 5 - n; S_{44}$

16. $a_n = (-2)^{n+2}; S_{12}$

17. $\sum_{n=1}^{\infty} 4\left(-\frac{1}{2}\right)^{n-1}$

18. $\sum_{n=1}^{100} \left(2n - \frac{3}{2}\right)$

Evaluate.

19. $\frac{14!}{10!6!}$

20. $\binom{9}{7}$

21. Determine the sum of the first 48 positive odd integers.

22. The first row of seating in a theater consists of 14 seats. Each successive row consists of two more seats than the previous row. If there are 22 rows, how many total seats are there in the theater?

23. A ball bounces back to one-third of the height that it fell from. If dropped from 27 feet, approximate the total distance the ball travels.

Expand using the binomial theorem.

24. $(x - 5y)^4$

25. $(3a + b^2)^5$

ANSWERS

1. −9, −3, 3, 9, 15
3. 0, $\frac{1}{3}$, $\frac{2}{5}$, $\frac{3}{7}$, $\frac{4}{9}$
5. 70
7. Arithmetic
9. Geometric
11. $a_n = 15 - 5n$
13. $a_n = -\frac{1}{8}(4)^{n-1}$
15. −770
17. $\frac{8}{3}$
19. $\frac{1,001}{30}$
21. 2,304
23. 54 feet
25. $243a^5 + 405a^4b^2 + 270a^3b^4 + 90a^2b^6 + 15ab^8 + b^{10}$